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Worst Singularities of Plane Curves of Given Degree

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Abstract We prove that $\frac{2}{d}$, $\frac{2d-3}{(d-1)^2}$, $\frac{2d-1}{d(d-1)}$, $\frac{2d-5}{d^2-3d+1}$ and $\frac{2d-3}{d(d-2)}$ are the smallest log canonical thresholds of reduced plane curves of degree $d \geqslant 3$, and we describe reduced plane curves of degree d whose log canonical thresholds are these numbers. As an application, we prove that $\frac{2}{d}$, $\frac{2d-3}{(d-1)^2}$, $\frac{2d-1}{d(d-1)}$, $\frac{2d-5}{d^2-3d+1}$ and $\frac{2d-3}{d(d-2)}$ are the smallest values of the α-invariant of Tian of smooth surfaces in $\mathbb{P}^3$ of degree $d \geqslant 3$. We also prove that every reduced plane curve of degree $d \geqslant 4$ whose log canonical threshold is smaller than $\frac{5}{2d}$ is GIT-unstable for the action of the group $\mathrm{PGL}_3(\mathbb{C})$, and we describe GIT-semistable reduced plane curves with log canonical thresholds $\frac{5}{2d}$.

Keywords Log canonical threshold · Plane curve · GIT-stability · α-Invariant of Tian · Smooth surface

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All varieties are assumed to be algebraic, projective and defined over $\mathbb{C}$.

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1 Introduction

Let C_d be a *reduced* plane curve in $\mathbb{P}^2$ of degree $d \geqslant 3$, and let P be a point in C_d. The curve C_d can have *any* given plane curve singularity at P provided that its degree d is *sufficiently* big. Thus, it is natural to ask

Question 1.1 What is the *worst* singularity that C_d can have at P?

Denote by m_P the multiplicity of the curve C_d at the point P, and denote by $\mu(P)$ the Milnor number of the point P. If we use m_P to measure the singularity of C_d at the point P, then a union of d lines passing through P is an answer to Question 1.1, since $m_P \leqslant d$, and $m_P = d$ if and only if C_d is a union of d lines passing through P. If we use the Milnor number $\mu(P)$, then the answer would be the same, since $\mu(P) \leqslant (d-1)^2$, and $\mu(P) = (d-1)^2$ if and only if C_d is a union of d lines passing through P. Alternatively, we can use the number

$$\mathrm{lct}_P\big(\mathbb{P}^2, C_d\big) = \sup\Big\{\lambda \in \mathbb{Q} \;\Big|\; \text{the log pair } \big(\mathbb{P}^2, \lambda C_d\big) \text{ is log canonical at } P\Big\},$$

which is known as the *log canonical threshold* of the log pair $(\mathbb{P}^2, C_d)$ at the point P or the log canonical threshold of the curve C_d at the point P (see [4, Definition 6.34]). The smallest $\mathrm{lct}_P(\mathbb{P}^2, C_d)$ when P runs through all points in C_d is usually denoted by $\mathrm{lct}(\mathbb{P}^2, C_d)$. Note that

$$\frac{1}{m_P} \leqslant \mathrm{lct}_P\big(\mathbb{P}^2, C_d\big) \leqslant \frac{2}{m_P}.$$

This is well known (see, [4, Exercise 6.18] and [4, Lemma 6.35]). So, the smaller $\mathrm{lct}_P(\mathbb{P}^2, C_d)$, the worse singularity of the curve C_d at the point P is.

Example 1.2 Suppose that C_d is given by $x_1^{n_1}x_2^{n_2}(x_1^{m_1}+x_2^{m_2}) = 0$ up to analytic change of local coordinates, where m_1 and m_2 are positive integers, and $n_1, n_2 \in \{0, 1\}$. Then

$$\mathrm{lct}_P\big(\mathbb{P}^2, C_d\big) = \min\left\{1, \frac{\frac{1}{m_1}+\frac{1}{m_2}}{1+\frac{n_1}{m_1}+\frac{n_2}{m_2}}\right\}$$

by [8, Proposition 2.2].

Log canonical thresholds of plane curves have been intensively studied (see, for example, [8]). Surprisingly, they give the same answer to Question 1.1 by

Theorem 1.3 ([1, Theorem 4.1]) *One has* $\mathrm{lct}_P(\mathbb{P}^2, C_d) \geqslant \frac{2}{d}$. *Moreover,* $\mathrm{lct}(\mathbb{P}^2, C_d) = \frac{2}{d}$ *if and only if* C_d *is a union of* d *lines that pass through* P.

In this paper we want to address

Question 1.4 What is the *second worst* singularity that C_d can have at P?

To give a *reasonable* answer to this question, we have to disregard m_P by obvious reasons. Thus, we will use the numbers $\mu(P)$ and $\mathrm{lct}_P(\mathbb{P}^2, \mathbb{C}_d)$. For cubic curves, they give the same answer.

Example 1.5 Suppose that $d = 3, m_P < 3$ and P is a singular point of C_3. Then P is a singular point of type $\mathbb{A}_1$, $\mathbb{A}_2$ or $\mathbb{A}_3$. Moreover, if C_3 has singularity of type $\mathbb{A}_3$ at P, then $C_3 = L + C_2$, where C_2 is a smooth conic, and L is a line tangent to C_2 at P. Furthermore, we have

$$\mu(P) = \begin{cases} 1 \text{ if } C_3 \text{ has } \mathbb{A}_1 \text{ singularity at } P, \\ 2 \text{ if } C_3 \text{ has } \mathbb{A}_2 \text{ singularity at } P, \\ 3 \text{ if } C_3 \text{ has } \mathbb{A}_3 \text{ singularity at } P. \end{cases}$$

Similarly, we have

$$\mathrm{lct}_P\big(\mathbb{P}^2, C_3\big) = \begin{cases} 1 \text{ if } C_3 \text{ has } \mathbb{A}_1 \text{ singularity at } P, \\ \dfrac{5}{6} \text{ if } C_3 \text{ has } \mathbb{A}_2 \text{ singularity at } P, \\ \dfrac{3}{4} \text{ if } C_3 \text{ has } \mathbb{A}_3 \text{ singularity at } P. \end{cases}$$

For quartic curves, the numbers $\mu(P)$ and $\mathrm{lct}_P(\mathbb{P}^2, \mathbb{C}_d)$ give different answers to Question 1.4.

Example 1.6 Suppose that $d = 4, m_P < 4$ and P is a singular point of C_4. Going through the list of all possible singularities that C_P can have at P (see, for example, [6]), we obtain

$$\mu(P) = \begin{cases} 6 \text{ if } C_4 \text{ has } \mathbb{D}_6 \text{ singularity at } P, \\ 6 \text{ if } C_4 \text{ has } \mathbb{A}_6 \text{ singularity at } P, \\ 6 \text{ if } C_4 \text{ has } \mathbb{E}_6 \text{ singularity at } P, \\ 7 \text{ if } C_4 \text{ has } \mathbb{A}_7 \text{ singularity at } P, \\ 7 \text{ if } C_4 \text{ has } \mathbb{E}_7 \text{ singularity at } P, \end{cases}$$

and $\mu(P) < 6$ in all remaining cases. Similarly, we get

$$\mathrm{lct}_P\big(\mathbb{P}^2, C_4\big) = \begin{cases} \dfrac{5}{8} \text{ if } C_4 \text{ has } \mathbb{A}_7 \text{ singularity at } P, \\ \dfrac{5}{8} \text{ if } C_4 \text{ has } \mathbb{D}_5 \text{ singularity at } P, \\ \dfrac{3}{5} \text{ if } C_4 \text{ has } \mathbb{D}_6 \text{ singularity at } P, \\ \dfrac{7}{12} \text{ if } C_4 \text{ has } \mathbb{E}_6 \text{ singularity at } P, \\ \dfrac{5}{9} \text{ if } C_4 \text{ has } \mathbb{E}_7 \text{ singularity at } P, \end{cases}$$

and $\mathrm{lct}_P(\mathbb{P}^2, C_4) > \frac{5}{8}$ in all remaining cases.

Recently, Arkadiusz Płoski proved that $\mu(P) \leqslant (d-1)^2 - \lfloor \frac{d}{2} \rfloor$ provided that $m_P < d$. Moreover, he described C_d in the case when $\mu(P) = (d-1)^2 - \lfloor \frac{d}{2} \rfloor$. To present his description, we need

Definition 1.7 The curve C_d is an *even Płoski* curve if d is even, the curve C_d has $\frac{d}{2} \geqslant 2$ irreducible components that are smooth conics passing through P, and all irreducible components of C_d intersect each other pairwise at P with multiplicity 4. The curve C_d is an *odd Płoski* curve if d is odd, the curve C_d has $\frac{d+1}{2} \geqslant 2$ irreducible components that all pass through P, $\frac{d-1}{2}$ irreducible component of the curve C_d are smooth conics that intersect each other pairwise at P with multiplicity 4, and the remaining irreducible component is a line in $\mathbb{P}^2$ that is tangent at P to all other irreducible components. We say that C_d is *Płoski* curve if it is either an even Płoski curve or an odd Płoski curve.

Each Płoski curve has unique singular point. If $d = 4$, then C_4 is a Płoski curve if and only if it has a singular point of type $\mathbb{A}_7$. Thus, if $d = 4$, then $\mu(P) = (d-1)^2 - \lfloor \frac{d}{2} \rfloor = 7$ if and only if either C_4 is a Płoski curve and P is its singular point or C_4 has singularity $\mathbb{E}_7$ at the point P (see Example 1.6). For $d \geqslant 5$, Płoski proved

Theorem 1.8 ([10, Theorem 1.4]) *If $d \geqslant 5$, then $\mu(P) = (d-1)^2 - \lfloor \frac{d}{2} \rfloor$ if and only if C_d is a Płoski curve and P is its singular point.*

This result gives a *very good* answer to Question 1.4. The main goal of this paper is to give an answer to Question 1.4. using log canonical thresholds. Namely, we will prove that

$$\mathrm{lct}_P\big(\mathbb{P}^2, C_d\big) \geqslant \frac{2d-3}{(d-1)^2}$$

provided that $m_P < d$, and we will describe C_d in the case when $\mathrm{lct}_P(\mathbb{P}^2, C_d) = \frac{2d-3}{(d-1)^2}$. To present this description, we need

Definition 1.9 The curve C_d has singularity of type $\mathbb{T}_r$ (resp., $\mathbb{K}_r, \widetilde{\mathbb{T}}_r, \widetilde{\mathbb{K}}_r$) at the point P if the curve C_d can be given by $x_1^r = x_1 x_2^r$ (resp., $x_1^r = x_2^{r+1}, x_2 x_1^{r-1} = x_1 x_2^r, x_2 x_1^{r-1} = x_2^{r+1}$) up to analytic change of coordinates at the point P.

Note that $\mathbb{T}_2 = \mathbb{A}_3, \mathbb{K}_2 = \mathbb{A}_2, \widetilde{\mathbb{T}}_2 = \widetilde{\mathbb{K}}_2 = \mathbb{A}_1, \widetilde{\mathbb{K}}_3 = \mathbb{D}_5, \widetilde{\mathbb{T}}_3 = \mathbb{D}_6, \mathbb{K}_3 = \mathbb{E}_6$ and $\mathbb{T}_3 = \mathbb{E}_7$. Furthermore, since we assume that $d \geqslant 3$, the formula in Example 1.2 gives

$$\mathrm{lct}_P\big(\mathbb{P}^2, C_d\big) = \begin{cases} \dfrac{2d-3}{(d-1)^2} \text{ if } C_d \text{ has } \mathbb{T}_{d-1} \text{singularity at } P, \\ \dfrac{2d-1}{d(d-1)} \text{ if } C_d \text{ has } \mathbb{K}_{d-1} \text{singularity at } P, \\ \dfrac{2d-5}{d^2-3d+1} \text{ if } C_d \text{ has } \widetilde{\mathbb{T}}_{d-1} \text{ singularity at } P, \\ \dfrac{2d-3}{d(d-2)} \text{ if } C \text{ has } \widetilde{\mathbb{K}}_{d-1} \text{ singularity at } P, \end{cases}$$

where $\frac{2}{d} < \frac{2d-3}{(d-1)^2} < \frac{2d-1}{d(d-1)} < \frac{2d-5}{d^2-3d+1} \leqslant \frac{2d-3}{d(d-2)}$. In this paper we will prove

Theorem 1.10 *Suppose that* $d \geqslant 4$ *and* $\mathrm{lct}_P(\mathbb{P}^2, C_d) \leqslant \frac{2d-3}{d(d-2)}$. *Then one of the following holds:*

(1) $m_P = d$,
(2) *the curve* C_d *has singularity of type* $\mathbb{T}_{d-1}$, $\mathbb{K}_{d-1}$, $\widetilde{\mathbb{T}}_{d-1}$ *or* $\widetilde{\mathbb{K}}_{d-1}$ *at the point* P,
(3) $d = 4$ *and* C_d *is a Płoski quartic curve* (*in this case* $\mathrm{lct}_P(\mathbb{P}^2, C_d) = \frac{5}{8}$).

This result describes the *five worst* singularities that C_d can have at the point P. In particular, Theorem 1.10 answers Question 1.4. This answer is very different from the answer given by Theorem 1.8. Indeed, if C_d is a Płoski curve, $d > 3$ and P is its singular point, then

$$\mathrm{lct}_P\big(\mathbb{P}^2, C_d\big) = \frac{5}{2d} > \frac{2d-3}{(d-1)^2}.$$

The proof of Theorem 1.10 implies one result that is interesting on its own. To describe it, let us identify the curve C_d with a point in the space $|\mathcal{O}_{\mathbb{P}^2}(d)|$ that parameterizes all (not necessarily reduced) plane curves of degree d. Since the group $\mathrm{PGL}_3(\mathbb{C})$ acts on $|\mathcal{O}_{\mathbb{P}^2}(d)|$, it is natural to ask whether C_d is GIT-stable (resp., GIT-semistable) for this action or not. For small d, its answer is classical and immediately follows from the Hilbert–Mumford criterion (see [9, Chapter 2.1]).

Example 1.11 ([9, Chapter 4.2]) If $d = 3$, then C_3 is GIT-stable (resp., GIT-semistable) if and only if C_3 is smooth (resp., C_3 has at most $\mathbb{A}_1$ singularities). If $d = 4$, then C_4 is GIT-stable (resp., GIT-semistable) if and only if C_4 has at most $\mathbb{A}_1$ and $\mathbb{A}_2$ singularities (resp., C_4 has at most singular double points and C_4 is not a union of a cubic with an inflectional tangent line).

Paul Hacking, Hosung Kim and Yongnam Lee noticed that the log canonical threshold $\mathrm{lct}(\mathbb{P}^2, C_d)$ and GIT-stability of the curve C_d are closely related. In particular, they proved

Theorem 1.12 ([5, Propositions 10.2 and 10.4], [7, Theorem 2.3]) *If* $\mathrm{lct}(\mathbb{P}^2, C_d) \geqslant \frac{3}{d}$, *then the curve* C_d *is GIT-semistable. If* $d \geqslant 4$ *and* $\mathrm{lct}(\mathbb{P}^2, C_d) > \frac{3}{d}$, *then the curve* C_d *is GIT-stable.*

This gives a *sufficient* condition for the curve C_d to be GIT-stable (resp, GIT-semistable). However, this condition is not a *necessary* condition. Let us give two examples that illustrate this.

Example 1.13 ([13, p. 268], [5, Example 10.5]) Suppose that $d = 5$, the quintic curve C_5 is given by

$$x^5 + \Big(y^2 - xz\Big)^2\Big(\frac{x}{4} + y + z\Big) = x^2\Big(y^2 - xz\Big)\Big(x + 2y\Big),$$

and $P = [0:0:1]$. Then C_5 is irreducible and has singularity $\mathbb{A}_{12}$ at the point P. In particular, it is rational. Furthermore, the curve C_5 is GIT-stable (see, for example, [9, Chapter 4.2]). On the other hand, it follows from Example 1.2 that

$$\mathrm{lct}\big(\mathbb{P}^2, C_5\big) = \mathrm{lct}_P\big(\mathbb{P}^2, C_5\big) = \frac{1}{2} + \frac{1}{13} = \frac{15}{26} < \frac{3}{5}.$$

Example 1.14 Suppose that C_d is a Płoski curve. Let P be its singular point, and let L be a general line in $\mathbb{P}^2$. Then

$$\mathrm{lct}\big(\mathbb{P}^2, C_d + L\big) = \mathrm{lct}\big(\mathbb{P}^2, C_d\big) = \mathrm{lct}_P\big(\mathbb{P}^2, C_d\big) = \frac{5}{2d} < \frac{3}{d}.$$

On the other hand, if d is even, then C_d is GIT-semistable, and $C_d + L$ is GIT-stable. This follows from the Hilbert–Mumford criterion. Similarly, if d is odd, then C_d is GIT-unstable, and $C_d + L$ is GIT-semistable.

In this paper we will prove the following result that complements Theorem 1.12.

Theorem 1.15 *If* $\mathrm{lct}(\mathbb{P}^2, C_d) < \frac{5}{2d}$*, then* C_d *is GIT-unstable. Moreover, if* $\mathrm{lct}(\mathbb{P}^2, C_d) \leqslant \frac{5}{2d}$*, then* C_d *is not GIT-stable. Furthermore, if* $\mathrm{lct}(\mathbb{P}^2, C_d) = \frac{5}{2d}$*, then* C_d *is GIT-semistable if and only if* C_d *is an even Płoski curve.*

Example 1.14 shows that this result is *sharp*. Surprisingly, its proof is very similar to the proof of Theorem 1.10. In fact, we will give a combined proof of both these theorems in Section 3.

In this paper we will also prove one application of Theorem 1.10. To describe it, we need

Definition 1.16 ([12, Appendix A], [3, Definition 1.20]) For a given smooth variety V equipped with an ample $\mathbb{Q}$-divisor H_V, let $\alpha_V^{H_V}: V \to \mathbb{R}_{\geqslant 0}$ be a function defined as

$$\alpha_V^{H_V}(O) = \sup\left\{\lambda \in \mathbb{Q} \;\middle|\; \begin{array}{l}\text{the pair } (V, \lambda D_V) \text{ is log canonical at } O\\ \text{for every effective } \mathbb{Q}\text{-divisor } D_V \sim_{\mathbb{Q}} H_V\end{array}\right\}.$$

Denote its infimum by $\alpha(V, H_V)$.

Let S_d be a smooth surface in $\mathbb{P}^3$ of degree $d \geqslant 3$, let H_{S_d} be its hyperplane section, let O be a point in S_d, and let T_O be the hyperplane section of S_d that is singular at O. Similar to $\mathrm{lct}_P(\mathbb{P}^2, C_d)$, we can define

$$\mathrm{lct}_O\big(S_d, T_O\big) = \sup\Big\{\lambda \in \mathbb{Q} \;\Big|\; \text{the log pair } \big(S_d, \lambda T_O\big) \text{ is log canonical at } O\Big\}.$$

Then $\alpha_{S_d}^{H_{S_d}}(O) \leqslant \mathrm{lct}_O(S_d, T_O)$ by Definition 1.16. Note that T_O is reduced, since the surface S_d is smooth. In this paper we prove

Theorem 1.17 *If* $\alpha_{S_d}^{H_{S_d}}(O) < \frac{2d-3}{d(d-2)}$, *then*

$$\alpha_{S_d}^{H_{S_d}}(O) = \mathrm{lct}_O\big(S_d, T_O\big) \in \left\{\frac{2}{d}, \frac{2d-3}{(d-1)^2}, \frac{2d-1}{d(d-1)}, \frac{2d-5}{d^2-3d+1}\right\}.$$

Similarly, if $\alpha(S_d, H_{S_d}) < \frac{2d-3}{d(d-2)}$, *then*

$$\alpha\big(S_d, H_{S_d}\big) = \inf_{O\in S_d}\left\{\mathrm{lct}_O\big(S_d, T_O\big)\right\} \in \left\{\frac{2}{d}, \frac{2d-3}{(d-1)^2}, \frac{2d-1}{d(d-1)}, \frac{2d-5}{d^2-3d+1}\right\}.$$

If $d = 3$, then we can drop the condition $\alpha_{S_d}^{H_{S_d}}(O) < \frac{2d-3}{d(d-2)}$ in Theorem 1.17, since $\frac{2d-3}{d(d-2)} = 1$ in this case. Thus, Theorem 1.17 implies

Corollary 1.18 ([3, Corollary 1.24]) *Suppose that* $d = 3$. *Then* $\alpha_{S_3}^{H_{S_3}}(O) = \mathrm{lct}_O(S_3, T_O)$.

If $d \geqslant 4$, we cannot drop the condition $\alpha_{S_d}^{H_{S_d}}(O) < \frac{2d-3}{d(d-2)}$ in Theorem 1.17 in general. Let us give two examples that illustrate this.

Example 1.19 Suppose that $d = 4$. Let S_4 be a quartic surface in $\mathbb{P}^3$ that is given by

$$t^3x + t^2yz + xyz(y+z) = 0,$$

and let O be the point $[0 : 0 : 0 : 1]$. Then S_4 is smooth, and T_O has singularity $\mathbb{A}_1$ at O, which implies that $\mathrm{lct}_O(S_4, T_O) = 1$. Let L_y be the line $x = y = 0$, let L_z be the line $x = z = 0$, and let C_2 be the conic $y + z = xt + yz = 0$. Then L_y, L_z and C_2 are contained in S_4, and $O = L_y \cap L_z \cap C_2$. Moreover,

$$L_y + L_z + \frac{1}{2}C_2 \sim 2H_{S_4},$$

because the divisor $2L_y + 2L_z + C_2$ is cut out on S_4 by $tx + yz = 0$. Furthermore, the log pair $(S_4, L_y + L_z + \frac{1}{2}C_2)$ is not log canonical at O, so that $\alpha_{S_4}^{H_{S_4}}(O) < 1$ by Definition 1.16.

Example 1.20 Suppose that $d \geqslant 5$ and T_O has $\mathbb{A}_1$ singularity at O. Then $\mathrm{lct}_O(S_d, T_O) = 1$. Let $f\colon \widetilde{S}_d \to S_d$ be a blow up of the point O. Denote by E its exceptional curve. Then

$$\Big(f^*(H_{S_d}) - \frac{11}{5}E\Big)^2 = 5 - \frac{121}{25} > 0.$$

Hence, it follows from Riemann–Roch theorem there is an integer $n \geqslant 1$ such that the linear system $|f^*(5nH_{S_d}) - 11nE|$ is not empty. Pick a divisor $\widetilde{D}$ in this linear system, and denote by D its image on S_d. Then $(S_d, \frac{1}{5n}D)$ is not log canonical at P, since $\mathrm{mult}_P(D) \geqslant 11n$. On the other hand, $\frac{1}{5n}D \sim_{\mathbb{Q}} H_{S_d}$ by construction, so that $\alpha_{S_d}^{H_d}(O) < 1$ by Definition 1.16.

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2 Preliminaries

In this section, we present results that will be used in the proof of Theorems 1.10, 1.15, 1.17. Let S be a smooth surface, let D be an effective non-zero $\mathbb{Q}$-divisor on the surface S, and let P be a point in the surface S. Write

$$D = \sum_{i=1}^{r} a_i C_i,$$

where each C_i is an irreducible curve on the surface S, and each a_i is a non-negative rational number. Let us recall

Definition 2.1 ([4, § 6]) Let $\pi : \widetilde{S} \to S$ be a birational morphism such that $\widetilde{S}$ is smooth. Then π is a composition of blow ups of smooth points. For each C_i, denote by $\widetilde{C}_i$ its proper transform on the surface $\widetilde{S}$. Let $F_1, \ldots, F_n$ be π-exceptional curves. Then

$$K_{\widetilde{S}} + \sum_{i=1}^{r} a_i \widetilde{C}_i + \sum_{j=1}^{n} b_j F_j \sim_{\mathbb{Q}} \pi^*\big(K_S + D\big)$$

for some rational numbers $b_1, \ldots, b_n$. Suppose, in addition, that $\sum_{i=1}^{r} \widetilde{C}_i + \sum_{j=1}^{n} F_j$ is a divisor with simple normal crossings. Then the log pair (S, D) is said to be *log canonical* at P if and only if the following two conditions are satisfied:

- $a_i \leqslant 1$ for every C_i such that $P \in C_i$,
- $b_j \leqslant 1$ for every F_j such that $\pi(F_j) = P$.

Similarly, the log pair (S, D) is said to be *Kawamata log terminal* at P if and only if $a_i < 1$ for every C_i such that $P \in C_i$, and $b_j < 1$ for every F_j such that $\pi(F_j) = P$.

Using just this definition, one can easily prove

Lemma 2.2 *Suppose that $r = 3$, $P \in C_1 \cap C_2 \cap C_3$, the curves C_1, C_2 and C_3 are smooth at P, $a_1 < 1$, $a_2 < 1$ and $a_3 < 1$. Moreover, suppose that both curves C_1 and C_2 intersect the curve C_3 transversally at P. Furthermore, suppose that (S, D) is not Kawamata log terminal at P. Put $k = \mathrm{mult}_P(C_1 \cdot C_2)$. Then $k(a_1 + a_2) + a_3 \geqslant k + 1$.*

Proof Put $S_0 = S$ and consider a sequence of blow ups

$$S_k \xrightarrow{\pi_k} S_{k-1} \xrightarrow{\pi_{k-1}} \cdots \xrightarrow{\pi_3} S_2 \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} S_0,$$

where each π_j is the blow up of the intersection point of the proper transforms of the curves C_1 and C_2 on the surface S_{j-1} that dominates P (such point exists, since $k = \mathrm{mult}_P(C_1 \cdot C_2)$). For each π_j, denote by E_j^k the proper transform of its exceptional curve on S_k. For each C_i, denote by C_i^k its proper transform on the surface S_k. Then

$$K_{S_k} + \sum_{i=1}^{n} a_i C_i^k + \sum_{j=1}^{k} \Big(j\big(a_1 + a_2\big) + a_3 - j\Big) E_j^k \sim_{\mathbb{Q}} (\pi_1 \circ \pi_2 \circ \cdots \circ \pi_k)^* \Big(K_S + D\Big),$$

and $\sum_{i=1}^{n} C_i^k + \sum_{j=1}^{k} E_j$ is a simple normal crossing divisor in every point of $\cup_{j=1}^{k} E_j$. Thus, it follows from Definition 2.1 that there exists $l \in \{1, \ldots, k\}$ such that $l(a_1 + a_2) + a_3 \geqslant l + 1$, because (S, D) is not Kawamata log terminal at P. If $l = k$, then we are done. So, we may assume that $l < k$. If $k(a_1 + a_2) + a_3 < k + 1$, then $a_1 + a_2 < 1 + \frac{1}{k} - a_3 \frac{1}{k}$, which implies that

$$\begin{aligned} l + 1 \leqslant l\big(a_1 + a_2\big) + a_3 &< \left(l + \frac{l}{k} - a_3 \frac{l}{k}\right) + a_3 = l + \frac{l}{k} + a_3 \left(1 - \frac{l}{k}\right) \\ &\leqslant l + \frac{l}{k} + \left(1 - \frac{l}{k}\right) = l + 1, \end{aligned}$$

because $a_3 < 1$. Thus, the obtained contradiction shows that $k(a_1 + a_2) + a_3 \geqslant k + 1$. □

Corollary 2.3 *Suppose that* $r = 2$, $P \in C_1 \cap C_2$, *the curves* C_1 *and* C_2 *are smooth at* P, $a_1 < 1$ *and* $a_2 < 1$. *Put* $k = \mathrm{mult}_P(C_1 \cdot C_2)$. *If* (S, D) *is not Kawamata log terminal at* P, *then* $k(a_1 + a_2) \geqslant k + 1$.

The log pair (S, D) is called *log canonical* if it is log canonical at every point of S. Similarly, the log pair (S, D) is called *Kawamata log terminal* if it is Kawamata log terminal at every point of the surface S.

Remark 2.4 Let R be any effective $\mathbb{Q}$-divisor on S such that $R \sim_{\mathbb{Q}} D$ and $R \neq D$. Put

$$D_\epsilon = (1 + \epsilon) D - \epsilon R,$$

where ϵ is a non-negative rational number. Then $D_\epsilon \sim_{\mathbb{Q}} D$. Moreover, since $R \neq D$, there exists the greatest rational number $\epsilon_0 \geqslant 0$ such that the divisor D_{ϵ_0} is effective. Then $\mathrm{Supp}(D_{\epsilon_0})$ does not contain at least one irreducible component of $\mathrm{Supp}(R)$. Moreover, if (S, D) is not log canonical at P, and (S, R) is log canonical at P, then (S, D_{ϵ_0}) is not log canonical at P by Definition 2.1, because

$$D = \frac{1}{1 + \epsilon_0} D_{\epsilon_0} + \frac{\epsilon_0}{1 + \epsilon_0} R$$

and $\frac{1}{1+\epsilon_0} + \frac{\epsilon_0}{1+\epsilon_0} = 1$. Similarly, if the log pair (S, D) is not Kawamata log terminal at P, and (S, R) is Kawamata log terminal at P, then (S, D_{ϵ_0}) is not Kawamata log terminal at P.

The following result is well known.

Lemma 2.5 ([4, Exercise 6.18]) *If (S, D) is not log canonical at P, then* $\mathrm{mult}_P(D) > 1$. *Similarly, if (S, D) is not Kawamata log terminal at P, then* $\mathrm{mult}_P(D) \geqslant 1$.

Combining with

Lemma 2.6 ([4, Lemma 5.36]) *Suppose that S is a smooth surface in $\mathbb{P}^3$, and $D \sim_{\mathbb{Q}} H_S$, where H_S is a hyperplane section of S. Then each a_i does not exceed* 1.

Lemma 2.5 gives

Corollary 2.7 *Suppose that S is a smooth surface in $\mathbb{P}^3$, and $D \sim_{\mathbb{Q}} H_S$, where H_S is a hyperplane section of S. Then (S, D) is log canonical outside of finitely many points.*

The following result is a special case of a much more general result, which is known as Shokurov's connectedness principle (see, for example, [4, Theorem 6.3.2]).

Lemma 2.8 ([11, Theorem 6.9]) *If $-(K_S + D)$ is big and nef, then the locus where (S, D) is not Kawamata log terminal is connected.*

Corollary 2.9 *Let C_d be a reduced curve in $\mathbb{P}^2$ of degree d, and let O and Q be two points in C_d such that $O \neq Q$. If* $\mathrm{lct}_O(\mathbb{P}^2, C_d) < \frac{3}{d}$, *then* $\mathrm{lct}_Q(\mathbb{P}^2, C_d) \geqslant \frac{3}{d}$.

Let $\pi_1 \colon S_1 \to S$ be a blow up of the point P, and let E_1 be the π_1-exceptional curve. Denote by D^1 the proper transform of the divisor D on the surface S_1 via π_1. Then the log pair $(S_1, D^1 + (\mathrm{mult}_P(D) - 1)E_1)$ is often called *the log pull back* of the log pair (S, D), because

$$K_{S_1} + D^1 + \Big(\mathrm{mult}_P(D) - 1\Big)E_1 \sim_{\mathbb{Q}} \pi_1^*\big(K_S + D\big).$$

This $\mathbb{Q}$-rational equivalence implies that the log pair (S, D) is not log canonical at P provided that $\mathrm{mult}_P(D) > 2$. Similarly, if $\mathrm{mult}_P(D) \geqslant 2$, then the singularities of the log pair (S, D) are not Kawamata log terminal at the point P.

Remark 2.10 The log pair (S, D) is log canonical at P if and only if $(S_1, D^1 + (\mathrm{mult}_P(D) - 1)E_1)$ is log canonical at every point of the curve E_1. Similarly, the log pair (S, D) is Kawamata log terminal at P if and only if $(S_1, D^1 + (\mathrm{mult}_P(D) - 1)E_1)$ is Kawamata log terminal at every point of the curve E_1.

Let Z be an irreducible curve on S that contains P. Suppose that Z is smooth at P, and Z is not contained in $\mathrm{Supp}(D)$. Let μ be a non-negative rational number. The following result is a very special case of a much more general result known as *Inversion of Adjunction* (see, for example, [11, § 3.4] or [4, Theorem 6.29]).

Theorem 2.11 ([11, Corollary 3.12], [4, Exercise 6.31], [2, Theorem 7]) *Suppose that the log pair* $(S, \mu Z+D)$ *is not log canonical at* P *and* $\mu \leqslant 1$. *Then* $\mathrm{mult}_P(D\cdot Z) > 1$.

This result implies

Theorem 2.12 *Suppose that* $(S, \mu Z + D)$ *is not Kawamata log terminal at* P, *and* $\mu < 1$. *Then* $\mathrm{mult}_P(D \cdot Z) > 1$.

Proof The log pair $(S, Z+D)$ is not log canonical at P, because $\mu < 1$, and $(S, \mu Z+D)$ is not Kawamata log terminal at P. Then $\mathrm{mult}_P(D \cdot Z) > 1$ by Theorem 2.11. □

Theorems 2.11 and 2.12 imply

Lemma 2.13 *If* (S, D) *is not log canonical at* P *and* $\mathrm{mult}_P(D) \leqslant 2$, *then there exists a* unique *point in* E_1 *such that* $(S_1, D^1 + (\mathrm{mult}_P(D)-1)E_1)$ *is not log canonical at it. Similarly, if* (S, D) *is not Kawamata log terminal at* P, *and* $\mathrm{mult}_P(D) < 2$, *then there exists a* unique *point in* E_1 *such that* $(S_1, D^1 + (\mathrm{mult}_P(D) - 1)E_1)$ *is not Kawamata log terminal at it.*

Proof If $\mathrm{mult}_P(D) \leqslant 2$ and $(S_1, D^1 + (\lambda\mathrm{mult}_P(D) - 1)E_1)$ is not log canonical at two distinct points P_1 and $\widetilde{P}_1$, then

$$2 \geqslant \mathrm{mult}_P\big(D\big) = D^1 \cdot E_1 \geqslant \mathrm{mult}_{P_1}\Big(D^1 \cdot E_1\Big) + \mathrm{mult}_{\widetilde{P}_1}\Big(D^1 \cdot E_1\Big) > 2$$

by Theorem 2.11. By Remark 2.10, this proves the first assertion. Similarly, we can prove the second assertion using Theorem 2.12 instead of Theorem 2.11. □

The following result can be proved similarly to the proof of Lemma 2.5. Let us show how to prove it using Theorem 2.12.

Lemma 2.14 *Suppose that* (S, D) *is not Kawamata log terminal at* P, *and* (S, D) *is Kawamata log terminal in a punctured neighbourhood of the point* P, *then* $\mathrm{mult}_P(D) > 1$.

Proof By Remark 2.10, the log pair $(S_1, D^1 + (\mathrm{mult}_P(D) - 1)E_1)$ is not Kawamata log terminal at some point $P_1 \in E_1$. Moreover, if $\mathrm{mult}_P(D) < 2$, then $(S_1, D^1 + (\mathrm{mult}_P(D) - 1)E_1)$ is Kawamata log terminal at a punctured neighbourhood of the point P_1. Thus, if $\mathrm{mult}_P(D) \leqslant 1$, then $\mathrm{mult}_P\big(D\big) = D^1 \cdot E_1 > 1$ by Theorem 2.12, which is absurd. □

Let Z_1 and Z_2 be two irreducible curves on the surface S such that Z_1 and Z_2 are not contained in $\mathrm{Supp}(D)$. Suppose that $P \in Z_1 \cap Z_2$, the curves Z_1 and Z_2 are smooth at P, the curves Z_1 and Z_2 intersect each other transversally at P. Let μ_1 and μ_2 be non-negative rational numbers.

Theorem 2.15 ([2, Theorem 13]) *Suppose that the log pair* $(S, \mu_1 Z_1 + \mu_2 Z_2 + D)$ *is not log canonical at the point* P, *and* $\mathrm{mult}_P(D) \leqslant 1$. *Then either* $\mathrm{mult}_P(D \cdot Z_1) > 2(1 - \mu_2)$ *or* $\mathrm{mult}_P(D \cdot Z_2) > 2(1 - \mu_1)$ *(or both).*

This result implies

Theorem 2.16 *Suppose that* $(S, \mu_1 Z_1 + \mu_2 Z_2 + D)$ *is not Kawamata log terminal at* P, *and* $\mathrm{mult}_P(D) < 1$. *Then either* $\mathrm{mult}_P(D \cdot Z_1) \geqslant 2(1-\mu_2)$ *or* $\mathrm{mult}_P(D \cdot Z_2) \geqslant 2(1-\mu_1)$ *(or both).*

Proof Let λ be a rational number such that

$$\frac{1}{\mathrm{mult}_P(D)} \geqslant \lambda > 1.$$

Then $(S, D + \lambda\mu_1 Z_1 + \lambda\mu_2 Z_2)$ is not log canonical at P. Now it follows from Theorem 2.15 that either $\mathrm{mult}_P(D\cdot Z_1) > 2(1-\lambda\mu_2)$ or $\mathrm{mult}_P(D\cdot Z_2) > 2(1-\lambda\mu_1)$ (or both). Since we can choose λ to be as close to 1 as we wish, this implies that either $\mathrm{mult}_P(D\cdot Z_1) \geqslant 2(1-\mu_2)$ or $\mathrm{mult}_P(D\cdot Z_2) \geqslant 2(1-\mu_1)$ (or both). □

3 Reduced Plane Curves

The purpose of this section is to prove Theorems 1.10 and 1.15. Let C_d be a *reduced* plane curve in $\mathbb{P}^2$ of degree $d \geqslant 4$, and let P be a point in C_d. Put $\lambda_1 = \frac{2d-3}{d(d-2)}$ and $\lambda_2 = \frac{5}{2d}$. To prove Theorem 1.10, we have to show that if the log pair $(\mathbb{P}^2, \lambda_1 C_d)$ is not Kawamata log terminal at the point P, then one of the following assertions hold:

- $\mathrm{mult}_P(C_d) = d$,
- C_d has singularity $\mathbb{T}_{d-1}$, $\mathbb{K}_{d-1}$, $\widetilde{\mathbb{T}}_{d-1}$ or $\widetilde{\mathbb{K}}_{d-1}$ at the point P,
- $d = 4$ and C_4 is a Płoski curve (see Definition 1.7).

To prove Theorem 1.15, we have to show that if $(\mathbb{P}^2, \lambda_2 C_d)$ is not Kawamata log terminal, then either C_d is GIT-unstable or C_d is an even Płoski curve. In the rest of the section, we will do this simultaneously. Let us start with few preliminary results.

Lemma 3.1 *The following inequalities hold:*

(i) $\lambda_1 < \frac{2}{d-1}$,
(ii) $\lambda_1 < \frac{2k+1}{kd}$ *for every positive integer* $k \leqslant d-3$,
(iii) if $d \geqslant 5$, *then* $\lambda_1 < \frac{2k+1}{kd+1}$ *for every positive integer* $k \leqslant d-4$,
(iv) $\lambda_1 < \frac{3}{d}$,
(v) $\lambda_1 < \frac{2}{d-2}$,
(vi) $\lambda_1 < \frac{6}{3d-4}$,
(vii) if $d \geqslant 5$, *then* $\lambda_1 < \lambda_2$.

Proof The equality $\frac{2}{d-1} = \lambda_1 + \frac{d-3}{d(d-1)(d-2)}$ implies (i). Let k be positive integer. If $k = d-2$, then $\lambda_1 = \frac{2k+1}{kd}$. This implies (ii), because $\frac{2k+1}{kd} = \frac{2}{d} + \frac{1}{kd}$ is a decreasing function on k for $k \geqslant 1$. Similarly, if $k = d-4$ and $d \geqslant 4$, then $\lambda_1 = \frac{2k+1}{kd+1} - \frac{3}{d(d-2)(d^2-4d+1)} < \frac{2k+1}{kd+1}$. This implies (iii), since $\frac{2k+1}{kd+1} = \frac{2}{d} + \frac{d-2}{d(kd+1)}$ is a decreasing function on k for $k \geqslant 1$. The equality $\lambda_1 = \frac{3}{d} - \frac{d-3}{d(d-2)}$ proves (iv). Note that (v) follows from (i). Since $\frac{6}{3d-4} > \frac{2}{d-1}$, (vi) also follows from (i). Finally, the equality $\lambda_1 = \lambda_2 - \frac{d-4}{2d(d-2)}$ implies (vii). □

We may assume that $P = [0 : 0 : 1]$. Then C_d is given by $F_d(x, y, z) = 0$, where $F_d(x, y, z)$ is a homogeneous polynomial of degree d. Put $x_1 = \frac{x}{z}$, $x_2 = \frac{y}{z}$ and $f_d(x_1, x_2) = F_d(x_1, x_2, 1)$. Put $m_0 = \mathrm{mult}_P(C_d)$. Then

$$f_d\big(x_1, x_2\big) = \sum_{\substack{i\geqslant 0, j\geqslant 0,\\ m_0\leqslant i+j\leqslant d}} \epsilon_{ij} x_1^i x_2^j,$$

where each ϵ_{ij} is a complex number. For every positive integers a and b, define the weight of the polynomial $f_d(x_1, x_2)$ as

$$\mathrm{wt}_{(a,b)}\big(f_d(x_1, x_2)\big) = \min\Big\{ai + bj \;\Big|\; \epsilon_{ij} \neq 0\Big\}.$$

Then the Hilbert–Mumford criterion implies

Lemma 3.2 ([7, Lemma 2.1]) *Let a and b be positive integers. If C_d is GIT-stable, then*

$$\mathrm{wt}_{(a,b)}\Big(f_d\big(x_1, x_2\big)\Big) < \frac{d}{3}\big(a + b\big).$$

Similarly, if C_d is GIT-semistable, then $\mathrm{wt}_{(a,b)}(f_d(x_1, x_2)) \leqslant \frac{d}{3}(a + b)$.

Let $f_1 : S_1 \to \mathbb{P}^2$ be a blow up of the point P. Denote by E_1 the exceptional curve of the blow up f_1. Denote by C_d^1 the proper transform on S_1 of the curve C_d.

Lemma 3.3 *If* $\mathrm{mult}_P(C_d) > \frac{2d}{3}$, *then C_d is GIT-unstable. Let O be a point in E_1. If*

$$\mathrm{mult}_P(C_d) + \mathrm{mult}_O(C_d^1) > d,$$

then C_d is GIT-unstable.

Proof Since $\mathrm{mult}_P(C_d) = \mathrm{wt}_{(1,1)}(f_d(x_1, x_2))$, the first assertion follows from Lemma 3.2. Let us prove the second assertion. We may assume that O is contained in the proper transform of the line in $\mathbb{P}^2$ that is given by $x = 0$. Then

$$\mathrm{wt}_{(2,1)}\big(f_d(x_1, x_2)\big) = \mathrm{mult}_P(C_d) + \mathrm{mult}_O(C_d^1),$$

so that the second assertion also follows from Lemma 3.2. □

Now we are ready to prove Theorems 1.10 and 1.15. To do this, we may assume that C_d is not a union of d lines passing through the point P. Suppose, in addition, that

- **(A)** either $(\mathbb{P}^2, \lambda_1 C_d)$ is not Kawamata log terminal at P,
- **(B)** or $(\mathbb{P}^2, \lambda_2 C_d)$ is not Kawamata log terminal at P.

We will show that (**A**) implies that either C_d has singularity $\mathbb{T}_{d-1}, \mathbb{K}_{d-1}, \widetilde{\mathbb{T}}_{d-1}$ or $\widetilde{\mathbb{K}}_{d-1}$ at the point P, or C_d is a Płoski quartic curve. Similarly, we will show that (**B**) implies that either C_d is GIT-unstable (i.e. C_d is not GIT-semistable), or C_d is an even Płoski curve. If (**A**) holds, let $\lambda = \lambda_1$. If (**B**) holds, let $\lambda = \lambda_2$.

If $d = 4$, then $\lambda_1 = \lambda_2$. If $d \geqslant 5$, then $\lambda_1 < \lambda_2$ by Lemma 3.1(vii). Since C_d is reduced and $\lambda < 1$, the log pair $(\mathbb{P}^2, \lambda C_d)$ is Kawamata log terminal outside of finitely many points. Thus, it is Kawamata log terminal outside of P by Lemma 2.8.

Then the log pair $(S_1, \lambda C_d^1 + (\lambda m_0 - 1)E_1)$ is not Kawamata log terminal at some point $P_1 \in E_1$ by Remark 2.10. Note that we have

$$K_{S_1} + \lambda C_d^1 + \Big(\lambda m_0 - 1\Big)E_1 \sim_{\mathbb{Q}} f_1^*\Big(K_{\mathbb{P}^2} + \lambda C_d\Big).$$

Let $f_2\colon S_2 \to S_1$ be a blow up of the point P_1, and let E_2 be its exceptional curve. Denote by C_d^2 the proper transform on S_2 of the curve C_d, and denote by E_1^2 the proper transform on S_2 of the curve E_1. Put $m_1 = \mathrm{mult}_{P_1}(C_d^1)$. Then

$$K_{S_2} + \lambda C_d^2 + \big(\lambda m_0 - 1\big)E_1^2 + \big(\lambda(m_0 + m_1) - 2\big)E_2 \sim_{\mathbb{Q}} f_2^*\Big(K_{S_1} + \lambda C_d^1 + \big(\lambda m_0 - 1\big)E_1\Big).$$

By Remark 2.10, the log pair $(S_2, \lambda C_d^2 + (\lambda m_0 - 1)E_1^2 + (\lambda(m_0 + m_1) - 2)E_2)$ is not Kawamata log terminal at some point $P_2 \in E_2$. Let $f_3\colon S_3 \to S_2$ be a blow up of this point, and let E_3 be the f_3-exceptional curve. Denote by C_d^3 the proper transform on S_3 of the curve C_d, denote by E_1^3 the proper transform on S_3 of the curve E_1, and denote by E_2^3 the proper transform on S_3 of the curve E_2. Put $m_2 = \mathrm{mult}_{P_2}(C_d^2)$. Then

$$\begin{aligned} &K_{S_3} + \lambda_2 C_d^3 + \big(\lambda_2 m_0 - 1\big)E_1^3 \\ &\quad + \big(\lambda_2(m_0 + m_1) - 2\big)E_2^3 + \big(\lambda_2(2m_0 + m_1 + m_2) - 4\big)E_3 \sim_{\mathbb{Q}} \\ &\quad \sim_{\mathbb{Q}} f_3^*\Big(K_{S_2} + \lambda_2 C_d^2 + \big(\lambda_2 m_0 - 1\big)E_1^2 + \big(\lambda_2(m_0 + m_1) - 2\big)E_2\Big). \end{aligned}$$

Thus, the log pair $(S_3, \lambda_2 C_d^3 + (\lambda_2 m_0 - 1)E_1^3 + (\lambda_2(m_0 + m_1) - 2)E_2^3 + (\lambda_2(2m_0 + m_1 + m_2) - 4)E_3)$ is not Kawamata log terminal at some point $P_3 \in E_3$ by Remark 2.10. Note that the divisor $\lambda_2 C_d^3 + (\lambda_2 m_0 - 1)E_1^3 + (\lambda_2(m_0 + m_1) - 2)E_2^3 + (\lambda_2(2m_0 + m_1 + m_2) - 4)E_3$ is effective by Lemma 2.5.

Lemma 3.4 *One has* $\lambda m_0 < 2$.

Proof Since C_d is not a union of d lines passing through P, we have $m_0 \leqslant d - 1$. Thus, if (**A**) holds, then $\lambda m_0 < 2$ by Lemma 3.1(i), because $d \geqslant 4$. Similarly, if (**B**) holds, then $m_0 \leqslant \frac{2d}{3}$ by Lemma 3.3, which implies that $\lambda m_0 \leqslant \frac{10}{6} < 2$. ◻

Thus, the log pair $(S_1, \lambda C_d^1 + (\lambda m_0 - 1)E_1)$ is Kawamata log terminal outside of P_1 by Lemma 2.13. Note that $P_1 \in C_d^1$, because the log pair $(S_1, (\lambda m_0 - 1)E_1)$ is Kawamata log terminal at P_1. Thus, we have $m_1 > 0$.

Let L be the line in $\mathbb{P}^2$ whose proper transform on S_1 contains the point P_1. Such a line exists and it is unique. By a suitable linear change of coordinates, we may assume that L is given by $x = 0$. Denote by L^1 the proper transform of the line L on the surface S_1.

Lemma 3.5 *Suppose that* (***A***) *holds and* $m_0 = d-1$. *Then* C_d *has singularity* $\mathbb{K}_{d-1}$, $\widetilde{\mathbb{K}}_{d-1}$, $\mathbb{T}_{d-1}$ *or* $\widetilde{\mathbb{T}}_{d-1}$ *at the point* P.

Proof Suppose that L is not an irreducible component of the curve C_d. Then $m_0+m_1 \leqslant d$, because

$$d-1-m_0 = C_d^1 \cdot L^1 \geqslant m_1.$$

Since $m_0 = d-1$, this gives $m_1 = 1$. Then $P_1 \in C_d^1$ and the curve C_d^1 is smooth at P_1. Put $k = \mathrm{mult}_{P_1}(C_d^1 \cdot E_1)$. Applying Corollary 2.3 to the log pair $(S_1, \lambda_1 C_d^1 + (\lambda_1 m_0 - 1)E_1)$ at the point P_1, we get

$$k\lambda_1 m_0 \geqslant k+1,$$

which gives $\lambda_1 \geqslant \frac{2k+1}{kd}$. Then $k \geqslant d-2$ by Lemma 3.1(ii). Since

$$k \leqslant C_d^1 \cdot E_1 = m_0 = d-1,$$

either $k = d-1$ or $k = d-2$. If $k = d-1$, then C_d has singularity $\mathbb{K}_{d-1}$ at P. If $k = d-2$, then C_d has singularity $\widetilde{\mathbb{K}}_{d-1}$ at the point P.

To complete the proof, we may assume that L is an irreducible component of the curve C_d. Then $C_d = L + C_{d-1}$, where C_{d-1} is a reduced curve in $\mathbb{P}^2$ of degree $d-1$ such that L is not its irreducible component. Denote by C_{d-1}^1 its proper transform on S_1. Put $n_0 = \mathrm{mult}_P(C_{d-1})$ and $n_1 = \mathrm{mult}_{P_1}(C_{d-1}^1)$. Then $n_0 = m_0 - 1 = d-2$ and $n_1 = m_1 - 1$. This implies that $P_1 \in C_{d-1}^1$, since the log pair $(S_1, \lambda_1 L^1 + (\lambda_1 m_0 - 1)E_1)$ is Kawamata log terminal at P. Hence, $n_1 \geqslant 1$. One the other hand, we have

$$d-1-n_0 = C_{d-1}^1 \cdot L^1 \geqslant n_1,$$

which implies that $n_0 + n_1 \leqslant d-1$. Then $n_1 = 1$, since $n_0 = d-2$.

We have $P_1 \in C_{d-1}^1$ and C_{d-1}^1 is smooth at P_1. Moreover, since

$$1 = d-1-n_0 = L^1 \cdot C_{d-1}^1 \geqslant n_1 = 1,$$

the curve C_{d-1}^1 intersects the curve L^1 transversally at the point P_1. Put $k = \mathrm{mult}_{P_1}(C_{d-1}^1 \cdot E_1)$. Then $k \geqslant 1$. Applying Lemma 2.2 to the log pair $(S_1, \lambda_1 C_{d-1}^1 + \lambda_1 L^1 + (\lambda_1(n_0+1)-1)E_1)$ at the point P_1, we get

$$k\Big(\lambda_1(n_0+2)-1\Big) + \lambda_1 \geqslant k+1.$$

Then $\lambda_1 \geqslant \frac{2k+1}{kd+1}$. Then $k \geqslant d-3$ by Lemma 3.1(iii). Since

$$k \leqslant E_1 \cdot C^1_{d-1} = n_0 = d-2,$$

either $k = d-2$ or $k = d-3$. In the former case, C_d has singularity $\mathbb{T}_{d-1}$ at the point P. In the latter case, C_d has singularity $\widetilde{\mathbb{T}}_{d-1}$ at the point P. ◻

Lemma 3.6 *Suppose that* (***A***) *holds and* $m_0 \leqslant d-2$. *Then the line* L *is not an irreducible component of the curve* C_d.

Proof Suppose that L is an irreducible component of the curve C_d. Let us see for a contradiction. Put $C_d = L + C_{d-1}$, where C_{d-1} is a reduced curve in $\mathbb{P}^2$ of degree $d-1$ such that L is not its irreducible component. Denote by C^1_{d-1} its proper transform on S_1. Put $n_0 = \mathrm{mult}_P(C_{d-1})$ and $n_1 = \mathrm{mult}_{P_1}(C^1_{d-1})$. Then $(S_1, (\lambda_1(n_0+1)-1)E_1 + \lambda_1 L^1 + \lambda_1 C^1_{d-1})$ is not Kawamata log terminal at P_1 and is Kawamata log terminal outside of the point P_1. In particular, $n_1 \neq 0$, because $(S_1, (\lambda_1(n_0+1)-1)E_1 + \lambda_1 L^1)$ is Kawamata log terminal at P_1. On the other hand,

$$d-1-n_0 = L^1 \cdot C^1_{d-1} \geqslant n_1,$$

which implies that $n_0 + n_1 \leqslant d-1$. Furthermore, we have $n_0 = m_0 - 1 \leqslant d-3$.

Since $n_0 + n_1 \geqslant 2n_1$, we have $n_1 \leqslant \frac{d-1}{2}$. Then $\lambda n_1 < 1$ by Lemma 3.1(i). Thus, we can apply Theorem 2.16 to the log pair $(S_1, (\lambda_1(n_0+1)-1)E_1 + \lambda_1 L^1 + \lambda_1 C^1_{d-1})$ at the point P_1. This gives either

$$\lambda_1\big(d-1-n_0\big) = \lambda_1 C^1_{d-1} \cdot L^1 \geqslant 2\Big(2 - \lambda_1\big(n_0+1\big)\Big)$$

or

$$\lambda_1 n_0 = \lambda_1 C^1_{d-1} \cdot E_1 \geqslant 2\big(1-\lambda_1\big)$$

(or both). In the former case, we have $\lambda_1(d+1+n_0) \geqslant 4$. In the latter case, we have $\lambda_1(n_0+2) > 2$. Thus, in both cases we have $\lambda_1(d-1) \geqslant 2$, since $n_0 \leqslant d-3$. But $\lambda_1(d-1) < 2$ by Lemma 3.1(i). This is a contradiction. ◻

If the curve C_d is GIT-semistable, then $m_0 \leqslant d-2$ by Lemma 3.3. Thus, it follows from Lemma 3.5 that we may assume that

$$m_0 \leqslant d-2$$

in order to complete the proof of Theorems 1.10 and 1.15. Moreover, if L is not an irreducible component of the curve C_d, then

$$d - m_0 = C^1_d \cdot L^1 \geqslant m_1.$$

Thus, if (**A**) holds, then $m_0 + m_1 \leqslant d$ by Lemma 3.6. Similarly, if the curve C_d is GIT-semistable, then $m_0 + m_1 \leqslant d$ by Lemma 3.3. Thus, to complete the proof of Theorems 1.10 and 1.15, we may also assume that

$$m_0 + m_1 \leqslant d. \tag{3.1}$$

Then $\lambda(m_0 + m_1) < 3$ by Lemma 3.1(v), so that $(S_2, \lambda C_d^2 + (\lambda m_0 - 1)E_1^2 + (\lambda(m_0 + m_1) - 2)E_2)$ is Kawamata log terminal outside of the point P_2 by Lemma 2.13. Furthermore, we have

Lemma 3.7 *Suppose that $P_2 = E_1^2 \cap E_2$. Then (**A**) does not hold and C_d is GIT-unstable.*

Proof We have $m_0 - m_1 = E_1^2 \cdot C_d^2 \geqslant m_2$, so that

$$m_2 \leqslant \frac{m_0}{2}, \tag{3.2}$$

because $2m_2 \leqslant m_1 + m_2$. On the other hand, $m_0 \leqslant d - 2$ by assumption. Thus, we have $m_2 \leqslant \frac{d-2}{2}$.

Suppose that (**A**) holds. Then $\lambda = \lambda_1$ and $\lambda_1 m_2 < 1$ by Lemma 3.1(v). Thus, we can apply Theorem 2.16 to the log pair $(S_2, \lambda_1 C_d^2 + (\lambda_1 m_0 - 1)E_1^2 + (\lambda_1(m_0 + m_1) - 2)E_2)$. This gives either

$$\lambda_1\big(m_0 - m_1\big) = \lambda_1 C_d^2 \cdot E_1^2 \geqslant 2\Big(3 - \lambda_1\big(m_0 + m_1\big)\Big)$$

or

$$\lambda_1 m_1 = \lambda_1 C_d^2 \cdot E_2 \geqslant 2\big(2 - \lambda_1 m_0\big)$$

(or both). The former inequality implies $\lambda_1(3m_0 + m_1) \geqslant 6$. The latter inequality implies $\lambda_1(2m_0 + m_1) \geqslant 4$. On the other hand, $m_0 + m_1 \leqslant d$ by (3.1), and $m_0 \leqslant d - 2$ by assumption. Thus, $3m_0 + m_1 \leqslant 3d - 4$ and $2m_0 + m_1 \leqslant 2d - 2$. Then $\lambda_1(3m_0 + m_1) < 6$ by Lemma 3.1(vi), and $\lambda_1(2m_0 + m_1) < 4$ by Lemma 3.1(i). The obtained contradiction shows that (**A**) does not hold.

We see that (**B**) holds. We have to show that C_d is GIT-unstable. Suppose that this is not the case, so that C_d is GIT-semistable. Let us seek for a contradiction.

By Lemma 3.2, we have $2m_0 + m_1 + m_2 \leqslant \frac{5d}{3}$, because

$$\mathrm{wt}_{(3,2)}\Big(f_d\big(x_1, x_2\big)\Big) = 2m_0 + m_1 + m_2.$$

Thus, we have $\lambda_2(2m_0 + m_1 + m_2) - 4 < 1$ by Lemma 3.1(v). Hence, the log pair $(S_3, \lambda_2 C_d^3 + (\lambda_2 m_0 - 1)E_1^3 + (\lambda_2(m_0 + m_1) - 2)E_2^3 + (\lambda_2(2m_0 + m_1 + m_2) - 4)E_3)$ is Kawamata log terminal outside of the point P_3 by Remark 2.10.

If $P_3 = E_1^3 \cap E_3$, then it follows from Theorem 2.12 that

$$\lambda_2\big(m_0 - m_1 - m_2\big) = \lambda_2 C_d^3 \cdot E_1^3 > 5 - \lambda_2\big(2m_0 + m_1 + m_2\big),$$

which implies that $m_0 > \frac{5}{3\lambda_2} = \frac{2d}{3}$, which is impossible by Lemma 3.3. If $P_3 = E_2^3 \cap E_3$, then it follows from Theorem 2.12 that

$$\lambda_2\big(m_1 - m_2\big) = \lambda_2 C_d^3 \cdot E_2^3 > 5 - \lambda_2\big(2m_0 + m_1 + m_2\big),$$

which implies that $m_0 + m_1 > \frac{5}{2\lambda_2} = d$, which is impossible by Lemma 3.3. Thus, we see that $P_3 \notin E_1^3 \cup E_2^3$. Then the log pair $(S_3, \lambda_2 C_d^3 + (\lambda_2(2m_0 + m_1 + m_2) - 4)E_3)$ is not Kawamata log terminal at P_3. Hence, Theorem 2.12 gives

$$\lambda_2 m_2 = \lambda_2 C_d^3 \cdot E_3 > 1,$$

which implies that $m_2 > \frac{1}{\lambda_2} = \frac{2d}{5}$. Then $m_0 > \frac{4d}{5}$ by (3.2), which is impossible by Lemma 3.3. □

Thus, to complete the proof of Theorems 1.10 and 1.15, we may assume that

$$P_2 \neq E_1^2 \cap E_2.$$

Denote by L^2 the proper transform of the line L on the surface S_2.

Lemma 3.8 *One has $P_2 \neq L^2 \cap E_2$.*

Proof Suppose that $P_2 = L^2 \cap E_2$. If L is not an irreducible component of the curve C_d, then

$$d - m_0 - m_1 = L^2 \cdot E_2 \geqslant m_2,$$

which implies that $m_0 + m_1 + m_2 \leqslant d$. Thus, if (**A**) holds, then $\lambda = \lambda_1$ and L is not an irreducible component of the curve C_d by Lemma 3.6, which implies that

$$\lambda_1 d \geqslant \lambda_1\big(m_0 + m_1 + m_2\big) > 3$$

by Lemma 2.14. On the other hand, $\lambda_1 d < 3$ by Lemma 3.1(iv). This shows that (**B**) holds.

Since $\lambda = \lambda_2 = \frac{5}{2d} < \frac{3}{d}$ and $\lambda_2(m_0 + m_1 + m_2) > 3$ by Lemma 2.14, we have $m_0 + m_1 + m_2 > d$. In particular, the line L must be an irreducible component of the curve C_d.

Put $C_d = L + C_{d-1}$, where C_{d-1} is a reduced curve in $\mathbb{P}^2$ of degree $d-1$ such that L is not its irreducible component. Denote by C_{d-1}^1 its proper transform on S_1, and denote by C_{d-1}^2 its proper transform on S_2. Put $n_0 = \mathrm{mult}_P(C_{d-1})$, $n_1 = \mathrm{mult}_{P_1}(C_{d-1}^1)$ and $n_2 = \mathrm{mult}_{P_2}(C_{d-1}^2)$. Then $(S_2, (\lambda_2(n_0 + n_1 + 2) - 2)E_2 + \lambda_2 L^1 + \lambda_2 C_{d-1}^1)$ is not Kawamata log terminal at P_2 and is Kawamata log terminal outside of the point P_2. Then Theorem 2.12 implies

$$\begin{aligned}\lambda_2\big(d - 1 - n_0 - n_1\big) = \lambda_2 C_{d-1}^2 \cdot L^2 &> 1 - \big(\lambda_2(n_0 + n_1 + 2) - 2\big)\\ &= 3 - \lambda_2(n_0 + n_1 + 2),\end{aligned}$$

which implies that $\frac{5(d+1)}{2d} = \lambda_2(d+1) > 3$. Hence, $d = 4$. Then $\lambda = \lambda_2 = \frac{5}{8}$.

By (3.1), $n_0 + n_1 \leqslant 2$. Thus, $n_0 = n_1 = n_2 = 1$, since

$$\frac{5}{8}\big(n_0 + n_1 + n_2 + 3\big) = \lambda_2\big(m_0 + m_1 + m_2\big) > 3$$

by Lemma 2.14. Then C_3 is a irreducible cubic curve that is smooth at P, the line L is tangent to the curve C_3 at the point P, and P is an inflexion point of the cubic curve C_3. This implies that $\mathrm{lct}_P(\mathbb{P}^2, C_d) = \frac{2}{3}$. Since $\frac{2}{3} > \frac{5}{8} = \lambda_2$, the log pair $(\mathbb{P}^2, \lambda_2 C_d)$ must be Kawamata log terminal at the point P, which contradicts (**B**). □

Recall that $m_0 + m_1 \leqslant d$ by (3.1). Then $m_1 \leqslant \frac{d}{2}$, since $2m_1 \leqslant m_0 + m_1$. Thus, we have

$$\lambda\big(m_0 + m_1 + m_2\big) \leqslant \lambda\big(m_0 + 2m_1\big) \leqslant \lambda\frac{3d}{2} \leqslant \lambda_2\frac{3d}{2} = \frac{15}{4} < 4. \tag{3.3}$$

Therefore, the log pair $(S_3, \lambda C_d^3 + (\lambda(m_0+m_1)-2)E_2^3 + (\lambda(m_0+m_1+m_2)-3)E_3)$ is Kawamata log terminal outside of the point P_3 by Lemma 2.13.

Lemma 3.9 *One has* $P_3 \neq E_2^3 \cap E_3$.

Proof If $P_3 = E_2^3 \cap E_3$, then Theorem 2.12 gives

$$\lambda\big(m_1 - m_2\big) = \lambda C_d^3 \cdot E_2^3 > 1 - \Big(\lambda\big(m_0 + m_1 + m_2\big) - 3\Big) = 4 - \lambda\big(m_0 + m_1 + m_2\big),$$

which implies that $\lambda(m_0 + 2m_1) > 4$. But $\lambda(m_0 + 2m_1) < 4$ by (3.3). □

Let $f_4\colon S_4 \to S_3$ be a blow up of the point P_3, and let E_4 be its exceptional curve. Denote by C_d^4 the proper transform on S_4 of the curve C_d, denote by E_3^4 the proper transform on S_4 of the curve E_3, and denote by L^4 the proper transform of the line L on the surface S_4. Then $(S_4, \lambda C_d^4 + (\lambda(m_0+m_1+m_2)-3)E_3^4 + (\lambda(m_0+m_1+m_2+m_3)-4)E_4)$ is not Kawamata log terminal at some point $P_4 \in E_4$ by Remark 2.10. Moreover, we have

$$2L^4 + E_1^4 + 2E_2^4 + E_3^4 \sim (f_1 \circ f_2 \circ f_3 \circ f_4)^*\Big(\mathcal{O}_{\mathbb{P}^2}(2)\Big) - (f_2 \circ f_3 \circ f_4)^*\big(E_1\big)$$
$$-(f_3 \circ f_4)^*\big(E_2\big) - f_4^*\big(E_3\big) - E_4.$$

Lemma 3.10 *The linear system* $|2L^4 + E_1^4 + 2E_2^4 + E_3^4|$ *is a pencil that does not have base points. Moreover, every divisor in* $|2L^4 + E_1^4 + 2E_2^4 + E_3^4|$ *that is different from* $2L^4 + E_1^4 + 2E_2^4 + E_3^4$ *is a smooth curve whose image on* $\mathbb{P}^2$ *is a smooth conic that is tangent to L at the point* P.

Proof All assertions follows from $P_2 \notin E_1^2 \cup L^2$ and $P_3 \notin E_2^3$. □

Let C_2^4 be a general curve in $|2L^4 + E_1^4 + 2E_2^4 + E_3^4|$. Denote by C_2 its image on $\mathbb{P}^2$, and denote by $\mathcal{L}$ the pencil generated by $2L$ and C_2. Then P is the only base point of the pencil $\mathcal{L}$, and every conic in $\mathcal{L}$ except $2L$ and C_2 intersects C_2 at P with multiplicity 4 (cf. [3, Remark 1.14]).

Lemma 3.11 *One has* $m_0 + m_1 + m_2 + m_3 \leqslant m_0 + m_1 + 2m_2 \leqslant \frac{5}{\lambda}$. *If* $m_0 + m_1 + m_2 + m_3 = \frac{5}{\lambda}$, *then* d *is even and* C_d *is a union of* $\frac{d}{2} \geqslant 2$ *smooth conics in* $\mathcal{L}$, *where* $d = 4$ *if* (**A**) *holds.*

Proof By (3.1), we have $m_2 + m_3 \leqslant 2m_2 \leqslant m_0 + m_1 \leqslant d$. This gives

$$m_0 + m_1 + m_2 + m_3 \leqslant m_0 + m_1 + 2m_2 \leqslant 2d = \frac{5}{\lambda_2} \leqslant \frac{5}{\lambda}.$$

To complete the proof, we may assume that $m_0 + m_1 + m_2 + m_3 = \frac{5}{\lambda}$. Then all inequalities above must be equalities. Thus, we have $m_2 = m_3 = \frac{d}{2}$ and $\lambda_1 = \lambda_2$. In particular, if (**A**) holds, then $d = 4$, because $\lambda_1 < \lambda_2 = \frac{5}{2d}$ for $d \geqslant 5$ by Lemma 3.1(vii). Moreover, since $m_0 \geqslant m_1 \geqslant m_2 = \frac{d}{2}$ and $m_0 + m_1 \leqslant d$, we see that $m_0 = m_1 = \frac{d}{2}$. Thus, d is even and

$$C_d^4 \sim \frac{d}{2}\Big(2L^4 + E_1^4 + 2E_2^4 + E_3^4\Big),$$

where $d = 4$ if (**A**) holds. Since $|2L^4 + E_1^4 + 2E_2^4 + E_3^4|$ is a free pencil and C_d^4 is reduced, it follows from Lemma 3.10 that C_d^4 is a union of $\frac{d}{2}$ smooth curves in $|2L^4 + E_1^4 + 2E_2^4 + E_3^4|$. In particular, L^4 is not an irreducible component of C_d^4. Thus, the curve C_d is a union of $\frac{d}{2}$ smooth conics in $\mathcal{L}$, where $d = 4$ if (**A**) holds. □

We see that $m_0 + m_1 + m_2 + m_3 \leqslant \frac{5}{\lambda}$. Moreover, if $m_0 + m_1 + m_2 + m_3 = \frac{5}{\lambda}$, then C_d is an even Płoski curve. Furthermore, if $m_0 + m_1 + m_2 + m_3 = \frac{5}{\lambda}$ and (**A**) holds, then $d = 4$. Thus, to prove Theorems 1.10 and 1.15, we may assume that

$$m_0 + m_1 + m_2 + m_3 < \frac{5}{\lambda}.$$

Let us show that this assumption leads to a contradiction. By Lemma 2.13, this inequality implies that the log pair $(S_4, \lambda C_d^4 + (\lambda(m_0 + m_1 + m_2) - 3)E_3^4 + (\lambda(m_0 + m_1 + m_2 + m_3) - 4)E_4)$ is Kawamata log terminal outside of the point P_4.

Lemma 3.12 *One has* $P_4 \neq E_3^4 \cap E_4$.

Proof By Lemma 3.11, $m_0 + m_1 + 2m_2 \leqslant \frac{5}{\lambda}$. If $P_4 = E_3^4 \cap E_4$, then Theorem 2.12 gives

$$\lambda\big(m_2 - m_3\big) = \lambda C_d^4 \cdot E_3^4 > 5 - \lambda\big(m_0 + m_1 + m_2 + m_3\big),$$

which implies that $m_0 + m_1 + 2m_2 > \frac{5}{\lambda}$. This shows that $P_4 \neq E_3^4 \cap E_4$. □

Thus, the log pair $(S_4, \lambda C_d^4 + (\lambda(m_0 + m_1 + m_2 + m_3) - 4)E_4)$ is not Kawamata log terminal at P_4 and is Kawamata log terminal outside of the point P_4.

Let Z^4 be the curve in $|2L^4+E_1^4+2E_2^4+E_3^4|$ that passes through the point P_4. Then Z^4 is a smooth irreducible curve by Lemma 3.8. Denote by Z the proper transform of this curve on $\mathbb{P}^2$. Then Z is a smooth conic in the pencil $\mathcal{L}$ by Lemma 3.10. If Z is not an irreducible component of the curve C_d, then

$$2d-\big(m_0+m_1+m_2+m_3\big)=Z^4\cdot C_d^4\geqslant \mathrm{mult}_{P_4}(C_d^4).$$

On the other hand, it follows from Lemma 2.14 that

$$\mathrm{mult}_{P_4}(C_d^4)+m_0+m_1+m_2+m_3>\frac{5}{\lambda}.$$

This shows that Z is an irreducible component of the curve C_d, since $\lambda\leqslant\lambda_2=\frac{5}{2d}$.

Put $C_d=Z+C_{d-2}$, where C_{d-2} is a reduced curve in $\mathbb{P}^2$ of degree $d-2$ such that Z is not its irreducible component. Denote by $C_{d-2}^1, C_{d-2}^2, C_{d-2}^3$ and C_{d-2}^4 its proper transforms on the surfaces S_1, S_2, S_3 and S_4, respectively. Put $n_0=\mathrm{mult}_P(C_{d-2}), n_1=\mathrm{mult}_{P_1}(C_{d-2}^1), n_2=\mathrm{mult}_{P_2}(C_{d-2}^2), n_3=\mathrm{mult}_{P_3}(C_{d-2}^3)$ and $n_4=\mathrm{mult}_{P_4}(C_{d-2}^4)$. Then

$$\Big(S_4, \lambda C_{d-2}^4+\lambda Z^4+(\lambda(n_0+n_1+n_2+n_3+4)-4)E_4\Big)$$

is not Kawamata log terminal at P_4 and is Kawamata log terminal outside of the point P_4. Thus, applying Theorem 2.12, we get

$$\lambda\Big(2(d-2)-n_0-n_1-n_2-n_3\Big)=\lambda C_{d-2}^4\cdot Z^4>5-\lambda\big(n_0+n_1+n_2+n_3+4\big),$$

which implies that $\lambda>\frac{5}{2d}$. This is impossible, since $\lambda\leqslant\lambda_2=\frac{5}{2d}$.

The obtained contradiction completes the proof of Theorems 1.10 and 1.15.

4 Smooth Surfaces in $\mathbb{P}^3$

The purpose of this section is to prove Theorem 1.17. Let S be a smooth surface in $\mathbb{P}^3$ of degree $d\geqslant 3$, let H_S be its hyperplane section, let P be a point in S, and let T_P be the hyperplane section of the surface S that is singular at P. Note that T_P is reduced by Lemma 2.6. Put $\lambda=\frac{2d-3}{d(d-2)}$. Then Theorem 1.17 follows from Theorem 1.10, Remark 2.4 and

Proposition 4.1 *Let D be any effective $\mathbb{Q}$-divisor on S such that $D\sim_{\mathbb{Q}} H_S$. Suppose that* Supp(D) *does not contain at least one irreducible component of the curve T_P. Then $(S,\lambda D)$ is log canonical at P.*

For $d=3$, this result is just [3, Corollary 1.13]. In the remaining part of the section, we will prove Proposition 4.1. Note that we will do this *without* using [3, Corollary 1.13]. Let us start with

Lemma 4.2 *The following assertions hold:*

(i) $\lambda \leqslant \frac{2}{d-1}$,
(ii) *if* $d \geqslant 5$, *then* $\lambda \leqslant \frac{3}{d+1}$,
(iii) *if* $d \geqslant 5$, *then* $\lambda \leqslant \frac{4}{d+3}$,
(iv) *If* $d \geqslant 6$, *then* $\lambda \leqslant \frac{3}{d+2}$,
(v) $\lambda \leqslant \frac{4}{d+1}$,
(vi) $\lambda \leqslant \frac{3}{d}$.

Proof The equality $\frac{2}{d-1} = \lambda + \frac{d-3}{d(d-1)(d-2)}$ implies (i), $\frac{4}{d+1} = \lambda + \frac{d^2-5d+3}{d(d+1)(d-2)}$ implies (ii), and $\frac{4}{d+3} = \lambda + \frac{2d^2-11d+9}{d(d+3)(d-2)}$ implies (iii). Similarly, (iv) follows from $\frac{3}{d+2} = \lambda + \frac{d^2-7d+6}{d(d^2-4)}$, (v) follows from $\frac{4}{d+1} = \lambda + \frac{2d^2-7d+3}{d(d+1)(d-2)}$, and (vi) follows from $\frac{3}{d} = \lambda + \frac{d-3}{d(d-2)}$. □

Let n be the number of irreducible components of the curve T_P. Write

$$T_P = T_1 + \cdots + T_n,$$

where each T_i is an irreducible curve on the surface S. For every curve T_i, we denote its degree by d_i, and we put $t_i = \mathrm{mult}_P(T_i)$.

Lemma 4.3 *Suppose that* $n \geqslant 2$. *Then*

$$T_i \cdot T_i = -d_i(d - d_i - 1)$$

for every T_i, *and* $T_i \cdot T_j = d_i d_j$ *for every* T_i *and* T_j *such that* $T_i \neq T_j$.

Proof The curve T_P is cut out on S by a hyperplane $H \subset \mathbb{P}^3$. Then $H \cong \mathbb{P}^2$. Hence, for every T_i and T_j such that $T_i \neq T_j$, we have $(T_i \cdot T_j)_S = (T_i \cdot T_j)_H = d_i d_j$. In particular, we have

$$d_1 = T_P \cdot T_1 = T_1^2 + \sum_{i=2}^{n} T_i \cdot T_1 = T_1^2 + \sum_{i=2}^{n} d_i d_1 = T_1^2 + (d - d_1)d_1,$$

which gives $T_1 \cdot T_1 = -d_1(d - d_1 - 1)$. Similarly, we see that $T_i \cdot T_i = -d_i(d - d_i - 1)$ for every curve T_i. □

Let D be any effective $\mathbb{Q}$-divisor on S such that $D \sim_{\mathbb{Q}} H_S$. Write

$$D = \sum_{i=1}^{n} a_i T_i + \Delta,$$

where each a_i is a non-negative rational number, and Δ is an effective $\mathbb{Q}$-divisor on S whose support does not contain the curves $T_1, \ldots, T_n$. To prove Proposition 4.1, it is enough to show that the log pair $(S, \lambda D)$ is log canonical at P provided that at least one number among $a_1, \ldots, a_n$ vanishes.

Without loss of generality, we may assume that $a_n = 0$. Suppose that the log pair $(S, \lambda D)$ is not log canonical at P. Let us seek for a contradiction.

Lemma 4.4 *Suppose that $n \geqslant 2$. Then*

$$\sum_{i=1}^{k} a_i d_i d_n \leqslant d_n - t_n \mathrm{mult}_P(\Delta).$$

In particular, $\sum_{i=1}^{k} a_i d_i \leqslant 1$ and each a_i does not exceed $\frac{1}{d_i}$.

Proof One has

$$d_n = T_n \cdot D = T_n \cdot \left(\sum_{i=1}^{n} a_i T_i + \Delta\right) = \sum_{i=1}^{n} a_i d_i d_n + T_n \cdot \Delta$$
$$\geqslant \sum_{i=1}^{n} a_i d_i d_n + t_n \mathrm{mult}_P(\Delta),$$

which implies the required inequality. □

Put $m_0 = \mathrm{mult}_P(D)$.

Lemma 4.5 *Suppose that $P \in T_n$. Then $d_n > \frac{d-1}{2}$. If $n \geqslant 2$, then T_n is smooth at P.*

Proof Since T_n is not contained in the support of the divisor D, we have

$$d \geqslant d_n = T_n \cdot D \geqslant t_n m_0,$$

which implies that $m_0 \leqslant \frac{d_n}{t_n}$. Since $m_0 > \frac{1}{\lambda}$ by Lemma 2.5, we have $d_n > \frac{d-1}{2}$ by Lemma 4.2(i). Moreover, if $n \geqslant 2$ and $t_n \geqslant 2$, then it follows from Lemma 2.5 that

$$\frac{1}{\lambda} < m_0 \leqslant \frac{d_n}{t_n} \leqslant \frac{d-1}{t_n} \leqslant \frac{d-1}{2},$$

which is impossible by Lemma 4.2(i). □

Now we are going to use Theorem 2.15 to prove

Lemma 4.6 *Suppose that $n \geqslant 3$ and P is contained in at least two irreducible components of the curve T_P that are different from T_n and that are both smooth at P. Then they are tangent to each other at P.*

Proof Without loss of generality, we may assume that $P \in T_1 \cap T_2$ and $t_1 = t_2 = 1$. Suppose that T_1 and T_2 are not tangent to each other at P. Put $\Omega = \sum_{i=3}^{n} a_i T_i + \Delta$, so that $D = a_1 T_1 + a_2 T_2 + \Omega$. Then $a_1 d_1 + a_2 d_2 \leqslant 1$ by Lemma 4.4.

Put $k_0 = \mathrm{mult}(\Omega)$. Then

$$d_1 + a_1 d_1\big(d - d_1 - 1\big) - a_2 d_1 d_2 = \Omega \cdot T_1 \geqslant k_0$$

by Lemma 4.3. Similarly, we have

$$d_2 - a_1 d_1 d_2 + a_2 d_2\big(d - d_2 - 1\big) = \Omega \cdot T_2 \geqslant k_0.$$

Adding these two inequalities together and using $a_1 d_1 + a_2 d_2 \leqslant 1$, we get

$$\begin{aligned} 2k_0 &\leqslant d_1 + d_2 + \big(a_1 d_1 + a_2 d_2\big)\big(d - d_1 - d_2 - 1\big) \\ &\leqslant d_1 + d_2 + \big(d - d_1 - d_2 - 1\big) = d - 1. \end{aligned}$$

Thus, $k_0 \leqslant \frac{1}{\lambda}$ by Lemma 4.2(i).

Since $\lambda k_0 \leqslant 1$, we can apply Theorem 2.15 to the log pair $(S, \lambda a_1 T_1 + \lambda a_2 T_2 + \lambda\Omega)$ at the point P. This gives either $\lambda\Omega \cdot T_1 > 2(1 - \lambda a_2)$ or $\lambda\Omega \cdot T_2 > 2(1 - \lambda a_1)$. Without loss of generality, we may assume that $\lambda\Omega \cdot T_2 > 2(1 - \lambda a_1)$. Then

$$d_2 + a_2 d_2\big(d - d_2 - 1\big) - a_1 d_1 d_2 = \Omega \cdot T_2 > \frac{2}{\lambda} - 2a_1. \tag{4.1}$$

Applying Theorem 2.12 to the log pair $(S, \lambda a_1 T_1 + \lambda b_1 T_2 + \lambda\Omega)$ and the curve T_1 at the point P, we get

$$d_1 + a_1 d_1\big(d - d_1 - 1\big) = \Big(\lambda a_2 T_2 + \lambda\Omega\Big) \cdot T_1 > \frac{1}{\lambda}.$$

Adding this inequality to (4.1), we get

$$d + 1 \geqslant d - 1 + 2a_1 \geqslant d_1 + d_2 + \big(a_1 d_1 + a_2 d_2\big)\big(d - d_1 - d_2 - 1\big) + 2a_1 > \frac{3}{\lambda},$$

because $a_1 d_1 + a_2 d_2 \leqslant 1$. Thus, it follows from Lemma 4.2(ii) that either $d = 3$ or $d = 4$.

If $d = 3$, then $n = 3$ and $d_1 = d_2 = d_3 = \lambda = 1$, which implies that $a_1 + a_2 > 1$ by (4.1). On the other hand, we know that $a_1 d_1 + a_2 d_2 \leqslant 1$, so that $a_1 + a_2 \leqslant 1$. This shows that $d \neq 3$.

We see that $d = 4$. Then $\lambda = \frac{5}{8}$ and $d_1 + d_2 \leqslant 3$. If $d_1 = d_1 = 1$, then (4.1) gives $2a_2 + a_1 > \frac{11}{5}$. If $d_1 = 1$ and $d_2 = 2$, then (4.1) gives $a_2 > \frac{3}{5}$. If $d_1 = 2$ and $d_2 = 1$, then (4.1) gives $a_2 > \frac{11}{5}$. All these three inequalities are inconsistent, because $a_1 d_1 + a_2 d_2 \leqslant 1$. The obtained contradiction completes the proof of the lemma. □

Note that every line contained in the surfaces S that passes through P must be an irreducible component of the curve T_P. Moreover, the curve T_n cannot be a line by Lemma 4.5. Thus, Lemma 4.6 implies that there exists at most one line in S that passes through P. In particular, we see that $n < d$.

Lemma 4.7 *Suppose that $n \geqslant 3$ and P is contained in at least two irreducible components of the curve T_P that are different from T_n. Then these curves are smooth at P.*

Proof Without loss of generality, we may assume that $P \in T_1 \cap T_2$ and $t_1 \leqslant t_2$. We have to show that $t_1 = t_2 = 1$. We may assume that $d \geqslant 5$, because the required assertion is obvious in the cases $d = 3$ and $d = 4$.

Put $\Omega = \sum_{i=3}^{n} a_i T_i + \Delta$ and put $k_0 = \mathrm{mult}_P(\Omega)$. Then $m_0 = k_0 + a_1 t_1 + a_2 t_2$. Moreover, we have $a_1 d_1 + a_2 d_2 \leqslant 1$ by Lemma 4.4. On the other hand, it follows from Lemma 4.3 that

$$d - 1 \geqslant d_1 + d_2 + \big(a_1 d_1 + a_2 d_2\big)\big(d - d_1 - d_2 - 1\big) = \Omega \cdot \Big(T_1 + T_2\Big) \geqslant k_0\big(t_1 + t_2\big),$$

because $a_1 d_1 + a_2 d_2 \leqslant 1$. Thus, we have $k_0 \leqslant \frac{d-1}{t_1+t_2}$. Hence, if $t_1 + t_2 \geqslant 4$, then

$$m_0 = k_0 + a_1 t_1 + a_2 t_2 \leqslant k_0 + a_1 d_1 + a_2 d_2 \leqslant \frac{d-1}{t_1 + t_2} + a_1 d_1 + a_2 d_2 \leqslant \frac{d-1}{t_1 + t_2} + 1 \leqslant \frac{d+3}{4}$$

because $a_1 d_1 + a_2 d_2 \leqslant 1$. Since $m_0 > \frac{1}{\lambda}$ by Lemma 2.5, the inequality $m_0 \leqslant \frac{d+3}{4}$ gives $\lambda > \frac{d+3}{4}$, which is impossible by Lemma 4.2(iii). Thus, $t_1 + t_2 \leqslant 3$. Since $t_1 \leqslant t_2$, we have $t_1 = 1$ and $t_2 \leqslant 2$.

To complete the proof of the lemma, we have to prove that $t_2 = 1$. Suppose $t_2 \neq 1$. Then $t_2 = 2$, since $t_1 + t_2 \leqslant 3$. Since $k_0 \leqslant \frac{d-1}{t_1+t_2} = \frac{d-1}{3}$ and $a_1 d_1 + a_2 d_2 \leqslant 1$, we have

$$m_0 = k_0 + a_1 t_1 + a_2 t_2 \leqslant k_0 + a_1 d_1 + a_2 d_2 \leqslant \frac{d-1}{32} + a_1 d_1 + a_2 d_2 \leqslant \frac{d-1}{t_1 + t_2} + 1 = \frac{d+2}{3}.$$

On the other hand, $m_0 > \frac{1}{\lambda}$ by Lemma 2.5, so that $\lambda > \frac{3}{d+2}$. Then $d = 5$ by Lemma 4.2(iv).

Since $d = 5, t_1 = 1$ and $t_2 = 2$, we have $n = 3, d_1 = 1, d_2 = 3$ and $d_3 = 1$. Applying Theorem 2.12 to the log pair $(S, \lambda a_1 T_1 + \lambda a_2 T_2 + \lambda\Omega)$, we get

$$1 + 3a_1 = d_1 + a_2 d_1\big(d - d_1 - 1\big) = \Big(\lambda a_2 T_2 + \lambda\Omega\Big) \cdot T_1 > \frac{1}{\lambda} = \frac{15}{7},$$

which gives $a_1 > \frac{8}{21}$. On the other hand, $a_1 + 3a_2 \leqslant 1$, because $a_1 d_1 + a_2 d_2 \leqslant 1$. Since $m_0 > \frac{1}{\lambda} = \frac{15}{7}$ by Lemma 2.5, we see that

$$\begin{aligned}\frac{15}{7}-\frac{1}{9}=\frac{128}{63}>\frac{8-5a_1}{3}&=\frac{3-a_1+\frac{7(1-a_1)}{3}}{2}=\frac{3-a_1+7a_2}{2}\\&=\frac{3-3a_1+3a_2}{2}+a_1+2a_2\\&=\frac{\Delta\cdot T_2}{2}+a_1+2a_2\geqslant\frac{\mathrm{mult}_P\big(\Delta\cdot T_2\big)}{2}+a_1+2a_2\geqslant\frac{t_2k_0}{2}+a_1+2a_2\\&=k_0+a_1+2a_2=m_0>\frac{15}{7},\end{aligned}$$

which is absurd. □

Now we are ready to prove

Lemma 4.8 *One has* $m_0\leqslant\frac{d+1}{2}$.

Proof Suppose that $m_0>\frac{d+1}{2}$. Let us seek for a contradiction. If $n=1$, then

$$d=T_n\cdot D\geqslant 2m_0,$$

which implies that $m_0\leqslant\frac{d}{2}$. Thus, have $n\geqslant 2$. Then $a_1\leqslant\frac{1}{d_1}$ by Lemma 4.4. Moreover, either $t_n=0$ or $t_n=1$ by Lemma 4.5. Hence, there is an irreducible component of T_P that passes through P and is different from T_n, because T_P is singular at P. Without loss of generality, we may assume that $t_1\geqslant 1$.

Put $\Upsilon=\sum_{i=2}^{n}a_iT_i+\Delta$, so that $D=a_1T_1+\Upsilon$. Put $n_0=\mathrm{mult}_P(\Upsilon)$, so that $m_0=n_0+a_1t_1$. Then $t_nn_0\leqslant d_n-a_1d_1d_n$ by Lemma 4.4, and

$$d_1+a_1d_1(d-d_1-1)=\Upsilon\cdot T_1\geqslant t_1n_0 \tag{4.2}$$

by Lemma 4.3. Adding these two inequalities, we get $(t_1+t_n)n_0\leqslant d_1+d_n+a_1d_1(d-d_1-d_n-1)$. Hence, if $n\geqslant 3$ and $t_n=1$, then

$$2n_0\leqslant\big(t_1+t_n\big)n_0\leqslant d_1+d_n+a_1d_1\big(d-d_1-d_n-1\big)\leqslant d-1\leqslant d-a_1d_1,$$

because $a_1\leqslant\frac{1}{d_1}$. Similarly, if $n=2$ and $t_n=1$, then

$$\begin{aligned}2n_0&\leqslant\big(t_1+t_n\big)n_0\leqslant d_1+d_2+a_1d_1\big(d-d_1-d_2-1\big)\\&=d_1+d_2-a_1d_1=d-a_1d_1.\end{aligned}$$

Thus, if $t_n=1$, then $n_0\leqslant\frac{d-a_1d_1}{2}$, which is impossible. Indeed, the inequality $n_0\leqslant\frac{d-a_1d_1}{2}$ gives

$$\frac{d+1}{2}<m_0=n_0+a_1t_1\leqslant n_0+a_1d_1\leqslant\frac{d-a_1d_1}{2}+a_1d_1=\frac{d+a_1d_1}{2}\leqslant\frac{d+1}{2},$$

because $a_1\leqslant\frac{1}{d_1}$. This shows that $t_n=0$.

If $t_1 \geqslant 2$, then it follows from (4.2) that

$$\frac{d+1}{2} < m_0 \leqslant n_0 + a_1 d_1 \leqslant \frac{d_1 + a_1 d_1 (d - d_1 - 1)}{2} + a_1 d_1 = \frac{d_1 + a_1 d_1 (d - d_1 + 1)}{2} \leqslant \frac{d+1}{2},$$

because $a_1 \leqslant \frac{1}{d_1}$. This shows that $t_1 = 1$.

Since $t_1 = 1$ and $t_n = 0$, there exists an irreducible component of the curve T_P that passes through P and is different from T_1 and T_n. In particular, we have $n \geqslant 3$. Without loss of generality, we may assume $P \in T_2$. Then T_2 is smooth at P by Lemma 4.7.

Put $\Omega = \sum_{i=3}^{n} a_i T_i + \Delta$ and put $k_0 = \mathrm{mult}_P(\Omega)$. Then $a_1 d_1 + a_2 d_2 \leqslant 1$ by Lemma 4.4. Thus, it follows from Lemma 4.3 that

$$2k_0 \leqslant \Omega \cdot \Big(T_1 + T_2\Big) = d_1 + d_2 + \big(a_1 d_1 + a_2 d_2\big)\big(d - d_1 - d_2 - 1\big) \leqslant d - 1,$$

which implies $k_0 \leqslant \frac{d-1}{2}$. Then

$$\frac{d+1}{2} < m_0 = k_0 + a_1 t_1 + a_2 t_2 \leqslant k_0 + a_1 d_1 + a_2 d_2 \leqslant \frac{d-1}{2} + a_1 d_1 + a_2 d_2 \leqslant \frac{d-1}{2} + 1 = \frac{d+1}{2},$$

because $a_1 d_1 + a_2 d_2 \leqslant 1$. The obtained contradiction completes the proof of the lemma. □

Let $f_1 \colon S_1 \to S$ be a blow up of the point P, and let E_1 be its exceptional curve. Denote by D^1 the proper transform of the $\mathbb{Q}$-divisor D on the surface S_1. Then

$$K_{S_1} + \lambda D^1 + \big(\lambda m_0 - 1\big) E_1 \sim_{\mathbb{Q}} f_1^*\Big(K_S + \lambda D\Big),$$

which implies that $(S_1, \lambda D^1 + (\lambda m_0 - 1) E_1)$ is not log canonical at some point $P_1 \in E_1$.

By Lemma 4.8, we have $m_0 \leqslant \frac{d+1}{2}$. By Lemma 4.2(v), we have $\lambda \leqslant \frac{4}{d+1}$. This gives $\lambda m_0 \leqslant 2$. Thus, the log pair $(S_1, \lambda D^1 + (\lambda m_0 - 1) E_1)$ is log canonical at every point of the curve E_1 that is different from P_1 by Lemma 2.13.

Put $m_1 = \mathrm{mult}_{P_1}(D^1)$. Then Lemma 2.5 gives

$$m_0 + m_1 > \frac{2}{\lambda}. \tag{4.3}$$

For each curve T_i, denote by T_i^1 its proper transform on S_1. Put $T_P^1 = \sum_{i=1}^{n} T_i^1$.

Lemma 4.9 *One has $P_1 \notin T_P^1$.*

Proof Suppose that $P_1 \in T_P^1$. Let us seek for a contradiction. If T_P is irreducible, then

$$d - 2m_0 = T_P^1 \cdot D^1 \geqslant m_1,$$

so that $m_1 + 2m_0 \leqslant d$. This inequality gives

$$\frac{3}{\lambda} < m_1 + 2m_0 \leqslant d,$$

because $2m_0 \geqslant m_0 + m_1 > \frac{2}{\lambda}$ by (4.3). This shows that T_P is reducible, because $\lambda \leqslant \frac{3}{d}$ by Lemma 4.2(vi).

We see that $n \geqslant 2$. If $P_1 \in T_n^1$, then

$$d - 1 - m_0 \geqslant d_n - m_0 = d_n - m_0 t_n = T_n^1 \cdot D^1 \geqslant m_1,$$

which is impossible, because $m_0 + m_1 > \frac{2}{\lambda}$ by (4.3), and $\lambda \leqslant \frac{2}{d-1}$ by Lemma 4.2(i). Thus, we see that $P_1 \notin T_n^1$.

Without loss of generality, we may assume that $P_1 \in T_1^1$. Put $\Upsilon = \sum_{i=2}^{n} a_i T_i + \Delta$, and denote by Υ^1 the proper transform of the $\mathbb{Q}$-divisor Ω on the surface S_1. Put $n_0 = \mathrm{mult}_P(\Upsilon)$, put $n_1 = \mathrm{mult}_{P_1}(\Omega^1)$ and put $t_1^1 = \mathrm{mult}_{P_1}(T_1^1)$. Then

$$d_1 + a_1 d_1 \big(d - d_1 - 1\big) - n_0 t_1 = T_1^1 \cdot \Upsilon^1 \geqslant t_1^1 n_1,$$

which implies that $n_0 t_1 + n_1 t_1^1 \leqslant d_1 + a_1 d_1 (d - d_1 - 1)$.

Note that $t_1^1 \leqslant t_1$. Moreover, we have $a_1 \leqslant \frac{1}{d_1}$ by Lemma 4.4. Thus, if $t_1^1 \geqslant 2$, then

$$\begin{aligned} 2\big(n_0 + n_1\big) &\leqslant t_1^1 \big(n_0 + n_1\big) \leqslant n_0 t_1 + n_1 t_1^1 \leqslant d_1 + a_1 d_1 \big(d - d_1 - 1\big) \\ &\leqslant d_1 + \big(d - d_1 - 1\big) = d - 1, \end{aligned}$$

which implies that $n_0 + n_1 \leqslant \frac{d-1}{2}$. Moreover, if $n_0 + n_1 \leqslant \frac{d-1}{2}$, then it follows from (4.3) that

$$\begin{aligned} \frac{d+3}{2} &= 2 + \frac{d-1}{2} \geqslant 2a_1 d_1 + \frac{d-1}{2} \geqslant 2a_1 t_1 + \frac{d-1}{2} \geqslant a_1 \big(t_1 + t_1^1\big) + n_0 + n_1 \\ &= m_0 + m_1 > \frac{2}{\lambda} \end{aligned}$$

which gives $d \leqslant 4$ by Lemma 4.2(iii). Thus, if $d \geqslant 5$, then $t_1^1 = 1$. Furthermore, if $d \leqslant 4$, then $d_1 \leqslant 3$, which implies that $t_1^1 \leqslant 1$. This shows that $t_1^1 = 1$ in all cases. Thus, the curve T_1^1 is smooth at P_1.

Applying Theorem 2.11 to the log pair $(S_1, \lambda\Upsilon^1+\lambda a_1 T_1^1+(\lambda(n_0+a_1t_1)-1)E_1)$, we see that

$$\lambda\big(d-1-n_0t_1\big)\geqslant\lambda\Big(d_1+a_1d_1\big(d-d_1-1\big)-n_0t_1\Big)$$
$$=\lambda\Omega^1\cdot T_1^1>2-\lambda\big(n_0+a_1t_1\big),$$

because $a_1\leqslant\frac{1}{d_1}$. Thus, we have $d-1+a_1t_1-n_0(t_1-1)>\frac{2}{\lambda}$. But $m_0=a_1t_1+n_0>\frac{1}{\lambda}$ by Lemma 2.5. Adding these inequalities together, we obtain

$$d-1+2a_1t_1-n_0(t_1-2)>\frac{3}{\lambda}. \tag{4.4}$$

If $t_1\geqslant 2$, this gives

$$d+1\geqslant d-1+2a_1d_1\geqslant d-1+2a_1t_1\geqslant d-1+2a_1t_1-n_0(t_1-2)>\frac{3}{\lambda}.$$

because $a_1\leqslant\frac{1}{d_1}$. One the other hand, if $d\geqslant 5$, then $\lambda\leqslant\frac{3}{d+1}$ by Lemma 4.2(ii). Thus, if $d\geqslant 5$, then $t_1=1$. Moreover, if $d=3$, then $d_1\leqslant 2$, which implies that $t_1=1$ as well. Furthermore, if $d=4$ and $t_1\neq 1$, then $d_1=3$, $t_1=2$, $\lambda=\frac{5}{8}$, which implies that

$$\frac{1}{3}=\frac{1}{d_1}\geqslant a_1>\frac{9}{20}$$

by (4.4). Thus, we see that $t_1=1$ in all cases. This simply means that the curve T_1 is smooth at the point P.

Since $a_1\leqslant\frac{1}{d_1}$, we have

$$d-1-n_0\geqslant d_1+a_1d_1\big(d-d_1-1\big)-n_0=\Omega^1\cdot T_1^1\geqslant n_1,$$

which implies that $n_1\leqslant\frac{n_0+n_1}{2}\leqslant\frac{d-1}{2}$. Then $\lambda n_1\leqslant 1$ by Lemma 4.2(i). Hence, we can apply Theorem 2.15 to the log pair $(S_1, \lambda\Upsilon^1+\lambda a_1T_1^1+(\lambda(n_0+a_1t_1)-1)E_1)$ at the point P_1. This gives either

$$\Upsilon^1\cdot T_1^1>\frac{4}{\lambda}-2(n_0+a_1)$$

or $\Upsilon^1\cdot E_1>\frac{2}{\lambda}-2a_1$ (or both). Since $a_1\leqslant\frac{1}{d_1}$, the former inequality gives

$$d-1-n_0\geqslant d_1+a_1d_1\big(d-d_1-1\big)-n_0=\Upsilon^1\cdot T_1^1>\frac{4}{\lambda}-2(n_0+a_1).$$

Similarly, the latter inequality gives

$$n_0=\lambda\Upsilon^1\cdot E_1>\frac{2}{\lambda}-2a_1.$$

Thus, either $d-1+2a_1+n_0>\frac{4}{\lambda}$ or $2a_1+n_0>\frac{2}{\lambda}$ (or both).

If $t_n\geqslant 1$, then $d_n\neq 1$ by Lemma 4.5. Thus, if $t_n\geqslant 1$, then

$$d-1\geqslant d_n\geqslant a_1d_1d_n+n_0\geqslant 2a_1+n_0$$

by Lemma 4.4. Therefore, if $t_n\geqslant 1$, then

$$2(d-1)\geqslant d-1+2a+n_0>\frac{4}{\lambda}$$

or $d-1\geqslant 2a+n_0>\frac{2}{\lambda}$, because $d-1+2a+n_0>\frac{4}{\lambda}$ or $2a+n_0>\frac{2}{\lambda}$. In both cases, we get $\lambda>\frac{d-1}{2}$, which is impossible by Lemma 4.2(i). This shows that $t_n=0$, so that $P\notin T_n$.

Since T_1 is smooth at P and $P\notin T_n$, there must be another irreducible component of T_P passing through P that is different from T_1 and T_n. In particular, we see that $n\geqslant 3$. Without loss of generality, we may assume that $P\in T_2$. Then T_2 is smooth at P by Lemma 4.7, so that $t_2=1$. Moreover, the curves T_1 and T_2 are tangent at P by Lemma 4.6, which implies that $d\geqslant 4$. Since $P_1\in T_1^1$, we see that $P_1\in T_2^1$ as well.

Put $\Omega=\sum_{i=3}^{n}a_iT_i+\Delta$ and $k_0=\mathrm{mult}_P(\Omega)$, so that $m_0=k_0+a_1+a_2$. Then $a_1d_1+a_2d_2\leqslant 1$ by Lemma 4.4.

Denote by Ω^1 the proper transform of the $\mathbb{Q}$-divisor Ω on the surface S_1. Put $k_1=\mathrm{mult}_{P_1}(\Omega^1)$. Then

$$\begin{aligned}d-1-2k_0&\geqslant d_1+d_2+\big(a_1d_1+a_2d_2\big)\big(d-d_1-d_2-1\big)-2k_0\\&=\Omega^1\cdot\Big(T_1^1+T_2^1\Big)\geqslant 2k_1\end{aligned}$$

because $a_1d_1+a_2d_2\leqslant 1$ and $d\geqslant d_1+d_2+d_n\geqslant d_1+d_2+1$. This gives $k_0+k_1\leqslant\frac{d-1}{2}$. On the other hand, we have

$$2a_1+2a_2+k_0+k_1=m_0+m_1>\frac{2}{\lambda}$$

by (4.3). Thus, we have

$$\begin{aligned}\frac{d+3}{2}&=2+\frac{d-1}{2}\geqslant 2\big(a_1d_1+a_2d_2\big)+\frac{d-1}{2}\geqslant 2a_1+2a_2+\frac{d-1}{2}\\&\geqslant 2a_1+2a_2+k_0+k_1>\frac{2}{\lambda}\end{aligned}$$

because $a_1d_1+a_2d_2\leqslant 1$. By Lemma 4.2(iii) this gives $d=4$. Thus, we have $\lambda=\frac{5}{8}$.

Since $d=4>n\geqslant 3$, we have $n=3$. Without loss of generality, we may assume that $d_1\leqslant d_2$. By Lemma 4.6, there exists at most one line in S that passes through P. This shows that $d_1=1$, $d_2=2$ and $d_3=1$. Thus, T_1 and T_3 are lines, T_2 is a conic, T_1 is tangent to T_2 at P, and T_3 does not pass through P. In particular, the curves T_1^1 and T_1^2 intersect each other transversally at P_1.

By Lemma 4.3, we have $T_1 \cdot T_1 = T_2 \cdot T_2 = -2$ and $T_1 \cdot T_2 = 2$. On the other hand, the log pair $(S_1, \lambda a_1 T_1^1 + \lambda a_2 T_2^1 + \lambda\Omega^1 + (\lambda(a_1 + a_2 + k_0) - 1)E_1)$ is not log canonical at the point P_1. Thus, applying Theorem 2.11 to this log pair and the curve T_1^1, we get

$$\lambda\big(1 + 2a_1 - 2a_2 - k_0\big) = \lambda\Omega^1 \cdot T_1^1 > 2 - \lambda(a_1 + a_2 + k_0) - \lambda a_2,$$

which implies that $3a_1 > \frac{2}{\lambda} - 1 = \frac{11}{5}$, because $\lambda = \frac{5}{8}$. Similarly, applying Theorem 2.11 to this log pair and the curve T_2^1, we get

$$\lambda\big(2 - 2a_1 + 2a_2 - k_0\big) = \lambda\Omega^1 \cdot T_2^1 > 2 - \lambda(a_1 + a_2 + k_0) - \lambda a_1,$$

which implies that $3a_2 > \frac{2}{\lambda} - 2 = \frac{6}{5}$. Hence, we have $a_1 > \frac{11}{15}$ and $a_2 > \frac{2}{5}$, which is impossible, since $a_1 + 2a_2 = a_1 d_1 + a_2 d_2 \leqslant 1$. The obtained contradiction completes the proof of the lemma. □

Now we are going to show that the curve T_P has at most two irreducible components. This follows from

Lemma 4.10 *One has* $n \geqslant 2$ *and* $\mathrm{mult}_P(T_P) = 2$. *Moreover, if* $n = 2$, *then* $P \in T_1 \cap T_2$, *both curves* T_1 *and* T_2 *are smooth at* P, *and* $d_1 \leqslant d_2$.

Proof If T_P is irreducible and $\mathrm{mult}_P(T_P) \geqslant 3$, then Lemma 2.5 gives

$$d = T_P \cdot D \geqslant 3m_0 > \frac{3}{\lambda},$$

which is impossible by Lemma 4.2(vi). Thus, if $n = 1$, then $\mathrm{mult}_P(T_P) = 2$.

To complete the proof, we may assume that $n \geqslant 2$. Then $t_n = 0$ or $t_n = 1$ by Lemma 4.5. In particular, there exists an irreducible component of the curve T_P different from T_n that passes through P. Without loss of generality, we may assume that $P \in T_1$.

Put $\Upsilon = \sum_{i=2}^{n} a_i T_i + \Delta$, and denote by Υ^1 the proper transform of the $\mathbb{Q}$-divisor Ω on the surface S_1. Put $n_0 = \mathrm{mult}_P(\Upsilon)$. Then the log pair $(S_1, \lambda\Upsilon^1 + (\lambda(n_0 + a_1 t_1) - 1)E_1)$ is not log canonical at P_1, since $P_1 \notin T_1^1$ by Lemma 4.9. In particular, it follows from Theorem 2.12 that

$$\lambda n_0 = \lambda\Upsilon^1 \cdot E_1 > 1,$$

which implies that $n_0 > \frac{1}{\lambda}$. Thus, if $t_1 \geqslant 2$, then it follows from Lemma 4.3 that

$$\frac{1}{\lambda} \geqslant \frac{d-1}{2} \geqslant \frac{d_1 + a_1 d_1(d - d_1 - 1)}{2} = \frac{\Upsilon \cdot T_1}{2} \geqslant \frac{t_1 n_0}{2} \geqslant n_0 > \frac{1}{\lambda},$$

because $a_1 \leqslant \frac{1}{d_1}$ by Lemma 4.4, and $\lambda \leqslant \frac{2}{d-1}$ by Lemma 4.2(i). This shows that $t_1 = 1$, so that the curve T_1 is smooth at P.

If $t_n = 1$ and $n \geqslant 3$, then

$$\frac{2}{\lambda} \geqslant d - 1 \geqslant d_1 + d_n + ad_1(d - d_1 - d_n - 1) = \Upsilon \cdot \Big(T_1 + T_n\Big) \geqslant 2n_0 > \frac{2}{\lambda}.$$

Thus, if $t_n = 1$, then $n = 2$. Vice versa, if $n = 2$, then $t_n = 1$, because T_1 is smooth at P. Furthermore, if $n = 2$, then $d_1 \leqslant d_n$, because $d_n > \frac{d-1}{2}$ by Lemma 4.5. Therefore, to complete the proof, we must show that $n = 2$.

Suppose that $n \geqslant 3$. Let us seek for a contradiction. We know that $P \notin T_n$, so that $t_n = 0$. Then every irreducible component of the curve T_P that contain P is smooth at P by Lemma 4.7. Hence, there should be at least one irreducible component of the curve T_P containing P that is different from T_1 and T_n. Without loss of generality, we may assume that $P \in T_2$.

Put $\Omega = \sum_{i=3}^{n} a_i T_i + \Delta$ and $k_0 = \mathrm{mult}_P(\Omega)$. By Lemma 4.4, we have $a_1 d_1 + a_2 d_2 \leqslant 1$. Thus, it follows from Lemma 4.3 that

$$\begin{aligned} 2k_0 \leqslant \Omega \cdot \Big(T_1 + T_2\Big) &= d_1 + d_2 + \Big(a_1 d_1 + a_2 d_2\Big)\Big(d - d_1 - d_2 - 1\Big) \\ &\leqslant d_1 + d_2 + \Big(d - d_1 - d_2 - 1\Big) = d - 1. \end{aligned}$$

Hence, we have $k_0 \leqslant \frac{d-1}{2}$.

Denote by Ω^1 the proper transform of the $\mathbb{Q}$-divisor Ω on the surface S_1. Then the log pair $(S_1, \lambda\Omega^1 + (\lambda(k_0 + a_1 + a_2) - 1)E_1)$ is not log canonical at P_1, because $P_1 \notin T_1^1$ and $P_1 \notin T_2^1$ by Lemma 4.9. In particular, it follows from Theorem 2.11 that

$$\lambda k_0 = \lambda\Omega^1 \cdot E_1 > 1,$$

which implies that $k_0 > \frac{1}{\lambda}$. This contradicts Lemma 4.2(i), because $k_0 \leqslant \frac{d-1}{2}$. □

Later, we will need the following simple

Lemma 4.11 *Suppose that $d = 4$. Then $m_0 \leqslant \frac{11}{5}$.*

Proof If $n = 1$, then

$$2t_n \geqslant d_n = T_n \cdot D \geqslant t_n m_0,$$

so that $m_0 \leqslant 2 < \frac{11}{5}$. Thus, we may assume that $n \neq 1$. Then it follows from Lemma 4.10 that $n = 2$, $P \in T_1 \cap T_2$, both curves T_1 and T_2 are smooth at P, and $d_1 \leqslant d_2$.

If $d_2 = 2$, then $m_0 \leqslant 2 < \frac{11}{5}$, because

$$2 = T_2 \cdot D \geqslant m_0.$$

Thus, we may assume that $d_2 \neq 2$. Then $d_1 = 1$ and $d_2 = 3$. Then $\mathrm{mult}_P(\Delta) + 3a_1 \leqslant 3$ by Lemma 4.4. Moreover, we have

$$1 + 2a_1 = T_1 \cdot \Delta \geqslant \mathrm{mult}_P(\Delta).$$

The obtained inequalities give $m_0 = \mathrm{mult}_P(\Delta) + a_1 \leqslant \frac{11}{5}$. □

Let $f_2 \colon S_2 \to S_1$ be a blow up of the point P_1. Denote by E_2 the f_2-exceptional curve, denote by E_1^2 the proper transform of the curve E_1 on the surface S_2, and denote by D^2 the proper transform of the $\mathbb{Q}$-divisor D on the surface S_2. Then

$$K_{S_2} + \lambda D^2 + \big(\lambda m_0 - 1\big)E_1^2 + \Big(\lambda\big(m_0 + m_1\big) - 2\Big)E_2 \sim_{\mathbb{Q}} f_2^*\Big(K_{S_1} + \lambda D^1 + \big(\lambda m_0 - 1\big)E_1\Big).$$

By Remark 2.10, the log pair $(S_2, \lambda D^2 + (\lambda m_0 - 1)E_1^2 + (\lambda(m_0 + m_1) - 2)E_2)$ is not log canonical at some point $P_2 \in E_1$.

Lemma 4.12 *One has* $m_0 + m_1 \leqslant \frac{3}{\lambda}$.

Proof Suppose that $m_0 + m_1 > \frac{3}{\lambda}$. Then $2m_0 \geqslant m_0 + m_1 > \frac{3}{\lambda}$. But $m_0 \leqslant \frac{d+1}{2}$ by Lemma 4.8. Then $\lambda > \frac{3}{d+1}$. Thus, we have $d \leqslant 4$ by Lemma 4.2(ii). Moreover, if $d = 4$, then

$$\frac{22}{5} \geqslant 2m_0 \geqslant m_0 + m_1 > \frac{3}{\lambda} = \frac{24}{5}$$

by Lemma 4.11. This shows that $d = 3$.

We have $\lambda = 1$. If $n = 1$, then

$$3 = T_P \cdot D \geqslant 2m_0 \geqslant m_1 + m_0 > \frac{3}{\lambda} = 3,$$

which is absurd. Hence, it follows from Lemma 4.10 that $n = 2$, $d_1 = 1$, $d_2 = 2$ and $P \in T_1 \cap T_2$.

We have $m_0 = \mathrm{mult}_P(\Delta) + a_1$. On the other hand, we have $\mathrm{mult}_P(\Delta) + 2a_1 \leqslant 2$ by Lemma 4.4. Moreover, we have

$$1 + a_1 = T_1 \cdot \Omega \geqslant \mathrm{mult}_P(\Delta),$$

which implies that $\mathrm{mult}_P(\Delta) - a_1 \leqslant 1$. Adding these inequalities, we get

$$3 \geqslant 2\mathrm{mult}_P(\Delta) + a = \mathrm{mult}_P(\Delta) + m_0 \geqslant m_1 + m_0 > \frac{3}{\lambda} = 3,$$

because $\mathrm{mult}_P(\Delta) \geqslant m_1$, since $P_1 \notin T_1^1$ by Lemma 4.9. □

Thus, the log pair $(S_2, \lambda D^2 + (\lambda m_0 - 1)E_1^2 + (\lambda(m_0 + m_1) - 2)E_2)$ is log canonical at every point of the curve E_2 that is different from the point P by Lemma 2.13.

Lemma 4.13 *One has* $P_2 \neq E_1^2 \cap E_2$.

Proof Suppose that $P_2 = E_1^2 \cap E_2$. Then Theorem 2.11 gives

$$\lambda\big(m_0 - m_1\big) = \lambda D^2 \cdot E_1^2 > 3 - \lambda\big(m_0 + m_1\big),$$

which implies that $m_0 > \frac{3}{2\lambda}$. But $m_0 \leqslant \frac{d+1}{2}$ by Lemma 4.8. Therefore, we have $\lambda > \frac{3}{d+1}$, which implies that $d \leqslant 4$ by Lemma 4.2(ii). If $d = 4$, then

$$\frac{12}{5} = \frac{3}{2\lambda} < m_0 \leqslant \frac{11}{5}$$

by Lemma 4.11. Thus, we have $d = 3$.

One has $\lambda = 1$. If $n = 1$, then

$$3 = T_P \cdot D \geqslant 2m_0 > \frac{3}{\lambda} = 3,$$

which is absurd. Hence, it follows from Lemma 4.10 that $n = 2$, $d_1 = 1$, $d_2 = 2$ and $P \in T_1 \cap T_2$.

We have $m_0 = \mathrm{mult}_P(\Delta) + a_1$. Moreover, we have $\mathrm{mult}_P(\Delta) + 2a_1 \leqslant 2$ by Lemma 4.4, Then $2\mathrm{mult}_P(\Delta) + a_1 \leqslant 3$, because

$$1 + a_1 = T_1 \cdot \Delta \geqslant \mathrm{mult}_P(\Delta).$$

Denote by Δ^1 the proper transform of the divisor Δ on the surface S_1, and denote by Δ^2 the proper transform of the divisor Δ on the surface S_2. Then $m_1 = \mathrm{mult}_{P_1}(\Delta^1)$, because $P_1 \notin T_1^1$ by Lemma 4.9. Thus, the log pair $(S_2, \lambda\Delta^2 + (m_0 - 1)E_1^2 + (m_0 + m_1 - 2)E_2)$ is not log canonical at P_2. Applying Theorem 2.11 to this pair and the curve E_1^2, we get

$$\mathrm{mult}_P(\Delta) - m_1 = \Delta^2 \cdot E_1^2 > 3 - m_0 - m_1,$$

which implies that $2\mathrm{mult}_P(\Delta) + a_1 > 3$. The latter is impossible, because we already proved that $2\mathrm{mult}_P(\Delta) + a_1 \leqslant 3$. □

Thus, the log pair $(S_2, \lambda D^2 + (\lambda(m_0 + m_1) - 2)E_2)$ is not log canonical at P_2. Then Lemma 2.5 gives

$$m_0 + m_1 + m_2 > \frac{3}{\lambda}. \tag{4.5}$$

Denote by T_P^2 the proper transform of the curve T_P on the surface S^2. Then

$$T_P^2 + E_1^2 \sim (f_1 \circ f_2)^*(\mathcal{O}_S(1)) - f_2^*(E_1) - E_2,$$

because $T_P^1 \sim f_1^*(\mathcal{O}_S(1)) - 2E_1$ by Lemma 4.10, and $P_1 \notin T_P^1$ by Lemma 4.9.

Lemma 4.14 *The linear system $|T_P^2+E_1^2|$ is a pencil that does not have base points in E_2.*

Proof Since $|T_P^1+E_1|$ is a two-dimensional linear system that does not have base points, $|T_P^2+E_1^2|$ is a pencil. Let C be a curve in $|T_P^1+E_1|$ that passes through P_1 and is different from $T_P^1+E_1$. Then C is smooth at P, since $P\in f_1(C)$ and $f_1(C)$ is a hyperplane section of the surface S that is different from T_P. Since $C\cdot E_1=1$, we see that $T_P^1+E_1$ and C intersect transversally at P_1. Thus, the proper transform of the curve C on the surface S_2 is contained in $|T_P^1+E_1|$ and have no common points with $T_P^2+E_1^2$ in E_2. This shows that the pencil $|T_P^1+E_1|$ does not have base points in E_2. □

Let Z^2 be the curve in $|T_P^2+E_2|$ that passes through the point P_2. Then

$$Z^2\neq T_P^2+E_1^2,$$

because $P_2\neq E_1^2\cap E_2$ by Lemma 4.13. Then Z_2 is smooth at P_2. Put $Z=f_1\circ f_2(Z^2)$ and $Z^1=f_2(Z^2)$. Then $P\in Z$ and $P_1\in Z^1$. Moreover, the curve Z is smooth at P, and the curve Z_1 is smooth at P_1. Furthermore, the curve Z is reduced by Lemma 2.6.

The log pair $(S,\lambda Z)$ is log canonical at P, because Z is smooth at P. Note that

$$Z\sim_{\mathbb{Q}} D.$$

Thus, we may assume that $\mathrm{Supp}(D)$ does not contain at least one irreducible component of the curve Z by Remark 2.4. Denote this irreducible component by $\overline{Z}$, and denote its degree in $\mathbb{P}^3$ by $\bar{d}$. Then $\bar{d}\leqslant d$.

Lemma 4.15 *One has $P\notin\overline{Z}$.*

Proof Suppose that $P\in\overline{Z}$. Let us seek for a contradiction. Denote by $\overline{Z}^2$ the proper transform of the curve $\overline{Z}$ on the surface S_2. Then

$$d-m_0-m_1\geqslant\bar{d}-m_0-m_1=\overline{Z}^2\cdot D^2\geqslant m_2,$$

which implies that $m_0+m_1+m_2\leqslant d$. One the other hand, $m_0+m_1+m_2>\frac{3}{\lambda}$ by (4.5). This gives $\lambda>\frac{3}{d}$, which is impossible by Lemma 4.2(vi). □

In particular, the curve Z is reducible. Denote by $\widehat{Z}$ its irreducible component that passes through P, denote its proper transform on the surface S_1 by $\widehat{Z}^1$, and denote its proper transform on the surface S_2 by $\widehat{Z}^2$. Then $\overline{Z}\neq\widehat{Z}$, $P_1\in\widehat{Z}^1$ and $P_2\in\widehat{Z}^2$. Denote by $\hat{d}$ the degree of the curve $\widehat{Z}$ in $\mathbb{P}^3$. Then $\hat{d}+\bar{d}\leqslant d$. Moreover, the intersection form of the curves $\widehat{Z}$ and $\overline{Z}$ on the surface S is given by

Lemma 4.16 *One has $\overline{Z}\cdot\overline{Z}=-\bar{d}(d-\bar{d}-1)$, $\widehat{Z}\cdot\widehat{Z}=-\hat{d}(d-\hat{d}-1)$ and $\overline{Z}\cdot\widehat{Z}=\bar{d}\hat{d}$.*

Proof See the proof of Lemma 4.3. □

Put $D = a\widehat{Z} + \Omega$, where a is a positive rational number, and Ω is an effective $\mathbb{Q}$-divisor on the surface S whose support does not contain the curve $\widehat{Z}$. Denote by Ω^1 the proper transform of the divisor Ω on the surface S_1, and denote by Ω^2 the proper transform of the divisor Ω on the surface S_2. Put $n_0 = \mathrm{mult}_P(\Omega)$, $n_1 = \mathrm{mult}_{P_1}(\Omega^1)$ and $n_2 = \mathrm{mult}_{P_2}(\Omega^2)$. Then $m_0 = n_0 + a$, $m_1 = n_1 + a$ and $m_2 = n_2 + a$. Then the log pair $(S_2, \lambda a\widehat{Z}^2 + \lambda\Omega^2 + (\lambda(n_0 + n_1 + 2a) - 2)E_2)$ is not log canonical at P_2, because $(S_2, \lambda D^2 + (\lambda(m_0 + m_1) - 2)E_2)$ is not log canonical at P_2. Thus, applying Theorem 2.11, we see that

$$\lambda\Big(\Omega \cdot \widehat{Z} - n_0 - n_1\Big) = \lambda\Omega^2 \cdot Z^2 > 1 - \Big(\lambda\big(n_0 + n_1 + 2a\big) - 2\Big) = 3 - \lambda\big(n_0 + n_1 + 2a\big),$$

which implies that

$$\Omega \cdot \widehat{Z} > \frac{3}{\lambda} - 2a. \tag{4.6}$$

On the other hand, we have

$$\bar{d} = D \cdot \overline{Z} = \Big(a\widehat{Z} + \Omega\Big) \cdot \overline{Z} \geqslant a\widehat{Z} \cdot \overline{Z} = a\hat{d}\bar{d}$$

by Lemma 4.16. This gives

$$a \leqslant \frac{1}{\hat{d}}. \tag{4.7}$$

Thus, it follows from (4.6), (4.7) and Lemma 4.16 that

$$\frac{3}{\lambda} - 2 \leqslant \frac{3}{\lambda} - 2a < \Omega \cdot \widehat{Z} = \hat{d} + a\hat{d}\Big(d - \hat{d} - 1\Big) \leqslant d - 1,$$

which implies that $\lambda > \frac{3}{d+1}$. Then $d \leqslant 4$ by Lemma 4.2(ii).

Lemma 4.17 *One has $d \neq 4$.*

Proof Suppose that $d = 4$. Then $\lambda = \frac{5}{8}$ and $\hat{d} \leqslant 3$. By Lemma 4.9, $\widehat{Z}$ is not a line, since every line passing through P must be an irreducible component of the curve T_P. Thus, either $\widehat{Z}$ is a conic or $\widehat{Z}$ is a plane cubic curve. If $\widehat{Z}$ is a conic, then $\widehat{Z}^2 = -2$ and $a \leqslant \frac{1}{2}$ by (4.7). Thus, if $\widehat{Z}$ is a conic, then

$$2 + 2a = \Omega \cdot \widehat{Z} > \frac{3}{\lambda} - 2a = \frac{24}{5} - 2a,$$

which implies that $\frac{1}{2} \geqslant a > \frac{7}{10}$. This shows that $\widehat{Z}$ is a plane cubic curve. Then $\widehat{Z}^2 = 0$. Since $a \leqslant \frac{1}{3}$ by (4.7), we have

$$3 = \Omega \cdot \widehat{Z} > \frac{3}{\lambda} - 2a = \frac{24}{5} - 2a \geqslant \frac{24}{5} - \frac{2}{3} = \frac{62}{15},$$

which is absurd. □

Thus, we see that $d = 3$. Then $\widehat{Z}$ is either a line or a conic. But every line passing through P must be an irreducible component of T_P. Since $\widehat{Z}$ is not an irreducible component of T_P by Lemma 4.9, the curve $\widehat{Z}$ must be a conic. Then $\widehat{Z}^2 = 0$. Therefore, it follows from (4.6) that

$$3 - 2a = \frac{3}{\lambda} - 2a < \Omega \cdot \widehat{Z} = \hat{d} + a\hat{d}\Big(d - \hat{d} - 1\Big) = \hat{d} = 2,$$

which implies that $a > \frac{1}{2}$. But $a \leqslant \frac{1}{\hat{d}} = \frac{1}{2}$ by (4.7). The obtained contradiction completes the proof of Theorem 1.17.

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