Rendiconti del Circolo Matematico di Palermo Series 2
https://doi.org/10.1007/s12215-024-01043-1

# On some pre-orders and partial orders of linear operators on infinite dimensional vector spaces

**Diego Alba Alonso**[1]

Received: 31 January 2024 / Accepted: 17 April 2024


**Abstract**

This article is devoted to the generalization of the Drazin pre-order and the G-Drazin partial order to Core-Nilpotent endomorphisms over arbitrary $k$-vector spaces, namely, infinite dimensional ones. The main properties of this orders are described, such as their respective characterizations and the relations between these orders and other existing ones, generalizing the existing theory for finite matrices. In order to do so, G-Drazin inverses are also studied in this framework. Also, it includes a generalization of the space pre-order to linear operators over arbitrary $k$-vector spaces.

**Keywords** Generalized inverse · Matrix partial order · Core-Nilpotent endomorphism · Space pre-order · Drazin pre-order · G-Drazin partial order

**Mathematics Subject Classification** 15A09 · 15A03 · 15A04 · 06A06

## Contents



## 1 Introduction

The theory of Matrix Partial Orders has been flourishing alongside advances in Matrix Generalized Inverses, with numerous applications during the past years. See for instance, [9], with the use of the sharp order for the study of autonomous systems, [20], where matrix partial orders are used to study matrix equations and inequations or [5], in which control systems are

✉ Diego Alba Alonso
daa29@usal.es

[1] Departamento de Matemáticas, Universidad de Salamanca, Plaza de la Merced 1-4, 37008 Salamanca, Spain

Published online: 10 May 2024

studied using this theories. The vast amount of its applications makes this a topic in which it is specially important to obtain as much information from each of the partial orders as possible. Therefore, following the philosophy of [1], this article presents the generalization of some of the most used pre-orders and partial orders to arbitrary vector spaces, usually infinite dimensional ones, over fields of arbitrary characteristic. This new insight has turned to be a powerful tool in such a way that, after generalizing the known theories, one can obtain new results for finite square matrices by specialization and one is able to proof the existing ones on more natural ways.

Let us consider an arbitrary $(n \times n)$-matrix A with entries in a general field $k$. In [11, Theorem 2.2.21], it has been proven that A can be written as the sum of two matrices $A_1$ and $A_2$ such that

- rk $(A_1) =$ rk $(A_1^2)$ (where rk denotes the rank of the matrix);
- $A_2$ is nilpotent;
- $A_1 A_2 = A_2 A_1 = 0$.

The matrices $A_1$ and $A_2$ are called the core and nilpotent parts of $A$, respectively, and this decomposition is unique. This ideas were generalized to arbitrary vector spaces (in general, of infinite dimension, see [13, 14], for details), leading to a new theory of Core-Nilpotent endomorphisms. The main idea in this notes is to generalize: the theory presented in [11, Section 3.2], about the space pre-order, to linear operators over arbitrary vector spaces, some of the results in [11, Section 4.4] dealing with the Drazin pre-order; to Core-Nilpotent endomorphisms and finally the G-Drazin partial order, [21, Section 3], to Core-Nilpotent endomorphisms.

The paper is organized as follows.

In Sect. 2, basic definitions and results of the theory of finite potent endomorphisms, Core-Nilpotent endomorphisms, Generalized Inverses and Matrix Partial Orders are summarized.

In Sect. 3, the space pre-order is generalized to linear operators over arbitrary $k$-vector spaces. It is worth noting that in Theorem 3.18 a characterization of the space pre-order is presented, showing its strong relation to the study of 1-inverses.

Section 4 deals with the generalization of the Drazin pre-order to Core-Nilpotent endomorphisms. Briefly, it contains the relation of this pre-order with the sharp order (and to $g$-commuting inverses), with the Drazin inverse and the class of all endomorphisms above a Core-Nilpotent endomorphism from this order is presented. Also, the relations between the AST-decompositions of two Core-Nilpotent endomorphisms related by this order is studied in depth, reaching a complete description of how the $k$-vector spaces are decomposed, see Propositions 4.16 and 4.19.

As one of the main objectives of this work was to study the G-Drazin partial order in the framework of Core-Nilpotent endomorphisms, the study of G-Drazin inverses for Core-Nilpotent endomorphisms was mandatory. Section 5 is related to this task. It contains a characterization of G-Drazin inverses in terms of the AST-decomposition induced in a vector space by a Core-Nilpotent endomorphism.

In Sect. 6 the G-Drazin partial order is generalized to Core-Nilpotent endomorphisms. Among other results, it contains the relation between this partial order and other studied pre-orders with constraints, several characterizations of the order, the relation between the G-Drazin inverses of two Core-Nilpotent endomorphisms related by the order... The proof that this order is indeed a partial order on the set of Core-Nilpotent endomorphisms (Theorem 6.14) relies heavily on all the previous study of the space pre-order, G-Drazin inverses and the exhaustive description of the AST-decompositions.

In Sect. 7 some remarks and considerations are included, as well as some open questions dealing with the previous work.

As far as the author knows, the study of all these topics (using linear algebra techniques) on infinite dimensional vector spaces is not present in literature. It is also worth mentioning that all the generalizations done here are compatible with the theory exposed in literature for finite matrices.

## 2 Preliminaries

Firstly, let us point out some notations that will appear all through the monograph. The use of $k$ will denote a field of arbitrary characteristic, $V$ will stand for an arbitrary $k$-vector space (in general, infinite dimensional) and, fixing an endomorphism $\varphi \in \mathrm{End}_k(V)$, then $X_\varphi(1)$, $X_\varphi(1,2)$, $X_\varphi(g_-)$, $X_\varphi(GD)$, denote the sets of 1-inverses, reflexive generalized inverses, $g$-commuting inverses and G-Drazin inverses of the endomorphism $\varphi$, in whatever way one could define them.

### 2.1 Finite potent endomorphisms

Let $k$ be an arbitrary field and let $V$ be a $k$-vector space. Let us now consider an endomorphism $\varphi$ of $V$. We say that $\varphi$ is "finite potent" if $\varphi^n V$ is finite dimensional for some $n$. This definition was introduced by Tate [19] as a basic tool for his elegant definition of Abstract Residues.

In 2007, M. Argerami, F. Szechtman and R. Tifenbach showed in [2] that an endomorphism $\varphi$ is finite potent if and only if $V$ admits a $\varphi$-invariant decomposition $V = U_\varphi \oplus W_\varphi$ such that $\varphi_{|U_\varphi}$ is nilpotent, $W_\varphi$ is finite dimensional and $\varphi_{|W_\varphi} : W_\varphi \overset{\sim}{\longrightarrow} W_\varphi$ is an isomorphism.

Indeed, if $k[x]$ is the algebra of polynomials in the variable $x$ with coefficients in $k$, we may view $V$ as a $k[x]$-module via $\varphi$, and the explicit definition of the above $\varphi$-invariant subspaces of $V$ is:

- $U_\varphi = \{v \in V \text{ such that } \varphi^m(v) = 0 \text{ for some } m\}$;
- $W_\varphi = \left\{\begin{array}{l} v \in V \text{ such that } p(\varphi)(v) = 0 \text{ for some } p(x) \in k[x] \\ \text{relatively prime to } x \end{array}\right\}$.

Note that if the annihilator polynomial of $\varphi$ is $x^m \cdot p(x)$ with $(x, p(x)) = 1$, then $U_\varphi = \mathrm{Ker}\, \varphi^m$ and $W_\varphi = \mathrm{Ker}\, p(\varphi)$.

Hence, this decomposition is unique. We shall call this decomposition the $\varphi$-invariant AST-decomposition of $V$.

Moreover, we shall call "index of $\varphi$", $i(\varphi)$, to the nilpotent order of $\varphi_{|U_\varphi}$. One has that $i(\varphi) = 0$ if and only if $V$ is a finite-dimensional vector space and $\varphi$ is an automorphism.

Basic examples of finite potent endomorphisms are all endomorphisms of a finite-dimensional vector spaces and finite rank or nilpotent endomorphisms of infinite-dimensional vector spaces.

### 2.2 Core-Nilpotent endomorphisms

Let us consider a square matrix $A$ with entries in a general field $k$. The index of $A$, $i(A) \geq 0$, is the smallest integer such that $\mathrm{rk}(A^{i(A)}) = rk(A^{i(A)+1})$. In [11, Theorem 2.2.21], it has been proven that $A$ can be written as the sum of two matrices $A_1$ and $A_2$ such that

- $\mathrm{rk}(A_1) = \mathrm{rk}(A_1^2)$ i.e. $i(A_1) \leq 1$;
- $A_2$ is nilpotent;
- $A_1 A_2 = A_2 A_1 = 0$.

The matrices $A_1$ and $A_2$ are called the core and nilpotent parts of $A$, respectively, and this decomposition is unique.

If $k$ is a field, given a finite-dimensional $k$-vector space $E$ we can define the index of an endomorphism $f \in \mathrm{End}_k(E)$, $i(f)$, as the smallest integer such that $\mathrm{Im}\, f^{i(f)} = \mathrm{Im}\, f^{i(f)+1}$. Bearing in mind the well known relationship between endomorphisms and matrices one can easily translate the previous theory to endomorphisms on a finite dimensional $k$-vector space.

Let us now consider an arbitrary $k$-vector space $V$ (in general, infinite dimensional). In [14], the general theory of core-nilpotent endomorphisms of arbitrary vector spaces was developed.

If we denote by $Aut_k(V)$ the group of $k$-linear automorphisms of a vector space $V$ then one has the following:

**Definition 2.1** [14, Definition 3.2] An endomorphism $\varphi \in \mathrm{End}_k(V)$ has index 0 when $\varphi \in Aut_k(V)$. Given $m \in \mathbb{N}$, an endomorphism $\varphi \in \mathrm{End}_k(V)$ has index $m$ when $\mathrm{Ker}\, \varphi^m \neq \mathrm{Ker}\, \varphi^{m-1}$ or $\mathrm{Im}\, \varphi^m \neq \mathrm{Im}\, \varphi^{m-1}$, $\mathrm{Ker}\, \varphi^m = \mathrm{Ker}\, \varphi^{m+1}$ and $\mathrm{Im}\, \varphi^m = \mathrm{Im}\, \varphi^{m+1}$. The index of an endomorphism $\varphi$ is denoted by $i(\varphi)$.

**Definition 2.2** [14, Definition 3.4] We say that an endomorphism $\varphi \in \mathrm{End}_k(V)$ is core-nilpotent (CN-endomorphism) when there exist two endomorphisms $\varphi_1, \varphi_2 \in \mathrm{End}_k(V)$ such that

- $\varphi = \varphi_1 + \varphi_2$;
- $i(\varphi_1) \leq 1$;
- $\varphi_2$ is nilpotent;
- $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1 = 0$.

Basic examples of CN-endomorphisms are all endomorphisms of a finite dimensional vector space, finite potent endomorphisms, automorphisms and nilpotent endomorphisms of infinite-dimensional vector spaces. Several results were given, as the characterization of CN-endomorphisms:

**Theorem 2.3** [14, Theorem 3.6] *If $\varphi \in \mathrm{End}_k(V)$, then the following conditions are equivalent:*

(1) *$\varphi$ is a CN-endomorphism.*
(2) $\mathrm{Ker}\, \varphi^m = \mathrm{Ker}\, \varphi^{m+1}$ *and* $\mathrm{Im}\, \varphi^m = \mathrm{Im}\, \varphi^{m+1}$ *for a certain* $m \in \mathbb{N}$.
(3) $V = \mathrm{Ker}\, \varphi^m \oplus \mathrm{Im}\, \varphi^m$ *for a certain* $m \in \mathbb{N}$.
(4) *There exists a unique decomposition $V = W_\varphi \oplus U_\varphi$, where $W_\varphi$ and $U_\varphi$ are $\varphi$-invariant $k$-subspaces of $V$, $\varphi_{|W_\varphi} \in Aut_k(W_\varphi)$ and $\varphi_{|U_\varphi}$ is nilpotent.*

One obtains as a corollary that the core-nilpotent decomposition of a core-nilpotent endomorphism is unique.

Following the parallelism with matrix theory, we will refer to $\varphi_1$ or $\varphi_{|W_\varphi}$ as the **core part** of the endomorphism and to $\varphi_2$ or $\varphi_{|U_\varphi}$ as the **nilpotent part** of the endomorphism.

From this characterization, several properties of Core-Nilpotent endomorphisms were studied, in particular, it was shown that:

**Lemma 2.4** [14, Lemma 3.13] *The index of an endomorphism $\varphi \in \mathrm{End}_k(V)$ exists if and only if $\varphi$ is a Core-Nilpotent endomorphism.*

### 2.2.1 CN decomposition of a finite potent endomorphism

Let $V$ be again an arbitrary $k$-vector space. Given a finite potent endomorphism $\varphi \in \operatorname{End}_k(V)$, there exists a unique decomposition $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1, \varphi_2 \in \operatorname{End}_k(V)$ are finite potent endomorphisms satisfying that:

- $i(\varphi_1) \leq 1$;
- $\varphi_2$ is nilpotent;
- $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1 = 0$.

Also, the following hold:

$$\varphi = \varphi_1 \iff U_\varphi = \operatorname{Ker} \varphi \iff W_\varphi = \operatorname{Im} \varphi \iff i(\varphi) \leq 1. \tag{2.1}$$

Moreover, if $V = W_\varphi \oplus U_\varphi$ is the AST-decomposition of $V$ induced by $\varphi$, then $\varphi_1$ and $\varphi_2$ are the unique linear maps such that:

$$\varphi_1(v) = \begin{cases} \varphi(v) & \text{if } v \in W_\varphi \\ 0 & \text{if } v \in U_\varphi \end{cases} \quad \text{and} \quad \varphi_2(v) = \begin{cases} 0 & \text{if } v \in W_\varphi \\ \varphi(v) & \text{if } v \in U_\varphi \end{cases}. \tag{2.2}$$

### 2.2.2 Jordan bases of a nilpotent endomorphism

Let $V$ be a vector space over an arbitrary field $k$ and let $g \in \operatorname{End}_k(V)$ be a nilpotent endomorphism. If $m$ is the nilpotency index of $g$, according to the statements of [12], setting $U_i^g = \operatorname{Ker} g^i/[\operatorname{Ker} g^{i-1} + g(\operatorname{Ker} g^{i+1})]$ with $i \in \{1, 2, \dots, m\}$, $\mu_i(V, g) = \dim_k U_i^g$ and $S_{\mu_i(V,g)}$ a set such that $\# S_{\mu_i(V,g)} = \mu_i(V, g)$ with $S_{\mu_i(V,g)} \cap S_{\mu_j(V,g)} = \emptyset$ for all $i \neq j$, one has that there exists a family of vectors $\{v_{s_i}\}$ that determines a Jordan basis of $g$:

$$B = \bigcup_{\substack{s_i \in S_{\mu_i(V,g)} \\ 1 \leq i \leq m}} \{v_{s_i}, g(v_{s_i}), \dots, g^{i-1}(v_{s_i})\}. \tag{2.3}$$

Moreover, if we write $H_{s_i}^g = \langle v_{s_i}, g(v_{s_i}), \dots, g^{i-1}(v_{s_i}) \rangle$, the basis $B$ induces a decomposition

$$V = \bigoplus_{\substack{s_i \in S_{\mu_i(V,g)} \\ 1 \leq i \leq m}} H_{s_i}^g. \tag{2.4}$$

### 2.2.3 Bases of a finite potent endomorphism

Let us now consider a finite potent endomorphism $\varphi \in \operatorname{End}_k(V)$ with CN-decomposition $\varphi = \varphi_1 + \varphi_2$ and that induces the AST-decomposition $V = U_\varphi \oplus W_\varphi$. Keeping the above notation, if $m$ is the nilpotency order of $\varphi_2$, we can construct a basis $B_V = B_{W_\varphi} \cup B_{U_\varphi}$ of $V$ where

$$B_{W_\varphi} = \{w_1, \dots, w_r\}$$

is a basis of $W_\varphi$ ($r = \dim_k W_\varphi$) and

$$B_{U_\varphi} = \bigcup_{\substack{s_i \in S_{\mu_i(U_\varphi,\varphi)} \\ 1 \le i \le m}} \{v_{s_i}, \varphi(v_{s_i}), \dots, \varphi^{i-1}(v_{s_i})\}$$

is a Jordan basis of $U_\varphi$ determined by $\varphi_{|U_\varphi}$.

If $\varphi = \varphi_1 + \varphi_2$ is the CN-decomposition of $\varphi$, it is clear that

$$B_{U_\varphi} = \bigcup_{\substack{s_i \in S_{\mu_i(U_\varphi,\varphi)} \\ 1 \le i \le m}} \{v_{s_i}, \varphi_2(v_{s_i}), \dots, \varphi_2^{i-1}(v_{s_i})\}$$

and

$$\operatorname{Ker}\varphi = \bigoplus_{\substack{s_i \in S_{\mu_i(U_\varphi,\varphi)} \\ 1 \le i \le m}} \langle \varphi^{i-1}(v_{s_i})\rangle = \bigoplus_{\substack{s_i \in S_{\mu_i(U_\varphi,\varphi)} \\ 1 \le i \le m}} \langle \varphi_2^{i-1}(v_{s_i})\rangle.$$

### 2.3 Generalized inverses

If $A \in \operatorname{Mat}_{n\times m}(k)$ is a matrix with entries in an arbitrary field $k$, a matrix $A^- \in \operatorname{Mat}_{m\times n}(k)$ is a 1-inverse of $A$ when $AA^-A = A$. Moreover, we say that a matrix $A^+ \in \operatorname{Mat}_{m\times n}(k)$ is a reflexive generalized inverse of $A$ when $A^+$ is a 1-inverse of $A$ and $A$ is a 1-inverse of $A^+$, this is, $AA^+A = A$ and $A^+AA^+ = A^+$. Similarly, given two $k$-vector spaces $V$ and $W$ and a linear map $\varphi\colon V \to W$, we will say thay a morphism $\varphi^-\colon W \to V$ is a 1-inverse of $\varphi$ when $\varphi \circ \varphi^- \circ \varphi = \varphi$ and that a linear map $\varphi^+\colon W \to V$ is a reflexive generalized inverse of $\varphi$ when $\varphi^+$ is a 1-inverse of $\varphi$ and $\varphi$ is a 1-inverse of $\varphi^+$.

#### 2.3.1 Drazin generalized inverse

Given a matrix $A \in \operatorname{Mat}_{n\times n}(k)$, M.P Drazin studied, in [4], the existence of an unique matrix that he denoted $A^D \in \operatorname{Mat}_{n\times n}(k)$ and later was named in his honour as Drazin inverse, satisfying the following equations:

$$\begin{aligned} A^D A A^D &= A^D; \\ A^D A &= A A^D; \\ A^{m+1} A^D &= A^m, \text{ for } i(A) = m. \end{aligned} \tag{2.5}$$

S.L. Campbell, in 1976, approached firstly the study of a Drazin inverse for ”infinite-dimensional” complex matrices in [3]. Considering an arbitrary $k$-vector space and a finite potent endomorphism, F. Pablos Romo proved the existence and uniqueness of the Drazin inverse of a finite potent endomorphism in [16], as well as some important properties that are briefly recalled here:

**Theorem 2.5** [16, Theorem 3.4] *For each finite potent endomorphism* $\varphi \in \mathrm{End}_k(V)$, *there exists an unique finite potent endomorphism* $\varphi^D \in \mathrm{End}_k(V)$ *satisfying:*

$$\begin{aligned} \varphi^D \varphi \varphi^D &= \varphi^D; \\ \varphi^D \varphi &= \varphi \varphi^D; \\ \varphi^{m+1} \varphi^D &= \varphi^m, . \end{aligned} \tag{2.6}$$

*for* $i(\varphi) = m$ *the index of* $\varphi$ *(Definition 2.1).*

Thus, if the AST-decomposition induced by $\varphi$ is $V = W_\varphi \oplus U_\varphi$, the unique linear map $\varphi^D$ that could satisfy the three conditions of the previous theorem is:

$$\varphi^D(v) = \begin{cases} (\varphi_{|W_\varphi})^{-1}(v) & \text{if } \ v \in W_\varphi \\ 0 & \text{if } \ v \in U_\varphi \end{cases} \tag{2.7}$$

for any $v \in V$.

Indeed:

**Theorem 2.6** [16, Theorem 4.8] *An endomorphism* $\varphi \in \mathrm{End}_k(V)$ *has a Drazin inverse if and only if* $\varphi$ *is a Core-Nilpotent endomorphism.*

As a direct result of this theorem the uniqueness of the Drazin inverse for Core-Nilpotent endomorphisms is guaranteed.

### 2.3.2 Group inverse

Given a matrix $A \in \mathrm{Mat}_{n\times n}(k)$, it is known that the system of equations:

$$\begin{aligned} AXA &= A; \\ XAX &= X; \\ AX &= XA, \end{aligned} \tag{2.8}$$

has a solution if and only if $i(A) \leq 1$ and the solution is unique. The solution to this system is known as the group inverse of $A$ and it is denoted as $A^\#$. Following the philosophy of the previous section, the author of [15], generalized the notion of group inverse of a matrix to arbitrary $k$-vector spaces. Consequently, given an arbitrary $k$-vector space, he proved that given a finite potent endomorphism $\varphi \in \mathrm{End}_k(V)$, if there exists a group inverse $\varphi^\# \in \mathrm{End}_k(V)$, then $i(\varphi) \leq 1$, see [15] for details. Moreover, the following theorem was given:

**Theorem 2.7** *If* $\varphi \in \mathrm{End}_k(V)$ *is a finite potent endomorphism with* $i(\varphi) \leq 1$, *then* $\varphi^D = \varphi^\#$ *is the unique group inverse of* $\varphi$, *where* $\varphi^D$ *is its Drazin inverse.*

### 2.3.3 G-Drazin inverse of a finite matrix

Given $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ with $i(A) = m$, H.Wang and X.Liu introduced in [21] the concept of *G-Drazin inverse* of $A$ as a solution $X$ of the system

$$\begin{aligned} AXA &= A; \\ XA^{m+1} &= A^m; \\ A^{m+1}X &= A^m, \end{aligned} \tag{2.9}$$

where $X$ is a $(n \times n)$-matrix with entries in $\mathbb{C}$.

Given the following matricial system:

$$\begin{aligned} AXA &= A; \\ XA^m &= A^m X \end{aligned} \tag{2.10}$$

C. Coll, M.Lattanzi and N.Thome proved in [6] that a matrix $X \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ is a solution of the system (2.9) if and only if it is a solution of system (2.10).

In fact, recall that if $J$ is the Jordan matrix associated with $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$, such that $A = B \cdot J \cdot B^{-1}$, with $B$ being a non-singular matrix and

$$J = \begin{pmatrix} J_1 & 0 \\ 0 & J_0 \end{pmatrix},$$

with $J_0$ and $J_1$ being the parts of $J$ corresponding to zero and non-zero eigenvalues respectively, one can obtain a $G$-Drazin inverse as

$$A^{GD} = B \cdot \begin{pmatrix} J_1^{-1} & 0 \\ 0 & J_0^{-} \end{pmatrix} \cdot B^{-1},$$

where $J_0^-$ is a 1-inverse of $J_0$.

### 2.3.4 1-inverses

Recently, F. Pablos Romo and the author of the present paper have been able to offer a method for the explicit computation of all 1-inverses of an arbitrary square matrix. The main tool to obtain this algorithm was the characterization of the set of 1-inverses of a finite potent endomorphism. Let us briefly recall one result presented in [17] that will enable us to generalize the characterization of the set of 1-inverses for Core-Nilpotent endomorphisms.

Let us consider a finite potent endomorphism $\varphi \in \mathrm{End}_k(V)$ that induces an AST-decomposition $V = W_\varphi \oplus U_\varphi$. Bearing in mind the basis of a finite potent endomorphism, Sect. 2.2.3, and the well known definition of 1-inverse (to witness, $\varphi^- \in \mathrm{End}_k(V)$ is a 1-inverse of $\varphi$ if $\varphi\varphi^-\varphi = \varphi$):

**Proposition 2.8** [17, Proposition 3.3] *If $\varphi \in \mathrm{End}_k(V)$ is a finite potent endomorphism, then an endomorphism $\varphi^- \in \mathrm{End}_k(V)$ is a 1-inverse of $\varphi$ if and only if $\varphi^-$ satisfies that*

- $\varphi^-(w_h) = (\varphi_{|W_\varphi})^{-1}(w_h) + u_h$ *for each* $h \in \{1, \dots, r\}$;
- $\varphi^-(\varphi^j(v_{s_i})) = \varphi^{j-1}(v_{s_i}) + u^j_{s_i}$ *for every* $s_i \in S_{\mu_i(U_\varphi,\varphi)}$ *and* $j \in \{1, \dots, i-1\}$;
- $\varphi^-(v_{s_i}) = \tilde{v}_{s_i}$ *for every* $s_i \in S_{\mu_i(U_\varphi,\varphi)}$;

*where $\tilde{v}_{s_i} \in V$ and $u_h, u^j_{s_i} \in \mathrm{Ker}\,\varphi$ for each $h \in \{1, \dots, r\}$ and for every $s_i \in S_{\mu_i(U_\varphi,\varphi)}$ and $j \in \{1, \dots, i-1\}$.*

Let us point that Core-Nilpotent endomorphisms have not been classified. More precisely, bearing in mind the characterization of Core-Nilpotent endomorphisms presented in statement (4) of Theorem 2.3 and the fact that nilpotent endomorphisms over arbitrary $k$-vector spaces have been classified (recall Sect. 2.2.2), arbitrary isomorphisms of $k$-vector spaces have not been classified. In other terms, we do not have a Jordan basis for them. For instance, if we consider the $\mathbb{R}$-vector space of countable dimension

$$V = \bigoplus_{i\in\mathbb{N}} < v_i >;$$

then the endomorphism $\varphi\colon k^{\mathbb{N}} \to k^{\mathbb{N}}$ defined as:

$$\varphi(v_i) = \begin{cases} i \cdot v_i & \text{if } i > 1 \\ 0 & \text{if } i = 1 \end{cases},$$

is a Core-Nilpotent endomorphism as $V = \operatorname{Ker}\varphi \oplus \operatorname{Im}\varphi$ but it does not admit annihilator polynomial as it has infinite eigenvalues.

Despite this fact, the idea that $\varphi_{|W_\varphi}$ still remains an isomorphism of $k$-vector spaces is still a powerful tool to work. In particular, it enables to characterize the 1-inverses of a Core-Nilpotent endomorphism in an analogous way as the mentioned Proposition 2.8.

**Proposition 2.9** *If $\varphi \in \operatorname{End}_k(V)$ is a Core-Nilpotent endomorphism, then an endomorphism $\varphi^- \in \operatorname{End}_k(V)$ is a 1-inverse of $\varphi$ if and only if $\varphi^-$ satisfies that*

- $\varphi^-(w_z) = (\varphi_{|W_\varphi})^{-1}(w_z) + u_z$ *for each* $w_z \in W_\varphi$*;*
- $\varphi^-(\varphi^j(v_{s_i})) = \varphi^{j-1}(v_{s_i}) + u^j_{s_i}$ *for every* $s_i \in S_{\mu_i(U_\varphi,\varphi)}$ *and* $j \in \{1, \dots, i-1\}$*;*
- $\varphi^-(v_{s_i}) = \tilde{v}_{s_i}$ *for every* $s_i \in S_{\mu_i(U_\varphi,\varphi)}$*;*

*where $\tilde{v}_{s_i} \in V$ and $u_z, u^j_{s_i} \in \operatorname{Ker}\varphi$ for every $s_i \in S_{\mu_i(U_\varphi,\varphi)}$ and $j \in \{1, \dots, i-1\}$.*

***Remark 2.10*** However, we must point out that one can not obtain results in the line of [17, Theorem 3.4], about the structure of the set of 1-inverses of Core-Nilpotent endomorphisms, because for obvious reasons we do not have tools to give information about the cardinal of $\varphi_{|W_\varphi}$ in a general way.

### 2.3.5 1-Inverses of nilpotent endomorphisms on arbitrary vector spaces

Let $V$ denote an arbitrary $k$-vector space and let $\varphi \in \operatorname{End}_k(V)$ be a nilpotent endomorphism, this is, there exists some non-negative integer $m$ such that $\varphi^m = 0$ (where we denote $\varphi^m = \varphi \circ \overset{m}{\cdots} \circ \varphi$). With this conditions, the AST decomposition of $V$ is $V = U_\varphi$. Hence, following the notations of Sect. 2.2.2, we know that a basis of $V$ is:

$$B_V = \bigcup_{\substack{s_i \in S_{\mu_i(V,g)} \\ 1 \leq i \leq m}} \{v_{s_i}, \varphi(v_{s_i}), \dots, \varphi^{i-1}(v_{s_i})\}.$$

for certain family of vectors $\{v_{s_i}\}$. Note that nilpotent endomorphisms over arbitrary $k$-vector spaces are examples of Core-Nilpotent endomorphisms and finite potent endomorphisms. Therefore, from Proposition 2.9 we immediately deduce that:

**Corollary 2.11** *If $\varphi \in \operatorname{End}_k(V)$ is a nilpotent endomorphism, then an endomorphism $\varphi^- \in \operatorname{End}_k(V)$ is a 1-inverse of $\varphi$ if and only if $\varphi^-$ satisfies the following conditions:*

- $\varphi^-(\varphi^j(v_{s_i})) = \varphi^{j-1}(v_{s_i}) + u^j_{s_i}$;
- $\varphi^-(v_{s_i}) = \tilde{v}_{s_i}$;

*for every $s_i \in S_{\mu_i(U_\varphi,\varphi)}$, where $\tilde{v}_{s_i} \in U_\varphi$ and $u^j_{s_i} \in \operatorname{Ker}\varphi$ for every $s_i \in S_{\mu_i(U_\varphi,\varphi)}$ and $j \in \{1, \dots, i-1\}$.*

### 2.4 Matrix partial orders

Let us recall some of the main matrix partial orders that are going to be present all along this article. Note that a binary relation is called a pre-order if it is reflexive and transitive and it is called a partial order if it is reflexive, transitive and antisymmetric on a non-empty set. As it is usual in literature, we will denote by $\mathcal{I}_{1,m}$ the set of all $m \times m$ matrices of index $\leq 1$. For matrices $A, B \in \mathrm{Mat}_{m\times m}(k)$, we say:

- $A$ is below $B$ under the minus partial order and it is usually denoted as $A <^{-} B$, if $A^{-}A = A^{-}B$ and $AA^{=} = BA^{=}$, for some 1-inverses of $A$ $A^{-}$, $A^{=}$. [8].
- $A$ is below $B$ under the sharp partial order i.e. $A <^{\#} B$, if $A, B \in \mathcal{I}_{1,m}$, $A^{\#}A = A^{\#}B$ and $AA^{\#} = BA^{\#}$ [1].
- $A$ is below $B$ under the C-N partial order i.e. $A <^{\#,-} B$, if $A, B \in \mathcal{I}_{1,m}$, $A_1 <^{\#} B_1$ and $A_2 <^{-} B_2$, in which $A = A_1 + A_2$ and $B = B_1 + B_2$ are the core-nilpotent decompositions of $A$ and $B$, respectively. This order was introduced by Mitra and Hartwig [10], and it is known that it implies the minus partial order [10, 11].
- $A$ is below $B$ under the Drazin pre-order i.e. $A <^{D} B$, if $A^{D}A = A^{D}B$, $AA^{D} = BA^{D}$, where $A^{D}$ denotes de Drazin inverse of $A$ (see [11] for details).

#### 2.4.1 G-Drazin partial order for matrices

Let us briefly recall the main ideas of the $G$-Drazin partial order, which was introduced in [21, Section 3]. It will be later studied in depth for Core-Nilpotent endomorphisms in Sect. 6.

**Definition 2.12** Let $A, B \in \mathrm{Mat}_{m\times m}(\mathbb{C})$. Then $A$ is said to be below $B$ under the $G$-Drazin order if there exist $G$-Drazin inverses $A^{-}_{GD}$ and $A^{=}_{GD}$ of $A$ such that

$$\begin{aligned} A^{-}_{GD}A &= A^{-}_{GD}B;\\ AA^{=}_{GD} &= BA^{=}_{GD}. \end{aligned} \tag{2.11}$$

When $A$ is below $B$ under the $G$-Drazin order, we denote $A <^{GD} B$. It can be seen that if $A^{-}_{GD}, A^{=}_{GD} \in A\{GD\}$ (where we are denoting by $A\{GD\}$ the set of $G$-Drazin inverses of matrix $A$) then

$$A^{GD} = A^{=}_{GD} \cdot A \cdot A^{-}_{GD} \in A\{GD\}.$$

This result enables to rewrite the conditions of Definition 2.11 and to obtain the first characterization of the order. It follows that $A <^{GD} B$ if and only if there exists a $G$-Drazin inverse $A^{GD}$ such that $A^{GD}A = A^{GD}B$ and $AA^{GD} = BA^{GD}$.

Moreover, the following characterization of the $G$-Drazin order was offered:

**Theorem 2.13** [21, Theorem 3.1] *Let $A, B \in \mathrm{Mat}_{m\times m}(\mathbb{C})$ and let the Jordan decomposition of $A$ be expressed as:*

$$A = P \cdot \begin{pmatrix} C & 0 \\ 0 & N \end{pmatrix} P^{-1},$$

*where $P$ and $C$ are nonsingular and $N$ is nilpotent. Then $A <^{GD} B$ if and only if*

$$B = P \cdot \begin{pmatrix} C & 0 \\ 0 & B_4 \end{pmatrix} \cdot P^{-1},$$

*where $N <^{-} B_4$.*

Moreover, in the same paper, it was proven that the order given in Definition 2.12 was indeed a matrix partial order, for details see [21, Theorem 3.3].

### 2.4.2 Sharp partial order for Core-Nilpotent endomorphisms

In [1], the author of the present paper generalized the study of the sharp partial order of matrices to Core-Nilpotent endomorphisms. Firstly, in order to do so, the $g$-commuting inverses of Core-Nilpotent endomorphisms where characterized and studied in depth.

**Definition 2.14** An endomorphism $\psi \in \operatorname{End}_k(V)$ is a $g$-commuting inverse of $\varphi$ when:

$$\varphi \circ \psi \circ \varphi = \varphi;$$
$$\varphi \circ \psi = \psi \circ \varphi.$$

Some of the results that are present in that article and that should be recalled now are the following:

**Definition 2.15** [1, Definition 4.2] Let us consider two Core-Nilpotent endomorphisms $\varphi, \psi \in \operatorname{End}_k(V)$ with $i(\varphi) = i(\psi) \leq 1$. We say that $\varphi$ is under $\psi$ for the sharp order if there exist Core-Nilpotent $g$-commuting inverses $g_1, g_2$ of $\varphi$ such that $\varphi g_1 = \psi g_1$ and $g_2\varphi = g_2\psi$.

When $\varphi$ is below $\psi$ under the sharp order, we write, as usual, $\varphi <^{\#} \psi$.

Some characterizations of the sharp order that were offered in [1] are:

**Proposition 2.16** [1, Proposition 4.5, Corollary 4.6] *Let $\varphi$ and $\psi$ be two Core-Nilpotent endomorphisms. Then,*

- $\varphi <^{\#} \psi$ *if and only if* $\varphi \circ \varphi^{\#} = \psi \circ \varphi^{\#} = \varphi^{\#} \circ \psi = \varphi^{\#} \circ \varphi$;
- $\varphi <^{\#} \psi$ *if and only if* $\varphi^2 = \varphi\psi = \psi\varphi$.

Moreover, one has that:

**Theorem 2.17** [1, Theorem 4.18] *The relation $<^{\#}$ (Definition 2.15) defines a partial order in the set of Core-Nilpotent endomorphisms of index* $\leq 1$.

## 3 Space pre-order for linear operators on arbitrary vector spaces

This section contains a generalization of the Space pre-order to linear operators over arbitrary $k$-vector spaces. This matrix pre-order was introduced in [11, Section 3.2] as a tool to study most on the matrix partial orders that include 1-inverses on their respective definitions. The original definition of the order involves the transpose matrix, therefore, as our interest deals with linear operators on arbitrary vector spaces (in general, infinite dimensional ones), we shall give a slightly different definition in the more general setting that specializes to the one offered in [11, Definition 3.2.1].

**Definition 3.1** [11, Definition 3.2.1] Let $A$ and $B$ be matrices (possibly rectangular) having the same order. Then $A$ is said to be below $B$ under the space pre-order, if $\mathcal{C}(A) \subseteq \mathcal{C}(B)$ and $\mathcal{C}(A^t) \subseteq \mathcal{C}(B^t)$ (where by $\mathcal{C}(A)$ we denote the column space of matrix $A$ and by $A^t$ we mean the usual transpose matrix). We denote the space pre-order by $<^s$ and write $A <^s B$, whenever $A$ is below $B$ under $<^s$ .

Let $V$ be an arbitrary $k$-vector space.

**Definition 3.2** Let us consider two linear operators $\varphi, \psi \in \mathrm{End}_k(V)$. The linear operator $\varphi$ is said to be below the linear operator $\psi$ under the space pre-order if:

$$\mathrm{Im}(\varphi) \subseteq \mathrm{Im}(\psi);$$
$$\mathrm{Ker}(\psi) \subseteq \mathrm{Ker}(\varphi).$$

When this happens we write $\varphi <^s \psi$.

***Remark 3.3*** If $E$ is a $k$-vector space of finite dimension and $f \in \mathrm{End}_k(E)$ is a linear operator, then we know that

- $\mathrm{Im}\, f^* = (\mathrm{Ker}\, f)^\circ$.
- Given two subspaces of a vector space of finite dimension (in fact, this statement is true in arbitrary dimension) $H_1, H_2 \subseteq E$, if $H_1 \subseteq H_2$ then $(H_2)^\circ \subseteq (H_1)^\circ$.
- For any subspace of a vector space of finte dimension, $H \subseteq E$, then $((H)^\circ)^\circ = H$,

where $f^*$ is the transpose of $f$ and $H^\circ$ is the incident subspace of $H \subseteq E$.

It is now easy to check that given two linear operators $f, g \in \mathrm{End}_k(E)$ then

$$\mathrm{Im}\, f^* \subseteq \mathrm{Im}\, g^* \text{ if and only if } \mathrm{Ker}\, g \subseteq \mathrm{Ker}\, f.$$

Accordingly, Definition 3.2 generalizes Definition 3.1 to arbitrary $k$-vector spaces.

**Lemma 3.4** *Let $\varphi, \psi \in \mathrm{End}_k(V)$ be two Core-Nilpotent endomorphisms with $V = W_\varphi \oplus U_\varphi$ the AST decomposition induced by $\varphi$. If $\varphi <^s \psi$ then:*

$$W_\varphi \subseteq \mathrm{Im}(\psi);$$
$$\mathrm{Ker}(\psi) \subseteq U_\varphi.$$

***Proof*** This is an immediate consequence of Definition 3.2 and the following chains of inclusions that are satisfied by every endomorphism. For $i(\varphi)$ the index (Definition 2.1),

$$\mathrm{Ker}\,\varphi \subseteq \mathrm{Ker}\,\varphi^2 \subseteq \cdots \subseteq \mathrm{Ker}\,\varphi^{i(\varphi)} = \mathrm{Ker}\,\varphi^{i(\varphi)+1} = U_\varphi$$

and

$$\mathrm{Im}\,\varphi \supseteq \mathrm{Im}\,\varphi^2 \supseteq \cdots \supseteq \mathrm{Im}\,\varphi^{i(\varphi)} = \mathrm{Im}\,\varphi^{i(\varphi)+1} = W_\varphi.$$

□

**Lemma 3.5** [17, Lemma 3.2] *Given a linear operator $\varphi \in \mathrm{End}_k(V)$ one has that $\varphi^- \in \mathrm{End}_k(V)$ is a 1-inverse of $\varphi$ if and only if for every $v \in V$ we have that*

$$\varphi^-(\varphi(v)) = v + u,$$

*with $u \in \mathrm{Ker}(\varphi)$.*

**Lemma 3.6** *Let $\varphi, \psi \in \mathrm{End}_k(V)$ be two linear operators. Then*

$$\mathrm{Im}(\varphi) \subseteq \mathrm{Im}(\psi) \textit{ if and only if } \varphi = \psi\psi^-\varphi,$$

*where $\psi^- \in X_\psi(1)$.*

***Proof*** Notice that if $\varphi = \psi\psi^-\varphi$ then, if $v \in \mathrm{Im}(\varphi)$, with $v = \varphi(v')$, we can write $v = \psi(\psi^-\varphi(v'))$ and $v \in \mathrm{Im}(\psi)$. Conversely, if $\mathrm{Im}(\varphi) \subseteq \mathrm{Im}(\psi)$, then every $v \in \mathrm{Im}(\varphi) \subseteq \mathrm{Im}(\psi)$ satisfies that $v = \varphi(v') = \psi(\bar{v})$ for some $v', \bar{v} \in V$. Hence,

$$\psi\psi^-\varphi(v') = \psi\psi^-\psi(\bar{v}) = \psi(\bar{v}) = \varphi(v'),$$

and the claim is deduced. □

**Lemma 3.7** *Let $\varphi, \psi \in \mathrm{End}_k(V)$ be two linear operators. Then*

$$\mathrm{Ker}(\psi) \subseteq \mathrm{Ker}(\varphi) \textit{ if and only if } \varphi = \varphi\psi^-\psi,$$

*where $\psi^- \in X_\psi(1)$.*

***Proof*** Firstly, let us suppose that $\mathrm{Ker}(\psi) \subseteq \mathrm{Ker}(\varphi)$. For any $v \in V$, one has that

$$\psi^-\psi(v) = v + u$$

with $u \in \mathrm{Ker}(\psi) \subseteq \mathrm{Ker}(\varphi)$, so

$$\varphi(\psi^-\psi(v)) = \varphi(v)$$

and we conclude. Conversely, let us suppose that $\varphi = \varphi\psi^-\psi$, and $v \in \mathrm{Ker}(\psi)$. Clearly,

$$\varphi(v) = \varphi\psi^-\psi(v) = 0,$$

so $v \in \mathrm{Ker}(\varphi)$ and the statement is proved. □

**Theorem 3.8** (Weak characterization of space pre-order) *Let $\varphi, \psi \in \mathrm{End}_k(V)$ be two linear operators. Then*

$$\varphi <^s \psi \textit{ if and only if } \varphi = \psi\psi^-\varphi = \varphi\psi^-\psi,$$

*where $\psi^- \in X_\psi(1)$.*

***Proof*** The proof is a direct consequence of Lemma 3.6 and Lemma 3.7. □

By substitution in the expression of $\varphi$ in the last theorem one obtains the following:

**Corollary 3.9** *Let $\varphi, \psi \in \mathrm{End}_k(V)$ be two linear operators. Then*

$$\varphi <^s \psi \textit{ if and only if } \varphi = \psi\psi^-\varphi\psi^-\psi,$$

*where $\psi^- \in X_\psi(1)$.*

***Remark 3.10*** Let $\varphi, \psi \in \mathrm{End}_k(V)$ be two linear operators and suppose that $X_\psi(1) \subseteq X_\varphi(1)$. Then

$$\varphi <^s \psi \text{ if and only if } \varphi = \psi\widehat{\varphi}\psi,$$

with $\widehat{\varphi} = \psi^-\varphi\psi^- \in X_\varphi(1, 2)$.

**Lemma 3.11** *Let $\varphi, \phi \in \mathrm{End}_k(V)$ be linear operators over an arbitrary k-vector space. If $\varphi\psi\phi = 0$ for every $\psi \in \mathrm{End}_k(V)$ and $\phi \neq 0$, then $\varphi = 0$.*

***Proof*** Let $v \in V$ such that $\varphi(v) \neq 0$. We choose a suitable $\psi$ such that $v = \psi(w)$ for $w \in \mathrm{Im}\,\phi$. Then, by hypothesis, it is

$$0 = \varphi\psi\phi(z) = \varphi\psi(w) = \varphi(v) \neq 0,$$

and we conclude by contradiction. □

**Corollary 3.12** *If $\varphi, \phi \in \mathrm{End}_k(V)$ are two linear operators of a $k$-vector space $V$, such that $\varphi\psi\phi = 0$ for every $\psi \in \mathrm{End}_k(V)$ and $\varphi \neq 0$, then $\phi = 0$.*

***Proof*** It is a direct consequence of Lemma 3.11. □

**Theorem 3.13** *The relation $<^s$ defines a pre-order in the set of linear operators.*

***Proof*** Reflexivity holds trivially. For transitivity, if $\varphi, \psi, \gamma \in \mathrm{End}_k(V)$ are three linear operators such that $\varphi <^s \psi$ and $\psi <^s \gamma$ then clearly

$$\mathrm{Im}(\varphi) \subseteq \mathrm{Im}(\psi) \subseteq \mathrm{Im}(\gamma);$$
$$\mathrm{Ker}(\gamma) \subseteq \mathrm{Ker}(\psi) \subseteq \mathrm{Ker}(\varphi),$$

and from Definition 3.2 we deduce that $\varphi <^s \gamma$. □

***Remark 3.14*** Readers can find a counterexample for the anti-symmetric property in [11, Section 3.2] that works for matrices and hence for linear operators.

**Proposition 3.15** *Let $\varphi \in \mathrm{End}_k(V)$. Then, for every fixed $\varphi^- \in X_\varphi(1)$ one has a bijection*

$$\begin{array}{rcl} \eta_{\varphi^-}\colon \mathrm{End}_k(V) & \simeq & X_\varphi(1) \\ \phi & \mapsto & \varphi^- + \phi - \varphi^-\varphi\phi\varphi\varphi^- \\ \varphi_2^- - \varphi^- & \leftarrow & \varphi_2^- \end{array}$$

***Proof*** Let us consider any $\phi \in \mathrm{End}_k(V)$ and let us fix $\varphi^- \in X_\varphi(1)$. It is clear that

$$\varphi(\varphi^- + \phi - \varphi^-\varphi\phi\varphi\varphi^-)\varphi = \varphi + \varphi\phi\varphi - \varphi\phi\varphi = \varphi,$$

from where we deduce that $\varphi^- + \phi - \varphi^-\varphi\phi\varphi\varphi^-$ is a 1-inverse of $\varphi$ for any $\phi$ and $\eta_{\varphi^-}$ is well defined. Let $\varphi_2^- \in X_\varphi(1)$ be an arbitrary 1-inverse of $\varphi$. Now, taking $\phi = \varphi_2^- - \varphi^-$ one obtains:

$$\varphi_2^- = \varphi^- + (\varphi_2^- - \varphi^-) - \varphi^-\varphi(\varphi_2^- - \varphi^-)\varphi\varphi^-;$$

and hence

$$\varphi_2^- = \varphi^- + \phi - \varphi^-\varphi\phi\varphi\varphi^-$$

as we wanted to show. By the election of $\phi$ we also conclude that $\phi$ is unique for every $\varphi$. □

If we denote by $Id$ to the identity endomorphism $Id\colon V \to V$, then one can prove the following result.

**Corollary 3.16** *Let $\varphi \in \mathrm{End}_k(V)$. Then for every fixed $\varphi^- \in X_\varphi(1)$, one has that the map*

$$\begin{array}{rcl} \Gamma_{\varphi^-}\colon \mathrm{End}_k(V) \times \mathrm{End}_k(V) & \to & X_\varphi(1) \\ (\gamma, \beta) & \mapsto & \varphi^- + (Id - \varphi^-\varphi)\gamma + \beta(Id - \varphi\varphi^-). \end{array}$$

*is surjective.*

***Proof*** Firstly, let us check that $\Gamma_{\varphi^-}$ is well defined. For any $\gamma, \beta \in \mathrm{End}_k(V)$, one has that

$$\varphi(\varphi^- + (Id - \varphi^-\varphi)\gamma + \beta(Id - \varphi\varphi^-))\varphi = \varphi + \varphi\gamma\varphi - \varphi\gamma\varphi + \varphi\beta\varphi - \varphi\beta\varphi = \varphi;$$

so $\Gamma_{\varphi^-}(\gamma, \beta) \in X_\varphi(1)$ for any $\gamma, \beta \in \mathrm{End}_k(V)$. For surjectivity, let us consider any $\varphi_2^- \in X_\varphi(1)$. By Proposition 3.15, we know that there exists a unique $\phi \in \mathrm{End}_k(V)$ such that $\varphi_2^- = \eta_{\varphi^-}(\phi)$. Taking $\gamma = \phi\varphi\varphi^-$ and $\beta = \phi$, we get that

$$(Id - \varphi^-\varphi)\gamma + \beta(Id - \varphi\varphi^-) = \phi\varphi\varphi^- - \varphi^-\varphi\phi\varphi\varphi^- + \phi - \phi\varphi\varphi^- = \phi - \varphi^-\varphi\phi\varphi\varphi^-;$$

so

$$\Gamma_{\varphi^-}(\phi\varphi\varphi^-, \phi) = \eta_{\varphi^-}(\phi) = \varphi_2^-,$$

and we conclude.

In general, $\Gamma_{\varphi^-}$ is not bijective. A counterexample is the following: given $\widehat{\varphi} \in X_\varphi(1, 2)$, then

$$\Gamma_{\varphi^-}(\widehat{\varphi}, \widehat{\varphi}) = \Gamma_{\varphi^-}(0, 0) = \widehat{\varphi}.$$

□

**Theorem 3.17** *Let $\varphi, \psi, \phi \in \mathrm{End}_k(V)$ be linear operators with $\psi \neq 0 \neq \phi$. Then, for any $\varphi^- \in X_\varphi(1)$, $\psi\varphi^-\phi$ is invariant under the election of $\varphi^-$ if and only if* $\mathrm{Im}\,\phi \subseteq \mathrm{Im}\,\varphi$ *and* $\mathrm{Ker}\,\varphi \subseteq \mathrm{Ker}\,\psi$.

***Proof*** Let $\varphi^- \in X_\varphi(1)$. Notice that

$$\varphi^- + (Id - \varphi^-\varphi)\gamma + \beta(Id - \varphi\varphi^-)$$

is a 1-inverse of $\varphi$ for any $\gamma, \beta \in \mathrm{End}_k(V)$, as it was shown in the proof of Corollary 3.16. By hypothesis, one obtains that

$$\psi(\varphi^- + (Id - \varphi^-\varphi)\gamma + \beta(Id - \varphi\varphi^-))\phi = \psi\varphi^-\phi$$

for any $\gamma, \beta \in \mathrm{End}_k(V)$, and hence:

$$\psi((Id - \varphi^-\varphi)\gamma + \beta(Id - \varphi\varphi^-))\phi = 0; \qquad (3.1)$$

for any $\gamma, \beta \in \mathrm{End}_k(V)$. Thus, taking $\beta = 0$ in (3.1) yields

$$\psi(Id - \varphi^-\varphi)\gamma\phi = 0$$

for any $\gamma \in \mathrm{End}_k(V)$. As $\phi$ is not null then it must be $\psi(Id - \varphi^-\varphi) = 0$ by Lemma 3.11 and therefore

$$\psi = \psi\varphi^-\varphi,$$

so by Lemma 3.7 one gets that $\mathrm{Ker}\,\varphi \subseteq \mathrm{Ker}\,\psi$. Similarly, by taking $\gamma = 0$ in (3.1), from Corollary 3.12, we get that

$$\phi = \varphi\varphi^-\phi$$

and by Lemma 3.6 one concludes that $\mathrm{Im}\,\phi \subseteq \mathrm{Im}\,\varphi$ and the claim is proved.

Conversely, let us suppose that $\mathrm{Im}\,\phi \subseteq \mathrm{Im}\,\varphi$ and $\mathrm{Ker}\,\varphi \subseteq \mathrm{Ker}\,\psi$. Then, in virtue of Lema 3.5 and as $\mathrm{Im}\,\phi \subseteq \mathrm{Im}\,\varphi$, for any $v = \phi(v') = \varphi(\bar{v})$ one gets:

$$\psi\varphi^-(v) = \psi\varphi^-(\phi(v)) = \psi\varphi^-(\varphi(\bar{v})) = \psi(v + u) = \psi(v), \text{ as } u \in \mathrm{Ker}\,\varphi \subseteq \mathrm{Ker}\,\psi,$$

for any 1-inverse $\varphi^- \in X_\varphi(1)$. □

**Theorem 3.18** (Characterization of the space pre-order) *Let $\varphi, \psi \in \mathrm{End}_k(V)$ be two linear operators. The following are equivalent:*

(I) $\varphi <^s \psi$,
(II) $\varphi = \psi\psi^-\varphi = \varphi\psi^-\psi$ *for a* $\psi^- \in X_\psi(1)$,
(III) $\varphi = \psi\psi^-\varphi = \varphi\psi^-\psi$ *for any* $\psi^- \in X_\psi(1)$.

***Proof*** Firstly notice that $I$ if and only if $III$ is a consequence of Theorem 3.8 and Theorem 3.17.

The fact that $III$ implies $II$ is obvious. Finally, let us see that $II$ implies $III$. In order to do so, let us suppose that the following equations hold for a fixed $\psi_\varphi^- \in X_\psi(1)$ :

$$\varphi = \psi\psi_\varphi^-\varphi = \varphi\psi_\varphi^-\psi. \tag{3.2}$$

In virtue of Proposition 3.15, we know that for any $\psi^- \in X_\psi(1)$, it is

$$\psi^- = \psi_\varphi^- + \phi - \psi_\varphi^-\psi\phi\psi\psi_\varphi^-,$$

for an arbitrary $\phi \in \mathrm{End}_k(V)$. Hence, on one side we get

$$\psi\psi^-\varphi = \psi\psi_\varphi^-\varphi + \psi\phi\varphi - \psi\phi\psi\psi_\varphi^-\varphi, \tag{3.3}$$

and, on the other side

$$\varphi\psi^-\psi = \varphi\psi_\varphi^-\psi + \varphi\phi\psi - \varphi\psi_\varphi^-\psi\phi\psi. \tag{3.4}$$

Now, using (3.2) in (3.3) one gets:

$$\psi\psi^-\varphi = \varphi + \psi\phi\psi\psi_\varphi^-\varphi - \psi\phi\psi\psi_\varphi^-\varphi = \varphi.$$

Similarly, using again (3.2) in (3.4), it is:

$$\varphi\psi^-\psi = \varphi + \varphi\psi_\varphi^-\psi\phi\psi - \varphi\psi_\varphi^-\psi\phi\psi = \varphi.$$

So, in short, it is

$$\varphi = \varphi\psi^-\psi = \psi\psi^-\varphi$$

for any $\psi^- \in X_\psi(1)$ as we wanted to prove. □

## 4 Drazin pre-order for Core-Nilpotent endomorphisms

The aim of the present section is to generalize the study of the Drazin pre-order that was presented in [11, Section 4.4] for finite matrices to arbitrary $k$-vector spaces, in general, infinite dimensional, using Core-Nilpotent endomorphisms.

Firstly, let us recall the basic definition of this order for matrices:

**Definition 4.1** [11, Definition 4.4.1] Let $A$ and $B$ be square matrices of the same order. Let $A = A_1 + A_2$ and $B = B_1 + B_2$ be the core-nilpotent decompositions of $A$ and $B$ respectively, where $A_1$, $B_1$ are the core parts of $A$ and $B$ respectively and $A_2$, $B_2$ are the nilpotent parts of $A$ and $B$ respectively. The matrix $A$ is said to be below matrix $B$ under the Drazin pre-order if $A_1 <^\# B_1$.
When this happens we write $A <^d B$.

Let $V$ be an arbitrary $k$-vector space.

**Definition 4.2** Let us consider two Core-Nilpotent endomorphisms $\varphi, \psi \in \mathrm{End}_k(V)$. Let $\varphi = \varphi_1 + \varphi_2$ and $\psi = \psi_1 + \psi_2$ be the core-nilpotent decomposition of $\varphi$ and $\psi$ respectively (as presented in Definition 2.2). The endomorphism $\varphi$ is said to be below endomorphism $\psi$ under the Drazin pre-order if $\varphi_1 <^{\#} \psi_1$ (Definition 2.15).

When this happens we write $\varphi <^d \psi$.

Notice that this definition makes sense with the one given for the Sharp Partial order in [1, Definition 4.2], as, by construction, the core-nilpotent decomposition of any endomorphism guarantees the index of the core part, in this case, $\varphi_1$ and $\psi_1$ respectively, is less or equal to 1 in both cases (recall Definition 2.2).

***Remark 4.3*** Let $\varphi, \psi \in \mathrm{End}_k(V)$ be Core-Nilpotent endomorphisms with $\varphi_1$ and $\psi_1$ being their respective core parts. Then, in particular, $\varphi^D = \varphi_1^{\#}$, as it was stated in Theorem 2.7 and $(\varphi^D)^{\#} = \varphi_1$. Analogously, $(\psi^D)^{\#} = \psi_1$, and therefore:

$$\varphi <^d \psi \text{ if and only if } \varphi^D <^{\#} \psi^D,$$

this is, the Drazin order is the study of the Sharp order for the Drazin inverse.

Although Definition 4.2 is useful in such a way that it enables to study the Drazin order, when working in the theory of matrix partial orders and, in general, in partial orders on arbitrary $k$-vector spaces, the natural definitions of the orders are those that are stated in terms of one (or more, but only) generalized inverses. Nevertheless, one fundamental problem in this theory is to characterize the orders in terms of other orders with constraints, Definition 4.2 goes more on the line of this thinking and the following proposition aims to establish the natural (and studied) definition of the Drazin order.

**Proposition 4.4** *Let $\varphi, \psi \in \mathrm{End}_k(V)$ be two Core-Nilpotent endomorphisms. Then:*

$$\varphi <^d \psi \textit{ if and only if } \varphi\varphi^D = \psi\varphi^D = \varphi^D\psi = \varphi^D\varphi.$$

***Proof*** Notice that $\varphi\varphi^D = \varphi_1\varphi_1^{\#}$. By hypothesis, $\varphi_1 <^{\#} \psi_1$, so the following holds:

$$\varphi_1^{\#}\psi_1 = \varphi_1^{\#}\varphi_1 = \varphi_1\varphi_1^{\#} = \psi_1\varphi_1^{\#},$$

(recall Proposition 2.16). Since

$$\varphi\varphi^{\#}(V) = (\varphi_1\varphi_1^{\#})(V) = W_\varphi = \mathrm{Im}\,\varphi_1$$

and

$$\varphi_1\varphi_1^{\#}(V) = (\psi_1\varphi_1^{\#})(V) = \psi_1(W_\varphi) \subseteq \mathrm{Im}\,\psi_1,$$

we get that

$$\mathrm{Im}\,\varphi_1 \subseteq \mathrm{Im}\,\psi_1.$$

As $\mathrm{Im}\,\varphi_1^{\#} = \mathrm{Im}\,\varphi_1$, and by construction of the core-nilpotent decomposition ($\varphi_1\varphi_2 = \varphi_2\varphi_1 = \psi_1\psi_2 = \psi_2\psi_1 = 0$), one obtains that

$$\psi_{2_{|\mathrm{Im}\,\varphi_1}} = 0 \text{ and } \varphi^{\#}_{1_{|\mathrm{Im}\,\psi_2}} = 0$$

this is, $\psi_2\varphi_1^{\#} = 0 = \varphi_1^{\#}\psi_2$. Hence,

$$\varphi\varphi^D = \varphi_1\varphi_1^{\#} = \psi_1\varphi_1^{\#} = \psi\varphi_1^{\#} = \psi\varphi^D.$$

Similarly,

$$\varphi_1^{\#}\psi_1 = \varphi_1^{\#}\varphi_1 = \varphi_1\varphi_1^{\#},$$

so $\varphi^D\psi = \varphi^D\varphi = \varphi\varphi^D$. Therefore,

$$\psi\varphi^D = \varphi\varphi^D = \varphi^D\varphi = \varphi^D\psi.$$

□

**Proposition 4.5** *Let $\varphi, \psi \in \mathrm{End}_k(V)$ be two Core-Nilpotent endomorphisms with $i(\varphi) = m$ the index of $\varphi$ (Definition 2.1). Then:*

$$\varphi <^d \psi \textit{ if and only if } \varphi^{m+1} = \varphi^m\psi = \psi\varphi^m.$$

***Proof*** Firstly, let us suppose $\varphi <^d \psi$ with the characterization that Proposition 4.4 gives. By definition of Drazin inverse $\varphi^{m+1}\varphi^D = \varphi^m$. Then,

$$\varphi^{m+1} = \varphi^{m+1}\varphi\varphi^D = \varphi^{m+1}\varphi^D\psi = \varphi^m\psi.$$

Analogously, one has that

$$\varphi^{m+1} = \varphi^D\varphi\varphi^{m+1} = \psi\varphi^D\varphi^{m+1} = \psi\varphi^m.$$

Conversely,

$$(\varphi^D)^{m+1}\varphi^m\psi = (\varphi^D)^{m+1}\varphi^{m+1} = \varphi^D\varphi,$$

using the definition of Drazin inverse. Reasoning in the same way we obtain $(\varphi^D)^{m+1}\varphi^m = \varphi^D$. Therefore, we get that

$$\varphi\varphi^D = \varphi^D\varphi = \varphi^D\psi.$$

Similarly, from $\psi\varphi^m = \varphi^{m+1}$ we obtain that

$$\psi\varphi^D = \varphi\varphi^D$$

and the claim is proved. □

The next step is to prove that the Drazin order is a pre-order in the set of Core-Nilpotent endomorphisms.

**Theorem 4.6** *The relation $<^d$ (Definition 4.2) defines a pre-order in the set of Core-Nilpotent endomorphisms.*

***Proof*** Reflexivity holds directly by definition of Drazin inverse and Proposition 4.4. Transitivity is a direct consequence of Definition 4.2 and Theorem 2.17. □

***Remark 4.7*** One can check in [11, Example 4.4.5] a counterexample for the anti-symmetric property for finite matrices which is valid for Core-Nilpotent endomorphisms bearing in mind the well known relation between finite matrices and endomorphisms.

Let us clarify even more the relationship between the Drazin pre-order and the sharp order.

**Corollary 4.8** *Let $\varphi, \psi \in \mathrm{End}_k(V)$ be Core-Nilpotent endomorphisms. Then $\varphi <^d \psi$ if and only if $\psi$ is a g-commuting inverse of $\varphi^D$ (Definition 2.14).*

***Proof*** If $\varphi <^d \psi$, using Proposition 4.4, we know that $\varphi\varphi^D = \psi\varphi^D = \varphi^D\varphi = \varphi^D\psi$, so: $\varphi^D = \varphi^D\psi\varphi^D$ and $\psi\varphi^D = \varphi^D\psi$. Conversely, we have that $\varphi^D\psi\varphi^D = \varphi^D$ and $\varphi^D\psi = \psi\varphi^D$. Therefore, composing with $\varphi$ and by definition of Drazin inverse, we can obtain:

$$\varphi\varphi^D = \varphi(\varphi^D\psi\varphi^D) = \varphi^D\varphi\psi\varphi^D = \varphi^D\varphi\varphi^D\psi = \varphi^D\psi.$$

Similarly, one proves that $\varphi^D\varphi = \psi\varphi^D$ and the result is proved. □

**Corollary 4.9** *Let $\varphi, \psi \in \mathrm{End}_k(V)$ be two Core-Nilpotent endomorphisms. Then:*

- *If $\psi(\varphi^D)^{r+1} = (\varphi^D)^{r+1}\psi = (\varphi^D)^r$ for a positive integer $r$, then $\varphi <^d \psi$.*
- *If $\varphi <^d \psi$, then $\psi(\varphi^D)^{r+1} = (\varphi^D)^{r+1}\psi = (\varphi^D)^r$ for every positive integer $r$.*

***Proof*** It follows directly from Corollary 4.8. □

**Corollary 4.10** *Let $\varphi, \psi \in \mathrm{End}_k(V)$ be two Core-Nilpotent endomorphisms. Then $\varphi <^d \psi$ if and only if $\psi(\varphi^D)^{r+1} = (\varphi^D)^{r+1}\psi = (\varphi^D)^r$ for a positive integer $r$.*

Now, let us fix a Core-Nilpotent endomorphism $\varphi$ and let us study the class of all endomorphisms $\psi$ above $\varphi$ for the Drazin pre-order, following the ideas of [11].

**Lemma 4.11** *Let $\varphi, \psi \in \mathrm{End}_k(V)$ be two Core-Nilpotent endomorphisms, being $V = W_\varphi \oplus U_\varphi$ the AST- decomposition induced by $\varphi$. If $\varphi <^d \psi$ then:*

$$\varphi_{|W_\varphi} = \psi_{|W_\varphi}.$$

***Proof*** It follows directly from $\varphi\varphi^D = \psi\varphi^D$ using the characterization of the Drazin inverse of a Core-Nilpotent endomorphism. □

**Theorem 4.12** *Let $\varphi \in \mathrm{End}_k(V)$ be a Core-Nilpotent endomorphism and let $V = W_\varphi \oplus U_\varphi$ be the AST-decomposition it induces. Then, the class of all endomorphisms $\psi$ above $\varphi$ for the Drazin pre-order, $\varphi <^d \psi$, are the ones satisfying:*

$$\psi(v) = \begin{cases} \varphi(v) & \text{if } v \in W_\varphi \\ \tilde{v} & \text{if } v \in U_\varphi \end{cases},$$

*with $\tilde{v} \in U_\varphi$.*

***Proof*** If $\varphi <^d \psi$ then

$$\varphi_{|W_\varphi} = \psi_{|W_\varphi}$$

as it follows from Lemma 4.11. As $\varphi^D\psi = \varphi^D\varphi$, if $v \in U_\varphi$ then

$$\varphi^D(\psi(v)) = \varphi^D(\varphi(v)) = \varphi(\varphi^D(v)) = 0,$$

hence

$$\psi(v) \in \mathrm{Ker}\,\varphi^D = U_\varphi.$$

Conversely, given any $\psi \in \mathrm{End}_k(V)$ as in the statement, then it is clear that, firstly; given $w \in W_\varphi$ and using the characterization of the Drazin inverse presented in (2.7) we have:

$$\psi\varphi^D(w) = \psi(\varphi_{|W_\varphi})^{-1}(w) = w = \varphi^D\varphi(w) = \varphi\varphi^D(w),$$

and that if $u \in U_\varphi$, then:

$$\varphi^D\psi(u) = 0 = \psi\varphi^D(u) \text{ and } \varphi\varphi^D(u) = 0 = \varphi^D\varphi(u).$$

□

As a direct consequence of this characterization, we obtain the following two corollaries:

**Corollary 4.13** *Let $\varphi, \psi \in \operatorname{End}_k(V)$ be two Core-Nilpotent endomorphisms such that $\varphi <^d \psi$. Then the AST-decomposition induced by $\varphi$ in $V$ is $\psi$-invariant.*

**Corollary 4.14** *Let $\varphi, \psi \in \operatorname{End}_k(V)$ be two Core-Nilpotent endomorphisms. The class of $\psi$ such that $\varphi <^d \psi$ is given by the endomorphisms satisfying:*

$$\psi_{|W_\varphi} = \varphi_{|W_\varphi} \text{ and } (\varphi^D \psi)_{|U_\varphi} = 0.$$

### 4.1 Relation between the AST decompositions in the Drazin pre-order

Let us recall that given a Core-Nilpotent endomorphism $\varphi \in \operatorname{End}_k(V)$ it induces the AST decomposition $V = W_\varphi \oplus U_\varphi$. In particular, if $i(\varphi) = m$ then $W_\varphi = \operatorname{Im} \varphi^m$ and $U_\varphi = \operatorname{Ker} \varphi^m$. Let us suppose $\varphi, \psi \in \operatorname{End}_k(V)$ are two Core-Nilpotent endomorphisms such that $\varphi <^d \psi$. The objective of the present section is to obtain the explicit relation between the AST decompositions of $\varphi$ and $\psi$ under this hypothesis.

Before continuing, let us point out the following property of Core-Nilpotent endomorphisms. Once fixed a Core-Nilpotent endomorphism $\varphi \in \operatorname{End}_k(V)$, then the $k$-subspace $W_\varphi$ is the largest $k$-subspace of $V$ such that when we restrict $\varphi$ to it, we obtain an automorphism. The use of the word largest in this context must be clarified. It means that there does not exist any other $k$-subspace, for instance $\widehat{W} \subseteq V$, such that $\varphi_{|\widehat{W}} \in \operatorname{Aut}_k(\widehat{W})$ with $W_\varphi \subset \widehat{W}$. If any $\widehat{W}$ with the previous properties did exist, then, in virtue of the AST decomposition: $\widehat{W} \cap U_\varphi \neq \{0\}$ and therefore $\varphi_{|W_{\widehat{W}}}$ would contain nilpotent elements and so contradicting the fact that $\varphi_{|\widehat{W}}$ is an automorphism. Henceforth, we will refer to this as the **maximality** property of the core part of a Core-Nilpotent endomorphism. Notice that we can also state this property by saying that if $W' \subseteq V$ is a $k$-subspace such that $\varphi_{|W'} \in \operatorname{Aut}_k(W')$ then necessarily $W' \subseteq W_\varphi$ (equality holds precisely when $W' + U_\varphi = V$).

**Lemma 4.15** *Let $\varphi, \psi \in \operatorname{End}_k(V)$ be two Core-Nilpotent endomorphisms, being $V = W_\varphi \oplus U_\varphi = W_\psi \oplus U_\psi$ their respective AST decompositions. If $\varphi <^d \psi$ then:*

$$W_\varphi \subseteq W_\psi.$$

***Proof*** Let us suppose that $\varphi <^d \psi$, then, by Lemma 4.11, $\varphi_{|W_\varphi} = \psi_{|W_\varphi}$ and moreover $(\varphi_{|W_\varphi})^{-1} = (\psi_{|W_\varphi})^{-1}$. Therefore, for every element $w \in W_\varphi$ there exists a unique (by definition of $W_\varphi$) $w' \in W_\varphi$ such that

$$\psi(w) = \varphi(w) = w';$$
$$\psi^{-1}(w') = \varphi^{-1}(w') = w.$$

Hence, $W_\psi \subseteq W_\varphi$ in virtue of the maximality property of the core part of $\psi$ and the previous calculation. □

**Proposition 4.16** *Let $\varphi, \psi \in \operatorname{End}_k(V)$ be two Core-Nilpotent endomorphisms, being $V = W_\varphi \oplus U_\varphi = W_\psi \oplus U_\psi$ their respective AST decompositions. If $\varphi <^d \psi$ then:*

$$W_\psi = W_\varphi \oplus (U_\varphi \cap W_\psi).$$

***Proof*** We shall see that $W_\varphi + (U_\varphi \cap W_\psi) = W_\psi$ and $W_\varphi \cap (U_\varphi \cap W_\psi) = \{0\}$. Let $\bar{w} \in W_\psi$, using the AST decomposition induced by $\varphi$, we write $\bar{w} = w + u$ with $w \in W_\varphi$ and $u \in U_\varphi$. Therefore, $u = \bar{w} - w \in U_\varphi \cap W_\psi$ because $W_\varphi \subseteq W_\psi$ by Lemma 4.15. Finally, $W_\varphi \cap (U_\varphi \cap W_\psi) \subseteq W_\varphi \cap U_\varphi = \{0\}$, and the claim is proved. □

**Proposition 4.17** *Let* $\varphi, \psi \in \text{End}_k(V)$ *be two Core-Nilpotent endomorphisms with* $i(\psi) = s$. *If* $\varphi <^d \psi$ *then*

$$\varphi^D \varphi^s = \varphi^D \psi^s = \varphi^s \varphi^D = \psi^s \varphi^D.$$

***Proof*** Since $\varphi <^d \psi$, we have that $\varphi\varphi^D = \psi\varphi^D = \varphi^D\psi = \varphi^D\varphi$ using Proposition 4.4. Then, it is:

$$\begin{aligned}\varphi^D \varphi^s &= \varphi^D \psi \varphi^{s-1} = \psi \varphi^D \varphi^{s-1} = \psi \varphi^D \varphi \varphi^{s-2} = \cdots = \\ &= \psi^s \varphi^D.\end{aligned}$$

Analogously, one can check that $\varphi^s \varphi^D = \psi^s \varphi^D$. To conclude it suffices to use the definition of Drazin inverse and check that $\varphi^D \varphi^s = \varphi^s \varphi^D$. □

**Lemma 4.18** *Let* $\varphi, \psi \in \text{End}_k(V)$ *be two Core-Nilpotent endomorphisms, being* $V = W_\varphi \oplus U_\varphi = W_\psi \oplus U_\psi$ *their respective AST decompositions with* $i(\psi) = s$. *If* $\varphi <^d \psi$ *then:*

$$U_\psi \subseteq U_\varphi.$$

***Proof*** Bearing in mind that $U_\psi = \text{Ker}\, \psi^s$, then, using Proposition 4.17

$$0 = \varphi^D \psi^s (U_\psi) = \psi^s \varphi^D (U_\psi),$$

so $\varphi^D(U_\psi) \subseteq U_\psi$. Moreover from the expression of the Drazin inverse of a Core-Nilpotent endomorphism (recall (2.7)) and Lemma 4.15, one has that

$$\text{Im}\, \varphi^D = W_\varphi \subseteq W_\psi.$$

Therefore, $\varphi^D(U_\psi) = 0$ and one concludes that

$$U_\psi \in \text{Ker}\, \varphi^D = U_\varphi,$$

as desired. □

**Proposition 4.19** *Let* $\varphi, \psi \in \text{End}_k(V)$ *be two Core-Nilpotent endomorphisms, being* $V = W_\varphi \oplus U_\varphi = W_\psi \oplus U_\psi$ *their respective AST decompositions with* $i(\psi) = s$. *If* $\varphi <^d \psi$ *then:*

$$U_\varphi = U_\psi \oplus (U_\varphi \cap W_\psi).$$

***Proof*** Again, let us recall that we shall prove that $U_\psi + (U_\varphi \cap W_\psi) = U_\varphi$ and $U_\psi \cap (U_\varphi \cap W_\psi) = \{0\}$. Let us express any $\bar{u} \in U_\varphi$ using the AST decomposition induced by $\psi$ as $\bar{u} = w + u$ with $w \in W_\psi$ and $u \in U_\psi$. Therefore, using Lemma 4.18 one has:

$$w = u - w_2 \in U_\varphi \cap W_\psi.$$

Finally, it is clear that

$$U_\psi \cap (U_\varphi \cap W_\psi) \subseteq U_\psi \cap W_\psi = \{0\},$$

and the claim is proved. □

**Proposition 4.20** *Let* $\varphi, \psi \in \text{End}_k(V)$ *be two Core-Nilpotent endomorphisms with* $i(\varphi) = m$ *and* $i(\psi) = s$. *Then*

$$\varphi <^d \psi \textit{ if and only if } \varphi_{|W_\varphi} = \psi_{|W_\varphi} \textit{ and } U_\varphi \textit{ is } \psi - \textit{invariant.}$$

***Proof*** Let us suppose that $\varphi\varphi^D = \psi\varphi^D = \varphi^D\psi = \varphi^D\varphi$, in virtue of Proposition 4.4. Then, bearing in mind the expression of the Drazin inverse for Core-Nilpotent endomorphisms, if $w \in W_\varphi$ and $u \in U_\varphi$, it is:

$$w = \varphi\varphi^D(w) = \psi\varphi^D(w);$$
$$0 = \psi(\varphi^D(u)) = \varphi^D(\psi(u)),$$

and the claim is deduced because $\psi(u) \in \operatorname{Ker}\varphi^D = U_\varphi$.
Conversely, let us suppose that $\varphi_{|W_\varphi} = \psi_{|W_\varphi}$ and that $U_\varphi$ is $\psi$-invariant. Then, for $w \in W_\varphi$ and $u \in U_\varphi$ :

$$\psi\varphi^D(w) = (\psi(\varphi_{|W_\varphi})^{-1})(w) = (\varphi(\varphi_{|W_\varphi})^{-1})(w) = w;$$
$$\varphi^D\psi(w) = (\varphi_{|W_\varphi})^{-1}\psi(w) = (\varphi_{|W_\varphi})^{-1}\varphi(w) = w;$$
$$\psi\varphi^D(u) = 0 = \varphi\varphi^D(u);$$
$$\varphi^D\psi(u) = 0 = \varphi^D\varphi(u);$$

using the expression of the Drazin inverse for Core-Nilpotent endomorphisms (recall (2.7)) and the statement is deduced. □

## 5 G-Drazin inverse for Core-Nilpotent endomorphisms

The aim of this section is to characterize the set of G-Drazin inverses of a Core-Nilpotent endomorphism, this is, to state explicitly the conditions required for them to exist and their expression.

### 5.1 G-Drazin inverse for Core-Nilpotent endomorphisms

Let $V$ be an arbitrary $k$-vector space. We define the G-Drazin inverse for a Core-Nilpotent endomorphism on an analogous way as the author of [18] did in his study of the G-Drazin inverse for finite potent endomorphisms. Moreover, his work is followed in order to reach the desired characterization.

**Definition 5.1** Given a Core-Nilpotent endomorphism $\varphi \in \operatorname{End}_k(V)$, we say that an endomorphism $\varphi^{GD} \in \operatorname{End}_k(V)$ is a $G$-Drazin inverse of $\varphi$ when it satisfies that

$$\varphi \circ \varphi^{GD} \circ \varphi = \varphi;$$
$$\varphi^{GD} \circ \varphi^m = \varphi^m \circ \varphi^{GD},$$

where $i(\varphi) = m$.

In 2018, C. Coll, M. Lattanzi and N. Thome proved in [6, Theorem 2.2] that there are several equivalent definitions of the G-Drazin inverse of a matrix, this is, that the G-Drazin inverse is the solution to two different, but equivalent, systems of equations. In the theory of Core-Nilpotent endomorphisms, this idea is generalized naturally using the AST-decomposition as it is shown in the following lemma.

**Lemma 5.2** *Let $\varphi \in \operatorname{End}_k(V)$ be a Core-Nilpotent endomorphism of index $m$, then $\varphi^{GD} \in \operatorname{End}_k(V)$ is a G-Drazin inverse of $\varphi$ (in the sense of Definition 5.1) if and only if $\varphi^{GD}$ satisfies* *the following conditions:*

$$\begin{aligned}\varphi\circ\varphi^{GD}\circ\varphi&=\varphi;\\ \varphi^{GD}\circ\varphi^{m+1}&=\varphi^{m};\\ \varphi^{m+1}\circ\varphi^{GD}&=\varphi^{m}.\end{aligned}$$

***Proof*** Firstly, let us suppose that $\varphi^{GD}$ is a G-Drazin inverse in the sense of Definition 5.1. Then:

$$\varphi^{GD}\varphi^{m+1}=\varphi^{GD}\varphi^{m}\varphi=\varphi^{m}\varphi^{GD}\varphi=\varphi^{m-1}(\varphi\varphi^{GD}\varphi)=\varphi^{m},$$

and, similarly,

$$\varphi^{m+1}\varphi^{GD}=\varphi\varphi^{m}\varphi^{GD}=\varphi\varphi^{GD}\varphi^{m}=(\varphi\varphi^{GD}\varphi)\varphi^{m-1}=\varphi^{m}.$$

Conversely, let us suppose that $\varphi^{GD}$ satisfies the three equations of the statement. By hypothesis $V=W_\varphi\oplus U_\varphi$. If $u\in U_\varphi$ it is clear that:

$$0=\varphi^{GD}\varphi^{m}(u)=\varphi^{GD}\varphi^{m+1}\varphi^{GD}(u)=\varphi^{m}\varphi^{GD}(u).$$

On the other hand, if $w\in W_\varphi$ with $w=\varphi^{m}(w')$ then:

$$\begin{aligned}\varphi^{GD}\varphi^{m}(w)&=\varphi^{GD}\varphi^{m+1}\varphi^{m-1}(w')=\varphi^{m}(\varphi^{m-1}(w'))=\varphi^{m-1}(w);\\ \varphi^{m}\varphi^{GD}(w)&=\varphi^{m}\varphi^{GD}\varphi^{m}(w')=\varphi^{m-1}\varphi\varphi^{GD}\varphi\varphi^{m-1}(w')=\varphi^{m-1}(w),\end{aligned}$$

so we conclude. □

**Lemma 5.3** *Let $\varphi\in \mathrm{End}_k(V)$ be a Core-Nilpotent endomorphism with $i(\varphi)=m$ and let $V=W_\varphi\oplus U_\varphi$ be the AST-decomposition determined by $\varphi$. If $\psi\in\mathrm{End}_k(V)$ is an endomorphism such that $\psi\varphi^{m}=\varphi^{m}\psi$, then $W_\varphi$ and $U_\varphi$ are both $\psi$-invariant.*

***Proof*** By definition $\varphi_{|W_\varphi}\in \mathrm{Aut}_k(W_\varphi)$. If $w\in W_\varphi$, (recall that as $i(\varphi)=m$ then $W_\varphi=\mathrm{Im}\,\varphi^{m}$) with $\varphi^{m}(w')=w$, then

$$\psi(w)=(\psi\varphi^{m})(w')=(\varphi^{m}\psi)(w')\in W_\varphi.$$

So, $W_\varphi$ is $\psi$-invariant.

Now, if $u\in U_\varphi$, then

$$(\psi\varphi^{m})(u)=(\varphi^{m}\psi)(u)=0$$

and therefore $\varphi(u)\in U_\varphi$. Hence, $U_\varphi$ is also $\psi$ invariant. □

**Theorem 5.4** *Given a Core-Nilpotent endomorphism $\varphi\in\mathrm{End}_k(V)$, with $i(\varphi)=m$, then $\varphi^{GD}\in\mathrm{End}_k(V)$ is a G-Drazin inverse of $\varphi$ if and only if both $U_\varphi$ and $W_\varphi$ are $\varphi^{GD}$- invariant and $\varphi^{GD}$ is characterized by:*

$$\varphi^{GD}(v)=\begin{cases}(\varphi_{|W_\varphi})^{-1}(v) & if\;\; v\in W_\varphi\\ (\varphi_{|U_\varphi})^{-}(v) & if\;\; v\in U_\varphi\end{cases},$$

*where $(\varphi_{|U_\varphi})^{-}$ is a 1-inverse of $\varphi_{|U_\varphi}$.*

***Proof*** Let $\varphi^{GD} \in \mathrm{End}_k(V)$ be a G-Drazin inverse of $\varphi$. By definition, it is a direct consequence of Lemma 5.3 that both $W_\varphi$ and $U_\varphi$ are $\varphi^{GD}$-invariant. Since $\varphi\varphi^{GD}\varphi = \varphi$, it is clear that $(\varphi^{GD})_{|_{W_\varphi}} = (\varphi_{|_{W_\varphi}})^{-1}$ and $(\varphi^{GD})_{|_{U_\varphi}} = (\varphi_{|_{U_\varphi}})^{-}$, where $(\varphi_{|_{U_\varphi}})^{-}$ is a 1-inverse of $\varphi_{U_\varphi}$. Conversely, let $\psi \in \mathrm{End}_k(V)$ be an endomorphism satisfying that both $W_\varphi$ and $U_\varphi$ are $\psi$-invariant, $\psi_{|_{W_\varphi}} = (\varphi_{|_{W_\varphi}})^{-1}$ and $\psi_{|_{U_\varphi}} = (\varphi_{|_{U_\varphi}})^{-}$ with $(\varphi_{|_{U_\varphi}})^{-}$ a 1-inverse of $\varphi_{|_{U_\varphi}}$. If $w \in W_\varphi$ with $w = \varphi^m(w')$ and $u \in U_\varphi$, bearing in mind Lemma 5.2, the following hold:

$$
\begin{aligned}
&\varphi\psi\varphi(w) = \varphi\psi\varphi^m(\varphi(w')) = \varphi\varphi^{m-1}(\varphi(w')) = \varphi(\varphi^m(w')) = \varphi(v);\\
&\varphi\psi\varphi(u) = \varphi(u) \text{ by construction.}
\end{aligned}
$$

Moreover, by definition, $(\varphi^r)_{|_{U_\varphi}} = 0$, so, by the properties of the aforementioned $\psi$, if $w \in W_\varphi$ with $w = \varphi^m(w')$ we deduce that:

$$\psi\varphi^m(w) = \varphi^{m-1}(w) = \varphi^m(\varphi^{m-1}(w')) = \varphi^m\psi(\varphi^m(w')) = \varphi^m\psi,$$

from where we get that

$$\psi\varphi^m = \varphi^m\psi$$

and the statement is proved bearing in mind Definition 5.1. □

**Corollary 5.5** *An arbitrary endomorphism $\varphi \in \mathrm{End}_k(V)$ admits G-Drazin inverses if and only if $\varphi$ is a Core-Nilpotent endomorphism.*

Recall that if $\varphi$ is a Core-Nilpotent endomorphism of index $i(\varphi) = m$ then it induces a decomposition $V = \mathrm{Ker}\,\varphi^m \oplus \mathrm{Im}\,\varphi^m$ which is precisely the AST decomposition presented in (4) of Theorem 2.3 where $U_\varphi = \mathrm{Ker}\,\varphi^m$ and $W_\varphi = \mathrm{Im}\,\varphi^m$. Therefore, bearing in mind the Jordan Bases of a nilpotent endomorphism, recall Sect. 2.2.2, using the characterization of 1-inverses presented in Proposition 2.9 and the recently proved Theorem 5.4 we are able to explicit the characterization of G-Drazin inverses:

**Proposition 5.6** (Characterization of G-Drazin inverses) *Let $\varphi \in \mathrm{End}_k(V)$ be a Core-Nilpotent endomorphism of index $i(\varphi) = m$. Then, $\varphi^{GD} \in \mathrm{End}_k(V)$ is a G-Drazin inverse of $\varphi$ if and only if $\varphi^{GD}$ verifies:*

- $\varphi^{GD}(w) = (\varphi_{|_{W_\varphi}})^{-1}(w)$ *for any* $w \in W_\varphi$;
- $\varphi^{GD}(\varphi^j(v_{s_i})) = \varphi^{j-1}(v_{s_i}) + u^j_{s_i}$, *with* $u^j_{s_i} \in \mathrm{Ker}\,\varphi$ *for every* $s_i \in S_{\mu_i(U_\varphi,\varphi)}$, $j \in \{1, \dots, i-1\}$;
- $\varphi^{GD}(v_{s_i}) = \tilde{v}_{s_i}$ *for every* $s_i \in S_{\mu_i(U_\varphi,\varphi)}$;

*where* $\tilde{v}_{s_i} \in U_\varphi$.

***Remark 5.7*** The proposition that has just been proved shows how to understand the subset of G-Drazin inverses on the set of 1-inverses. The set of G-Drazin inverses of a given Core-Nilpotent endomorphism $\varphi \in \mathrm{End}_k(V)$ is precisely the set of 1-inverses of $\varphi$ that leave the AST-decomposition induced by $\varphi$ invariant, as it can be deduced immediately from the structure of the AST-decomposition of a Core-Nilpotent endomorphism ((4) of Theorem 2.3) and the structure of the nilpotent basis of an arbitrary endomorphism (Sect. 2.2.2) together with Proposition 2.9 and Proposition 5.6.

Analogously to the results proven in [18]:

**Corollary 5.8** *(Equation for G-Drazin inverses). Let $\varphi \in \text{End}_k(V)$ be a Core-Nilpotent endomorphism. If $\varphi^{GD}$ is a G-Drazin inverse of $\varphi$, then one has that*

$$\varphi^{GD} = \varphi^D + \varphi^-_{GD},$$

*where $\varphi^D$ is the Drazin inverse of $\varphi$, and $\varphi^-_{GD} \in \text{End}_k(V)$ is the unique linear map satisfying that*

$$\varphi^-_{GD}(v) = \begin{cases} 0 & \text{if } v \in W_\varphi \\ (\varphi^{GD})_{|U_\varphi}(v) & \text{if } v \in U_\varphi \end{cases},$$

*which is a G-Drazin inverse of $\varphi_2$.*

**Corollary 5.9** *If $\varphi = \varphi_1 + \varphi_2$ is the Core-Nilpotent decomposition of a Core-Nilpotent endomorphism $\varphi \in \text{End}_k(V)$, then the Drazin inverse $\varphi^D$ is a G-Drazin inverse of $\varphi_1$.*

## 6 G-Drazin Partial Order for Core-Nilpotent endomorphisms

The main goal of this section is to generalize the theory presented in [21, Section 3] about the G-Drazin order for finite matrices to Core-Nilpotent endomorphisms. This theory will be valid for arbitrary vector spaces over fields of arbitrary characteristic, in particular, infinite dimensional ones.

### 6.1 G-Drazin Partial order for Core-Nilpotent endomorphisms

Let $V$ denote an arbitrary $k$-vector space.

**Definition 6.1** Let us consider two Core-Nilpotent endomorphisms $\varphi, \psi \in \text{End}_k(V)$. We say $\varphi$ is under $\psi$ for the G-Drazin order if there exist $\varphi^{GD}_-, \varphi^{GD}_= \in \text{End}_k(V)$ G-Drazin inverses of $\varphi$ such that:

$$\varphi\varphi^{GD}_- = \psi\varphi^{GD}_-,$$
$$\varphi^{GD}_=\varphi = \varphi^{GD}_=\psi.$$

When $\varphi$ is under $\psi$ for the G-Drazin order, it will be denoted as $\varphi <^{GD} \psi$.

**Lemma 6.2** *Let $\varphi \in \text{End}_k(V)$ be a Core-Nilpotent endomorphism and let $\varphi^{GD}_-, \varphi^{GD}_= \in \text{End}_k(V)$ be two G-Drazin inverses of $\varphi$. Then:*

$$\varphi^{GD} = \varphi^{GD}_=\varphi\varphi^{GD}_-$$

*is a G-Drazin inverse of $\varphi$.*

***Proof*** Let us show that the three conditions of Proposition 5.6 hold. Bearing in mind that $\varphi^{GD}$ leaves both $W_\varphi$ and $U_\varphi$ invariant, then, if $w \in W_\varphi$ :

$$\varphi^{GD}_=\varphi\varphi^{GD}_-(w) = \varphi^{GD}_=\varphi(\varphi_{|W_\varphi})^{-1}(w) = \varphi^{GD}_=(w) = (\varphi_{|W_\varphi})^{-1}(w).$$

Now, if $v_{s_i} \in U_\varphi$, one gets:

$$\varphi^{GD}_=\varphi\varphi^{GD}_-(\varphi^j(v_{s_i})) = \varphi^{GD}_=\varphi(\varphi^{j-1}(v_{s_i}) + u^j_{s_i}) = \varphi^{GD}_=(\varphi^j(v_{s_i})) = \varphi^{j-1}(v_{s_i}) + u^j_{s_i};$$
$$\varphi^{GD}_=\varphi\varphi^{GD}_-(v_{s_i}) = \varphi^{GD}_=\varphi(\tilde{v}_{s_i}) = \tilde{v}_{s_i} + \tilde{u}_{s_i} \in U_\varphi;$$

with $\tilde{u}_{s_i}, u_{s_i}^j \in \operatorname{Ker}\varphi \subseteq U_\varphi$ for every $s_i \in S_{\mu_i(U_\varphi,\varphi)}$ and $j \in \{1, \dots, i-1\}$. Therefore, the statement is proved. □

The next corollary will be taken as the definition of the G-Drazin order in the rest of the monograph. Simply, we can state Definition 5.1 in terms of the same G-Drazin inverse.

**Corollary 6.3** *Let $\varphi, \psi \in \operatorname{End}_k(V)$ be two Core-Nilpotent endomorphisms. Then $\varphi <^{GD} \psi$ (in the sense of Definition 5.1) if and only if there exists a G-Drazin inverse $\varphi^{GD} \in \operatorname{End}_k(V)$ of $\varphi$ such that:*

$$\varphi\varphi^{GD} = \psi\varphi^{GD},$$
$$\varphi^{GD}\varphi = \varphi^{GD}\psi.$$

***Proof*** If $\varphi <^{GD} \psi$ then, using both G-Drazin inverses from Definition 6.1 one gets the desired result by applying Lemma 6.2. Conversely, is immediate. □

**Proposition 6.4** *Let $\varphi, \psi \in \operatorname{End}_k(V)$ be two Core-Nilpotent endomorphisms. If $\varphi <^{GD} \psi$, then the AST-decomposition induced by $\varphi$ is $\psi$-invariant. In particular:*

- $\psi(W_\varphi) = W_\varphi$;
- $\psi(\varphi^{j-1}(v_{s_i} + u_{s_i}^j)) = \varphi^j(v_{s_i})$, *with* $u_{s_i}^j \in \operatorname{Ker}\varphi$ *for every* $s_i \in S_{\mu_i(U_\varphi,\varphi)}$ *and* $j \in \{1, \dots, i-1\}$.

***Proof*** Bearing in mind the characterization given for G-Drazin inverses of Core-Nilpotent endomorphisms in Proposition 5.6, then, if $\varphi <^{GD} \psi$ and $w \in W_\varphi$, it is clear that:

$$\psi\varphi^{GD}(w) = \varphi\varphi^{GD}(w) = w$$

therefore:

$$(\psi\varphi^{GD})_{|W_\varphi} = Id_{|W_\varphi}.$$

With the same reasoning,

$$\psi\varphi^{GD}(v_{s_i}) = \varphi\varphi^{GD}(v_{s_i}) = \varphi(\tilde{v}_{s_i}) \in U_\varphi;$$
$$\psi(\varphi^{j-1}(v_{s_i}) + u_{s_i}^j) = \psi\varphi^{GD}(\varphi^j(v_{s_i})) = \varphi\varphi^{GD}\varphi^j(v_{s_i}) = \varphi^j(v_{s_i});$$

with $u_{s_i}^j \in \operatorname{Ker}\varphi$ for every $s_i \in S_{\mu_i(U_\varphi,\varphi)}$ and $j \in \{1, \dots, i-1\}$, so all the claims are proved. □

Let us now offer the characterization of the G-Drazin order for Core-Nilpotent endomorphisms, which is presented in the following theorem.

**Theorem 6.5** *Let $\varphi, \psi \in \operatorname{End}_k(V)$ be two Core-Nilpotent endomorphisms and let $V = W_\varphi \oplus U_\varphi$ be the AST-decomposition induced by $\varphi$. Then $\varphi <^{GD} \psi$ if and only if*

- $\varphi_{|W_\varphi} = \psi_{|W_\varphi}$;
- *There exists a 1-inverse $\varphi^- \in \operatorname{End}_k(U_\varphi)$ of $\varphi$ such that:*

$$(\varphi\varphi^-)_{|U_\varphi} = (\psi\varphi^-)_{|U_\varphi};$$
$$(\varphi^-\varphi)_{|U_\varphi} = (\varphi^-\psi)_{|U_\varphi}.$$

***Proof*** Let $\varphi <^{GD} \psi$ with the conditions aforementioned. The first claim was proved in Lemma 4.11. Recalling Theorem 5.4, one has that, in particular, $\varphi^{GD}$ is a 1-inverse of $\varphi$ when restricted to $U_\varphi$, so taking as $\varphi^- = (\varphi^{GD})_{|U_\varphi}$ enables us to conclude the first claim.

Conversely, let us define:

$$\tilde{\varphi}^{GD}(v) = \begin{cases} (\varphi_{|W_\varphi})^{-1}(v) & \text{if } v \in W_\varphi \\ \varphi^-(v) & \text{if } v \in U_\varphi \end{cases},$$

with $\varphi^- \in \text{End}_k(U_\varphi)$ the endomorphism satisfying the second condition in the statement of the theorem.

In particular, one deduces that $\tilde{\varphi}^{GD}$ leaves the AST-decomposition induced by $\varphi$ invariant and in virtue of the Proposition 5.6, $\tilde{\varphi}^{GD}$ is in fact a G-Drazin inverse of $\varphi$.

Let us check that the conditions of the order are satisfied (precisely, the ones given by Corollary 6.3), using $\tilde{\varphi}^{GD}$ and the $\psi$ in the hypothesis. In $U_\varphi$ it is automatic by the construction of $\tilde{\varphi}^{GD}$ and $\psi$. On the other hand, if $w \in W_\varphi$ we get:

$$\varphi\tilde{\varphi}^{GD}(w) = \varphi(\varphi_{|W_\varphi})^{-1}(w) = w;$$
$$\psi\tilde{\varphi}^{GD}(w) = \psi(\varphi_{|W_\varphi})^{-1}(w) = w,$$

where in the second calculation we are using that $(\varphi_{|W_\varphi})^{-1}(w) \in W_\varphi$ and that by hypothesis $\varphi_{|W_\varphi} = \psi_{|W_\varphi}$. Similarly, one checks that $\tilde{\varphi}^{GD}\varphi(w) = w$ and, using Proposition 6.4 and again the hypothesis, $\tilde{\varphi}^{GD}(\psi(w)) = w$, so we get what desired. □

The main task from now on is to prove that the G-Drazin order is indeed a partial order in the set of Core-Nilpotent endomorphisms. In order to do so, we must solve two main problems. Firstly, to enunciate the G-Drazin order in terms of other orders with constraints. Secondly, if $\varphi, \psi \in \text{End}_k(V)$ are two Core-Nilpotent endomorphisms such that $\varphi <^{GD} \psi$, we shall understand the relations between the AST-decompositions of both endomorphisms. This will show the existing relation between the G-Drazin inverses of $\psi$ and those of $\varphi$.

Let us start illustrating these ideas to the reader with the following theorem, which comes motivated by [21, Theorem 3.4, (IV,V)].

**Theorem 6.6** *Let $\varphi, \psi \in \text{End}_k(V)$ be two Core-Nilpotent endomorphisms with $i(\varphi) = m$. Then, the following conditions are equivalent:*

(I) $\varphi <^{GD} \psi$;
(II) $\varphi^{m+1} = \varphi^m\psi = \psi\varphi^m$ *and there exists a* $1 - inverse$ $\varphi^- \in \text{End}_k(U_\varphi)$ *of* $\varphi$ *such that:*

$$(\varphi\varphi^-)_{|U_\varphi} = (\psi\varphi^-)_{|U_\varphi};$$
$$(\varphi^-\varphi)_{|U_\varphi} = (\varphi^-\psi)_{|U_\varphi}.$$

III.) $\varphi <^d \psi$ *and there exists a* $1 - inverse$ $\varphi^- \in \text{End}_k(U_\varphi)$ *of* $\varphi$ *such that:*

$$(\varphi\varphi^-)_{|U_\varphi} = (\psi\varphi^-)_{|U_\varphi};$$
$$(\varphi^-\varphi)_{|U_\varphi} = (\varphi^-\psi)_{|U_\varphi}.$$

***Proof*** Firstly, let us check that $I$ implies $II$. Let us suppose that $\varphi <^{GD} \psi$ with the conditions given by Corollary 6.3, this is $\varphi\varphi^{GD} = \psi\varphi^{GD}$ and $\varphi^{GD}\varphi = \varphi^{GD}\psi$. Then:

$$\varphi^{m+1} = \varphi\varphi^{GD}\varphi^{m+1} = \psi\varphi^{GD}\varphi^{m+1} = \psi\varphi^m;$$
$$\varphi^{m+1} = \varphi^{m+1}\varphi^{GD}\varphi = \varphi^{m+1}\varphi^{GD}\psi = \varphi^m\psi,$$

using the equivalent definitions of the G-Drazin inverse given by Lemma 5.2. The existence of a 1-inverse satisfying the conditions of the statement is deduced immediately by the characterization of Theorem 6.5.

Let us now prove that $II$ occurs if and only if $III$ occurs. This affirmation is exactly Proposition 4.5.

Finally, let us show that $III$ imply $I$. Let us suppose that $\varphi <^d \psi$. Then, by Proposition 4.4 we know that $\varphi\varphi^D = \psi\varphi^D$ and using the expression of the Drazin inverse (see, (2.7)) we know that

$$\varphi_{|W_\varphi} = \psi_{|W_\varphi}.$$

Hence, the rest follows directly from the characterization of the G-Drazin order given in Theorem 6.5. □

**Lemma 6.7** *Let $\varphi, \psi \in \mathrm{End}_k(V)$ be two Core-Nilpotent endomorphisms, being $V = W_\varphi \oplus U_\varphi = W_\psi \oplus U_\psi$ their respective AST decompositions. If $\varphi <^{GD} \psi$, then $U_\varphi \cap W_\psi$ is $\psi$-invariant.*

***Proof*** The fact that $W_\psi$ is $\psi$-invariant is by definition of AST decomposition. Moreover, as $\varphi <^{GD} \psi$ then we know that the AST decomposition induced by $\varphi$ is $\psi$-invariant in virtue of Proposition 6.4, so we conclude. □

**Proposition 6.8** *Let $\varphi, \psi \in \mathrm{End}_k(V)$ be two Core-Nilpotent endomorphisms, being $V = W_\varphi \oplus U_\varphi = W_\psi \oplus U_\psi$ their respective AST decompositions. If $\varphi <^{GD} \psi$, then the restriction map:*

$$\psi : U_\varphi \cap W_\psi \longrightarrow U_\varphi \cap W_\psi$$

*is an automorphism of $k$-vector spaces.*

***Proof*** The restriction mentioned in the statement makes sense due to Lemma 6.7. Since $\mathrm{Ker}\, \psi_{|W_\varphi} = 0$, then $\mathrm{Ker}\, \psi_{|(U_\varphi \cap W_\psi)} = 0$ and $\psi_{|(U_\varphi \cap W_\psi)}$ is injective. For surjectivity let us recall that if $\varphi <^{GD} \psi$, then, for $m = i(\varphi)$, we have that $\varphi^{m+1} = \varphi^m\psi = \psi\varphi^m$ by $II$ of Theorem 6.6. Let us consider any $\bar{v} \in (U_\varphi \cap W_\psi)$, such that $\varphi^m(\bar{v}) = 0$ and $\bar{v} = \psi(v')$ for a unique $v' \in W_\psi$. Therefore, it is:

$$0 = \varphi^m(\bar{v}) = \varphi^m(\psi(v')) = \psi(\varphi^m(v')),$$

so $\varphi^m(v') \subseteq \mathrm{Ker}\, \psi \subseteq U_\psi \subseteq U_\varphi$ by Lemma 4.18. Then $v' \in U_\varphi$ and $v' \in U_\varphi \cap W_\psi$, hence surjectivity is proved. □

**Corollary 6.9** *Let $\varphi, \psi \in \mathrm{End}_k(V)$ be two Core-Nilpotent endomorphisms, being $V = W_\varphi \oplus U_\varphi = W_\psi \oplus U_\psi$ their respective AST decompositions. If $\varphi <^{GD} \psi$ then $\psi_{|U_\varphi}$ is a Core-Nilpotent endomorphism such that $i(\psi_{|U_\varphi}) = i(\psi)$.*

***Proof*** Notice that as $\varphi <^{GD} \psi$ then $U_\varphi = U_\psi \oplus (U_\varphi \cap W_\psi)$ as it was proved in Proposition 4.19. Therefore, the statement is a direct consequence of Proposition 6.8 as $\psi_{|U_\psi}$ is nilpotent by definition of the AST decomposition induced by $\psi$. □

Given a Core-Nilpotent endomorphism $\varphi \in \mathrm{End}_k(V)$, recall that $X_\varphi(1)$ and $X_\varphi(GD)$ denote the sets of 1-inverses of $\varphi$ and G-Drazin inverses of $\varphi$ respectively.

Let $\varphi, \psi \in \text{End}_k(V)$ be two Core-Nilpotent endomorphisms and suppose that there exists a 1-inverse $\varphi^- \in \text{End}_k(U_\varphi)$ of $\varphi$ such that:

$$(\varphi\varphi^-)_{|U_\varphi} = (\psi\varphi^-)_{|U_\varphi};$$
$$(\varphi^-\varphi)_{|U_\varphi} = (\varphi^-\psi)_{|U_\varphi}.$$

With this conditions, we have that

$$\varphi_{|U_\varphi} = (\varphi\varphi^-\varphi)_{|U_\varphi} = (\psi\varphi^-\varphi)_{|U_\varphi} = (\varphi\varphi^-\psi)_{|U_\varphi} = (\psi\varphi^-\psi)_{|U_\varphi}. \tag{6.1}$$

**Lemma 6.10** *Let $\varphi, \psi \in \text{End}_k(V)$ be two Core-Nilpotent endomorphisms and suppose that there exists a 1-inverse $\varphi^- \in \text{End}_k(U_\varphi)$ of $\varphi$ such that:*

$$(\varphi\varphi^-)_{|U_\varphi} = (\psi\varphi^-)_{|U_\varphi};$$
$$(\varphi^-\varphi)_{|U_\varphi} = (\varphi^-\psi)_{|U_\varphi}.$$

*If $\psi^- \in \text{End}_k(U_\varphi)$, is such that $\psi^- \in X_{\psi_{|U_\varphi}}(1)$ then*

$$\psi^- \in X_{\varphi_{|U_\varphi}}(1).$$

***Proof*** Let $\psi^- \in X_{\psi|_{U_\varphi}}(1)$. Then, applying the equalities of (6.1), when restricting to $U_\varphi$, one gets:

$$\begin{aligned}\varphi\psi^-\varphi &= (\varphi\varphi^-\psi)\psi^-(\psi\varphi^-\varphi) = \\ &= \varphi\varphi^-(\psi\psi^-\psi)\varphi^-\varphi = \\ &= \varphi\varphi^-\psi\varphi^-\varphi = (\varphi\varphi^-\psi)\varphi^-\varphi = \varphi\varphi^-\varphi = \varphi,\end{aligned}$$

from where the claim is deduced. □

Let us consider a Core-Nilpotent endomorphism $\psi \in \text{End}_k(V)$. In virtue of Theorem 5.4, we know the expression of any G-Drazin inverse of $\psi$, which is:

$$\psi^{GD}(v) = \begin{cases} (\psi_{|W_\psi})^{-1}(v) & \text{if } v \in W_\psi \\ (\psi_{|U_\psi})^-(v) & \text{if } v \in U_\psi \end{cases}, \tag{6.2}$$

with $(\psi_{|U_\psi})^- \in X_{\psi_{|U_\psi}}(1)$. Moreover, let us suppose that $\varphi \in \text{End}_k(V)$ is another Core-Nilpotent endomorphism such that $\varphi <^{GD} \psi$. By Proposition 4.16, $W_\psi = W_\varphi \oplus (U_\varphi \cap W_\psi)$. Therefore, bearing in mind the description of the G-Drazin inverses of $\psi$ in (6.2), their expression can be restated as:

$$\psi^{GD}(v) = \begin{cases} (\psi_{|W_\varphi})^{-1}(v) & \text{if } v \in W_\varphi \\ (\psi_{|(U_\varphi\cap W_\psi)})^{-1}(v) & \text{if } v \in (U_\varphi \cap W_\psi) \\ (\psi_{|U_\psi})^-(v) & \text{if } v \in U_\psi \end{cases}, \tag{6.3}$$

with $(\psi_{|U_\psi})^- \in X_{\psi_{|U_\psi}}(1)$.

**Lemma 6.11** *Let $\varphi, \psi \in \text{End}_k(V)$ be two Core-Nilpotent endomorphisms with $V = W_\varphi \oplus U_\varphi = W_\psi \oplus U_\psi$ their respective AST decompositions. Let us define $\tilde{\psi} \in \text{End}_k(U_\varphi)$ as follows:*

$$\tilde{\psi}(v) = \begin{cases} (\psi_{|(U_\varphi\cap W_\psi)})^{-1}(v) & \text{if } v \in U_\varphi \cap W_\psi \\ (\psi_{|U_\psi})^-(v) & \text{if } v \in U_\psi \end{cases},$$

*with* $(\psi_{|U_\psi})^- \in X_{\psi_{|U_\psi}}(1)$. *If* $\varphi <^{GD} \psi$, *then* $\tilde{\psi} \in X_{\psi_{|U_\varphi}}(1)$.

***Proof*** By Proposition 4.19 we know that $U_\varphi = U_\psi \oplus (U_\varphi \cap W_\psi)$. Hence, by Theorem 5.4 it is immediate that $\tilde{\psi} \in X_{\psi_{|U_\varphi}}(GD) \subseteq X_{\psi_{|U_\varphi}}(1)$. □

**Theorem 6.12** *Let* $\varphi, \psi \in \mathrm{End}_k(V)$ *be two Core-Nilpotent endomorphisms. If*

$$\varphi <^{GD} \psi \text{ then } X_\psi(GD) \subseteq X_\varphi(GD).$$

***Proof*** Let us suppose that $\varphi <^{GD} \psi$. By $III$ of Theorem 6.6, we know that $\varphi <^d \psi$ and there exists a $1-inverse$ $\varphi^- \in \mathrm{End}_k(U_\varphi)$ of $\varphi$ such that:

$$(\varphi\varphi^-)_{|U_\varphi} = (\psi\varphi^-)_{|U_\varphi};$$
$$(\varphi^-\varphi)_{|U_\varphi} = (\varphi^-\psi)_{|U_\varphi}.$$

By Proposition 4.20, we know that:

$$\varphi_{|W_\varphi} = \psi_{|W_\varphi}, \tag{6.4}$$

and in particular, $(\varphi_{|W_\varphi})^{-1} = (\psi_{|W_\varphi})^{-1}$. Moreover, by Propositions 4.16 and 4.19 the following decompositions hold:

$$W_\psi = W_\varphi \oplus (U_\varphi \cap W_\psi) \text{ and } U_\varphi = U_\psi \oplus (U_\varphi \cap W_\psi). \tag{6.5}$$

The expression of any G-Drazin inverse of $\psi$ in this conditions is the one presented in (6.3). In fact, by the condition mentioned in (6.4), the explicit expression of any G-Drazin inverse of $\psi$ can be written as:

$$\psi^{GD}(v) = \begin{cases} (\varphi_{|W_\varphi})^{-1}(v) & \text{if } v \in W_\varphi \\ (\psi_{|(U_\varphi \cap W_\psi)})^{-1}(v) & \text{if } v \in (U_\varphi \cap W_\psi) \\ (\psi_{|U_\psi})^{-}(v) & \text{if } v \in U_\psi \end{cases}. \tag{6.6}$$

Accordingly, for any $v = w + u \in V = W_\varphi \oplus U_\varphi$, we have that

$$\psi^{GD}(v) = \varphi^D(w) + \psi^-_{GD}(u), \tag{6.7}$$

(with the notations used in Corollary 5.8) with $\psi^-_{GD} \in \mathrm{End}_k(U_\varphi)$ and satisfying that $\psi^-_{GD} \in X_{\psi_{|U_\varphi}}(1)$ by Lemma 6.11. Moreover, in virtue of Lemma 6.10 one gets that $\psi^-_{GD} \in X_{\varphi_{|U_\varphi}}(1)$. Therefore, bearing in mind the expression of (6.7), one concludes that $\psi^{GD} \in X_\varphi(GD)$ as desired. □

The following theorem establishes the relationship between the space pre-order (Definition 3.2) and the G-Drazin order. Although there was an entire theorem dedicated to the relation of the G-Drazin order with other pre-orders and orders, this result is strongly related to the recent discussion dealing with G-Drazin inverses so its late appearance is justified.

**Theorem 6.13** *Let* $\varphi, \psi \in \mathrm{End}_k(V)$ *be two Core-Nilpotent endomorphisms. Then,*

$$\varphi <^{GD} \psi \text{ if and only if } \varphi <^s \psi \text{ and } X_\psi(GD) \subseteq X_\varphi(GD).$$

***Proof*** Firstly, let us suppose that $\varphi <^{GD} \psi$. The fact that $X_\psi(GD) \subseteq X_\varphi(GD)$ was proven in Theorem 6.12. We already know that $X_\psi(GD) \subseteq X_\varphi(GD)$, so let us consider any $\psi^{GD} \in X_\psi(GD)$ such that, as $\varphi <^{GD} \psi$, then

$$\varphi\psi^{GD} = \psi\psi^{GD} \tag{6.8}$$

and

$$\psi^{GD}\varphi = \psi^{GD}\psi. \tag{6.9}$$

From (6.8), we get that

$$\varphi = \psi\psi^{GD}\varphi$$

and from (6.9) one obtains

$$\varphi = \varphi\psi^{GD}\psi.$$

Therefore, it is

$$\varphi = \psi\psi^{GD}\varphi = \varphi\psi^{GD}\psi,$$

and $\varphi <^s \psi$ by $II$ of Theorem 3.18 as, in particular $X_\psi(GD) \subseteq X_\varphi(GD) \subseteq X_\varphi(1)$.

Conversely, let us suppose that $\varphi <^s \psi$ and $X_\psi(GD) \subseteq X_\varphi(GD)$. We are going to prove that $\varphi <^{GD} \psi$ using Definition 6.1, inspired by the proof of [7, Theorem 2.4]. Let us fix $\varphi^{GD} \in X_\varphi(GD)$ and $\psi^{GD} \in X_\psi(GD)$ and let us consider: $\tilde{\varphi}^{GD} = \varphi^{GD}\varphi\psi^{GD}$ and $\hat{\varphi}^{GD} = \psi^{GD}\varphi\varphi^{GD}$. As $X_\psi(GD) \subseteq X_\varphi(GD)$, if $i(\varphi) = m$ is the index of $\varphi$, one has:

$$\begin{aligned}
\varphi\tilde{\varphi}^{GD}\varphi &= \varphi\varphi^{GD}\varphi\psi^{GD}\varphi = \varphi\psi^{GD}\varphi = \varphi;\\
\varphi^m\tilde{\varphi}^{GD} &= \varphi^m\varphi^{GD}\varphi\psi^{GD} = \varphi^{GD}\varphi^m\varphi\psi^{GD} = \varphi^{GD}\varphi\varphi^m\psi^{GD} =\\
&= \varphi^{GD}\varphi\psi^{GD}\varphi^m = \tilde{\varphi}^{GD}\varphi^m,
\end{aligned}$$

so $\tilde{\varphi}^{GD} \in X_\varphi(GD)$ by Definition 5.1. An analogous calculation shows that $\hat{\varphi}^{GD} \in X_\varphi(GD)$. If $\varphi <^s \psi$, then for any 1-inverse of $\psi$ it is $\varphi = \varphi\psi^-\psi = \psi\psi^-\varphi$ as it was proved in $III$ of Theorem 3.18. Bearing this in mind, using that $\varphi = \varphi\psi^-\psi$, we get that:

$$\begin{aligned}
\tilde{\varphi}^{GD}\varphi &= \varphi^{GD}(\varphi\psi^{GD}\varphi) = \varphi^{GD}\varphi = \varphi^{GD}\varphi\psi^-\psi =\\
&= \varphi^{GD}\varphi\psi^-\psi\psi^{GD}\psi = \varphi^{GD}\varphi\psi^{GD}\psi = \tilde{\varphi}^{GD}\psi.
\end{aligned}$$

On a similar way, using that $\varphi = \psi\psi^-\varphi$, we obtain:

$$\begin{aligned}
\varphi\hat{\varphi}^{GD} &= \varphi\psi^{GD}\varphi\varphi^{GD} = \varphi\varphi^{GD} = \psi\psi^-\varphi\varphi^{GD} =\\
&= \psi\psi^{GD}\psi\psi^-\varphi\varphi^{GD} = \psi\psi^{GD}\varphi\varphi^{GD} = \psi\hat{\varphi}^{GD},
\end{aligned}$$

and the claim is proved. □

**Theorem 6.14** *The relation $<^{GD}$ defines a partial order in the set of Core-Nilpotent endomorphisms.*

***Proof*** Reflexivity holds immediately.

Antisymmetry. If $\varphi <^{GD} \psi$ and $\psi <^{GD} \varphi$ then, by Theorem 6.12 we know $X_\varphi(GD) = X_\psi(GD)$. Moreover, we know that

$$\begin{aligned}
\varphi\phi^{GD} &= \psi\phi^{GD},\\
\phi^{GD}\varphi &= \phi^{GD}\psi,
\end{aligned}$$

for a G-Drazin inverse $\phi^{GD} \in \operatorname{End}_k(V)$ of both $\varphi, \psi$. Hence:

$$\varphi = \varphi\phi^{GD}\varphi = \psi\phi^{GD}\varphi = \psi\phi^{GD}\psi = \psi,$$

as every G-Drazin inverse is in particular a 1-inverse.

Transitivity. Let $\varphi <^{GD} \psi$ and $\psi <^{GD} \gamma$ for $\varphi, \psi, \gamma \in \mathrm{End}_k(V)$ Core-Nilpotent endomorphisms. In virtue of Theorem 6.13 we know that

$$\varphi <^{s} \psi, \psi <^{s} \gamma \text{ and that } X_\gamma(GD) \subseteq X_\psi(GD) \subseteq X_\varphi(GD).$$

Therefore, we conclude because the relation $<^s$ is a pre-order, and in particular; it is transitive, as it was shown in Theorem 3.13. We get $\varphi <^s \gamma$ and $X_\gamma(GD) \subseteq X_\varphi(GD)$ and again by Theorem 6.13 $\varphi <^{GD} \gamma$. □

## 7 Final remarks and considerations

Let us conclude with some considerations.

***Remark 7.1*** Notice that if $\varphi, \psi \in \mathrm{End}_k(V)$ are two Core-Nilpotent endomorphisms with $i(\varphi) = i(\psi) \leq 1$ then the Drazin pre-order and the sharp order coincide. This is a direct consequence of Theorem 2.7. Therefore, the Drazin pre-order is a partial order in the set of Core-Nilpotent endomorphisms of index less or equal than 1. In this conditions, the AST-decompositions induced by $\varphi$ and $\psi$ are exactly:

$$V = \mathrm{Im}\,\varphi \oplus \mathrm{Ker}\,\varphi = \mathrm{Im}\,\psi \oplus \mathrm{Ker}\,\psi.$$

Moreover, one can check that the Lemmas 4.15,4.18 and the Propositions 4.11, 4.16, 4.19 just proved before make sense with [1, Lemmas 4.9, 4.10] and Propositions [1, Propositions 4.8,4.13,4.14]. This parallelism goes further as Proposition 4.20 restricts exactly to [1, Proposition 4.5] when both endomorphisms involved are of index 1 and Proposition 4.5 is again [1, Corollary 4.6] when (with the notations used in this paper) $m \leq 1$.

Let us consider $\varphi \in \mathrm{End}_k(V)$ a Core-Nilpotent endomorphism with $i(\varphi) \leq 1$. From the definition of G-Drazin inverse (Definition 6.1), we deduce that the set $X_\varphi(GD)$ of G-Drazin inverses of $\varphi$ coincides with the set of $g$-commuting inverses of $\varphi$, $X_\varphi(g_-)$ (Definition 2.14). In particular, in the frame of Core-Nilpotent endomorphisms, we know that these sets are not empty, as the Drazin inverse exists (Theorem 2.6) and belongs to them. Thus, if $\varphi, \psi \in \mathrm{End}_k(V)$ are two Core-Nilpotent endomorphisms with $i(\varphi) = i(\psi) \leq 1$, the G-Drazin order turns out to be equivalent to the Sharp partial order and hence, to the Drazin pre-order by the previous observations, namely Theorem 2.7.

fact, it is easy to see that given two Core-Nilpotent endomorphisms $\varphi, \psi \in \mathrm{End}_k(V)$ with $i(\varphi) = i(\psi) \leq 1$, if $\varphi^2 = \varphi$ (commonly stated as $\varphi$ being a projector), then $\varphi <^{GD} \psi$ if and only if $\psi \in X_\varphi(GD)$.

***Proof*** This is a direct consequence of the previous discussion and the result proven by the author of this paper in [1, Lemma 4.21]. □

***Remark 7.2*** Bearing in mind the definition of Core-Nilpotent endomorphism (Definition 2.2), all the results that have been presented in the article can be used to obtain immediately the theory of G-Drazin inverses and the G-Drazin partial order for nilpotent endomorphisms over arbitrary vector spaces, finite potent endomorphisms and automorphisms over arbitrary $k$-vector spaces.

***Remark 7.3*** Moreover, the obtained results generalize the theory of the G-Drazin partial order for finite square matrices (see, for instance [21]) in such a way that using the well known theory that relates square matrices with endomorphisms over vector spaces one can obtain new results from the main properties of the G-Drazin order and new proofs of the known ones. For example, among others: the class of all endomorphisms $\psi$ above a Core-Nilpotent endomorphism $\varphi$ for the Drazin pre-order is calculated on a natural way using the AST-decomposition in Theorem 4.12; the relation between the AST decompositions in the Drazin pre-order presented in Sect. 4.1, the relation between the G-Drazin partial order and other pre-orders and orders stated in Theorem 6.6 and finally the relation between the G-Drazin order and the G-Drazin inverses of the endomorphisms involved stated in Theorem 6.12.

***Remark 7.4*** Some remaining questions that the author think could be of interest, that are not present in literature (as far as the author knows) and that can be approached using the theory of Core-Nilpotent endomorphisms are the following:

- To give an algorithm to calculate explicitly all the endomorphisms $\psi$ above a Core-Nilpotent $\varphi$ for the Drazin pre-order.
- To enunciate clearly the class of all endomorphisms $\psi$ above a Core-Nilpotent $\varphi$ for the G-Drazin partial order.
- Accordingly, once the previous statement is achieved, to give an algorithm to calculate explicitly all the endomorphisms $\psi$ above a Core-Nilpotent $\varphi$ for the G-Drazin partial order.
- Specializing the previous points to the theory of finite square matrices.

**Author Contributions** Not applicable.

**Funding** Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature. No funding was received to assist with the preparation of this manuscript.

**Availability of data and materials** Not applicable.

**Code availability** Not applicable.

## Declarations

**Conflict of interest** The author has no relevant financial or non-financial interests to disclose.

**Ethics approval** Not applicable.

**Consent to participate** Not applicable.

**Consent for publication** The author authorises the publication of the previous notes.



## References

1. Alba Alonso, D.: Sharp order of Core-Nilpotent endomorphisms. Filomat **37**, 10,3167-3181 (2023). https://doi.org/10.2298/FIL2310167A

2. Argerami, M., Szechtman, F., Tifenbach, R.: On Tate's trace. Linear Multilinear Algebra **55**(6), 515–520 (2007)
3. Campbell, S.L.: The Drazin inverse of an infinite matrix. SIAM J. Appl. Math. **31**, 492–503 (1976)
4. Drazin, M.P.: Pseudo-inverses in associative rings and semigroups. Am. Math. Mon. **40**(7), 506–514 (1958)
5. Coll, C., Herrero, A., Sánchez, E., Thome, N.: On the minus partial order in control systems. Appl. Math. Comput. **125529**, 529 (2020)
6. Coll, C., Lattanzi, M., Thome, N.: Weighted G-Drazin inverses and a new pre-order on rectangular matrices. Appl. Math. Comput. **317**, 12–24 (2018)
7. Ferreyra, D.E., Lattanzi, M., Levis, F.E., et al.: Solving an open problem about the G-Drazin partial order. Electron. J. Linear Algebra **36**, 55–66 (2020)
8. Hartwig, R.E.: How to partially order regular elements. Math. Jpn. **25**(1), 1–13 (1980)
9. Herrero, A., Thome, N.: Sharp partial order and linear autonomous systems. Appl. Math. Comput. **366**, 124736 (2020)
10. Mitra, S.K., Hartwig, R.E.: Partial orders based on outer inverses. Linear Algebra Appl. **176**, 3–20 (1992)
11. Mitra, S.K., Bhimasankaram, P., Malik, S.B.: Matrix Partial Orders, Shorted Operators and Applications. World Scientific (2010)
12. Pablos Romo, F.: Classification of finite potent endomorphisms. Linear Algebra Appl. **440**, 266–277 (2014)
13. Pablos Romo, F.: Core-nilpotent decomposition and new generalized inverses of finite potent endomorphisms. Linear Multilinear Algebra **68**(11), 2254–2275 (2020)
14. Pablos Romo, F.: Core-Nilpotent decomposition of infinite dimensional vector spaces. Mediterr. J. Math.(2021)
15. Pablos Romo, F.: Group inverse of finite potent endomorphisms on arbitrary vector spaces. Oper. Matrices **14**(4), 1029–1042 (2020)
16. Pablos Romo, F.: On the Drazin inverse of finite potent endomorphisms. Linear Multilinear Algebra **67**(10), 627–647 (2019)
17. Pablos Romo, F., Alba Alonso, D.: On the explicit computation of the set of 1-inverses of a square matrix. Linear Multilinear Algebra **71**(18), 2869–2876 (2023)
18. Pablos Romo, F.: On G-Drazin inverses of finite potent endomorphisms and arbitrary square matrices. Linear Multilinear Algebra **70**(12), 2227–2247 (2019). https://doi.org/10.1080/03081087.2020.1791790
19. Tate, J.: Residues of differentials on curves. Ann. Scient. Éc. Norm. Sup. 1, 4a série, 149–159 (1968)
20. Tian, Y.: Solutions of the matrix inequalities in the minus partial ordering and löwner partial ordering. Math. Inequal. Appl. **16**, 861–872 (2013)
21. Wang, X., Liu, H.: Partial orders based on core-nilpotent decomposition. Linear Algebra Appl. **488**, 235–248 (2016)