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Positive solutions for a class of nonlinear parametric Robin problems

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Abstract
We consider a nonlinear Robin problem driven by the p-Laplacian and a parametric concave-convex reaction with the parameter multiplying the convex (superlinear) term. We prove a multiplicity result for positive solutions which is global in the parameter $\lambda > 0$ (bifurcation-type theorem). We also show the existence of a minimal positive solution u_λ^* and determine the monotonicity and continuity properties of the map $\lambda \longmapsto u_\lambda^*$

Keywords Concave-convex nonlinearities · Positive solutions · Truncation · Nonlinear regularity · Nonlinear maximum principle · Minimal positive solution

Mathematics Subject Classification 35J20 · 35J60

1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. We study the following parametric Robin problem

$$\begin{cases} -\Delta_p u(z) + \xi(z)u(z)^{p-1} = g(z, u(z)) + \lambda f(z, u(z)) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)u^{p-1} = 0 \quad \text{on } \partial\Omega, \ u \geq 0, \ \lambda > 0. \end{cases} \tag{P_λ}$$

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By Δ_p we denote the p-Laplacian differential operator defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2} Du) \quad \forall u \in W^{1,p}(\Omega).$$

There is also a potential term $\xi(z)u^{p-1}$ with $\xi \in L^\infty(\Omega)$, $\xi(z) \geq 0$ for almost all $z \in \Omega$. In the reaction we have the combined effects of two nonlinearities $g(z, x)$ and $\lambda f(z, x)$ with $\lambda > 0$ being a parameter. Both functions are Carathéodory. We assume that $g(z, \cdot)$ is strictly $(p-1)$-sublinear as $x \to +\infty$, while $f(z, \cdot)$ is $(p-1)$-superlinear as $x \to +\infty$, but need not satisfy the usual in such cases Ambrosetti-Rabinowitz condition. So, in the reaction we have the combined effect of concave and convex terms. However, in our case this parameter multiplies the convex (superlinear) term, while in the classical "concave-convex problem", the parameter multiplies the concave (sublinear) term. This changes the structure of the equation and consequently the approach is different.

In the Robin boundary condition $\frac{\partial u}{\partial n_p}$ denotes the conormal derivative of u corresponding to the p-Laplacian and if $u \in C^1(\overline{\Omega})$, then

$$\frac{\partial u}{\partial n_p} = |Du|^{p-1}(Du, n)_{\mathbb{R}^N} = |Du|^{p-2}\frac{\partial u}{\partial n},$$

with n being the outward unit normal on $\partial\Omega$. For general u, the boundary condition is understood using the nonlinear Green's identity (see Papageorgiou-Rădulescu-Repovš [19, p. 35]). The boundary coefficient β is nonnegative.

Our aim is to prove a multiplicity theorem for the positive solutions of (P_λ) which is global with respect to the parameter $\lambda > 0$, that is, our result gives a precise description of the changes in the set of positive solutions as the parameter λ varies in $(0, +\infty)$ (bifurcation-type theorem). So, our main result in the paper (Theorem 3.8) establishes the existence of a critical parameter value $\lambda^* > 0$ such that
- for all $\lambda \in (0, \lambda^*)$ problem (P_λ) has at least two distinct positive solutions;
- for $\lambda = \lambda^*$ problem (P_λ) has at least one positive solutions;
- for $\lambda > \lambda^*$ problem (P_λ) has no positive solutions.

This global multiplicity result reveals an interesting discontinuity property for the "spectrum" of (P_λ). This is better illustrated when we consider the standard "concave-convex" reaction

$$x \longmapsto x^{q-1} + \lambda x^{r-1},$$

with $1 < q < p < r < p^*$, where

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } N \leq p. \end{cases}$$

According to our global multiplicity result described above, for all $\lambda > 0$ small problem (P_λ) has at least two positive solutions. On the other hand in the limit case $\lambda = 0$, the problem becomes

$$\begin{cases} -\Delta_p u(z) + \xi(z)u(z)^{p-1} = u^{q-1} & \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)u^{p-1} = 0 & \text{on } \partial\Omega,\ u \geq 0,\ \lambda > 0. \end{cases}$$

This problem has a unique positive solution (see Proposition 2.6 in Sect. 2). For the other case where the parameter $\lambda > 0$ multiplies the concave term, the limit problem (that is for $\lambda = 0$) always has a positive solution which is not unique (see Papageorgiou-Rădulescu [14]). This illustrates the different structure of the two concave-convex problems. We mention also the recent works of Papageorgiou-Rădulescu-Repovš [16] and Papageorgiou-Vetro-Vetro [20], where the reader can find instances of such discontinuities in the "spectrum" of parametric problems.

In the past most the works on concave-convex problems, focused on Dirichlet problems with the parameter multiplying the concave (sublinear) term. Everything started with the paper of Ambrosetti-Brézis-Cerami [2], which deals with semilinear equations driven by the Laplacian. Their work was extended to nonlinear Dirichlet problems driven by the p-Laplacian by García Azorero-Manfredi-Peral Alonso [5] and Guo-Zhang [9]. All the aforementioned works deal with problems having the classical concave-convex reaction

$$u \longmapsto \lambda u^{q-1} + u^{r-1},$$

with $1 < q < p < r < p^*$. More general differential operators and/or reactions can be found in the works of Papageorgiou-Rădulescu-Repovš [15], Rădulescu-Repovš [23] (semilinear equations) and El Manouni-Papageorgiou-Winkert [3], Papageorgiou-Rădulescu-Repovš [18], Papageorgiou-Vetro-Vetro [21], Winkert [24] (nonlinear equations). We also mention the recent work of Papageorgiou-Zhang [22], where the "concave" contribution comes from the boundary condition. The only "concave-convex" work with the parameter multiplying the convex (superlinear) term, is that of Marano-Marino-Papageorgiou [12]. There the problem is a Dirichlet (p, q)-equation, with the concave contribution being of the power form ($g(z, u) = u^{q-1}$) and the condition on $f(z, \cdot)$ are more restrictive (see hypotheses (h_1)–(h_4) in [12]).

2 Mathematical background: hypotheses

The main spaces in the study of (P_λ) are the Sobolev space $W^{1,p}(\Omega)$, the Banach space $C^1(\overline{\Omega})$ and the "boundary" Lebesgue space $L^s(\partial\Omega)$ $(1 \leq s < +\infty)$.

By $\|\cdot\|$ we denote the norm of $W^{1,p}(\Omega)$ defined by

$$\|u\| = \left(\|u\|_p^p + \|Du\|_p^p\right)^{\frac{1}{p}} \quad \forall u \in W^{1,p}(\Omega).$$

The Banach space $C^1(\overline{\Omega})$ is ordered with positive (order) cone

$$C_+ = \{u \in C^1(\overline{\Omega}) : \ u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\mathrm{int} C_+ = \{u \in C_+ : \ u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

We will also use another open cone in $C^1(\overline{\Omega})$ which is defined by

$$D_+ = \left\{u \in C^1(\overline{\Omega}) : \ u(z) > 0 \text{ for all } z \in \Omega, \ \frac{\partial u}{\partial n}|_{\partial\Omega \cap u^{-1}(0)} < 0\right\}.$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure σ. Using this measure we can define in the usual way the boundary Lebesgue space $L^s(\partial\Omega)$ $(1 \leq s \leq +\infty)$. From the theory of Sobolev spaces, we know that there exists a unique continuous linear operator $\widehat{\gamma}_0 \colon W^{1,p}(\Omega) \longrightarrow L^p(\partial\Omega)$ known as the "trace operator" such that

$$\widehat{\gamma}_0(u) = u|_{\partial\Omega} \quad \forall u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}).$$

So, the trace operator extends the notion of "boundary values" to all Sobolev functions. In the sequel for the sake of notational simplicity, we drop the use of the trace operator

$\widehat{\gamma}_0$. All restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces. We mention that using the trace operator, we have that $W^{1,p}(\Omega) \subseteq L^s(\partial\Omega)$ continuously for all $1 \leq s \leq \frac{(N-1)p}{N-p}$ if $p < N$ and for all $1 \leq s < +\infty$ if $N \leq p$. Also $W^{1,p}(\Omega) \subseteq L^s(\partial\Omega)$ compactly for all $1 \leq s < \frac{(N-1)p}{N-p}$ if $p < N$ and for all $1 \leq s < +\infty$ if $N \leq p$.

If $u \colon \Omega \longrightarrow \mathbb{R}$ is a measurable function, we define

$$u^{\pm}(z) = \max\{\pm u(z), 0\} \quad \forall z \in \Omega.$$

If $u \in W^{1,p}(\Omega)$, then $u^{\pm} \in W^{1,p}(\Omega)$ and we have $u = u^+ - u^-$, $|u| = u^+ + u^-$. Also, given two measurable functions $u, v \colon \Omega \longrightarrow \mathbb{R}$ such that $u(z) \leq v(z)$ for all $z \in \Omega$, we define

$$\begin{aligned} [u, v] &= \{h \in W^{1,p}(\Omega) : \ u(z) \leq h(z) \leq v(z) \text{ for a.a. } z \in \Omega\}, \\ [u) &= \{h \in W^{1,p}(\Omega) : u(z) \leq h(z) \text{ for a.a. } z \in \Omega\}. \end{aligned}$$

We introduce the hypotheses on the potential function ξ and on the boundary coefficient β.

$\underline{H_0}$: $\xi \in L^{\infty}(\Omega)$, $\xi(z) \geq 0$ for almost $z \in \Omega$, $\beta \in C^{0,\alpha}(\partial\Omega)$ with $0 < \alpha < 1$, $\beta(z) \geq 0$ for all $z \in \partial\Omega$ and $\xi \not\equiv 0$ or $\beta \not\equiv 0$.

Remark 2.1 With this hypotheses, we cover also the case of Neumann problem, which corresponds to the case $\beta \equiv 0$.

Let $\gamma_p \colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ be defined by

$$\gamma_p(u) = \|Du\|_p^p + \int_{\Omega} \xi(z)|u|^p \, dz + \int_{\partial\Omega} \beta(z)|u|^p \, d\sigma \quad \forall u \in W^{1,p}(\Omega).$$

Hypotheses H_0 together with Lemma 4.11 of Mugnai-Papageorgiou [13] and Proposition 2.4 of Gasiński-Papageorgiou [8], imply that there exists $c_0 > 0$ such that

$$c_0\|u\|^p \leq \gamma_p(u) \quad \forall u \in W^{1,p}(\Omega) \tag{2.1}$$

(that is, $\gamma_p(\cdot)$ is equivalent norm on $W^{1,p}(\Omega)$).

Let $\widehat{\lambda}_1$ be the first eigenvalue of

$$\begin{cases} -\Delta_p u + \xi(u)|u|^{p-2}u = \widehat{\lambda}|u|^{p-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0. \end{cases}$$

On account of (2.1), we have $\widehat{\lambda}_1 > 0$. We know that

$$\widehat{\lambda}_1 = \inf_{u \in W^{1,p}(\Omega)\setminus\{0\}} \frac{\gamma_p(u)}{\|u\|_p^p}.$$

This infimum is realized on the corresponding eigenspace, the elements of which have fixed sign. Let $\widehat{u}_1$ be the positive, L^p-normalized (that is, $\|\widehat{u}_1\|_p = 1$) eigenfunction for $\widehat{\lambda}_1$. We know that $\widehat{u}_1 \in \operatorname{int} C_+$. Note that $\widehat{\lambda}_1$ is the only eigenvalue with eigenfunctions of constant sign (see Fragnelli-Mugnai-Papageorgiou [4]).

Let $A \colon W^{1,p}(\Omega) \longrightarrow W^{1,p}(\Omega)^*$ be defined by

$$\langle A(u), h\rangle = \int_{\Omega} |Du|^{p-2}(Du, Dh)_{\mathbb{R}^N} \, dz \quad \forall u, h \in W^{1,p}(\Omega).$$

From Gasiński-Papageorgiou [7, p. 279], we have the following property.

Proposition 2.2 *If hypotheses H_0 hold, then A is continuous, monotone (thus maximal monotone too) and of type $(S)_+$, that is "if $u_n \xrightarrow{w} u$ in $W^{1,p}(\Omega)$ and*

$$\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \le 0,$$

then $u_n \longrightarrow u$ in $W^{1,p}(\Omega)$."

Next we introduce the hypotheses on the two functions involved in the reaction (right hand side of (P_λ)).
$\underline{H_1}$: $g\colon \Omega\times\mathbb{R}\longrightarrow\mathbb{R}$ is a Carathéodory function such that $g(z,0)=0$ for almost all $z\in\Omega$ and

(i) for every $\varrho>0$, there exists $a_\varrho\in L^\infty(\Omega)$ such that

$$|g(z,x)|\le a_\varrho(z)\quad \text{for a.a. } z\in\Omega,\ \text{all } 0\le x\le\varrho;$$

(ii) $\displaystyle\lim_{x\to+\infty}\frac{g(z,x)}{x^{p-1}}=0$ uniformly for almost all $z\in\Omega$;
(iii) there exist $q\in(1,p)$ and $\delta>0, \widehat{c}>0$ such that

$$\widehat{c}x^{q-1}\le g(z,x)\quad \text{for a.a. } z\in\Omega,\ \text{all } 0\le x\le\delta.$$

Remark 2.3 Since we are interested on positive solutions and the above hypotheses concern the positive semiaxis $\mathbb{R}_+=[0,+\infty)$, without any loss of generality we may assume that $g(z,x)=0$ for almost all $z\in\Omega$, all $x\le 0$. Hypothesis $H_1(ii)$ implies that $g(z,\cdot)$ is strictly $(p-1)$-sublinear as $x\to+\infty$ ("concave" nonlinearity). Hypothesis $H_1(iii)$ implies that $g(z,\cdot)$ is $(p-1)$-sublinear as $x\to 0^+$. We point out that we do not assume that $g\ge 0$. It can change sign. The following functions satisfy hypotheses H_1 (for the sake of simplicity we drop the z-dependence):

$$g_1(x)=(x^+)^{q-1},$$
$$g_2(x)=(x^+)^{q-1}-2(x^+)^{\tau-1},$$

with $1<q<\tau<p$. Note that g_2 is sign changing.

$\underline{H_2}$: $f\colon\Omega\times\mathbb{R}\longrightarrow\mathbb{R}$ is a Carathéodory function such that $f(z,0)=0$ for almost all $z\in\Omega$ and

(i) $f(z,x)\le a(z)(1+x^{r-1})$ for almost all $z\in\Omega$, all $x\ge 0$, with $a\in L^\infty(\Omega)$, $p<r<p^*$;
(ii) if $F(z,x)=\int_0^x f(z,s)\,ds$, then $\lim_{x\to+\infty}\frac{F(z,x)}{x^p}=+\infty$ uniformly for almost all $z\in\Omega$;
(iii) if $G(z,x)=\int_0^x g(z,s)\,ds$ and

$$e_\lambda(z,x)=\big(g(z,x)+\lambda f(z,x)\big)-p\big(G(z,x)+\lambda F(z,x)\big),\quad \lambda>0,$$

then there exists $\widetilde{\vartheta}_\lambda\in L^!(\Omega)$ such that

$$e_\lambda(z,x)\le e_\lambda(z,y)+\widetilde{\vartheta}_\lambda(z)\quad \text{for a.a. } z\in\Omega,\ \text{all } 0\le x\le y;$$

(iv) $\displaystyle\lim_{x\to 0^+}\frac{f(z,x)}{x^{p-1}}=0$ uniformly for almost all $z\in\Omega$ and for every $s>0$, there exists $\eta_s>0$ such that $\eta_s\le f(z,x)$ for almost all $z\in\Omega$, all $x\ge s$.

Remark 2.4 Again we assume that $f(z,x)=0$ for almost all $z\in\Omega$, all $x\le 0$. Evidently in hypothesis $H_2(iii)$ we can assume that $\lambda\to\|\widetilde{\vartheta}_\lambda\|_1$ is increasing. Also, if in $H_2(iii)$ we let $x=0$ and use hypotheses $H_1(ii)$ and $H_2(ii)$, we see that

$$\lim_{x\to+\infty}\frac{f(z,x)}{x^{p-1}}=+\infty \quad \text{uniformly for a.a. } z\in\Omega.$$

Therefore $f(z,\cdot)$ is $(p-1)$-superlinear ("convex" nonlinearity). Usually problems with superlinear reaction are treated using the so-called Ambrosetti-Rabinowith condition. We recall that this condition (unilateral version since $g(z,x)=f(z,x)=0$ for almost all $z\in\Omega$, all $x\le 0$), says that there exist $\eta>p$ and $M>0$ such that

$$\begin{cases} 0<\eta F(z,x)\le f(z,x)x & \text{for a.a. } z\in\Omega,\ \text{all } x\ge M,\\ 0<\operatorname*{ess\,inf}_{\Omega}F(\cdot,M). \end{cases}$$

Integrating, we obtain the weaker requirement that

$$c_1x^\eta\le F(z,x)\quad \text{for a.a. } z\in\Omega,\ \text{all } x\ge M,$$

for some $c_1>0$, thus

$$c_2x^{\eta-1}\le f(z,x)\quad \text{for a.a. } z\in\Omega,\ \text{all } x\ge M,$$

for some $c_2>0$. So, we see that the Ambrosetti-Rabinowitz condition imposes at least $(\eta-1)$-polynomial growth on $f(z,\cdot)$. This way we exclude superlinear functions with slower growth as $x\to+\infty$. Consider the functions (as before we drop the z-dependence):

$$\begin{aligned} f_1(x)&=(x^+)^{\eta-1},\\ f_2(x)&=(x^+)^{p-1}\ln(1+x^+), \end{aligned}$$

with $1<p<\eta<p^*$. Then these two functions combined with any of g_1 or g_2 satisfy hypothesis $H_2(iii)$. Note that f_2 does not satisfy the Ambrosetti-Rabinowitz condition.

$\underline{H_3}$: For every $\varrho>0$ and every $J\subseteq(0,+\infty)$ finite, there exists $\widehat{\xi}_\varrho^J>0$ such that for almost all $z\in\Omega$, all $\lambda\in J$, the function

$$x\longmapsto g(z,x)+\lambda f(z,x)+\widehat{\xi}_\varrho^J x^{p-1}$$

is nondecreasing on $[0,\varrho]$.

Remark 2.5 Any pair of functions g,f formed by the collections $\{g_1,g_2\}$ and $\{f_1,f_2\}$ satisfies H_3. In general, if $g(z,\cdot)$ and $f(z,\cdot)$ are differentiable and for every $J\subseteq(0,+\infty)$ finite, we have

$$\big(g_x'(z,x)+\lambda f_x'(z,x)\big)x\ge-\widehat{\xi}_\varrho^J x^{p-1}$$

for almost all $z\in\Omega$, all $0\le x\le\varrho$, all $\lambda\in J$, with $\widehat{\xi}_\varrho^J>0$, then hypothesis H_3 is satisfied.

We introduce the following sets related to problem (P_λ):

$$\begin{aligned} \mathcal{L}&=\{\lambda>0:\ \text{problem}(P_\lambda)\text{has a positive solution}\},\\ S_\lambda&=\{u:\ u\text{is a positive solutions of}(P_\lambda)\}. \end{aligned}$$

Also we set

$$\lambda^*=\sup\mathcal{L}.$$

Note that on account of hypotheses $H_1(i)$, (ii), we have

$$|g(z,x)| \leq c_3(1 + x^{p-1}) \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0, \tag{2.2}$$

for some $c_3 > 0$. Also hypothesis $H_1(iii)$ and the fact that $f \geq 0$ imply that for all $\lambda > 0$, we have

$$g(z,x) + \lambda f(z,x) \geq \widehat{c} x^{q-1} \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \delta. \tag{2.3}$$

This unilateral growth restriction on the reaction, leads to the following auxiliary Robin problem

$$\begin{cases} -\Delta_p u(z) + \xi(z) u(z)^{p-1} = \widehat{c} u(z)^{q-1} & \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z) u^{p-1} = 0 & \text{on } \partial\Omega, \ u \geq 0. \end{cases} \tag{2.4}$$

Proposition 2.6 *If hypotheses H_0 hold, then problem (2.4) admits a unique positive solution $\widetilde{u} \in \mathrm{int} C_+$.*

Proof First we show the existence of a positive solution. To this end, let $\psi_0 \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ be the C^1-functional defined by

$$\psi_0(u) = \frac{1}{p}\gamma_p(u) - \frac{\widehat{c}}{q}\|u^+\|_q^q \geq \frac{c_0}{p}\|u\|^p - c_4\|u\|^q,$$

for some $c_4 > 0$ (see (2.1) and recall that the embedding $W^{1,p}(\Omega) \subseteq L^q(\Omega)$ is continuous). So ψ_0 is coercive (recall that $q < p$).

Also from the Sobolev embedding theorem and the compactness of the trace operator, we infer that ψ_0 is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $\widetilde{u} \in W^{1,p}(\Omega)$ such that

$$\psi_0(\widetilde{u}) = \inf_{u \in W^{1,p}(\Omega)} \psi_0(u). \tag{2.5}$$

Recall that $\widehat{u}_1 \in \mathrm{int} C_+$ and let $t > 0$. Then

$$\psi_0(t\widehat{u}_1) = \frac{t^p}{p}\widehat{\lambda}_1 - \frac{\widehat{c}t^q}{q}\|\widehat{u}_1\|_q^q.$$

Since $1 < q < p$, choosing $t \in (0,1)$ small, we have

$$\psi_0(t\widehat{u}_1) < 0,$$

so

$$\psi_0(\widetilde{u}) < 0 = \psi_0(0)$$

(see (2.5)) and thus $\widetilde{u} \neq 0$. From (2.5) we have

$$\psi_0'(\widetilde{u}) = 0,$$

so

$$\begin{aligned} &\langle A(\widetilde{u}), h\rangle + \int_\Omega \xi(z)|\widetilde{u}|^{p-2}\widetilde{u} h \, dz + \int_{\partial\Omega} \beta(z)|\widetilde{u}|^{p-2}\widetilde{u} h \, d\sigma \\ &= \int_\Omega \widehat{c}(\widetilde{u})^{q-1} h \, dz \quad \forall h \in W^{1,p}(\Omega). \end{aligned} \tag{2.6}$$

In (2.6) we choose $h = -\widetilde{u}^- \in W^{1,p}(\Omega)$ and obtain

$$\gamma(\widetilde{u}^-) = 0,$$

so

$$\widetilde{u} \geq 0, \quad \widetilde{u} \neq 0 \tag{2.7}$$

(see (2.1)). Then from (2.6) and (2.7) we have

$$\begin{cases} -\Delta_p \widetilde{u}(z) + \xi(z)\widetilde{u}(z)^{p-1} = \widehat{c}\widetilde{u}^{q-1} & \text{in } \Omega, \\ \frac{\partial \widetilde{u}}{\partial n_p} + \beta(z)\widetilde{u}^{p-1} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.8}$$

From (2.8) and Proposition 2.10 of Papageorgiou-Rădulescu [14], we have that

$$\widetilde{u} \in L^\infty(\Omega).$$

Applying Theorem 2 of Lieberman [11], we infer that

$$\widetilde{u} \in C_+ \setminus \{0\}.$$

From (2.8) we have that

$$\Delta_p \widetilde{u} \leq \|\xi\|_\infty \widetilde{u}^{p-1} \quad \text{in } \Omega,$$

so $\widetilde{u} \in \operatorname{int} C_+$ (see Gasiński-Papageorgiou [6]).

Now we show the uniqueness of this positive solution of (2.4). So, suppose that $\widetilde{v}$ is another positive solution of (2.4). Again we have $\widetilde{v} \in \operatorname{int} C_+$. Consider the function

$$R(\widetilde{u}, \widetilde{v}) = |D\widetilde{u}|^p - |D\widetilde{v}|^{p-2}\left(D\widetilde{v}, D\left(\frac{\widetilde{u}^p}{\widetilde{v}^{p-1}}\right)\right)_{\mathbb{R}^N}.$$

Using the nonlinear Picone's identity of Allegretto-Huang [1], we have

$$\begin{aligned} 0 &\leq \int_\Omega R(\widetilde{u}, \widetilde{v}) \\ &= \|D\widetilde{u}\|_p^p - \int_\Omega (-\Delta_p \widetilde{v}) \frac{\widetilde{u}^p}{\widetilde{v}^{p-1}}\, dz + \int_{\partial\Omega} \beta(z)\widetilde{u}^p\, d\sigma \\ &= \gamma_p(\widetilde{u}) - \widehat{c}\int_\Omega \frac{\widetilde{u}}{\widetilde{v}^{p-q}}\, dz \\ &= \int_\Omega \widehat{c}\frac{\widetilde{u}^q}{\widetilde{v}^{p-q}}\left(\widetilde{v}^{p-q} - \widetilde{u}^{p-q}\right) dz \end{aligned} \tag{2.9}$$

(using the nonlinear Green's identity; see Gasiński-Papageorgiou [6, p. 211]). Interchanging the roles of $\widetilde{u}$ and $\widetilde{v}$ in the above argument we also have

$$0 \leq \int_\Omega \widehat{c}\frac{\widetilde{v}^q}{\widetilde{u}^{p-q}}\left(\widetilde{u}^{p-q} - \widetilde{v}^{p-q}\right) dz. \tag{2.10}$$

Adding (2.9) and (2.10) we have

$$\begin{aligned} 0 &\leq \int_\Omega \widehat{c}\left(\frac{\widetilde{v}^q}{\widetilde{u}^{p-q}} - \frac{\widetilde{u}^q}{\widetilde{v}^{p-q}}\right)\left(\widetilde{u}^{p-q} - \widetilde{v}^{p-q}\right) dz \\ &= \int_\Omega \frac{\widehat{c}}{\widetilde{u}^{p-q}\widetilde{v}^{p-q}}\left(\widetilde{v}^p - \widetilde{u}^p\right)\left(\widetilde{u}^{p-q} - \widetilde{v}^{p-q}\right) dz \leq 0, \end{aligned}$$

so $\widetilde{u} = \widetilde{v}$ (recall that $1 < q < p$).

This proves the uniqueness of the positive solution $\widetilde{u} \in \operatorname{int} C_+$ of problem (2.4). ◻

Since $\widetilde{u} \in \mathrm{int}C_+$, we can find $t \in (0,1)$ small such that

$$\overline{u}(z) = t\widetilde{u}(z) \in (0, \delta] \quad \forall z \in \overline{\Omega},$$

with $\delta > 0$ as in the hypothesis $H_1(iii)$. Then $\overline{u} \in \mathrm{int}C_+$ and

$$\begin{aligned} -\Delta_p \overline{u} + \xi(z)\overline{u}^{p-1} &= t^{p-1}\big(-\Delta_p \widetilde{u} + \xi(z)\widetilde{u}^{p-1}\big) \\ &= t^{p-1}\widehat{c}\widetilde{u}^{q-1} \le \widehat{c}\overline{u}^{q-1} \quad \text{in } \Omega \end{aligned} \tag{2.11}$$

(since $t \in (0,1)$ and $1 < q < p$).

3 Positive solutions

First we show the nonemptiness of $\mathcal{L}$ and determine the regularity of the elements of the solution set S_λ.

Proposition 3.1 *If hypotheses H_0, H_1, H_2 and H_3 hold, then $\mathcal{L} \neq \emptyset$ and for every $\lambda > 0$ we have $S_\lambda \subseteq \mathrm{int}C_+$.*

Proof Let $\overline{u} \in \mathrm{int}C_+$ be as above. For $\lambda > 0$ we consider the Carathéodory function $k_\lambda : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$k_\lambda(z,x) = \begin{cases} g(z, \overline{u}(z)) + \lambda f(z, \overline{u}(z)) & \text{if } x \le \overline{u}(z), \\ g(z,x) + \lambda f(z,x) & \text{if } \overline{u}(z) < x. \end{cases} \tag{3.1}$$

We set

$$K_\lambda(z,x) = \int_0^x k_\lambda(z,s)\,ds$$

and consider the C^1-functional $\widehat{\varphi}_\lambda : W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\widehat{\varphi}_\lambda(u) = \frac{1}{p}\gamma_p(u) - \int_\Omega K_\lambda(z,u)\,dz \quad \forall u \in W^{1,p}(\Omega).$$

Let $u \in \mathrm{int}C_+$ and choose $t \in (0,1)$ small so that $tu \le \overline{u}$ (recall that $\overline{u} \in \mathrm{int}C_+$). We have

$$\widehat{\varphi}_\lambda(tu) \le \frac{t^p}{p}\gamma_p(u) - t\int_\Omega \big(g(z,\overline{u}) + \lambda f(z,\overline{u})\big)u\,dz$$

(see (3.1)). Since $t \in (0,1)$ and $p > 1$, choosing $t \in (0,1)$ even smaller if necessary, we have

$$\widehat{\varphi}_\lambda(tu) < 0 \quad \forall t \in (0,1) \text{ small.} \tag{3.2}$$

On account of hypotheses H_1, given $\varepsilon > 0$, we can find $c_5 = c_5(\varepsilon) > 0$ such that

$$g(z,x) \le \varepsilon x^{p-1} + c_5 x^{q-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge 0,$$

so

$$G(z,x) \le \frac{\varepsilon}{p}x^p + \frac{c_5}{q}x^q \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge 0. \tag{3.3}$$

Then for $u \in W^{1,p}(\Omega)$ we have

$$\widehat{\varphi}_\lambda(u) = \frac{1}{p}\gamma_p(u) - \int_{\{u \le \overline{u}\}} \big(g(z,\overline{u}) + \lambda f(z,\overline{u})\big)u\,dz$$

$$
\begin{aligned}
&-\int_{\{\overline{u}<v\}}\big(G(z,u)-G(z,\overline{u})+\lambda(F(z,u)-F(z,\overline{u}))\big)\,dz\\
&\geq \frac{1}{p}\gamma_p(u)-\int_\Omega \big(g(z,\overline{u})+\lambda f(z,\overline{u})\big)u^+\,dz\\
&=-\int_\Omega G(z,u)\,dz-\lambda\int_{\{\overline{u}\leq u\}}F(z,u)\,dz\\
&\geq \frac{1}{p}\big(\gamma_p(u)-\varepsilon\|u\|^p\big)-c_6\big(\|u\|^q+\|u\|\big)-\lambda c_7\big(\|u\|+\|u\|^r\big)\\
&\geq c_8\|u\|^p-c_6\big(\|u\|^q+\|u\|\big)-\lambda c_7\big(\|u\|+\|u\|^r\big)
\end{aligned}
\tag{3.4}
$$

for some $c_6, c_7, c_8 > 0$ (see hypothesis $H_1(ii)$, (3.3), hypothesis $H_1(i)$ and recall that $\overline{u}(z) \leq \delta$ for all $z \in \overline{\Omega}$). Choose $\varrho_0 > 0$ such that

$$\xi_0 = c_6\varrho_0^p - c_6(\varrho_0^q + \varrho_0) > 0.$$

Having fixed $\varrho_0 > 0$ as above, choose $\lambda_0 > 0$ small so that

$$\xi_0 > \lambda c_7(\varrho_0 + \varrho_0^r) \quad \forall \lambda \in (0, \lambda_0).$$

Returning to (3.4), we have

$$\widehat{\varphi}_\lambda(u) > 0 \quad \forall \|u\| = \varrho_0,\ 0 < \lambda < \lambda_0. \tag{3.5}$$

Let $\overline{B}_0 = \overline{B}_{\varrho_0} = \{u \in W^{1,p}(\Omega) : \|u\| \leq \varrho_0\}$. The functional $\widehat{\varphi}_\lambda$ is sequentially weakly lower semicontinuous and $\overline{B}_0$ is sequentially weakly compact (from the reflexivity of $W^{1,p}(\Omega)$ and the Eberlein-Smulian theorem). So, we can find $u_\lambda \in W^{1,p}(\Omega)$ such that

$$\widehat{\varphi}_\lambda(u_\lambda) = \inf_{u\in\overline{B}_0} \widehat{\varphi}_\lambda(u), \tag{3.6}$$

so

$$0 < \|u_\lambda\| < \varrho_0 \quad \forall 0 < \lambda < \lambda_0 \tag{3.7}$$

(see (3.2) and (3.5)). From (3.6) and (3.7) it follows that

$$\widehat{\varphi}'_\lambda(u_\lambda) = 0,$$

so

$$
\begin{aligned}
&\langle A(u_\lambda, h) + \int_\Omega \xi(z)|u_\lambda|^{p-2}u_\lambda h\,dz + \int_{\partial\Omega}\beta(z)|u_\lambda|^{p-2}u_\lambda h\,d\sigma\\
&= \int_\Omega k_\lambda(z, u_\lambda)h\,dz \quad \forall h \in W^{1,p}(\Omega).
\end{aligned}
\tag{3.8}
$$

In (3.8) we choose $h = (\overline{u} - u_\lambda)^+ \in W^{1,p}(\Omega)$. Then

$$
\begin{aligned}
&\langle A(u_\lambda), (\overline{u}-u_\lambda)^+\rangle + \int_\Omega \xi(z)|u_\lambda|^{p-2}u_\lambda(\overline{u}-u_\lambda)^+\,dz\\
&+\int_{\partial\Omega}\beta(z)|u_\lambda|^{p-2}u_\lambda(\overline{u}-u_\lambda)^+\,d\sigma\\
&=\int_\Omega \big(g(z,\overline{u})+\lambda f(z,\overline{u})\big)(\overline{u}-u_\lambda)^+\,dz\\
&\geq \int_\Omega g(z,\overline{u})(\overline{u}-u_\lambda)^+\,dz\\
&\geq \int_\Omega \widehat{c}\,\overline{u}^{q-1}(\overline{u}-u_\lambda)^+\,dz
\end{aligned}
$$

$$\geq \langle A(\overline{u}), (\overline{u} - u_\lambda)^+ \rangle + \int_\Omega \xi(z)\overline{u}^{p-1}(\overline{u} - u_\lambda)^+ \, dz + \int_{\partial\Omega} \beta(z)\overline{u}^{p-1}(\overline{u} - u_\lambda) \, d\sigma$$

(see (3.1), (2.11) use hypothesis $H_1(iii)$ and recall that $f \geq 0$ and $\overline{u}(z) \leq \delta$ for all $z \in \overline{\Omega}$), so

$$\overline{u} \leq u_\lambda. \tag{3.9}$$

Then on account of (3.1), (3.9) and (3.8), we obtain

$$\begin{cases} -\Delta_p u_\lambda(z) + \xi(z)u_\lambda(z)^{p-1} = g(z, u_\lambda) + \lambda f(z, u_\lambda) & \text{in } \Omega, \\ \frac{\partial u_\lambda}{\partial n_p} + \beta(z)u_\lambda^{p-1} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.10}$$

As before the nonlinear regularity theorem (see Lieberman [11] and Papageorgiou-Rădulescu [14]) implies that $u_\lambda \in C_+\backslash\{0\}$. Let $\varrho = \|u_\lambda\|_\infty$ and with $J = \{\lambda\}$, let $\widehat{\xi}_\varrho^J > 0$ be as postulated by hypothesis H_3. Then from (3.10) we have

$$\Delta_p u_\lambda \leq \left(\|\xi\|_\infty + \widehat{\xi}_\varrho^J\right)u_\lambda^{p-1} \quad \text{in } \Omega,$$

so $u_\lambda \in \text{int}C_+$ (see Gasiński-Papageorgiou [6, p. 738]). Therefore we have proved that

$$(0, \lambda_0) \subseteq \mathcal{L} \neq \emptyset$$

and

$$S_\lambda \subseteq \text{int}C_+ \quad \forall\lambda > 0.$$

□

Next we show that $\mathcal{L}$ is an interval (connected).

Proposition 3.2 *If hypotheses H_0, H_1, H_2 and H_3 hold, $\lambda \in \mathcal{L}$ and $0 < \mu < \lambda$, then $\mu \in \mathcal{L}$.*

Proof Since $\lambda \in \mathcal{L}$ we can find $u_\lambda \in S_\lambda \subseteq \text{int}C_+$. We consider the following truncation of the reaction for the problem (P_μ):

$$d_\mu(z, x) = \begin{cases} g(z, x^+) + \mu f(z, x^+) & \text{if } x \leq u_\lambda(z) \\ g(z, u_\lambda(z)) + \mu f(z, u_\lambda(z)) & \text{if } u_\lambda(z) < x. \end{cases} \tag{3.11}$$

This is a Carathéodory function. We set

$$D_\mu(z, x) = \int_0^x d_\mu(z, s) \, ds$$

and consider the C^1-functional $\widehat{\psi}_\mu : W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\widehat{\psi}_\mu(u) = \frac{1}{p}\gamma_p(u) - \int_\Omega D_\mu(z, u) \, dz \quad \forall u \in W^{1,p}(\Omega).$$

From (3.11) and (2.1) it is clear that $\widehat{\psi}_\mu$ is coercive. Also using the Sobolev embedding theorem and the compactness of the trace map, we see that $\widehat{\psi}_\mu$ is sequentially weakly lower semicontinuous. So, we can find $u_\mu \in W^{1,p}(\Omega)$ such that

$$\widehat{\psi}_\mu(u_\mu) = \inf_{u \in W^{1,p}(\Omega)} \widehat{\psi}_\mu(u). \tag{3.12}$$

We choose $t \in (0, 1)$ small so that

$$t\widehat{u}_1 \leq u_\lambda \quad \text{and} \quad t\widehat{u}_1(z) \in (0, \delta] \quad \forall z \in \overline{\Omega}.$$

Since $\widehat{u}_1, u_\lambda \in \mathrm{int}C_+$ such a $t \in (0,1)$ exists. We have

$$\widehat{\psi}_\mu(t\widehat{u}_1) \le \frac{t^p}{p}\widehat{\lambda}_1 - t^q\frac{\widehat{c}}{q}\|\widehat{u}_1\|_q^q$$

(see hypothesis $H_1(iii)$ and recall that $\|\widehat{u}_1\|_p = 1$). Since $1 < q < p$, by choosing $t \in (0,1)$ even smaller if necessary, we have that

$$\widehat{\psi}_\mu(t\widehat{u}_1) < 0,$$

so

$$\widehat{\psi}_\mu(u_\mu) < 0 = \widehat{\psi}_\mu(0)$$

(see (3.12)) and thus $u_\mu \neq 0$.

From (3.12) we have

$$\widehat{\psi}'_\mu(u_\mu) = 0,$$

so

$$\begin{aligned}&\langle A(u_\mu), h\rangle + \int_\Omega \xi(z)|u_\mu|^{p-2}u_\mu h\,dz + \int_{\partial\Omega}\beta(z)|u_\mu|^{p-2}u_\mu h\,d\sigma\\ &= \int_\Omega d_\mu(z,u_\mu)h\,dz \quad \forall h \in W^{1,p}(\Omega).\end{aligned} \tag{3.13}$$

In (3.13) we first choose $h = -u_\mu^- \in W^{1,p}(\Omega)$. Then

$$\gamma_p(u_\mu^-) = 0$$

(see (3.11)), so

$$u_\mu \ge 0, \quad u_\mu \neq 0$$

(see (2.1)). Nest in (3.13) we choose $(u_\mu - u_\lambda)^+ \in W^{1,p}(\Omega)$. Then, using (3.10) and since $0 < \mu < \lambda$ and $f \ge 0$, we have

$$\begin{aligned}&\langle A(u_\mu), (u_\mu - u_\lambda)^+\rangle + \int_\Omega \xi(z)u_\mu^{p-1}(u_\mu - u_\lambda)^+\,dz\\ &\quad + \int_{\partial\Omega}\beta(z)u_\mu^{p-1}(u_\mu-u_\lambda)^+\,d\sigma\\ &= \int_\Omega \big(g(z,u_\lambda) + \mu f(z,u_\lambda)\big)(u_\mu - u_\lambda)^+\,dz\\ &\le \int_\Omega \big(g(z,u_\lambda) + \lambda f(z,u_\lambda)\big)(u_\mu - u_\lambda)^+\,dz\\ &= \langle A(u_\lambda), (u_\mu - u_\lambda)^+ + \int_\Omega \xi(z)u_\lambda^{p-1}(u_\mu - u_\lambda)^+\,dz\\ &\quad + \int_{\partial\Omega}\beta(z)u_\lambda^{p-1}(u_\mu - u_\lambda)^+\,d\sigma,\end{aligned}$$

so $u_\mu \le u_\lambda$ (see Proposition 2.2). So, we have proved that

$$u_\mu \in [0, u_\lambda], \quad u_\mu \neq 0. \tag{3.14}$$

Then (3.14), (3.11) and (3.13) imply that

$$u_\mu \in S_\mu \subseteq \mathrm{int}C_+,$$

and so $\mu \in \mathcal{L}$. □

Embedded in the above proof, is the following "monotonicity" property for S_λ as a function of the parameter $\lambda > 0$.

Corollary 3.3 *If hypotheses H_0, H_1, H_2 and H_3 hold, $\lambda \in \mathcal{L}$, $u_\lambda \in S_\lambda \subseteq \mathrm{int}C_+$ and $0 < \mu < \lambda$, then $\mu \in \mathcal{L}$ and we can find $u_\mu \in S_\mu \subseteq \mathrm{int}C_+$ such that*

$$u_\mu \leq u_\lambda.$$

We can improve the conclusion of this corollary.

Proposition 3.4 *If hypotheses H_0, H_1, H_2 and H_3 hold, $\lambda \in \mathcal{L}$, $u_\lambda \in S_\lambda \subseteq \mathrm{int}C_+$ and $0 < \mu < \lambda$, then $\mu \in \mathcal{L}$ and there exists $u_\mu \in S_\mu \subseteq C_+$ such that*

$$u_\lambda - u_\mu \in D_+.$$

Proof From Corollary 3.3 we already know that $\mu \in \mathcal{L}$ and that we can find $u_\mu \in S_\mu \subseteq \mathrm{int}C_+$ such that

$$0 < u_\mu \leq u_\lambda. \tag{3.15}$$

Let $\varrho = \|u_\lambda\|_\infty$, $J = \{\lambda, \mu\}$ and consider $\widehat{\xi}_\varrho^J > 0$ as postulated by hypothesis H_3, We have

$$\begin{aligned}
&-\Delta_p u_\mu + \big(\xi(z) + \widehat{\xi}_\varrho^J\big)u_\mu^{p-1} \\
&= g(z, u_\mu) + \mu f(z, u_\mu) + \xi_\varrho^J u_\mu^{p-1} \\
&= g(z, u_\mu) + \lambda f(z, u_\mu) - (\lambda - \mu) f(z, u_\mu) + \widehat{\xi}_\varrho^J u_\mu^{p-1} \\
&\leq g(z, u_\lambda) + \lambda f(z, u_\lambda) + \widehat{\xi}_\varrho^J u_\lambda^{p-1} \\
&= -\Delta_p u_\lambda \big(\xi(z) + \widehat{\xi}_\varrho^J\big)u_\lambda^{p-1}
\end{aligned} \tag{3.16}$$

(see (3.15) and hypothesis H_3), with $0 < m_\mu = \min_{\overline{\Omega}} u_\mu$ ($u_\mu \in \mathrm{int}C_+$) and η_{m_μ} as in $H_2(iv)$.

From (3.16) and using Proposition 2.10 of Papageorgiou-Rădulescu-Repovš [17], we obtain that

$$u_\lambda - u_\mu \in D_+.$$

□

Recall that $\lambda^* = \sup \mathcal{L}$. Next we show that $\lambda^* < +\infty$.

Proposition 3.5 *If hypotheses H_0, H_1, H_2 and H_3 hold, then $\lambda^* < +\infty$.*

Proof Let $\eta > \widehat{\lambda}_1$. Hypotheses $H_1(iii)$ and $H_2(ii)$, (iv) imply that we can find $\widetilde{\lambda} > 0$ big such that

$$g(z, x) + \widetilde{\lambda} f(z, x) \geq \eta x^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{3.17}$$

Let $\lambda > \widetilde{\lambda}$ and suppose that $\lambda \in \mathcal{L}$. We can find $u_\lambda \in S_\lambda \subseteq \mathrm{int}C_+$. We introduce the Carathéodory function $\vartheta_\lambda(z, x)$ defined by

$$\vartheta_\lambda(z, x) = \begin{cases} \eta (x^+)^{p-1} & \text{if } x \leq u_\lambda(z), \\ \eta u_\lambda(z)^{p-1} & \text{if } u_\lambda(z) < x. \end{cases} \tag{3.18}$$

We set

$$\Theta_\lambda(z, x) = \int_0^x \vartheta_\lambda(z, s)\, ds$$

and consider the C^1-functional $\tau_\lambda\colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\tau_\lambda(u) = \frac{1}{p}\gamma_p(u) - \int_\Omega \Theta_\lambda(z,u)\,dz \quad \forall u \in W^{1,p}(\Omega).$$

From (3.18) and (2.1) we see that τ_λ is coercive. Also it is sequentially weakly lower semicontinuous. So, we can find $\widehat{u}_* \in W^{1,p}(\Omega)$ such that

$$\tau_\lambda(\widehat{u}_*) = \inf_{u \in W^{1,p}(\Omega)} \tau_\lambda(u). \tag{3.19}$$

As before we choose $t \in (0,1)$ small so that

$$0 < t\widehat{u}_1 \le u_\lambda.$$

Then we have

$$\tau_\lambda(t\widehat{u}_1) = \frac{t^p}{p}(\widehat{\lambda}_1 - \eta) < 0$$

(recall that $\|\widehat{u}_1\|_p = 1$), so

$$\tau_\lambda(\widehat{u}_*) < 0 = \tau_\lambda(0)$$

(see (3.19)), thus

$$\widehat{u}_* \neq 0.$$

From (3.19) we have

$$\tau'_\lambda(\widehat{u}_*) = 0,$$

so

$$\begin{aligned} &\langle A(\widehat{u}_*), h\rangle + \int_\Omega \xi(z)|\widehat{u}_*|^{p-2}\widehat{u}_* h\,dz + \int_{\partial\Omega} \beta(z)|\widehat{u}_*|^{p-2}\widehat{u}_* h\,d\sigma \\ &= \int_\Omega \vartheta_\lambda(z,\widehat{u}_*)h\,dz \quad \forall h \in W^{1,p}(\Omega). \end{aligned} \tag{3.20}$$

In (3.20) first we choose $h = -\widehat{u}_*^- \in W^{1,p}(\Omega)$ and obtain

$$\gamma_p(\widehat{u}_*^-) = 0$$

(see (3.18)), so

$$\widehat{u}_* \ge 0, \quad \widehat{u}_* \neq 0$$

(see (2.1)).

Next in (3.20) we choose $h = (\widehat{u}_* - u_\lambda)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} &\langle A(\widehat{u}_*), (\widehat{u}_* - u_\lambda)^+\rangle + \int_\Omega \xi(z)\widehat{u}_*^{p-1}(\widehat{u}_* - u_\lambda)^+\,dz + \int_{\partial\Omega} \beta(z)\widehat{u}_*^{p-1}(\widehat{u}_* - u_\lambda)^+\,d\sigma \\ &= \int_\Omega \eta u_\lambda^{p-1}(\widehat{u}_* - u_\lambda)^+\,dz \\ &\le \int_\Omega \big(g(z,u_\lambda) + \lambda f(z,u_\lambda)\big)(\widehat{u}_* - u_\lambda)^+\,dz \\ &= \langle A(u_\lambda), (\widehat{u}_* - u_\lambda)^+\rangle + \int_\Omega \xi(z)u_\lambda^{p-1}(\widehat{u}_* - u_\lambda)^+\,dz \\ &\quad + \int_{\partial\Omega} \beta(z)u_\lambda^{p-1}(\widehat{u}_* - u_\lambda)^+\,d\sigma \end{aligned}$$

(see (3.18), (3.17) and recall that $\lambda > \widetilde{\lambda}$ and $u_\lambda \in S_\lambda$), so

$$\widehat{u}_* \leq u_\lambda.$$

So, we have proved that

$$\widehat{u}_* \in [0, u_\lambda], \quad \widehat{u}_* \neq 0. \tag{3.21}$$

From (3.21), (3.18) and (3.20), we infer that

$$\begin{cases} -\Delta_p \widehat{u}_* + \xi(z)\widehat{u}_*^{p-1} = \eta u_*^{p-1} & \text{in } \Omega, \\ \frac{\partial \widehat{u}_*}{\partial n_p} + \beta(z)\widehat{u}_*^{p-1} 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\widehat{u}_* \geq 0$, $\widehat{u}_* \neq 0$ and $\eta > \widehat{\lambda}_1$, we have a contradiction. Therefore $\lambda \in \mathcal{L}$ and so $\lambda^* \leq \widetilde{\lambda} < +\infty$. □

We show that the critical parameter λ^* is admissible (that is, $\lambda^* \in \mathcal{L}$).

Proposition 3.6 *If hypotheses H_0, H_1, H_2 and H_3 hold, then $\lambda^* \in \mathcal{L}$.*

Proof In what follows for every $\lambda > 0$ by $\varphi_\lambda \colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ we denote the energy functional for problem (P_λ) defined by

$$\varphi_\lambda(u) = \frac{1}{p}\gamma_p(u) - \int_\Omega \big(G(z,u) + \lambda F(z,u)\big)\,dz \quad \forall u \in W^{1,p}(\Omega).$$

We know that $\varphi_\lambda \in C^1(W^{1,p}(\Omega))$.

Let $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq \mathcal{L}$ be such that $\lambda_n \nearrow \lambda^*$ and $u_n \in S_{\lambda_n} \subseteq \operatorname{int}C_+$, $n \in \mathbb{N}$. According to Proposition 3.2 and its proof, we can have

$$\varphi_{\lambda_n}(u_n) < 0 \quad \forall n \in \mathbb{N},$$

so

$$\gamma_p(u_n) - \int_\Omega p\big(G(z,u_n) + \lambda_n F(z,u_n)\big)\,dz < 0 \quad \forall n \in \mathbb{N}. \tag{3.22}$$

Also we have

$$\begin{aligned} &\langle A(u_n), h\rangle + \int_\Omega \xi(z)u_n^{p-1}h\,dz + \int_{\partial\Omega} \beta(z)u_n^{p-1}h\,d\sigma \\ &= \int_\Omega \big(g(z,u_n) + \lambda_n f(z,u_n)\big)h\,dz \quad \forall h \in W^{1,p}(\Omega),\ n \in \mathbb{N}. \end{aligned}$$

We choose $h = u_n \in W^{1,p}(\Omega)$. Then

$$-\gamma_p(u_n) + \int_\Omega \big(g(z,u_n) + \lambda f(z,u_n)\big)u_n\,dz = 0 \quad \forall n \in \mathbb{N}. \tag{3.23}$$

We add (3.22) and (3.23) and obtain

$$\int_\Omega e_{\lambda_n}(z,u_n)\,dz < 0 \quad \forall n \in \mathbb{N}. \tag{3.24}$$

Claim. The sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega)$ is bounded.

We argue by contradiction. So, suppose that at least for a subsequence, we have

$$\|u_n\| \longrightarrow +\infty \quad \text{as } n \to +\infty.$$

Let $y_n = \frac{u_n}{\|u_n\|}$ for $n \in \mathbb{N}$. Then $\|y_n\| = 1$, $y_n \geq 0$ for all $n \in \mathbb{N}$ and so we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W^{1,p}(\Omega) \quad \text{and} \quad y_n \longrightarrow y \quad \text{in } L^r(\Omega) \text{ and in } L^p(\partial\Omega), \tag{3.25}$$

with $y \geq 0$.

First assume that $y \neq 0$. Let $\Omega_+ = \{z \in \Omega : \ y(z) > 0\}$. If by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$, then $|\Omega_+|_N > 0$ (recall that $y \geq 0$; see (3.25)) and we have

$$u_n(z) \longrightarrow +\infty \quad \text{for a.a. } z \in \Omega_+,$$

so

$$\frac{F(z, u_n(z))}{u_n(z)^p} \longrightarrow +\infty \quad \text{for a.a. } z \in \Omega_+$$

(see hypothesis $H_1(ii)$), thus

$$\frac{F(z, u_n(z))}{\|u_n\|^p} = \frac{F(z, u_n(z))}{u_n(z)^p} y_n(z)^p \longrightarrow +\infty \quad \text{for a.a. } z \in \Omega_+.$$

Then hypothesis $H_2(ii)$ and Fatou's lemma imply that

$$\frac{1}{\|u_n\|^p} \int_{\Omega_+} F(z, u_n)\, dz \longrightarrow +\infty. \tag{3.26}$$

Note that

$$\begin{aligned} \frac{1}{\|u_n\|^p} \int_{\Omega} F(z, u_n)\, dz &= \frac{1}{\|u_n\|^p} \int_{\Omega_+} F(z, u_n)\, dz + \frac{1}{\|u_n\|^p} \int_{\Omega\setminus\Omega_+} F(z, u_n)\, dz \\ &\geq \frac{1}{\|u_n\|^p} \int_{\Omega_+} F(z, u_n)\, dz \end{aligned}$$

(since $F \geq 0$), so

$$\frac{1}{\|u_n\|^p} \int_{\Omega} F(z, u_n)\, dz \longrightarrow +\infty \tag{3.27}$$

(see (3.26)).

On the other hand from hypotheses $H_1(i)$, (ii) we see that given $\varepsilon > 0$, we can find $c_8 = c_8(\varepsilon) > 0$ such that

$$|G(z, x)| \leq \varepsilon x^p + c_8 \quad \text{for a.a. } z \in \Omega, \ \text{all } x \geq 0.$$

Therefore we have

$$\lim_{n \to +\infty} \frac{1}{\|u_n\|^p} \int_{\Omega} |G(z, u_n)|\, dz \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$\frac{1}{\|u_n\|^p} \int_{\Omega} G(z, u_n)\, dz \longrightarrow 0 \quad \text{as } n \to +\infty. \tag{3.28}$$

From (3.27) and (3.28), we have

$$\frac{1}{\|u_n\|^p} \int_{\Omega} \big(G(z, u_n) + \lambda_n F(z, u_n)\big)\, dz \longrightarrow +\infty \quad \text{as } n \to +\infty. \tag{3.29}$$

Hypothesis $H_2(iii)$ implies that for all $\lambda > 0$ we have

$$0 \leq e_\lambda(z, x) + \widehat{\vartheta}_\lambda(z) \quad \text{for a.a. } z \in \Omega, \ \text{all } x \geq 0,$$

so

$$p\big(G(z,x)+\lambda F(z,x)\big)\leq\big(g(z,x)+\lambda f(z,x)\big)+\widehat{\vartheta}_\lambda(z)\quad\text{for a.a. } z\in\Omega,\ \text{all } x\geq 0. \tag{3.30}$$

Therefore

$$\begin{aligned}&\int_\Omega p\big(G(z,u_n)+\lambda_n F(z,u_n)\big)\,dz\\ &\leq\int_\Omega\big(g(z,u_n)+\lambda_n f(z,u_n)\big)u_n\,dz+\|\vartheta_{\lambda_n}\|_1\\ &\leq\gamma_p(u_n)+\|\vartheta_{\lambda^*}\|_1\end{aligned}$$

(see (3.30), (3.23)), so

$$\frac{1}{\|u_n\|^p}\int_\Omega p\big(G(z,u_n)+\lambda F(z,u_n)\big)\,dz\leq\gamma_p(y_n)+\frac{\|\vartheta_{\lambda^*}\|_1}{\|u_n\|^p}\leq c_9\quad\forall n\in\mathbb{N}, \tag{3.31}$$

for some $c_9<0$. Comparing (3.29) and (3.31), we have a contradiction.

Next we assume that $y\equiv 0$. Let $\eta>0$ and define

$$v_n=(p\eta)^{\frac{1}{p}}y_n\in W^{1,p}(\Omega)\quad\forall n\in\mathbb{N}.$$

We have

$$v_n\longrightarrow 0\quad\text{in } L^r(\Omega)$$

(see (3.25) and recall that $y=0$), so

$$\int_\Omega\big(G(z,u_n)+\lambda_n F(z,u_n)\big)\,dz\longrightarrow 0 \tag{3.32}$$

(see (2.2) and hypothesis $H_2(i)$).

Since $\|u_n\|\longrightarrow+\infty$, we can find $n_0\in\mathbb{N}$ such that

$$(p\eta)^{\frac{1}{p}}\frac{1}{\|u_n\|}\leq 1\quad\forall n\geq n_0. \tag{3.33}$$

Let $t_n\in[0,1]$ be such that

$$\varphi_{\lambda_n}(t_nu_n)=\max_{0\leq t\leq 1}\varphi_{\lambda_n}(tu_n). \tag{3.34}$$

We have

$$\begin{aligned}\varphi_{\lambda_n}(t_nu_n)&\geq\varphi_{\lambda_n}(v_n)\\ &=\frac{1}{p}\gamma_p(v_n)-\int_\Omega\big(G(z,v_n)+\lambda_n F(z,v_n)\big)\,dz\\ &\geq\eta\gamma_p(y_n)-\int_\Omega\big(G(z,v_n)+\lambda_n F(z,v_n)\big)\,dz\\ &\geq\eta\widehat{c}-\int_\Omega\big(G(z,v_n)+\lambda_n F(z,v_n)\big)\,dz\end{aligned}$$

(see (3.33), (3.34), (2.1) and recall that $\|y_n\|=1$), so

$$\varphi_{\lambda_n}(u_n)\geq\frac{\eta\widehat{c}}{2}\quad\forall n\geq n_1\geq n_0$$

(see (3.32)). Since $\eta > 0$ is arbitrary, we conclude that

$$\varphi_{\lambda_n}(t_b u_n) \longrightarrow +\infty \quad \text{as } n \to +\infty. \tag{3.35}$$

We know that

$$\varphi_{\lambda_n}(0) = 0 \quad \text{and} \quad \varphi_{\lambda_n}(u_n) < 0 \quad \forall n \in \mathbb{N}. \tag{3.36}$$

From (3.35) and (3.36) it follows that

$$t_n \in (0,1) \quad \forall n \geq n_2,$$

so

$$\frac{d}{dt}\varphi_{\lambda_n}(tu_n)\Big|_{t=t_n} = 0 \quad \forall n \geq n_2$$

(see (3.34)) and thus

$$\langle \varphi'_{\lambda_n}(t_n u_n), u_n \rangle = 0 \quad \forall n \geq n_2$$

(by the chain rule). Hence for $n \geq n_2$ we have

$$\gamma_p(t_n u_n) - \int_\Omega \big(g(z, t_n u_n) + \lambda_n f(z, t_n u_n)\big)(t_n u_n)\, dz = 0,$$

so

$$\begin{aligned}\gamma_p(t_n u_n) &= \int_\Omega e_{\lambda_n}(z, t_n u_n)\, dz + \int_\Omega p\big(G(z, t_n u_n) + \lambda_n F(z, t_n u_n)\big)\, dz \\ &\leq \int_\Omega e_{\lambda_n}(z, u_n)\, dz + \|\vartheta_{\lambda^*}\|_1 + \int_\Omega p\big(G(z, t_n u_n) + \lambda_n F(z, t_n u_n)\big)\, dz\end{aligned}$$

and thus

$$p\varphi_{\lambda_n}(t_n u_n) \leq \|\vartheta_{\lambda^*}\|_1 \quad \forall n \geq n_2 \tag{3.37}$$

(see (3.24)). Comparing (3.37) and (3.34) we have a contradiction. Therefore the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega)$ is bounded. This proves the Claim.

On account of the Claim we may assume that

$$u_n \xrightarrow{w} u_* \quad \text{in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \longrightarrow u_* \quad \text{in } L^r(\Omega) \text{ and in } L^p(\partial\Omega). \tag{3.38}$$

We have

$$\begin{aligned}&\langle A(u_n), u_n - u_* \rangle + \int_\Omega \xi(z) u_n^{p-1}(u_n - u_*)\, dz + \int_{\partial\Omega} \beta(z) u_n^{p-1}(u_n - u_*)\, d\sigma \\ &= \int_\Omega \big(g(z, u_n) + \lambda_n f(z, u_n)\big)(u_n - u_*)\, dz \quad \forall n \in \mathbb{N},\end{aligned}$$

so

$$\lim_{n\to+\infty} \langle A(u_n), u_n - u_* \rangle = 0$$

(see (3.38)) and thus

$$u_n \longrightarrow u_* \quad \text{in } W^{1,p}(\Omega) \tag{3.39}$$

(see Proposition 2.2), with $u \geq 0$.

Using (3.39) in the limit as $n \to +\infty$, we have

$$\langle A(u_*), h\rangle + \int_\Omega \xi(z) u_*^{p-1} h\, dz + \int_{\partial\Omega} \beta(z) u_*^{p-1} h\, d\sigma$$
$$= \int_\Omega \big(g(z, u_*) + \lambda^* f(f, u_*)\big) h\, dz \quad \forall h \in W^{1,p}(\Omega).$$

If we show that $u_* \neq 0$, then $u_* \in S_{\lambda^*} \subseteq \mathrm{int} C_+$ and so $\lambda^* \in \mathcal{L}$. Arguing by contradiction, suppose that $u_* = 0$. Then from (3.40), Proposition 2.10 of Papageorgiou-Rădulescu [14], Theorem 2 of Lieberman [11] and exploiting the compactness of the embedding $C^{1,\vartheta}(\overline{\Omega}) \subseteq C^1(\overline{\Omega})$ (with $0 < \vartheta < 1$), we have that

$$u_n \longrightarrow u_* \quad \text{in } C^1(\overline{\Omega}),$$

so

$$u_n(z) \in (0, \delta] \quad \forall z \in \overline{\Omega},\ n \geq \widehat{n}$$

(where $\delta > 0$ is as in hypothesis $H_1(iii)$), so

$$g(z, u_n(z)) + \lambda^* f(z, u_n(z)) \geq \widehat{c} u_n(z)^{q-1} \quad \text{for a.a. } z \in \Omega,\ \text{all } n \geq \widehat{n}$$

(see (3.20)). Then for $n \geq \widehat{n}$ we consider the Carathéodory function

$$l_n(z, x) = \begin{cases} \widehat{c}(x^+)^{q-1} & \text{if } x \leq u_n(z), \\ \widehat{c} u_n(z)^{q-1} & \text{if } u_n(z) < x. \end{cases} \tag{3.40}$$

We set

$$L_n(z, x) = \int_0^x l(z, s)\, ds$$

and consider the C^1-functional $\zeta_n \colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\zeta_n(u) = \frac{1}{p}\gamma_p(u) - \int_\Omega L_n(z, u)\, dz \quad \forall u \in W^{1,p}(\Omega).$$

Using the direct method od the calculus of variation and the fact that $q < p$, we can find $\widetilde{u}_* \in W^{1,p}(\Omega)$ such that

$$\zeta_n(\widetilde{u}_*) = \inf_{u \in W^{1,p}(\Omega)} \zeta_n(u) < 0 = \zeta_n(0),$$

so $\widetilde{u}_* \neq 0$. Then using (3.40) we show that

$$\widetilde{u}_* \in [0, u_n], \quad \widetilde{u}_* \neq 0,$$

so

$$\widetilde{u}_* = \widetilde{u} \in \mathrm{int} C_+$$

(see Proposition 2.6), thus

$$\widetilde{u} \leq u_n \quad \forall n \geq \widehat{n},$$

and finally

$$\widetilde{u} \leq u_*,$$

a contradiction. Therefore $u_* \neq 0$ and so $u_* \in S_{\lambda^*} \subseteq \mathrm{int} C_+$ and so $\lambda^* \in \mathcal{L}$. □

So, we have proved that

$$\mathcal{L} = (0, \lambda^*].$$

Finally we show that for $0 < \lambda < \lambda^*$ we have multiplicity of positive solutions.

Proposition 3.7 *If hypotheses H_0, H_1, H_2 and H_3 hold and $0 < \lambda < \lambda^*$, then problem (P_λ) has at least two positive solutions $u_0, \widehat{u} \in \operatorname{int} C_+$, $u_0 \neq \widehat{u}$.*

Proof Let $0 < \mu < \lambda < \eta < \lambda^*$. According to Corollary 3.3, we can find $u_\eta \in S_\eta \subseteq \operatorname{int} C_+$, $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$, $u_\mu \in S_\mu \subseteq \operatorname{int} C_+$ such that

$$u_\eta - u_\lambda \in D_+ \quad \text{and} \quad u_\lambda - u_\mu \in D_+,$$

so

$$u_\lambda \in \operatorname{int}_{C^1(\overline{\Omega})}[u_\mu, u_\eta] \tag{3.41}$$

(by $\operatorname{int}_{C^1(\overline{\Omega})}[u_\mu, u_\eta]$ we denote the interior in $C^1(\overline{\Omega})$ of $[u_\mu, u_\eta] \cap C^1(\overline{\Omega})$). We consider the following truncation of the reaction in problem (P_λ)

$$\widetilde{\tau}_\lambda(z, x) = \begin{cases} g(z, u_\mu(z)) + \lambda f(z, u_\mu(z)) & \text{if } x < u_\mu(z), \\ g(z, x) + \lambda f(z, x) & \text{if } u_\mu(z) \leq x \leq u_\eta(z), \\ g(z, u_\eta(z)) + \lambda f(z, u_\eta(z)) & \text{if } u_\eta(z) < x. \end{cases} \tag{3.42}$$

This is a Carathéodory function. We set

$$\widetilde{T}_\lambda(z, x) = \int_0^x \widetilde{\tau}_\lambda(z, s)\, ds$$

and consider the C^1-functional $\widetilde{\psi}_\lambda \colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\widetilde{\psi}_\lambda(u) = \frac{1}{p}\gamma_p(u) - \int_\Omega \widetilde{T}_\lambda(z, u)\, dz \quad \forall u \in W^{1,p}(\Omega)$$

Let

$$K_{\widetilde{\psi}_\lambda} = \{u \in W^{1,p}(\Omega) : \ \widetilde{\psi}'_\lambda(u) = 0\}$$

(the critical set of $\widetilde{\psi}_\lambda$).
Claim 1: $K_{\widetilde{\psi}_\lambda} \subseteq [u_\mu, u_\eta] \cap \operatorname{int} C_+$.
Let $u \in K_{\widetilde{\psi}_\lambda}$. We have

$$\widetilde{\psi}'_\lambda(u) = 0,$$

so

$$\begin{aligned} &\langle A(u), h\rangle + \int_\Omega \xi(z)|u|^{p-2}uh\, dz + \int_{\partial\Omega} \beta(z)|u|^{p-2}uh\, d\sigma \\ &= \int_\Omega \widetilde{\tau}_\lambda(z, u)h\, dz \quad \forall h \in W^{1,p}(\Omega). \end{aligned} \tag{3.43}$$

In (3.43) first we choose $h = (u_\mu - u)^+ \in W^{1,p}(\Omega)$. Then, using (3.42) and the facts that $\lambda > \mu$, $f \geq 0$ and $u_\mu \in S_\mu$, we have

$$\begin{aligned}
&\langle A(u), (u_\mu - u)^+\rangle + \int_\Omega \xi(z)|u|^{p-2}u(u_\mu - u)^+\,dz \\
&\quad + \int_{\partial\Omega} \beta(z)|u|^{p-2}u(u_\mu - u)^+\,d\sigma \\
&= \int_\Omega \big(g(z, u_\mu) + \lambda f(z, u_\mu)\big)(u_\mu - u)^+\,dz \\
&\geq \int_\Omega \big(g(z, u_\mu) + \mu f(z, u_\mu)\big)(u_\mu - u)^+\,dz \\
&= \langle A(u_\mu), (u_\mu - u)^+\rangle + \int_\Omega \xi(z)|u_\mu|^{[-2}u_\mu(u_\mu - u)^+\,dz \\
&\quad + \int_{\partial\Omega} \beta(z)|u_\mu|^{p-2}u_\mu(u_\mu - u)^+\,d\sigma,
\end{aligned}$$

so

$$u_\mu \leq u.$$

Similarly if in (3.43) we choose $h = (u - u_\eta)^+ \in W^{1,p}(\Omega)$, we show that

$$u \leq u_\eta,$$

so

$$u \in [u_\mu, u_\eta].$$

Moreover, the nonlinear regularity theory (see Lieberman [11]) implies that $u \in C^1(\overline{\Omega})$. Therefore

$$K_{\widetilde{\psi}_\lambda} \subseteq [u_\mu, u_\eta] \cap \mathrm{int} C_+.$$

This proves Claim 1.

Evidently $u_\lambda \in K_{\widetilde{\psi}_\lambda}$ (see (3.42)). We may assume that

$$K_{\widetilde{\psi}_\lambda} = \{u_\lambda\}. \tag{3.44}$$

Otherwise on account of Claim 1 and (3.42), we see that we already have a second positive solution of problem (P_λ) and so we are done.

From (2.1) and (3.42), we see that $\widetilde{\psi}_\lambda$ is coercive. Also it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}_\lambda \in W^{1,p}(\Omega)$ such that

$$\widetilde{\psi}_\lambda(\widetilde{u}_\lambda) = \inf_{u \in W^{1,p}(\Omega)} \widetilde{\psi}_\lambda(u),$$

so

$$\widetilde{u}_\lambda \in K_{\widetilde{\psi}_\lambda}$$

and thus

$$\widetilde{u}_\lambda = u_\lambda \in \mathrm{int}_{C^1(\overline{\Omega})}[u_\mu, u_\eta] \tag{3.45}$$

(see (3.44) and (3.41)). We introduce the Carathéodory function $\widetilde{k}_\lambda(z, x)$ defined by

$$\widetilde{k}_\lambda(z, x) = \begin{cases} g(z, u_\mu(z)) + \lambda f(z, u_\mu(z)) & \text{if } x \leq u_\mu(z), \\ g(z, x) + \lambda f(z, x) & \text{if } u_\mu(z) < x. \end{cases} \tag{3.46}$$

We set

$$\widetilde{K}_\lambda(z, x) = \int_0^x \widetilde{k}_\lambda(z, s)\,ds$$

and consider the C^1-functional $\widetilde{\varphi}_\lambda\colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\widetilde{\varphi}_\lambda(u) = \frac{1}{p}\gamma_p(u) - \int_\Omega \widetilde{K}_\lambda(z,u)\,dz \quad \forall u \in W^{1,p}(\Omega).$$

If

$$K_{\widetilde{\varphi}_\lambda} = \{u \in W^{1,p}(\Omega) : \widetilde{\varphi}_\lambda'(u) = 0\},$$

then using (3.46) we can check that

$$K_{\widetilde{\varphi}_\lambda} \subseteq [u_\mu) \cap \operatorname{int} C_+. \tag{3.47}$$

Moreover, from (3.42) and (3.46) it is clear that

$$\widetilde{\varphi}_\lambda|_{[u_\mu,u_\eta]} = \widetilde{\psi}_\lambda|_{[u_\mu,u_\eta]}. \tag{3.48}$$

From (3.45) and (3.48) it follows that

$$u_\lambda \text{ is a local } C^1(\overline{\Omega})\text{-minimizer of } \widetilde{\varphi}_\lambda,$$

so also

$$u_\lambda \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \widetilde{\varphi}_\lambda \tag{3.49}$$

(see Papageorgiou-Rădulescu [14]).

On account of (3.47) we may assume that $K_{\widetilde{\varphi}_\lambda}$ is finite (otherwise we already have an infinity of positive smooth solutions of (P_λ) and so we are done). Then (3.49) and Theorem 5.7.6 of Papageorgiou-Rădulescu-Repovš [19, p. 449], imply that we can find $\varrho \in (0,1)$ small such that

$$\widetilde{\varphi}_\lambda(u_\lambda) < \inf_{\|u-u_\lambda\|=\varrho} \widetilde{\varphi}_\lambda(u) = \widetilde{m}_\lambda. \tag{3.50}$$

Also, because of hypothesis $H_2(ii)$ we have

$$\widetilde{\varphi}_\lambda(t\widehat{u}_1) \longrightarrow -\infty \quad \text{as } t \to +\infty. \tag{3.51}$$

Claim 2. $\widetilde{\varphi}_\lambda$ satisfies the Cerami condition (see Papageorgiou-Rădulescu-Repovš [19, p. 366]).

We consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega)$ such that

$$|\widetilde{\varphi}_\lambda(u_n)| \le c_9 \quad \forall n \in \mathbb{N}, \tag{3.52}$$

for some $c_9 > 0$, and

$$(1 + \|u_n\|)\widetilde{\varphi}_\lambda'(u_n) \longrightarrow 0 \quad \text{in } W^{1,p}(\Omega)^* \quad \text{as } n \to +\infty. \tag{3.53}$$

From (3.53) we have

$$\begin{aligned}
&\Big|\langle A(u_n), h\rangle + \int_\Omega \xi(z)|u_n|^{p-2}u_n h\,dz \\
&\int_{\partial\Omega} \beta(z)|u_n|^{p-2}u_n h\,d\sigma - \int_\Omega \widetilde{k}_\lambda(z,u_n)h\,dz\Big| \\
&\le \frac{\varepsilon_n\|h\|}{1+\|u_n\|} \quad \forall h \in W^{1,p}(\Omega),
\end{aligned} \tag{3.54}$$

with $\varepsilon_n \to 0^+$. In (3.54) we choose $h = -u_n^- \in W^{1,p}(\Omega)$. Then

$$\gamma_p(u_n^-) \le c_{10} \quad \forall n \in \mathbb{N},$$

for some $c_{10} > 0$ (see (3.46)), so

$$\{u_n^-\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega) \text{ is bounded} \tag{3.55}$$

(see (2.1)). From (3.52) and (3.55) we have

$$|\widetilde{\varphi}_\lambda(u_n^+)| \le c_{11} \quad \forall n \in \mathbb{N}, \tag{3.56}$$

for some $c_{11} > 0$, so

$$\gamma_p(u_n^+) - \int_\Omega p\big(G(z, u_n^+) + \lambda F(z, u_n^+)\big)\, dz \le c_{12} \quad \forall n \in \mathbb{N}, \tag{3.57}$$

for some $c_{12} > 0$ (see (3.46)). In (3.54), we choose $h = u_n^+ \in W^{1,p}(\Omega)$ and obtain

$$-\gamma_p(u_n^+) + \int_\Omega \big(g(z, u_n^+) + \lambda f(z, u_n^+)u_n^+\, dz \le c_{13} \quad \forall n \in \mathbb{N}, \tag{3.58}$$

for some $c_{13} > 0$ (see (3.46)).

Adding (3.57) and (3.58) we obtain

$$\int_\Omega e_\lambda(z, u_n^+)\, dz \le c_{14} \quad \forall n \in \mathbb{N}, \tag{3.59}$$

for some $c_{14} > 0$. Using (3.59) and arguing as in the Claim of the proof of Proposition 3.6, we show that the sequence $\{u_n^+\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega)$ is bounded and so

$$\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega) \quad \text{is bounded} \tag{3.60}$$

(see (3.55)). Then from (3.60) as in the proof of Proposition 3.6, using the $(S)_+$-property of A (see Proposition 2.2), we show that at least for a subsequence, we have

$$u_n \longrightarrow u \quad \text{in } W^{1,p}(\Omega),$$

so $\widetilde{\varphi}_\lambda$ satisfied the Cerami condition. This proves Claim 2.

Then (3.50), (3.51) and Claim 2 permit the use of the mountain pass theorem. So, we can find $\widehat{u}_\lambda \subseteq W^{1,p}(\Omega)$ such that

$$\widehat{u}_\lambda \subseteq K_{\widetilde{\varphi}_\lambda} \subseteq [u_\lambda) \cap \mathrm{int} C_+$$

(see (3.47)), so

$$\widetilde{\varphi}_\lambda(u_\lambda) < \widetilde{m}_\lambda \le \widetilde{\varphi}_\lambda(\widehat{u}_\lambda)$$

(see (3.50)). We conclude that

$$\widehat{u}_\lambda \ne u_\lambda \quad \text{and} \quad \widehat{u}_\lambda \subseteq S_\lambda \subseteq \mathrm{int} C_+$$

(see (3.46)). □

So, summarizing we can have the following multiplicity theorem for problem (P_λ) which is global with respect to the parameter $\lambda > 0$ (bifurcation-type theorem).

Theorem 3.8 *If hypotheses H_0, H_1, H_2, H_3 hold, then there exists $\lambda^* > 0$ such that*
(a) *for all $\lambda \in (0, \lambda^*)$ problem (P_λ) has at least two positive solutions $u_\lambda, \widehat{u}_\lambda \subseteq \mathrm{int} C_+$, $u_\lambda \ne \widehat{u}_\lambda$;*
(b) *for $\lambda = \lambda^*$ problem (P_λ) has at least one positive solution $u_* \in \mathrm{int} C_+$;*
(c) *for all $\lambda > \lambda^*$ problem (P_λ) has no positive solutions.*

4 Minimal positive solution

In this section we show that for every admissible parameter $\lambda \in \mathcal{L} = (0, \lambda^*]$, problem (P_λ) has a smallest positive solution $u_\lambda^* \in \operatorname{int} C_+$ (that is, $u_\lambda^* \le u$ for all $u \in S_\lambda$) and study the monotonicity and continuity properties of the map $\lambda \longmapsto u_\lambda^*$.

From Papageorgiou-Rădulescu-Repovš [15] (see the proof of Proposition 3.5), we know that the set S_λ is downward directed (that is, if $u_1, u_2 \in S_\lambda$, then we can find $u \in S_\lambda$ such that $u \le u_1$, $u \le u_2$).

Proposition 4.1 *If hypotheses H_0, H_1, H_2 and H_3 hold and $\lambda \in \mathcal{L} = (0, \lambda^*]$, then problem (P_λ) has a smallest positive solution $u_\lambda^* \in \operatorname{int} C_+$.*

Proof Since S_λ is downward directed, using Lemma 3.10 of [10, p. 178], we can find a decreasing sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq S_\lambda$ such that

$$\inf_{n\in\mathbb{N}} u_n = \inf S_\lambda.$$

We have

$$\begin{aligned} &\langle A(u_n), h\rangle + \int_\Omega \xi(z) u_n^{p-1} h\, dz + \int_{\partial\Omega} \beta(z) u_n^{p-1} h\, d\sigma \\ &= \int_\Omega \big(g(z, u_n) + \lambda f(z, u_n)\big) h,\ dz \quad \forall h \in W^{1,p}(\Omega). \end{aligned} \tag{4.1}$$

Since $0 \le u_n \le u_1$ for all $n \in \mathbb{N}$, choosing $h = u_n \in W^{1,p}(\Omega)$ and using hypotheses $H_1(i)$ and $H_2(ii)$, we obtain that

$$\gamma_p(u_n) \le c_{15} \quad \forall n \in \mathbb{N},$$

for some $c_{15} > 0$, so

$$\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega) \quad \text{is bounded} \tag{4.2}$$

(see (2.1)). From (4.1), we have

$$\begin{cases} -\Delta_p u_n(z) + \xi(z) u(z)_n^{p-1} = g(z, u_n) + \lambda f(z, u_n) & \text{in } \Omega, \\ \frac{\partial u_n}{\partial n_p} + \beta(z) u_n^{p-1} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

Then from (4.2), (4.3) and Proposition 2.10 of [14], we can find $c_{16} > 0$ such that

$$\|u_n\|_\infty \le c_{16} \quad \forall n \in \mathbb{N}.$$

Invoking Theorem 2 of Lieberman [11], we can find $\alpha \in (0, 1)$ and $c_{17} > 0$ such that

$$u_n \in C^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \le c_{17} \quad \forall n \in \mathbb{N}. \tag{4.4}$$

Exploiting the compactness of the embedding $C^{1,\alpha}(\overline{\Omega}) \subseteq C^1(\overline{\Omega})$, we see that at least for a subsequence, we have

$$u_n \longrightarrow u_\lambda^* \quad \text{in } C^1(\overline{\Omega}). \tag{4.5}$$

As in the proof of Proposition 3.6, using hypothesis $H_1(iii)$ and Proposition 2.6, we show that $u_\lambda^* \ne 0$.

Passing to the limit as $n \to +\infty$ in (4.1) and using (4.5), we conclude that

$$u_\lambda^* \in S_\lambda \subseteq \operatorname{int} C_+ \quad \text{and} \quad u_\lambda^* = \inf S_\lambda.$$

$\square$

We consider the minimal solution map $\widehat{m}\colon \mathcal{L} = (0, \lambda^*] \longrightarrow S_\lambda \subseteq \mathrm{int}C_+$ defined by

$$\widehat{m}(\lambda) = u_\lambda^*.$$

We say that $\widehat{m}$ is "strictly increasing", if

$$0 < \mu < \lambda \leq \lambda^* \implies \widehat{m}(\lambda) - \widehat{m}(\mu) \in D_+.$$

Proposition 4.2 *If hypotheses H_0, H_1, H_2 and H_3 hold, then*
(a) *$\widehat{m}$ is strictly increasing;*
(b) *$\widehat{m}$ is right continuous.*

Proof **(a)** Let $0 < \mu < \lambda \leq \lambda^*$. According to Corollary 3.3, we can find $u_\mu \in S_\mu \subseteq \mathrm{int}C_+$, such that

$$u_\lambda^* - u_\mu \in D_+,$$

so

$$u_\lambda^* - u_\mu^* \in D_+$$

(since $u_\mu^* \leq u_\mu$) and thus $\widehat{m}$ is strictly increasing.
(b) Let $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq \mathcal{L}$ and suppose that $\lambda_n \nearrow \lambda$ ($\lambda \in \mathcal{L}$). As before (see the proof of Proposition 4.1), from the nonlinear regularity theory (see Lieberman [11]), we know that we can find $\lambda \in (0, 1)$ and $c_{18} > 0$ such that

$$u_{\lambda_n}^* \in C^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|u_{\lambda_n}^*\|_{C^{1,\alpha}(\overline{\Omega})} \leq c_{18} \quad \forall n \in \mathbb{N}.$$

The compactness of the embedding $C^{1,\alpha}(\overline{\Omega}) \subseteq C^1(\overline{\Omega})$ and part (a) imply that

$$u_{\lambda_n}^* \longrightarrow \widetilde{u}^* \quad \text{in } C^1(\overline{\Omega}). \tag{4.6}$$

We claim that $\widetilde{u}^* = u_\lambda^*$. If this is not true, then we can find $z_0 \in \overline{\Omega}$ such that

$$u_\lambda^*(z_0) < \widetilde{u}^*(z_0),$$

so

$$u_\lambda^*(z_0) < u_{\lambda_n}^*(z_0) \quad \forall n \geq n_0$$

(see (4.6)), which contradicts (a) (recall $\lambda_n < \lambda$ for all $n \in \mathbb{N}$). Therefore $\widehat{m}$ is right continuous. □

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