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# Surjective and closed range differentiation operator

**Tesfa Mengestie**[1]

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**Abstract**
We identify Fock-type spaces $\mathcal{F}_{(m,p)}$ on which the differentiation operator $D$ has closed range. We prove that $D$ has closed range only if it is surjective, and this happens if and only if $m = 1$. Moreover, since the operator is unbounded on the classical Fock spaces, we consider the modified or the weighted composition–differentiation operator, $D_{(u,\psi,n)} f = u \cdot \left(f^{(n)} \circ \psi\right)$, on these spaces and describe conditions under which the operator admits closed range, surjective, and order bounded structures.

**Keywords** Fock-type spaces · Closed range · Differentiation operator · Surjective · Order bounded

**Mathematics Subject Classification** Primary 47B37 · 30H20 · Secondary 46E15 · 46E20

## 1 Introduction

The differentiation operator, $Df = f'$, often appears as an example of unbounded linear operators in many Banach spaces, including Hardy spaces and Bergman spaces [11], Fock spaces and Fock-type spaces, where the weight decays faster than the Gaussian weight [17], and Fock–Sobolev spaces, where the weights decay slower than the Gaussian weight [16]. Inspired by all these developments, the question whether there could exist Fock-type spaces on which $D$ admits basic operator-theoretic structures was investigated in [14]. To answer the question, the author considered the space

$$\mathcal{F}_{(m,p)} := \left\{ f \in \mathcal{H}(\mathbb{C}) : \|f\|^p_{(m,p)} = \int_{\mathbb{C}} |f(z)|^p e^{-p|z|^m} dA(z) < \infty \right\},$$

where $\mathcal{H}(\mathbb{C})$ denotes the set of entire functions on the complex plane $\mathbb{C}$, $m > 0$, $1 \le p < \infty$, and $A$ is the usual Lebesgue area measure. Then the following basic property was proved.

**Theorem 1.1** *[14, Theorem 1.1] Let $1 \le p, q < \infty$ and $m > 0$.*

✉ Tesfa Mengestie
Tesfa.Mengestie@hvl.no

1 Mathematics Section, Western Norway University of Applied Sciences, Klingenbergvegen 8, N-5414 Stord, Norway

*(i)* *If* $p \leq q$, *then* $D : \mathcal{F}_{(m,p)} \to \mathcal{F}_{(m,q)}$ *is bounded if and only if*

$$m \leq 2 - \frac{pq}{pq + q - p}, \tag{1.1}$$

*and compact if and only if the inequality in (1.1) is strict.*

*(ii)* *If* $p > q$, *then* $D : \mathcal{F}_{(m,p)} \to \mathcal{F}_{(m,q)}$ *is bounded (compact) if and only if*

$$m < 1 - 2\Big(\frac{1}{q} - \frac{1}{p}\Big).$$

For $p = q$, the inequality in (1.1) simplifies to $m \leq 1$, which is stronger than the boundedness condition for $p < q$. On the corresponding growth type space,

$$\mathcal{F}_{(m,\infty)} := \Big\{ f \in \mathcal{H}(\mathbb{C}) : \|f\|_{(m,\infty)} = \sup_{z \in \mathbb{C}} |f(z)| e^{-|z|^m} < \infty \Big\},$$

the boundedness of $D$ was characterized by the same condition $m \leq 1$; see [2, 3, 10]. If $D$ acts between two different Fock-type spaces $\mathcal{F}_{(m,p)}$ and $\mathcal{F}_{(m,q)}$, where one of the spaces is growth type, then a simple variant of the proof of Theorem 1.1 in [14] gives the following result.

**Corollary 1.2** *Let* $1 \leq p < \infty$ *and* $m > 0$. *Then the operator*

*(i)* $D : \mathcal{F}_{(m,p)} \to \mathcal{F}_{(m,\infty)}$ *is bounded if and only if*

$$m \leq 2 - \frac{p}{p+1}, \tag{1.2}$$

*and compact if and only if the inequality in (1.2) is strict.*

*(ii)* $D : \mathcal{F}_{(m,\infty)} \to \mathcal{F}_{(m,p)}$ *is bounded (compact) if and only if* $m < 1 - \frac{2}{p}$.

For more related results, we refer the interested readers to [4, 14] and the references therein. One of the main objectives of this work is to identify Fock-type spaces on which the differentiation operator admits closed range structure. Our next main result shows there exists no closed range differentiation operator acting between two different Fock-type spaces.

**Theorem 1.3** *Let* $1 \leq p, q \leq \infty$, $m > 0$, *and* $D : \mathcal{F}_{(m,p)} \to \mathcal{F}_{(m,q)}$ *be bounded. Then the following statements are equivalent.*

*(i)* *D has closed range;*
*(ii)* $p = q$ *and* $m = 1$;
*(iii)* *D is surjective.*

The result identifies $\mathcal{F}_{(1,p)}$ as the only Fock-type space supporting closed range structure for the operator $D$. The proof of the result will be presented in Sect. 2.

As mentioned earlier, $D$ is not bounded on the classical Fock spaces. In [15], the author studied whether simply modulating the classical Gaussian weight function $|z|^2/2$ by positive parameters $\alpha$ would produce a bounded $D$ on the spaces

$$\mathcal{F}_\alpha^p := \Big\{ f \in \mathcal{H}(\mathbb{C}) : \int_{\mathbb{C}} |f(z)|^p e^{-\frac{p\alpha}{2}|z|^2} dA(z) < \infty \Big\}.$$

It follows that for positive parameters $\alpha$ and $\beta$, $D : \mathcal{F}_\alpha^p \to \mathcal{F}_\beta^q$ is bounded if and only if $\alpha < \beta$ and $p \leq q$. This condition equivalently describes the compactness of the operator. Consequently, as it will be explained later in the proof of Theorem 1.3, such modulated spaces support no closed range compact differentiation operator.

### 1.1 Weighted composition–differentiation operator

In the preceding section, we considered Fock-type spaces on which the differentiation operator is bounded, and we classified them based on whether they support closed range structure for $D$ or not. In this section, we modify the operator itself and study the closed range and surjectivity problems on the classical Fock spaces.

For each $n \in \mathbb{N}_0 = \{0, 1, 2, ...\}$ and $u, \psi$ in $\mathcal{H}(\mathbb{C})$, we define the weighted composition–differentiation operator $D_{(u,\psi,n)}$ by

$$D_{(u,\psi,n)} f = u \cdot \big(f^{(n)} \circ \psi\big),$$

where $f^{(n)}$ is the $n^{th}$ order derivative of the function $f$ and $f^{(0)} = f$. Then we investigate when the operator $D_{(u,\psi,n)}$ admits closed range structure on the classical Fock spaces. This class of operators has lately attracted a considerable amount of attention; see for example [22] and the references therein.

For $1 \leq p \leq \infty$, recall that the Fock spaces $\mathcal{F}_p$ are defined by

$$\mathcal{F}_p := \big\{f \in \mathcal{H}(\mathbb{C}) : \|f\|_p < \infty\big\},$$

where

$$\|f\|_p := \begin{cases} \Big(\frac{p}{2\pi}\int_{\mathbb{C}} |f(z)|^p e^{-\frac{p}{2}|z|^2} dA(z)\Big)^{\frac{1}{p}} < \infty, & 1 \leq p < \infty \\ \sup_{z\in\mathbb{C}} |f(z)| e^{-\frac{1}{2}|z|^2} < \infty, & p = \infty. \end{cases}$$

The space $\mathcal{F}_2$ is a reproducing kernel Hilbert space with kernel and normalized reproducing kernel functions

$$K_w(z) = e^{\overline{w}z} \text{ and } k_w(z) = \|K_w\|_2^{-1} K_w(z) = e^{\overline{w}z - \frac{|w|^2}{2}}$$

for all $z, w \in \mathbb{C}$. A straightforward calculation shows $\|k_w\|_p = 1$ for all $p$. For more details, we refer to the book [23].

Note that for each $f \in \mathcal{H}(\mathbb{C})$ and $p \neq \infty$, by [23, p. 37] the local estimate

$$|f(z)| \leq \frac{e^{\frac{|z|^2}{2}}}{r^2} \bigg(\int_{D(z,r)} |f(w)|^p e^{-\frac{p|w|^2}{2}} dA(w)\bigg)^{1/p} \tag{1.3}$$

holds, where $D(z, r)$ is a disc of center $z$ and radius $r$. For $r = 1$, this estimate gives

$$|f(z)| \leq e^{\frac{|z|^2}{2}} \|f\|_p. \tag{1.4}$$

By definition of the norm, the estimate in (1.4) holds for $p = \infty$ as well.

We note that the result in [17] addresses the unboundedness of only the first order differentiation operator $D$ on Fock spaces. A simple argument shows the $n^{th}$ order differentiation operator, $D^n f = f^{(n)}$, is not bounded for all $n \in \mathbb{N}$ either. Indeed, using the kernel function $K_w$, we observe

$$\frac{\|D^n K_w\|_q}{\|K_w\|_p} = |\overline{w}|^n \frac{\|K_w\|_q}{\|K_w\|_p} = |w|^n \to \infty,$$

when $|w| \to \infty$ independently of the exponents $p$ and $q$. On the other hand, an easy computation using the equivalent norms in (2.9) below shows the composed differentiation operator, $D^n C_\psi f = f^{(n)} \circ \psi$, is bounded on Fock spaces for every $\psi(z) = az$ and $0 <$

$|a| < 1$. Motivated by this, in [22] the bounded and compact structures of $D_{(u,\psi,n)}$ were described in terms of the function

$$L_{(u,\psi,n)}(z) := |u(z)||\psi(z)|^n e^{\frac{1}{2}(|\psi(z)|^2-|z|^2)}$$

for each $z \in \mathbb{C}$. For further referencing, we state the result below.

**Theorem 1.4** *[22, Theorem 1.3] Let $u, \psi \in \mathcal{H}(\mathbb{C})$, $n \in \mathbb{N}_0$, and $1 \le p, q \le \infty$.*

*(i) If $p \le q$, then $D_{(u,\psi,n)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded if and only if $L_n := \sup_{z\in\mathbb{C}} L_{(u,\psi,n)}(z) < \infty$, and compact if and only if $L_{(u,\psi,n)}(z) \to 0$ when $|z| \to \infty$.*
*(ii) If $p > q$, then $D_{(u,\psi,n)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded (compact) if and only if $L_{(u,\psi,n)} \in L^{\frac{pq}{p-q}}(\mathbb{C}, dA)$ for $p < \infty$ and $L_{(u,\psi,n)} \in L^q(\mathbb{C}, dA)$ for $p = \infty$.*

A bounded $D_{(u,\psi,n)}$ implies $\psi(z) = a_n z + b_n$, $a_n, b_n \in \mathbb{C}$ and $|a_n| \le 1$: see [22] for the details. For simplicity, we write $\psi(z) = az + b$, $a, b \in \mathbb{C}$. If $|a| = 1$, then

$$L_{(u,\psi,n)}(z) = |u(z)||\psi(z)|^n e^{\frac{1}{2}(|\psi(z)|^2-|z|^2)} = |u(z)\psi^n(z)K_{\overline{a}b}(z)|e^{\frac{|b|^2}{2}} \le L_n$$

for all $z \in \mathbb{C}$, $n \in \mathbb{N}_0$, and $L_n$ as in Theorem 1.4. Consequently,

$$|u(z)\psi^n(z)K_{\overline{a}b}(z)| \le L_n e^{-\frac{|b|^2}{2}}.$$

By Liouville's theorem, it follows that $u\psi^n K_{\overline{a}b}$ is a constant $C_n$ and hence $u(z)\psi^n(z) = C_n K_{-\overline{a}b}(z)$. Setting $z = 0$, we get $C_n = b^n u(0)$. Therefore,

$$u(z)\psi^n(z) = b^n u(0) K_{-\overline{a}b}(z). \tag{1.5}$$

The representation in (1.5) will be needed later.

For an $f \in \mathcal{H}(\mathbb{C})$ and a positive $r$, we set $M_f(r) = \max\{|f(z)| : |z| = r\}$. Then the order $\rho(f)$ of $f$ is defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\big(\log M_f(r)\big)}{\log r}.$$

Now by Theorem 1.4,

$$|u(z)||\psi(z)|^n \le L_n e^{\frac{1}{2}(|z|^2-|az+b|^2)} \tag{1.6}$$

for all $z \in \mathbb{C}$. It follows that $u\psi^n$ is of order at most 2. Consequently, a simple variant of the proof of [5, Theorem 3.2] gives the following representation whenever $u\psi^n$ is non-vanishing.

**Lemma 1.5** *Let $u, \psi \in \mathcal{H}(\mathbb{C})$, $n \in \mathbb{N}_0$, and $1 \le p \le \infty$. Let $D_{(u,\psi,n)}$ be bounded on $\mathcal{F}_p$ and hence $\psi(z) = az + b$ for some $a, b \in \mathbb{C}$ such that $|a| \le 1$. If $0 \le |a| < 1$ and $u\psi^n$ is non-vanishing, then*

*(i) $D_{(u,\psi,n)}$ is compact if and only if $u\psi^n$ has the form*

$$u(z)\psi^n(z) = e^{a_{0n}+a_{1n}z+a_{2n}z^2} \tag{1.7}$$

*for some constants $a_{0n}, a_{1n}, a_{2n}$ in $\mathbb{C}$ such that $|a_{2n}| < \frac{1-|a|^2}{2}$.*

*(ii)* $D_{(u,\psi,n)}$ *is bounded but not compact if and only if* $u\psi^n$ *has the form in (1.7) with* $|a_{2n}| = \frac{1-|a|^2}{2}$ *and either* $a_{1n} + a\overline{b} = 0$ *or* $a_{1n} + a\overline{b} \neq 0$ *and*

$$a_{2n} = -\frac{(1-|a|^2)(a_{1n} + a\overline{b})^2}{2|a_{1n} + a\overline{b}|^2}.$$

By Theorem 1.4 we get that for $u(z) = 1$ and $\psi(z) = az + b$ with $|a| < 1$ the operator $D_{(1,\psi,n)}$ is compact. In this case, the operator is quasinilpotent, that is, its spectral radius is zero. Indeed, if $\lambda$ is an eigenvalue, then

$$D_{(1,\psi,n)}f = f^{(n)}(az+b) = \lambda f(z)$$

for some nonzero function $f \in \mathcal{F}_p$. Differentiating $m$ times both sides of the equation gives

$$a^m f^{(n+m)}(az+b) = \lambda f^{(m)}(z).$$

It follows that

$$f^{(n+m)}(az+b) = D_{(1,\psi,n)})f^{(m)}(z) = \frac{\lambda}{a^m} f^{(m)}(z).$$

This shows $f$ cannot be a polynomial and $f^{(m)}$ cannot be zero. Thus, $\lambda/a^m$ forms a sequence of eigenvalues for $D_{(1,\psi,n)}$, which is a contradiction since $\lambda/a^m \to \infty$ as $m \to \infty$.

## 1.2 Order bounded $D_{(u,\psi,n)}$

Another interesting notion closely related to boundedness of an operator is order boundedness. The notion finds applications due to its close relation to absolutely summing operators [7, 8]. We say that an operator $T : \mathcal{F}_p \to \mathcal{F}_q$ is order bounded if there exists a positive function $h \in L^q\big(\mathbb{C}, e^{-\frac{q}{2}|z|^2} dA(z)\big)$ such that for all $f \in \mathcal{F}_p$ with $\|f\|_p \leq 1$

$$|T(f(z))| \leq h(z)$$

almost everywhere with respect to the measure $A$. This definition was introduced by Hunziker and Jarchow in [12]. For some recent work on the subject, we refer the interested reader to [19, 21] and the references therein. For the operator $D_{(u,\psi,n)}$, we provide the following characterization.

**Theorem 1.6** *Let* $u, \psi \in \mathcal{H}(\mathbb{C})$, $n \in \mathbb{N}_0$, *and* $1 \leq p, q \leq \infty$. *Then* $D_{(u,\psi,n)} : \mathcal{F}_p \to \mathcal{F}_q$ *is order bounded if and only if* $L_{(u,\psi,n)} \in L^q(\mathbb{C}, dA)$.

By the discussion after Theorem 1.4, we observe that the order bounded condition implies $\psi(z) = az + b$ and $|a| < 1$. Note that if $|a| = 1$, then by (1.5)

$$L_{(u,\psi,n)}(z) = |u(z)\psi^n(z)K_{\overline{a}b}(z)|e^{\frac{|b|^2}{2}} = |b^n u(0)||K_{-\overline{a}b}(z)K_{\overline{a}b}(z)|e^{\frac{|b|^2}{2}} = |b^n u(0)|e^{\frac{|b|^2}{2}}$$

and hence the condition in the theorem fails to hold. On the other hand, for $q = \infty$, the boundedness and order boundedness conditions coincide. This together with Theorem 1.4 implies every order bounded operator $D_{(u,\psi,n)}$ is compact. By [22, Theorem 1.8], we also observe that the order bounded $D_{(u,\psi,n)}$ are exactly those which are Hilbert-Schmidt in $\mathcal{F}_2$.

If $n = 0$, then $D_{(u,\psi,n)}$ reduces to the weighted composition operator $D_{(u,\psi,0)}f = u \cdot (f \circ \psi) = W_{(u,\psi)}$. Thus, Theorem 1.6 describes the order bounded weighted composition operators on Fock spaces, which we state it as follows.

**Corollary 1.7** *Let $u, \psi \in \mathcal{H}(\mathbb{C})$ and $1 \leq p, q \leq \infty$. Then $W_{(u,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is order bounded if and only if $m_{(u,\psi)} \in L^q(\mathbb{C}, dA)$, where*

$$m_{(u,\psi)}(z) = |u(z)|e^{\frac{1}{2}(|\psi(z)|^2-|z|^2)}$$

*for all $z \in \mathbb{C}$.*

If $u = 1$, then we set $D_{(\psi,n)} := D_{(1,\psi,n)}$, and Theorem 1.6 implies the following result about the composition-differentiation operator.

**Corollary 1.8** *Let $\psi \in \mathcal{H}(\mathbb{C})$ and $1 \leq p, q \leq \infty$. Then for each $n \in \mathbb{N}_0$, the following statements are equivalent.*

*(i)* $D_{(\psi,n)} : \mathcal{F}_p \to \mathcal{F}_q$ *is bounded;*
*(ii)* $D_{(\psi,n)} : \mathcal{F}_p \to \mathcal{F}_q$ *is order bounded;*
*(iii)* $\psi(z) = az + b$ *and* $|a| < 1$.

### 1.3 Closed range $D_{(u,\psi,n)}$

We now study the closed range property of $D_{(u,\psi,n)}$. If $\psi = b \in \mathbb{C}$, then $D_{(u,\psi,n)}f = uf^{(n)}(b)$, and hence the range,

$$\mathcal{R}(D_{(u,\psi,n)}) = \left\{uf^{(n)}(b) : f \in \mathcal{F}_p\right\},$$

is closed. Thus, we assume that $\psi$ is not a constant in the rest of the manuscript. The next proposition shows a nontrivial $D_{(u,\psi,n)}$ cannot have closed range if it acts between two different Fock spaces.

**Proposition 1.9** *Let $u, \psi \in \mathcal{H}(\mathbb{C})$ such that $\psi$ is not a constant, $1 \leq p, q \leq \infty$, and $n \in \mathbb{N}_0$. Then a bounded $D_{(u,\psi,n)} : \mathcal{F}_p \to \mathcal{F}_q$ has closed range only if $p = q$.*

The proof of the proposition will be given later in Sect. 2.3.

We now recall the notion of sampling sets for Banach spaces. The notion was introduced by Ghatage, Zheng and Zorboska [9] as a tool to study bounded below composition operators on Bloch spaces. Since then, it has been used to study both the bounded below and closed range properties of several operators on spaces of analytic functions. There have been also various ways of defining the notion; see for example [9, 18]. On Fock spaces, we provide the following unified and general definition. Let $1 \leq p \leq \infty$, $k \in \mathbb{N}_0$ and $M$ be a non-empty subset of $\mathcal{F}_p$. A subset $S$ of $\mathbb{C}$ is a $(p, k)$ sampling set for $M$ if there exists a positive constant $\delta_k$ such that

$$\delta_k\|f\|_p \leq \begin{cases} \sup_{z\in S} \frac{|f^{(k)}(z)|}{(1+|z|)^k}e^{-\frac{|z|^2}{2}}, & p = \infty \\ \left(\int_S \frac{|f^{(k)}(z)|^p}{(1+|z|)^{kp}}e^{-\frac{p}{2}|z|^2}dA(z)\right)^{\frac{1}{p}}, & p < \infty. \end{cases} \tag{1.8}$$

for all $f$ in $M$. For each positive $\epsilon_n$, $n \in \mathbb{N}_0$, we also define the sets

$$\Omega^{\epsilon_n}_{(u,\psi,n)} := \{z \in \mathbb{C} : L_{(u,\psi,n)}(z) > \epsilon_n\}, \quad G^{\epsilon_n}_{(u,\psi,n)} := \psi\big(\Omega^{\epsilon_n}_{(u,\psi,n)}\big).$$

We now state our next main result.

**Theorem 1.10** *Let $u, \psi \in \mathcal{H}(\mathbb{C})$ such that $\psi$ is not a constant, $1 \leq p \leq \infty$, $n \in \mathbb{N}_0$, and $D_{(u,\psi,n)}$ be bounded on $\mathcal{F}_p$. Then $D_{(u,\psi,n)}$ has closed range if and only if there exists $\epsilon_n > 0$ such that $G^{\epsilon_n}_{(u,\psi,n)}$ is a $(p, n)$ sampling set for $\mathcal{F}^0_{(p,n)}$, where*

$$\mathcal{F}^0_{(p,n)} := \{f \in \mathcal{F}_p : f(0) = f'(0) = ... = f^{(n-1)}(0) = 0\}.$$

### 1.4 Surjective $D_{(u,\psi,n)}$

In this section we consider the question of when the operator $D_{(u,\psi,n)}$ is surjective on Fock spaces. To state our result, we may first recall the notation of essential boundedness. We say a non-zero function $g$ in $\mathcal{H}(\mathbb{C})$ is essentially bounded away from zero if there exists a constant $\delta > 0$ such that the measure of the set $\{z \in \mathbb{C} : |g(z)| < \delta\}$ is zero.

**Theorem 1.11** *Let $u, \psi \in \mathcal{H}(\mathbb{C})$ such that $\psi$ is not a constant, $1 \leq p \leq \infty$, and $n \in \mathbb{N}_0$. Let $D_{(u,\psi,n)}$ be bounded on $\mathcal{F}_p$ and hence $\psi(z) = az + b$ for some $a, b \in \mathbb{C}$ and $|a| \leq 1$. Then the following statements are equivalent.*

*(i)* $L_{(u,\psi,n)}$ *is essentially bounded away from zero on* $\mathbb{C}$;
*(ii)* $D_{(u,\psi,n)}$ *is surjective;*
*(iii)* $|a| = 1$.

We close this section with a word on notation. The notion $U(z) \lesssim V(z)$ (or equivalently $V(z) \gtrsim U(z)$) means that there is a constant $C > 0$ such that $U(z) \leq CV(z)$ holds for all $z$ in the set of question. We write $U(z) \simeq V(z)$ if both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$.

## 2 Proof of the results

In this section, we present the proofs of the results. We begin by reminding the connection between the closed range problem and the bounded below property of a linear operator on Banach spaces. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Banach spaces. An operator $T : \mathcal{H}_1 \to \mathcal{H}_2$ is said to be bounded below if there exists a positive constant $C$ such that $\|Tf\|_{\mathcal{H}_2} \geq C\|f\|_{\mathcal{H}_1}$ for every $f \in \mathcal{H}_1$. As known from an application of the Open Mapping Theorem, an injective bounded operator on Banach spaces has closed range if and only if it is bounded below; see for example [1, Theorem 2.5]. The operator $D$ maps all constants to the zero function and fails to be injective unless its action is restricted to the spaces modulo the constants or

$$\mathcal{F}^0_{(m,p)} := \big\{f \in \mathcal{F}_{(m,p)} : f(0) = 0\big\}.$$

In the latter case, $D$ has closed range if and only if it is bounded below. On the other hand, for each $f \in \mathcal{F}_{(m,p)}$, the function $f - f(0)$ belongs to $\mathcal{F}^0_{(m,p)}$, and $D(f) = D(f - f(0)) = f'$. Thus, $D$ has closed range on $\mathcal{F}^0_{(m,p)}$ if and only if it has closed range on $\mathcal{F}_{(m,p)}$. For the sake of further referencing, we record this useful observation below.

**Lemma 2.1** *Let $1 \leq p, q \leq \infty$ and $m > 0$. Then $D : \mathcal{F}^0_{(m,p)} \to \mathcal{F}_{(m,q)}$ is bounded below if and only if $D : \mathcal{F}_{(m,p)} \to \mathcal{F}_{(m,q)}$ has closed range.*

The lemma will be used in our next proof.

### 2.1 Proof of Theorem 1.3

Note that (iii) obviously implies (i). Let us see (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii). From [6, 17], for each $f \in \mathcal{F}_{(m,p)}$ we have

$$\|f\|_{(m,p)} \simeq \begin{cases} \left(|f(0)|^p + \int_{\mathbb{C}} \frac{|f'(z)|^p e^{-p|z|^m}}{(1+|z|)^{p(m-1)}} dA(z)\right)^{\frac{1}{p}}, & p < \infty \\ |f(0)| + \sup_{z\in\mathbb{C}} \frac{|f'(z)|e^{-|z|^m}}{(1+|z|)^{(m-1)}}, & p = \infty. \end{cases} \tag{2.1}$$

We first consider the case when either $p > q$ or $p \leq q$ and $m < 2 - \frac{pq}{pq+q-p}$. Then by Theorem 1.1, the operator is compact. It is known that a compact operator has closed range if and only if its range is finite dimensional. On the other hand, $D$ is injective on an infinite dimensional set $\mathcal{F}^0_{(m,p)}$. Therefore, the operator has no closed range in this case.

Next, we consider $p \leq q < \infty$ and $m = 2 - \frac{pq}{pq+q-p}$. For this case, we prove the operator is not bounded below unless $p = q$. The norms of the monomials are estimated by

$$\|z^n\|_{(p,m)} \simeq \left(\frac{n}{me}\right)^{\frac{n}{m}+\frac{2}{mp}-\frac{1}{2p}}. \tag{2.2}$$

See [10] for the details. For $p = \infty$, the corresponding estimate becomes

$$\|z^n\|_{(\infty,m)} \simeq \left(\frac{n}{me}\right)^{\frac{n}{m}}. \tag{2.3}$$

We may now use Lemma 2.1 and suppose $p \leq q < \infty$ and the operator is bounded below. Then there exists a positive constant $\epsilon$ such that for all $n \in \mathbb{N}$

$$\|Dz^n\|_{(q,m)} = n\|z^{n-1}\|_{(q,m)} \geq \epsilon\|z^n\|_{(p,m)}. \tag{2.4}$$

This and (2.2) imply

$$\frac{n\|z^{n-1}\|_{(q,m)}}{\|z^n\|_{(p,m)}} \simeq \frac{n(n-1)^{\frac{n-1}{m}+\frac{2}{mq}-\frac{1}{2q}}}{n^{\frac{n}{m}+\frac{2}{mp}-\frac{1}{2p}}} \simeq \frac{n^{\frac{m-1}{m}+\frac{2}{mq}-\frac{1}{2q}}}{n^{\frac{2}{mp}-\frac{1}{2p}}} \gtrsim \epsilon \tag{2.5}$$

for all $n \in \mathbb{N}$. Now setting $m = 2 - \frac{pq}{pq+q-p}$ and simplifying further, the relation in (2.5) holds only when $n^{\frac{p-q}{2pq}} \gtrsim \epsilon$, which implies $p = q$ and hence $m = 2 - \frac{pq}{pq+q-p} = 1$. Similarly, if $p \leq q = \infty$, then

$$\frac{n\|z^{n-1}\|_{(\infty,m)}}{\|z^n\|_{(p,m)}} \simeq \frac{n(n-1)^{\frac{n-1}{m}}}{n^{\frac{n}{m}+\frac{2}{mp}-\frac{1}{2p}}} \simeq n^{1-\frac{1}{m}+\frac{1}{2p}-\frac{2}{mp}} = n^{-\frac{1}{2p}} \gtrsim \epsilon,$$

which implies $p = q$ and hence $m = 1$.

Next, we show (ii) implies (iii). Let now $p = q$ and $m = 1$. We need to show the range of the operator is $\mathcal{F}_{(1,p)}$. For each $f \in \mathcal{F}_{(1,p)}$, consider the entire function

$$h_f(z) = \int_0^z f(w)dw.$$

Applying (2.1),

$$\|h_f\|^p_{(1,p)} \simeq \int_{\mathbb{C}} |f(z)|^p e^{-p|z|} dA(z) \simeq \|f\|^p_{(1,p)} < \infty,$$

and hence $h_f \in \mathcal{F}_{(1,p)}$. Furthermore, $Dh_f = f$ and completes the proof.

**Remark 1**

The same argument used above to show that a compact $D$ cannot have closed range on the Fock-type spaces will be used in the sequel for the operator $D_{(u,\psi,n)}$.

## 2.2 Proof of Theorem 1.6

First note that for $n \in \mathbb{N}_0$ and $|z| \leq 1$, using the Cauchy integral formula and (1.4) we have

$$|f^{(n)}(z)| \leq \frac{n!}{2\pi} \int_{|w-z|=1} \frac{|f(w)|}{|w-z|^{n+1}} |dw| \leq n! \|f\|_p \max_{|w-z|=1} e^{\frac{|w|^2}{2}}$$
$$\leq n! e^{3/2} e^{\frac{|z|^2}{2}} \|f\|_p \tag{2.6}$$

for all $f \in \mathcal{F}_p$. Similarly, for $|z| > 1$,

$$|f^{(n)}(z)| \leq \frac{n!}{2\pi} \int_{|w-z|=1/|z|} \frac{|f(w)|}{|w-z|^{n+1}} |dw| \leq n! |z|^n \|f\|_p \max_{|w-z|=1/|z|} e^{\frac{|w|^2}{2}}$$
$$\leq n! e^{3/2} |z|^n e^{\frac{|z|^2}{2}} \|f\|_p. \tag{2.7}$$

Combining (2.6) and (2.7), we get

$$|f^{(n)}(z)| \leq n! e^{3/2} (1+|z|)^n e^{\frac{|z|^2}{2}} \|f\|_p,$$

and hence

$$|D_{(u,\psi,n)} f(z)| = |u(z) f^{(n)}(\psi(z))| \leq n! e^{3/2} |u(z)| (1+|\psi(z)|)^n e^{\frac{|\psi(z)|^2}{2}} \|f\|_p. \tag{2.8}$$

Suppose now that $D_{(u,\psi,n)}$ is order bounded. Then there exists a positive function $h_n \in L^q\big(\mathbb{C}, e^{-\frac{q}{2}|z|^2} dA(z)\big)$ such that

$$|D_{(u,\psi,n)} f(z)| \leq h_n(z)$$

for almost all $z \in \mathbb{C}$ and $\|f\|_p \leq 1$. Applying this inequality to the normalized kernel functions $k_w$, $w \in \mathbb{C}$, we get

$$|D_{(u,\psi,n)} k_w(z)| = |u(z) k_w^{(n)}(\psi(z))| = \big|u(z)\overline{w}^n\big| \big|e^{\overline{w}\psi(z) - |w|^2/2}\big| \leq h_n(z)$$

for almost all $z \in \mathbb{C}$. For $w = \psi(z)$ in particular,

$$\Big|u(z)\overline{w}^n e^{\overline{w}\psi(z) - \frac{|w|^2}{2}}\Big| = |u(z)\psi(z)^n| e^{\frac{|\psi(z)|^2}{2}} = L_{(u,\psi,n)}(z) e^{\frac{|z|^2}{2}} \leq h_n(z)$$

from which the necessity of the condition follows.

To prove the converse, setting

$$h_n(z) := n! e^2 |u(z)| (1+|\psi(z)|)^n e^{\frac{|\psi(z)|^2}{2}} \simeq n! e^2 L_{(u,\psi,n)}(z) e^{\frac{|z|^2}{2}},$$

we observe that the assumption on $L_{(u,\psi,n)}$ implies $h_n \in L^q\big(\mathbb{C}, e^{-\frac{q}{2}|z|^2} dA(z)\big)$. Furthermore, by (2.8)

$$|D_{(u,\psi,n)} f(z)| \leq n! e^{3/2} |u(z)| (1+|\psi(z)|)^n e^{\frac{|\psi(z)|^2}{2}} \leq h_n(z)$$

for any $f \in \mathcal{F}_p$ such that $\|f\|_p \leq 1$, and completes the proof of the theorem.

The next basic lemma connects the closed range problem and boundedness from below of the operator $D_{(u,\psi,n)}$.

**Lemma 2.2** *Let $u, \psi \in \mathcal{H}(\mathbb{C})$ such that $\psi$ is not a constant, $1 \leq p \leq \infty$, and $n \in \mathbb{N}_0$. Then a bounded $D_{(u,\psi,n)} : \mathcal{F}_p \to \mathcal{F}_p$ has closed range if and only if the restriction operator $D_{(u,\psi,n)} : \mathcal{F}^0_{(p,n)} \to \mathcal{F}_p$ is bounded from below, where*

$$\mathcal{F}^0_{(p,n)} := \big\{ f \in \mathcal{F}_p : f(0) = f'(0) = ... = f^{(n-1)}(0) = 0 \big\}.$$

***Proof*** Note that $D_{(u,\psi,n)}$ is injective on $\mathcal{F}^0_{(p,n)}$ but not on $\mathcal{F}_p$. Thus, as explained in the preceding section, $D_{(u,\psi,n)}$ has closed range on $\mathcal{F}^0_{(p,n)}$ if and only if it is bounded below. On the other hand, for each $f \in \mathcal{F}_p$, the function $f - S_n f$ belongs to $\mathcal{F}^0_{(p,n)}$, where $S_n f$ refers to the first $n$ terms of the Taylor series expansion of the function $f$. Now, $D_{(u,\psi,n)} f = D_{(u,\psi,n)}(f - S_n f)$, from which and the connection between the closed range problem and boundedness below, the claim follows.

## 2.3 Proof of Proposition 1.9

Let $1 \leq p \leq \infty$. From [13, 20], the estimate

$$\|f\|_p \simeq \begin{cases} \sum_{j=0}^{n-1} |f^{(j)}(0)| + \Big( \int_{\mathbb{C}} \frac{|f^{(n)}(z)|^p}{(1+|z|)^{np}} e^{-\frac{p}{2}|z|^2} dA(z) \Big)^{\frac{1}{p}}, & p < \infty \\ \sum_{j=0}^{n-1} |f^{(j)}(0)| + \sup_{z\in\mathbb{C}} \frac{|f^{(n)}(z)|}{(1+|z|)^n} e^{-\frac{1}{2}|z|^2}, & p = \infty \end{cases} \tag{2.9}$$

holds. We will appeal to this estimate several times in the sequel.

Let us consider first the case $p < q < \infty$ and assume $D_{(u,\psi,n)} : \mathcal{F}_p \to \mathcal{F}_q$ has closed range. By Lemma 2.2, the operator is bounded below on $\mathcal{F}^0_{(p,n)}$. We consider the sequence of the monomials $f_k(z) = z^k$, $k \in \mathbb{N}_0$, $k \geq n$. Using Stirling's approximation formula again

$$\|f_k\|_p^p = p \int_0^\infty r^{kp+1} e^{-pr^2/2} dr = \big(1/p\big)^{kp/2} \Gamma\big((kp+2)/2\big) \simeq \big(k/e\big)^{\frac{kp}{2}} \sqrt{k}. \tag{2.10}$$

See also [23, p. 40]. Now applying (2.9) and Theorem 1.4,

$$\begin{aligned} \|D_{(u,\psi,n)} f_k\|_q^q &= \frac{q}{2\pi} \int_{\mathbb{C}} |u(z)|^q |f_k^{(n)}(az+b)|^q e^{-\frac{q}{2}|z|^2} dA(z) \\ &= \frac{q}{2\pi} \int_{\mathbb{C}} |u(z)|^q (1 + |az+b|)^{nq} \frac{|f_k^{(n)}(az+b)|^q}{(1+|az+b|)^{nq}} e^{-\frac{q}{2}|z|^2} dA(z) \\ &\simeq \int_{\mathbb{C}} L^q_{(u,\psi,n)}(z) \frac{|f_k^{(n)}(az+b)|^q}{(1+|az+b|)^{nq}} e^{-\frac{q}{2}|az+b|^2} dA(z) \\ &\lesssim L_n^q \int_{\mathbb{C}} \frac{|f_k^{(n)}(az+b)|^q}{(1+|az+b|)^{nq}} e^{-\frac{q}{2}|az+b|^2} dA(z) \lesssim L_n^q \|f_k\|_q^q \lesssim \|f_k\|_q^q, \end{aligned}$$

where $L_n$ is a constant as in Theorem 1.4. This and boundedness below imply there exists $\epsilon_n > 0$ for which

$$\|f_k\|_q \geq \epsilon_n \|f_k\|_p. \tag{2.11}$$

Now, applying (2.10), the estimate in (2.11) holds only if $k^{\frac{1}{2q}-\frac{1}{2p}} \geq \epsilon_n$ for all $k \in \mathbb{N}_0, k \geq n$. This gives a contradiction when $k \to \infty$.
Similarly, for $p < q = \infty$, we have $\|f_k\|_\infty = \big(k/e\big)^{k/2}$. By (2.9) and Theorem 1.4,

$$\|D_{(u,\psi,n)} f_k\|_\infty \simeq \sup_{z\in\mathbb{C}} L_{(u,\psi,n)}(z) \frac{|f_k^{(n)}(az+b)|}{(1+|az+b|)^n} e^{-\frac{1}{2}|az+b|^2} \lesssim L_n \|f_k\|_\infty.$$

Therefore,

$$\left(\frac{k}{e}\right)^{\frac{k}{2}} = \|f_k\|_\infty \geq \epsilon_n \|f_k\|_p \simeq \epsilon_n \left(\frac{k}{e}\right)^{\frac{k}{2}} k^{\frac{1}{2p}}$$

for some $\epsilon_n > 0$. This gives a contradiction when $k \to \infty$ again. If $p > q$, then by Theorem 1.4, $D_{(u,\psi,n)} : \mathcal{F}_p \to \mathcal{F}_q$ is compact and injective on an infinite dimensional space $\mathcal{F}^0_{(p,n)}$. By Remark 1, it follows that the range of the operator cannot be closed.

### 2.4 Proof of Theorem 1.10

By Lemma 2.2, it is enough to show that $D_{(u,\psi,n)}$ has closed range on $\mathcal{F}^0_{(p,n)}$ if and only if there exists a constant $\epsilon_n > 0$ such that $G^{\epsilon_n}_{(u,\psi,n)}$ is a $(p, n)$ sampling set for $\mathcal{F}^0_{(p,n)}$.

Suppose $G^{\epsilon_n}_{(u,\psi,n)}$ is a $(p, n)$ sampling set for some $\epsilon_n > 0$. If $p < \infty$, then there exists $\delta_n > 0$ such that for each $f \in \mathcal{F}^0_{(p,n)}$,

$$\delta_n \|f\|_p^p \leq \int_{G^{\epsilon_n}_{(u,\psi,n)}} \frac{|f^{(n)}(z)|^p}{(1+|z|)^{np}} e^{-\frac{p}{2}|z|^2} dA(z). \tag{2.12}$$

It follows that

$$\begin{aligned}
\|D_{(u,\psi,n)} f\|_p^p &= \frac{p}{2\pi} \int_{\mathbb{C}} \big|u(z) f^{(n)}(\psi(z))\big|^p e^{-\frac{p}{2}|z|^2} dA(z) \\
&\geq \frac{p}{2\pi} \int_{\Omega^{\epsilon_n}_{(u,\psi,n)}} \big|u(z) f^{(n)}(\psi(z))\big|^p e^{-\frac{p}{2}|z|^2} dA(z) \\
&\geq \frac{p}{2\pi} \int_{\Omega^{\epsilon_n}_{(u,\psi,n)}} L^p_{(u,\psi,n)}(z) \frac{|f^{(n)}(\psi(z))|^p}{(1+|\psi(z)|)^{np}} e^{-\frac{p}{2}|\psi(z)|^2} dA(z).
\end{aligned}$$

By (2.12), the last right-hand integral above is bounded below by

$$\frac{p\epsilon_n^p}{2\pi|a|^2} \int_{G^{\epsilon_n}_{(u,\psi,n)}} \frac{|f^{(n)}(z)|^p}{(1+|z|)^{np}} e^{-\frac{p}{2}|z|^2} dA(z) \geq \frac{p\epsilon_n^p}{2\pi|a|^2} \delta_n \|f\|_p^p. \tag{2.13}$$

Similarly, for $p = \infty$, there exists $\delta_n$ such that for each $f \in \mathcal{F}^0_{(p,n)}$,

$$\begin{aligned}
\|D_{(u,\psi,n)} f\|_\infty &= \sup_{z\in\mathbb{C}} \big|u(z) f^{(n)}(\psi(z))\big| e^{-\frac{1}{2}|z|^2} \\
&\geq \sup_{z\in\Omega^{\epsilon_n}_{(u,\psi,n)}} L_{(u,\psi,n)}(z)(1+|\psi(z)|)^{-n} |f^{(n)}(\psi(z))| e^{-\frac{1}{2}|\psi(z)|^2} \\
&\geq \epsilon_n \sup_{z\in G^{\epsilon_n}_{(u,\psi,n)}} (1+|z|)^{-n} |f^{(n)}(z)| e^{-\frac{1}{2}|z|^2} \gtrsim \delta_n \epsilon_n \|f\|_\infty.
\end{aligned} \tag{2.14}$$

From (2.13), (2.14) and Lemma 2.2, the sufficiency of the condition follows.

Conversely, let $n$ be fixed and suppose $G^{\epsilon}_{(u,\psi,n)}$ is not a $(p,n)$ sampling set for each $\epsilon > 0$. Let $p < \infty$. Then there exists a unit norm sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathcal{F}^0_{(p,n)}$ such that

$$\int_{G^{1/k}_{(u,\psi,n)}} \frac{|f_k^{(n)}(z)|^p}{(1+|z|)^{np}} e^{-\frac{p}{2}|z|^2} dA(z) \to 0 \text{ as } k \to \infty.$$

It follows that

$$\begin{aligned}\|D_{(u,\psi,n)} f_k\|_p^p &= \frac{p}{2\pi}\int_{\mathbb{C}} |u(z)|^p |f_k^{(n)}(\psi(z))|^p e^{-\frac{p}{2}|z|^2} dA(z)\\ &= \frac{p}{2\pi}\int_{\mathbb{C}} L^p_{(u,\psi,n)}(z) \frac{|f_k^{(n)}(\psi(z))|^p}{(1+|\psi(z)|)^{np}} e^{-\frac{p}{2}|\psi(z)|^2} dA(z) = I_{1n} + I_{2n},\end{aligned}$$

where we set

$$\begin{aligned} I_{1n} &= \frac{p}{2\pi}\int_{\Omega^{1/k}_{(u,\psi,n)}} L^p_{(u,\psi,n)}(z) \frac{|f_k^{(n)}(\psi(z))|^p}{(1+|\psi(z)|)^{np}} e^{-\frac{p}{2}|\psi(z)|^2} dA(z)\\ &\le \frac{pL_n^p}{2\pi}\int_{\Omega^{1/k}_{(u,\psi,n)}} \frac{|f_k^{(n)}(\psi(z))|^p}{(1+|\psi(z)|)^{np}} e^{-\frac{p}{2}|\psi(z)|^2} dA(z)\\ &= \frac{pL_n^p}{2\pi|a|^2}\int_{G^{1/k}_{(u,\psi,n)}} \frac{|f_k^{(n)}(z)|^p}{(1+|z|)^{np}} e^{-\frac{p}{2}|z|^2} dA(z) \to 0\end{aligned}$$

as $k \to \infty$. To estimate the remaining integral, we eventually apply (2.9) and

$$\begin{aligned} I_{2n} &\lesssim \int_{\mathbb{C}\setminus\Omega^{1/k}_{(u,\psi,n)}} L^p_{(u,\psi,n)}(z) \frac{|f_k^{(n)}(\psi(z))|^p}{(1+|\psi(z)|)^{np}} e^{-\frac{p}{2}|\psi(z)|^2} dA(z)\\ &\le \frac{1}{k^p}\int_{\mathbb{C}\setminus\Omega^{1/k}_{(u,\psi,n)}} \frac{|f_k^{(n)}(\psi(z))|^p}{(1+|\psi(z)|)^{np}} e^{-\frac{p}{2}|\psi(z)|^2} dA(z)\\ &\le \frac{1}{k^p}\int_{\mathbb{C}} \frac{|f_k^{(n)}(\psi(z))|^p}{(1+|\psi(z)|)^{np}} e^{-\frac{p}{2}|\psi(z)|^2} dA(z)\\ &\simeq \frac{\|f_k\|_p^p}{k^p} \simeq \frac{1}{k^p} \to 0 \text{ as } k \to \infty.\end{aligned}$$

This contradicts the assumption that the operator is bounded below.

Next, consider $p = \infty$ and suppose $D_{(u,\psi,n)}$ is bounded below. Then there exists a constant $\delta_n > 0$ such that for each $f \in \mathcal{F}^0_{(\infty,n)}$

$$\sup_{z\in\mathbb{C}} |u(z)||f^{(n)}(\psi(z))|e^{-\frac{|z|^2}{2}} \ge \delta_n \|f\|_\infty.$$

Then, by definition of supremum for each $f$ there exists $w_f \in \mathbb{C}$ such that

$$|u(w_f)||f^{(n)}(\psi(w_f))|e^{-\frac{|w_f|^2}{2}} > \frac{\delta_n}{2}\|f\|_\infty. \tag{2.15}$$

On the other hand, by (2.9) again

$$|u(w_f)||f^{(n)}(\psi(w_f))|e^{-\frac{|w_f|^2}{2}} \lesssim L_{(u,\psi,n)}(w_f)\frac{|f^{(n)}(\psi(w_f))|}{(1+|\psi(w_f)|)^n}e^{-\frac{|\psi(w_f)|^2}{2}}$$
$$\leq L_{(u,\psi,n)}(w_f)\|f\|_\infty$$

and with (2.15), we deduce

$$L_{(u,\psi,n)}(w_f) > \frac{\delta_n}{2}.$$

Setting $\epsilon_n = \delta_n/2$, we observe that $w_f \in \Omega^{\epsilon_n}_{(u,\psi,n)}$ and using (2.15)

$$\begin{aligned}\|f\|_\infty &\leq \frac{2|u(w_f)|}{\delta_n}|f^{(n)}(\psi(w_f))|e^{-\frac{|w_f|^2}{2}}\\ &\lesssim \frac{2L_{(u,\psi,n)}(w_f)}{\delta_n}\frac{|f^{(n)}(\psi(w_f))|}{(1+|\psi(w_f)|)^n}e^{-\frac{|\psi(w_f)|^2}{2}}\\ &\lesssim \frac{2L_n}{\delta_n}\frac{|f^{(n)}(\psi(w_f))|}{(1+|\psi(w_f)|)^n}e^{-\frac{|\psi(w_f)|^2}{2}}\\ &\leq \frac{2L_n}{\delta_n}\sup_{z\in G^{\epsilon_n}_{(u,\psi,n)}}\frac{|f^{(n)}(z)|}{(1+|z|)^n}e^{-\frac{|z|^2}{2}}\end{aligned}$$

and completes the proof.

## 2.5 Proof of Theorem 1.11

Let $p < \infty$. We prove first the implication (i)⇒(ii) and suppose $\gamma_n$ is an essential lower bound for $L_{(u,\psi,n)}$. Then for each $f \in \mathcal{F}_p$, consider the function

$$h_f(z) = \begin{cases} g(z) & u(z) \neq 0\\ \lim_{w\to z} g(w), & u(z) = 0,\end{cases}$$

where we set

$$g(z) = \int_0^z\int_0^{z_1}\int_0^{z_2}\ldots\int_0^{z_{n-1}}\frac{f(\psi^{-1}(w))}{u(\psi^{-1}(w))}dA(w)dA(z_{n-1})\ldots dA(z_2)dA(z_1).$$

Clearly, $D_{(u,\psi,n)}h_f = f$. Since $u$ is entire and vanishes at most in a set of measure zero, we estimate

$$\begin{aligned}\|h_f\|_p^p &\simeq \int_{\mathbb{C}}\frac{|g^{(n)}(z)|^p}{(1+|z|)^{np}}e^{-\frac{p}{2}|z|^2}dA(z) \simeq \int_{\mathbb{C}}\frac{|f(\psi^{-1}(z))|^p e^{-\frac{p}{2}|z|^2}}{(1+|z|)^{np}|u(\psi^{-1}(z))|^p}dA(z)\\ &\leq \int_{\mathbb{C}}\frac{|f(z)|^p e^{-\frac{p}{2}|\psi(z)|^2}}{(1+|\psi(z)|)^{np}|u(z)|^p}dA(z)\\ &\lesssim \int_{\mathbb{C}} L^{-p}_{(u,\psi,n)}(z)|f(z)|^p e^{-\frac{p}{2}|z|^2}dA(z)\\ &\leq 2\pi\gamma_n{}^{-p}p^{-1}\|f\|_p^p < \infty,\end{aligned}$$

from which the statement in (ii) follows.

Next, we prove $(ii) \Rightarrow (iii)$, and suppose on the contrary $|a| < 1$. The surjectivity of the operator implies there exists some $f \in \mathcal{F}_p$ such that $1 = D_{(u,\psi,n)} f$. It follows that $u$ has no zeros in $\mathbb{C}$, and $u\psi^n$ has a zero set of measure zero which does not affect our integral approach below in (2.16). Thus, we can assume that $u\psi^n$ is non-vanishing. Then by Lemma 1.5, it follows that $u\psi^n(z) = e^{a_{0n}+a_{1n}z+a_{2n}z^2}$, for some constants $a_{0n}, a_{1n}, a_{2n} \in \mathbb{C}$ such that $|a_{2n}| \leq \frac{1-|a|^2}{2}$. Using this, we write

$$\begin{aligned} L_{(u,\psi,n)}(z) &= |u(z)\psi^n(z)|e^{\frac{1}{2}(|az+b|^2-|z|^2)} \\ &= e^{\Re(a_{0n}+a_{1n}z+a_{2n}z^2)}e^{\frac{1}{2}(|az+b|^2-|z|^2)} \\ &= Ce^{\Re((a_{1n}+a\overline{b})z)+\Re(a_{2n}z^2)+\frac{|a|^2-1}{2}|z|^2} \end{aligned}$$

for all $z \in \mathbb{C}$, where $C = e^{\Re(a_{0n})+\frac{|b|^2}{2}}$. By surjectivity, for each $h \in \mathcal{F}_p$, there exists some $f \in \mathcal{F}_p$ such that $D_{(u,\psi,n)} f(z) = u(z)f^{(n)}(az+b) = h(z)$ for all $z \in \mathbb{C}$. This implies

$$\int_{\mathbb{C}} \frac{|f^{(n)}(az+b)|^p e^{-\frac{p}{2}|az+b|^2}}{(1+|\psi(z)|)^{np}} dA(z) = \int_{\mathbb{C}} \frac{|h(z)|^p e^{-\frac{p}{2}|z|^2}}{L^p_{(u,\psi,n)}(z)} dA(z). \tag{2.16}$$

By (2.9), the right-hand integral in (2.16) should be finite for each $h \in \mathcal{F}_p$. The plan is now to show the existence of some functions $h$ in the space for which this integral diverges.

Now, if $|a_{2n}| < \frac{1-|a|^2}{2}$, then the operator is compact and by Remark 1, its range is not closed and the operator is not surjective. Thus, we set $|a_{2n}| = \frac{1-|a|^2}{2}$ and consider the following two cases following Lemma 1.5.

**Case 1.** For $a_{1n} + a\overline{b} = 0$, we have

$$L_{(u,\psi,n)}(z) = Ce^{\Re(a_{2n}z^2)+\frac{|a|^2-1}{2}|z|^2}. \tag{2.17}$$

We may also write $a_{2n} = |a_{2n}|e^{-2i\theta_{2n}}$, where $0 \leq \theta_{2n} < \pi$.

Replacing $z$ by $e^{i\theta_{2n}}w$ in (2.17)

$$L_{(u,\psi,n)}(e^{i\theta_{2n}}w) = Ce^{\frac{1-|a|^2}{2}(\Re(w^2)-|w|^2)} \tag{2.18}$$

for all $w \in \mathbb{C}$. Setting $w = x + iy$, the relation in (2.18) implies

$$\frac{e^{-\frac{1}{2}|e^{i\theta_{2n}}w|^2}}{L_{(u,\psi,n)}(e^{i\theta_{2n}}w)} = \frac{1}{C}e^{(\frac{1}{2}-|a|^2)y^2-\frac{1}{2}x^2}. \tag{2.19}$$

from which we observe that the integral in (2.16) diverges for every nonzero constant function $h$ in the space whenever $|a| < \frac{1}{\sqrt{2}}$. Thus, the question is when $|a| \geq \frac{1}{\sqrt{2}}$. We may consider a function $h(z) = h_\alpha(z) = e^{\alpha z^2}$ where $\alpha$ is a real number and $|\alpha| < \frac{1}{2}$. A suitable $\alpha$ will be chosen later. A straightforward calculation using (2.19) gives

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|h_\alpha(e^{i\theta_{2n}}(x+iy))|^p e^{-\frac{p}{2}|e^{i\theta_{2n}}(x+iy)|^2}}{L^p_{(u,\psi,n)}(e^{i\theta_{2n}}(x+iy))} dxdy = \frac{1}{C}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{p\beta(x,y)} dxdy, \tag{2.20}$$

where

$$\beta(x, y) := \left(\alpha \cos(2\theta_{2n}) - \frac{1}{2}\right) x^2 + \left(\frac{1}{2} - |a|^2 - \alpha \cos(2\theta_{2n})\right) y^2$$
$$- 2\alpha xy \sin(2\theta_{2n}).$$

Since $|\alpha| < 1/2$, it holds that $\alpha \cos(2\theta_{2n}) - \frac{1}{2} < 0$. Integrating with respect to $x$,

$$\int_{-\infty}^{\infty} e^{p(\alpha \cos(2\theta_{2n}) - \frac{1}{2})x^2 - 2p\alpha y \sin(2\theta_{2n})x} dx = \int_{-\infty}^{\infty} e^{-p(\frac{1}{2} - \alpha \cos(2\theta_{2n}))x^2 - 2p\alpha y \sin(2\theta_{2n})x} dx$$
$$= \sqrt{\pi}\left(\frac{p}{2} - p\alpha \cos(2\theta_{2n})\right)^{-\frac{1}{2}} e^{\frac{\alpha^2 y^2 p^2 \sin^2(2\theta_{2n})}{\frac{p}{2} - p\alpha \cos(2\theta_{2n})}}.$$

Taking this in (2.20), the coefficient of $y^2$ becomes

$$\frac{p}{2} - p|a|^2 - p\alpha \cos(2\theta_{2n}) + \frac{\alpha^2 p^2 \sin^2(2\theta_{2n})}{\frac{p}{2} - p\alpha \cos(2\theta_{2n})}$$
$$= \frac{p\alpha^2 + \frac{p}{4} - \frac{p|a|^2}{2} + p\alpha \cos(2\theta_{2n})(|a|^2 - 1)}{\frac{1}{2} - \alpha \cos(2\theta_{2n})}. \tag{2.21}$$

Now, if $\cos(2\theta_{2n}) \leq 0$, we choose a positive $\alpha$ such that

$$\frac{1}{4} > \alpha^2 > \frac{|a|^2}{2} - \frac{1}{4}. \tag{2.22}$$

Note that such a choice is possible since $|a| < 1$. For such $\alpha$, the expression in (2.21) is nonnegative and hence the integral in (2.20) diverges. On the other hand, if $\cos(2\theta_{2n}) > 0$, we can choose a negative $\alpha$ such that (2.22) holds and hence the integral in (2.20) diverges again.

**Case 2**. Let $a_{1n} + a\overline{b} \neq 0$ and

$$a_{2n} = -\frac{(1 - |a|^2)(a_{1n} + a\overline{b})^2}{2|a_{1n} + a\overline{b}|^2}.$$

Using this and $a_{2n} = |a_{2n}|e^{-2i\theta_{2n}}$ as above, we obtain

$$(a_{1n} + a\overline{b})e^{i\theta_{2n}} = \pm i|a_{1n} + a\overline{b}|,$$

which is a purely imaginary number. Setting $(a_{1n} + a\overline{b})e^{i\theta_{2n}} = iy_n$ for some $y_n \in \mathbb{R}$, $w = x + iy$, and $z = e^{i\theta_{2n}} w$

$$L_{(u,\psi,n)}(e^{i\theta_{2n}} w) = Ce^{-y_n y - (1-|a|^2)y^2}$$

and hence

$$\frac{e^{-\frac{1}{2}|e^{i\theta_{2n}} w|^2}}{L_{(u,\psi,n)}(e^{i\theta_{2n}} w)} = \frac{1}{C} e^{y_n y + (\frac{1}{2} - |a|^2)y^2 - \frac{1}{2}x^2}.$$

This shows that if $|a| \leq \frac{1}{\sqrt{2}}$, then the integral in (2.16) diverges for every nonzero constant function $h$ in the space again. For the rest, we consider a function $h_\alpha(z) = e^{\alpha z^2}$ and argue exactly in the same way as above.

It remains to show (iii)$\Rightarrow$ (i). For $|a|=1$, by (1.5) we have

$$L_{(u,\psi,n)}(z)=|u(z)\psi^n(z)|e^{\frac{1}{2}(|az+b|^2-|z|^2)}=|b^n u(0)|e^{\frac{|b|^2}{2}}.$$

Note that if $b^n u(0)=0$, then the function $u$ vanishes since $\psi$ is entire and nonzero. Hence $b^n u(0)\neq 0$ and $L_{(u,\psi,n)}$ is bounded away from zero.

For $p=\infty$, we simply replace the integral argument in the proof by the supremum.

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