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Mann–Dotson's algorithm for a countable family of non-self strict pseudo-contractive mappings

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Abstract

The aim of this paper to present some weak and strong convergence results for countable family of non-self mappings. More precisely, we employ the Mann–Dotson's algorithm to approximate common fixed points of a countable family of non-self k-strict Pseudocontractive mappings in q-uniformly smooth Banach spaces.

Keywords Pseudocontractive mapping · Inward condition · Banach space

Mathematics Subject Classification Primary 47H10 · 47H09

1 Introduction

Strict pseudocontractive mappings were introduced by Browder and Petryshyn [1] in 1967. They proved the first convergence result for k-strict pseudocontractive self-mappings in real Hilbert spaces. They proved weak and strong convergence theorems by using the Krasnosel'skiĭ-Mann algorithm with a constant control sequence. This class of mappings properly includes the class of nonexpansive mappings. Iterative algorithms for nonexpansive mappings have received a lot of attention from researchers, on the other hand, Iterative algorithms for strict pseudocontractive mappings are far less developed. Since pseudocontractive mappings have more applications in solving inverse problems than nonexpansive mappings, developing new iterative algorithms for strict pseudocontractive mappings is interesting [2]. Rhoades [3] generalized results presented in [1] and proved fixed point results using Krasnosel'skiĭ-Mann algorithm with a variable control sequence under some conditions. Recently, Marino and Xu

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[4] proved that the sequence generated by Krasnosel'skiĭ-Mann algorithm converges weakly to a fixed point of k-strict pseudocontractive self mapping under some assumptions.

Finding fixed points of nonexpansive mappings and k-strict pseudocontractive by Krasnosel'skiĭ-Mann algorithm [5–9] have been extensively studied in the last few decades for a self-mapping ([10, 11] see also). If the k-strict pseudocontractive mapping is non-self then most of the Krasnoselskii-Mann type algorithms are based on the nearest point projection technique. But in many applications, calculating the nearest point projection is not easy and it may require approximation algorithm by itself, even in the case of Hilbert spaces [12, 13]. To overcome this problem Colao and Marino [12] used inwardness condition on non-self mapping and introduced a new line of research and proved some fixed point results for non-self nonexpansive mapping using Mann–Dotson algorithm. Recently, Colao et al. [14] proved the following convergence results for non-self k-strict pseudocontractive mapping which satisfies inward condition in Hilbert spaces.

Theorem 1.1 *Suppose $\mathcal{E}$ be a closed, strictly convex, and nonempty subset of a given Hilbert space $\mathcal{M}$, $T : \mathcal{E} \to \mathcal{M}$ a k-strict pseudocontractive mapping which satisfies the inward condition with $\emptyset \neq F(T)$. Choose $\varepsilon > 0$ such that $\tilde{k} = k + \varepsilon < 1$. Then sequence $\{x_n\}$ given by*

$$\begin{cases} x_0 \in \mathcal{E}, \\ \alpha_0 = \max\{\tilde{k}, v(x_0)\}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)T(x_n), \\ \alpha_{n+1} = \max\{\alpha_n, v(x_{n+1})\}, \end{cases} \tag{1.1}$$

converges weakly to a fixed point of the mapping T. If $\sum_n (1-\alpha_n) < \infty$, then the convergence is strong.

Marino and Muglia [15] generalized Colao et al. [14] results from Hilbert space to q-uniformly smooth Banach space and proved the following theorem

Theorem 1.2 *Suppose $\mathcal{E}$ be a convex closed and nonempty subset of a given q-uniformly smooth Banach space $\mathcal{X}$, $T : \mathcal{E} \to \mathcal{X}$ a k-strict pseudocontractive mapping which satisfies the inward condition with $\emptyset \neq F(T)$. Then sequence $\{x_n\}$ given by*

$$\begin{cases} \alpha_0 = \min\{k, h_{\mathcal{E},T}(x_0)\}, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T(x_n), \\ \alpha_{n+1} = \min\{\alpha_n, h_{\mathcal{E},T}(x_{n+1})\}, \end{cases} \tag{1.2}$$

(a) converges strongly to a fixed point of the mapping T if $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\mathcal{E}$ is strictly convex (b) converges weakly to a fixed point of the mapping T if $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\mathcal{X}$ is uniformly convex and $\mathcal{E}$ is a nonexpansive retract of $\mathcal{X}$ and (c) if $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\mathcal{X}$ satisfies Opial's condition then the sequence converges weakly to a fixed point of the mapping T.

In recent years, there are many researchers who investigated fixed point results for strict pseudocontractive mappings with different settings and conditions [1, 4, 16–21].

In this paper, using inwardness condition we prove some weak and strong convergence results using Mann–Dotson's algorithm for a countable family of non-self k-strict pseudocontractive mappings. We ensure that Mann–Dotson's algorithm converges (strongly and weakly) to common fixed points under different conditions on countable family of nonself mappings.

2 Preliminaries

Let $(\mathcal{X}, \|\cdot\|)$ be a real Banach space, $\mathcal{E}$ a subset of $\mathcal{X}$. A non-self mapping $T : \mathcal{E} \to \mathcal{X}$ is said to be Lipschitz if for each $x, y \in \mathcal{E}$ there exists a real constant $L \geq 0$, such that

$$\|T(x) - T(y)\| \leq L\|x - y\|.$$

If Lipschitz constant $L = 1$, then the mapping T is said to be a non-self nonexpansive mapping. A point $p \in \mathcal{E}$ is said to be a fixed point of the mapping T if $T(p) = p$. The set of all fixed points for mapping T is denoted by $F(T)$.

Definition 2.1 Suppose J be a normalized duality mapping on a real Banach space $\mathcal{X}$, $J : \mathcal{X} \to 2^{\mathcal{X}^*}$ defined as $J(x) = \{h \in \mathcal{X}^* : \langle x, h\rangle = \|x\|\|h\|, \|h\| = \|x\|\}$, for all $x \in \mathcal{X}$, where $\mathcal{X}^*$ is dual space of $\mathcal{X}$ and the pair $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. If the dual space $\mathcal{X}^*$ is strictly convex, then the normalized duality J is single valued, and the single valued duality mapping is denoted by j.

1. A mapping $T : D(T) \subset \mathcal{X} \to \mathcal{X}$ is said to be pseudocontractive, if for all $x, y \in D(T)$, there is $j(x-y) \in J(x-y)$ such that

$$\langle T(x) - T(y), j(x-y)\rangle \leq \|x-y\|^2. \tag{2.1}$$

2. A mapping $T : D(T) \subset \mathcal{X} \to \mathcal{X}$ is said to be k-strict pseudocontractive in the Browder-Petryshyn sense, if there exists $k \in (0, 1)$ such that for each $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle T(x) - T(y), j(x-y)\rangle \leq \|x-y\|^2 - k\|x - y - (T(x) - T(y))\|^2. \tag{2.2}$$

Remark 2.2 It is also easy to get that every k-strict pseudocontractive mapping is L-Lipschitz mapping for $L = \frac{k+1}{k}$ [22].

Now we will recollect some basic concepts, definitions, and facts from the literature.

- We use $\rightharpoonup$ to denote weak convergence, $\to$ to denote strong convergence;
- $\omega_w(x_n)$ represents cluster points (ω-limit) set of a sequence $\{x_n\}$, that is, $\omega_w(x_n) := \{x : \exists\, x_{n_k} \rightharpoonup x\}$.

Lemma 2.3 [23]. *Let us assume that* $\{p_n\}$, $\{q_n\}$ *and* $\{r_n\}$ *be any sequences of positive real numbers in such a way that*

$$p_{n+1} \leq (1 + q_n)p_n + r_n \text{ for all } n \in \mathbb{N}.$$

If $\sum_{n=1}^{\infty} q_n < \infty$ *and* $\sum_{n=1}^{\infty} r_n < \infty$. *Then the limit* $\lim\limits_{n\to\infty} p_n$ *exists* [24].

Definition 2.4 [10]. A subset $\mathcal{E}$ of $\mathcal{X}$ is said to be strictly convex if it is convex as well as it satisfies the property

$$\delta x + (1-\delta)y \in \text{interior}(\mathcal{E})$$

for all $x, y \in \partial\mathcal{E}$ and $\delta \in (0, 1)$. Otherwise, there are no segments lies within the boundary $\partial\mathcal{E}$.

Definition 2.5 [10]. A Banach space $\mathcal{X}$ is said to be uniformly convex if for every $0 < \varepsilon \leq 2$ there exists a $\delta > 0$ in such a way that, for each $x, y \in \mathcal{X}$ satisfying $\|x\| = \|y\| = 1$, the condition $\|x - y\| \geq \varepsilon$ implies $\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$.

Definition 2.6 [25]. A Banach space $\mathcal{X}$ is said to satisfy the Opial condition if, for any given sequence $\{x_n\}$ which converges weakly and have the weak limit $x \in \mathcal{X}$ satisfies:

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

for all $y \in \mathcal{X}$ with $x \neq y$.

All the ℓ^p $(1 \leq p < \infty)$ spaces, all Hilbert spaces and all finite dimensional Banach spaces satisfy Opial condition. A Banach space having a weakly sequentially continuous duality mapping also satisfies the Opial condition. But L_p $(0 < p < \infty,\ p \neq 2)$ spaces do not satisfy Opial condition [10].

Definition 2.7 [13]. The norm defined on a given Banach space $\mathcal{X}$ is said to be Fréchet differentiable at any given unit vector x if for each y with $\|y\| = 1$, the limit

$$\lim_{\lambda\to 0} \frac{\|x + \lambda y\| - \|x\|}{\lambda} \tag{2.3}$$

exists uniformly. The norm defined on a Banach space $\mathcal{X}$ is said to be Fréchet differentiable if it is Fréchet differentiable at the each unit vector $x \in \mathcal{X}$.

In this case, for all bounded $u, v \in \mathcal{X}$ the following inequality holds (see [23]):

$$\frac{1}{2}\|u\|^2 + \langle v, j(u)\rangle \leq \frac{1}{2}\|u + v\|^2 \leq \frac{1}{2}\|u\|^2 + \langle v, j(u)\rangle + g(\|v\|) \tag{2.4}$$

where $j(\cdot) : \mathcal{X} \to \mathcal{X}^*$ is the normalized duality mapping of $\mathcal{X}$ and $g : [0, \infty) \to \mathbb{R}$ is a mapping in such a way that $\lim_{t\to 0} \frac{g(t)}{t} = 0$ (or $g(t) = o(t)$ as $t \to 0$).

Definition 2.8 [13] A Banach space $\mathcal{X}$ is said to be uniformly smooth if the limit defined in equation (2.3) is exists uniformly in x too.

Remark 2.9 [26] If $\mathcal{X}$ is a q-uniformly smooth Banach space, there exists a constant $C_q > 0$ then following holds

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + C_q\|y\|^q$$

for each $x, y \in \mathcal{X}$, where

$$j_q(x) = \{x^* \in \mathcal{X}^* : \langle x, x^*\rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\}.$$

Definition 2.10 [13] Suppose the space $\mathcal{X}$ is q-uniformly smooth with $(q \in (1, 2])$. The mapping $T : D(T) \subset \mathcal{X} \to \mathcal{X}$ is said to be k-strict pseudocontractive mapping with $k \in (0, 1)$ if for each $x, y \in D(T)$

$$\langle T(x) - T(y), j_q(x - y)\rangle \leq \|x - y\|^q - k\|(x - y) - (T(x) - T(y))\|^q.$$

Lemma 2.11 [16] *Suppose $\mathcal{X}$ be a real q-uniformly smooth Banach space and $\mathcal{E}$ a convex and nonempty subset of $\mathcal{X}$. Suppose $T : \mathcal{E} \to \mathcal{X}$ be a k-strict pseudocontractive mapping. Let $\mu \in \left(0, \min\left\{\frac{qk^{\frac{1}{k-1}}}{C_q}, 1\right\}\right)$. Then the mapping $S = \mu I + (1 - \mu)T$ is nonexpansive. If $F(T) \neq \emptyset$ then $F(S) = F(T)$.*

Lemma 2.12 [27]. *Suppose $\mathcal{X}$ be uniformly convex Banach space, $\{x_n\}$, $\{y_n\} \in \mathcal{X}$ are two sequences in such a way that $\limsup_{n\to\infty} \|x_n\| \leq \delta$, $\limsup_{n\to\infty} \|y_n\| \leq \delta$, and $\lim_{n\to\infty} \|\alpha_n x_n + (1 - \alpha_n)y_n\| = \delta$, where $\{\alpha_n\} \subseteq [a, b] \subset [0, 1]$ and $\delta \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$ (see also [28]).*

Lemma 2.13 [26]. *Suppose $\mathcal{X}$ be a given uniformly convex Banach space, $\mathcal{B}_s := \{x \in \mathcal{X} : \|x\| \leq s\}$, for any $s > 0$. Then there is a strictly increasing convex continuous function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$ such that*

$$\|\mu x + (1-\mu)y\|^2 \leq \mu\|x\|^2 + (1-\mu)\|y\|^2 - \mu(1-\mu)\psi(\|x-y\|) \tag{2.5}$$

for each $x, y \in \mathcal{B}_s$ and $\mu \in [0, 1]$.

Lemma 2.14 [29]. *Suppose $\mathcal{E}$ be a convex, closed, bounded and nonempty subset of a given uniformly rotund Banach space $\mathcal{X}$. Then there is a strictly increasing continuous convex function $\tau : \mathbb{R}^+ \to \mathbb{R}^+$ with $\tau(0) = 0$ in such a way that for every contraction mapping $T : \mathcal{E} \to \mathcal{X}$, for each $x, y \in \mathcal{E}$ and $\beta \in [0, 1]$, following holds:*

$$\tau(\|\beta T(x) + (1-\beta)T(y) - T\{\beta x + (1-\beta)y\}\|) \leq \|x-y\| - \|T(x) - T(y)\|.$$

Lemma 2.15 (*Demiclosedness principle*) [30]. *Suppose $\mathcal{X}$ be a given uniformly convex Banach space, $\mathcal{E}$ a convex closed and nonempty subset of $\mathcal{X}$, $T : \mathcal{E} \to \mathcal{X}$ a nonexpansive mapping. Suppose $\{x_n\}$ is a given sequence in $\mathcal{X}$ in such a way that $\{x_n\}$ converges weakly to x, $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$. Then $T(x) = x$. That is, $I - T$ is demiclosed at zero.*

Definition 2.16 [31]. Let $\mathcal{E} \neq \emptyset$ be a subset of a Hilbert space $\mathcal{H}$ and $\{T_n\}$ a family of mappings from $\mathcal{E}$ into $\mathcal{H}$ with $\mathcal{F} = \cap_{n=1}^{\infty} F(T_n) \neq \emptyset$. The family of mappings $\{T_n\}$ is said to be uniformly weakly closed if for any convergent sequence $\{x_n\} \subset \mathcal{E}$ such that

$$\lim_{n\to\infty} \|x_n - T_n(x_n)\| = 0 \text{ implies } \omega_w\{x_n\} \subset \cap_{n=1}^{\infty} F(T_n).$$

The following definitions are useful in dealing with countable family of mappings. Suppose $\{T_n\}$, $\mathfrak{T}$ are two families of non-self mappings from $\mathcal{E}$ into $\mathcal{X}$ with $\emptyset \neq F(\mathfrak{T}) = \cap_{n=1}^{\infty} F(T_n)$, where $F(T_n)$ denotes the set of all fixed points of mappings T_n, $F(\mathfrak{T})$ is the set of all common fixed points of all mappings in $\mathfrak{T}$.

(i) The family of mappings $\{T_n\}$ satisfies AKTT-condition (I) if for every bounded subset $\mathcal{B}$ of $\mathcal{E}$, $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}(x) - T_n(x)\| : x \in \mathcal{B}\} < \infty$ [32].

(ii) The family of mappings $\{T_n\}$ satisfies AKTT-condition (II) if for every bounded subset $\mathcal{B}$ of $\mathcal{E}$, and every increasing sequence $\{n_i\}$ of natural numbers $\exists$ a mapping $T : \mathcal{E} \to \mathcal{X}$ along with $I - T$ is demiclosed at 0 and a subsequence $\{n_{i_j}\}$ of $\{n_i\}$ in such a way that [33]

$$\lim_{j\to\infty} \sup\{\|T_{n_{i_j}}(x) - T(x)\| : x \in \mathcal{B}\} = 0, \cap_{n=1}^{\infty} F(T_n) = F(T).$$

(iii) The family of mappings $\{T_n\}$ satisfies NST-condition if for every bounded sequence $\{x_n\}$ in $\mathcal{E}$, [34]

$$\lim_{n\to\infty} \|x_n - T_n(x_n)\| = 0 \implies \omega_w\{x_n\} \subset \cap_{n=1}^{\infty} F(T_n).$$

(iv) The family of mappings $\{T_n\}$ satisfies NST-condition (I) along with $\mathfrak{T}$ if for every given bounded sequence $\{x_n\}$ in $\mathcal{E}$

$$\lim_{n\to\infty} \|x_n - T_n(x_n)\| = 0 \implies \lim_{n\to\infty} \|x_n - T(x_n)\| = 0$$

for each $T \in \mathfrak{T}$ [35].

(v) The family of mappings $\{T_n\}$ is said to satisfy NST-condition (II) if for every given bounded sequence $\{x_n\}$ in $\mathcal{E}$

$$\lim_{n\to\infty} \|x_{n+1} - T_n(x_n)\| = 0 \text{ implies } \lim_{n\to\infty} \|x_n - T_m(x_n)\| = 0$$

for each $m \in \mathbb{N}$ [35].

Motivated by the above conditions we consider a new type of following condition:

Definition 2.17 The family of mappings $\{T_n\}$ is said to satisfy AKTT*-condition with $\mathfrak{T}$ if the family of mappings satisfies AKTT-condition (I) and NST-condition (I) both.

Remark 2.18 It can be seen that if $\{T_n\}$ is weakly closed then the family of mappings $\{T_n\}$ satisfies NST-condition.

Lemma 2.19 [32]. *Suppose $\mathcal{X}$ be a given Banach space, $\mathcal{E}$ a closed, nonempty subset of $\mathcal{X}$. Suppose the family of mappings $\{T_n\} : \mathcal{E} \to \mathcal{X}$ satisfies AKTT-condition* (I). *Then, $\forall x \in \mathcal{E}$, $\{T_n(x)\}$ converges strongly to some point of $\mathcal{X}$. Further, suppose $T : \mathcal{E} \to \mathcal{X}$ be a mapping defined for $x \in \mathcal{E}$*

$$T(x) = \lim_{n\to\infty} T_n(x). \tag{2.6}$$

Then, $\lim_{n\to\infty} \sup\{\|T(x) - T_n(x)\| : x \in \mathcal{B}\} = 0$ for any bounded subset $\mathcal{B}$ of $\mathcal{E}$. In particular, if $I - T$ is demiclosed at 0, $\cap_{n=1}^{\infty} F(T_n) = F(T) \neq \emptyset$, then the given family of mappings $\{T_n\}$ satisfies AKTT-condition (II).

Definition 2.20 [36]. The mapping $T : \mathcal{E} \to \mathcal{X}$ is called a inward mapping if $\forall x \in \mathcal{E}$, we have

$$T(x) \in I_{\mathcal{E}}(x) := \{x + c(y - x) : c \geq 1 \text{ and } y \in \mathcal{E}\}.$$

Definition 2.21 [36]. The mapping $T : \mathcal{E} \to \mathcal{X}$ is called weakly inward if $\forall x \in \mathcal{E}$, we have

$$T(x) \in \overline{I_{\mathcal{E}}(x)}, \text{ where } \overline{I_{\mathcal{E}}(x)} \text{ is the closure of } I_{\mathcal{E}}(x). \tag{2.7}$$

For more details and properties of weakly inward mappings, one may refer to [37, 38].

Definition 2.22 [13]. Suppose $\mathcal{X}$ be a Banach space and $\mathcal{E}$ a convex, closed and nonempty subset of $\mathcal{X}$ and $T : \mathcal{E} \to \mathcal{X}$ a mapping, define a function $h_{\mathcal{E},T} : \mathcal{E} \to \mathbb{R}$ as

$$h_{\mathcal{E},T}(x) := \inf\{\Gamma \geq 0 : \Gamma x + (1 - \Gamma)T(x) \in \mathcal{E}\}. \tag{2.8}$$

Lemma 2.23 [13]. *Suppose $\mathcal{E}$ be a convex closed and nonempty subset of a given Banach space $\mathcal{X}$. Suppose $T : \mathcal{E} \to \mathcal{X}$ be a mapping and $h_{\mathcal{E},T} : \mathcal{E} \to \mathbb{R}$ a function defined in* (2.8). *Then the following properties hold:*

(Z1) $\forall\, x \in \mathcal{E}$ *and* $\forall\, \alpha \in [h_{\mathcal{E},T}(x), 1]$, $\alpha x + (1 - \alpha)T(x) \in \mathcal{E}$;
(Z2) $\forall\, x \in \mathcal{E}$ *and* $\forall\, \beta \in [0, h_{\mathcal{E},T}(x))$, $\beta x + (1 - \beta)T(x) \notin \mathcal{E}$;
(Z3) $\forall\, x \in \mathcal{E}$, $T(x) \in \mathcal{E}$ *if and only if* $h_{\mathcal{E},T}(x) = 0$;
(Z4) *If* $T(x) \notin \mathcal{E}$, *then* $h_{\mathcal{E},T}(x)x + (1 - h_{\mathcal{E},T}(x))T(x) \in \partial\mathcal{E}$.

Lemma 2.24 [39]. *Suppose $\mathcal{E}$ be a convex, closed and nonempty subset of a given Banach space $\mathcal{X}$ and $T : \mathcal{E} \to \mathcal{X}$ a mapping which is weakly inward. Then $h_{\mathcal{E},T}(x) < 1$ for all $x \in \mathcal{E}$.*

Definition 2.25 [40]. The mapping $T : \mathcal{E} \to \mathcal{X}$ is said to be semicompact if for each bounded sequence $\{x_n\} \in \mathcal{X}$ such that $x_n - T(x_n) \to y$ for some $y \in \mathcal{X}$, there exists a convergent subsequence.

Definition 2.26 [41]. The mapping $T : \mathcal{E} \to \mathcal{X}$ is said to be closed if a sequence $\{x_n\} \in \mathcal{E}$ satisfying $x_n \to x$ and $T(x_n) \to y$ then $x \in \mathcal{E}$ and $T(x) = y$.

Definition 2.27 [42] The mapping $T : \mathcal{E} \to \mathcal{X}$ with $F(T) \neq \emptyset$ satisfies condition (I) if $\exists$ a nondecreasing function $f : [0, \infty) \to [0, \infty)$ along with $f(r) > 0$, $f(0) = 0\ \forall\, r \in (0, \infty)$ in such a way that $f(d(x, F(T))) \leq \|x - T(x)\|\ \forall x \in \mathcal{E}$, where $d(x, F(T)) = \inf\{\|x - y\| : y \in F(T)\}$.

Definition 2.28 [41]. Any finite family of mappings $\mathfrak{T} : \mathcal{E} \to \mathcal{X}$ along with $\emptyset \neq F(\mathfrak{T})$ is said to satisfy condition (II) if $\exists$ a nondecreasing function $f : [0, \infty) \to [0, \infty)$ along with $f(r) > 0$, $f(0) = 0\ \forall r \in (0, \infty)$, in such a way that

$$\max\{\|x - T(x)\| : T \in \mathfrak{T}\} \geq f(d(x, F(\mathfrak{T})))\ \ \forall x \in \mathcal{E}.$$

Lemma 2.29 [16] *Suppose $\mathcal{E}$ be a convex and nonempty subset of a real q-uniformly smooth Banach space (in short q-USBS) $\mathcal{X}$. Suppose $C_q > 0$ and T be a k-strict pseudocontractive mapping. Suppose $\mu \in \left(0, \min\left\{\left(\frac{kq}{C_q}\right)^{\frac{1}{q-1}}, 1\right\}\right)$ then*

$$T_\mu = (1 - \mu)I + \mu T$$

is a nonexpansive mapping. If $F(T) \neq \emptyset$ then $F(T) = F(T_\mu)$.

Proposition 2.30 [15] *Suppose $\mathcal{E}$ be nonempty subset of a space $\mathcal{X}$ with $F(T) \neq \emptyset$. If the mapping T is inward, then for all $\mu \in (0, 1)$ the average mapping $T_\mu = (1 - \mu)I + \mu T$ is inward too.*

3 Weak convergence results

Suppose $\mathcal{E}$ be a convex closed and nonempty subset of a given q-uniformly smooth Banach space $\mathcal{X}$, $\{T_n\}$ a family of k_n-strict pseudocontractive weakly inward mappings from $\mathcal{E}$ into $\mathcal{X}$. Assume that $x_1 \in \mathcal{E}$, we can generate a sequence $\{x_n\}$ in $\mathcal{E}$ as follows:

$$\begin{cases} \alpha_1 = \max\{\frac{1}{2}, h_{\mathcal{E},T_{\mu_1}}(x_1)\} \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T_{\mu_n}(x_n) \\ \alpha_{n+1} = \max\{\alpha_n, h_{\mathcal{E},T_{\mu_{n+1}}}(x_{n+1})\}. \end{cases} \tag{3.1}$$

From Lemma 2.24, for any weakly inward mapping $T_\mu : \mathcal{E} \to \mathcal{X}$, we have $h_{\mathcal{E},T_\mu}(x) < 1$ $\forall x \in \mathcal{E}$. Now it can be seen that for each $n \in \mathbb{N}$, $\alpha_{n+1} \in [h_{\mathcal{E},T_{\mu_{n+1}}}(x_{n+1}), 1]$. Thus from Lemma 2.23 (Z1), we get

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T_{\mu_n}(x_n) \in \mathcal{E}.$$

Hence the algorithm (3.1) is well defined.

Now, we present some important lemmas which can be utilized to prove the main convergence results.

Lemma 3.1 *Suppose $\mathcal{E}$ be a convex closed and nonempty subset of a q-uniformly smooth Banach space (in short q-USBS) $\mathcal{X}$. Suppose $\{T_n\}$ be any given family of k_n-strict pseudo-contractive weakly inward mappings from $\mathcal{E}$ into $\mathcal{X}$ with $\emptyset \neq \cap_{n=1}^{\infty} F(T_n)$. Suppose $\{x_n\}$ be a sequence given by (3.1). Then for each $p \in \cap_{n=1}^{\infty} F(T_n)$, $\lim_{n\to\infty} \|x_n - p\|$ exists.*

Proof Suppose $p \in \cap_{n=1}^{\infty} F(T_n)$. From (3.1) $\forall n \in \mathbb{N}$, we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n x_n + (1-\alpha_n)T_{\mu_n}(x_n) - p\| \\ &\le \alpha_n\|x_n - p\| + (1-\alpha_n)\|T_{\mu_n}(x_n) - p\| \\ &= \alpha_n\|x_n - p\| + (1-\alpha_n)\|T_{\mu_n}(x_n) - T_{\mu_n}(p)\| \\ &\le \alpha_n\|x_n - p\| + (1-\alpha_n)\|x_n - p\| \\ &\le \|x_n - p\|. \end{aligned} \tag{3.2}$$

Hence we get the required conclusion.

Lemma 3.2 *Suppose $\mathcal{E}$ be a convex closed and nonempty subset of a given q-USBS $\mathcal{X}$. Suppose $\{T_n\}$ be a family of k_n-strict pseudocontractive weakly inward mappings from $\mathcal{E}$ into $\mathcal{X}$ with $\emptyset \neq \cap_{n=1}^{\infty} F(T_n)$. Suppose $\{x_n\}$ be a sequence given by (3.1). Then there exists a strictly increasing convex continuous function $\psi : [0,\infty) \to [0,\infty)$ along with $\psi(0) = 0$ such that $\sum_{n=1}^{\infty}(1-\alpha_n)\psi(\|x_n - T_{\mu_n}(x_n)\|) < \infty$.*

Proof By Lemma 3.1, both the sequences $\{x_n - p\}$ and $\{T_{\mu_n}(x_n) - p\}$ are bounded, so these are contained in $\mathcal{B}_s := \{x \in \mathcal{X} : \|x\| \le s\}$ for sufficiently large $s > 0$. In view of Lemma 2.13, there is a strictly increasing convex continuous function $\psi : [0,\infty) \to [0,\infty)$ along with $\psi(0) = 0$ such that (2.5) holds. Thus, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_n x_n + (1-\alpha_n)T_{\mu_n}(x_n) - p\|^2 \\ &= \|\alpha_n(x_n - p) + (1-\alpha_n)(T_{\mu_n}(x_n) - p)\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|T_{\mu_n}(x_n) - p\|^2 - \alpha_n(1-\alpha_n)\psi(\|x_n - T_{\mu_n}(x_n)\|) \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - \alpha_n(1-\alpha_n)\psi(\|x_n - T_{\mu_n}(x_n)\|) \\ &= \|x_n - p\|^2 - \alpha_n(1-\alpha_n)\psi(\|x_n - T_{\mu_n}(x_n)\|).\end{aligned}$$

So,

$$\alpha_n(1-\alpha_n)\psi(\|x_n - T_{\mu_n}(x_n)\|) \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2. \tag{3.3}$$

Since $\alpha_n \in \left[\frac{1}{2}, 1\right]$,

$$\frac{1}{2}(1-\alpha_n)\psi(\|x_n - T_{\mu_n}(x_n)\|) \le \alpha_n(1-\alpha_n)\psi(\|x_n - T_{\mu_n}(x_n)\|).$$

Therefore, from (3.3) and using the hypothesis of the Theorem, we get the desired conclusion.

Lemma 3.3 *Suppose $\mathcal{E}$ be a convex closed and nonempty subset of a given q-USBS $\mathcal{X}$. Suppose $\{T_n\}$ be a family of k_n-strict pseudocontractive weakly inward mappings from $\mathcal{E}$ into $\mathcal{X}$ with $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Suppose $\mathcal{E}$ is a nonexpansive retract of $\mathcal{X}$ and $\{x_n\}$ be a sequence given by (3.1). Then for any $\beta \in [0,1]$, $p, q \in \cap_{n=1}^{\infty} F(T_n)$, the limit $\lim_{n\to\infty}\|\beta x_n + (1-\beta)p - q\|$ exists.*

Proof In view of Lemma 3.1, the given sequence $\{x_n\}$ is bounded. Let $p, q \in \cap_{n=1}^{\infty} F(T_n)$ and set

$$\zeta_n(\beta) := \|\beta x_n + (1-\beta)p - q\|.$$

Then $\lim_{n\to\infty}\zeta_n(0) = \|p - q\|$. By Lemma 3.1, $\lim_{n\to\infty}\zeta_n(1) = \|x_n - q\|$ exists. Now it only remains to check for the case $\beta \in (0,1)$. Since, we compute the sequence $\{\alpha_n\}$ by (3.1), we have

$$\begin{cases}\alpha_1 = \max\{\frac{1}{2}, h_{\mathcal{E},T_{\mu_1}}(x_1)\} \\ \alpha_{n+1} = \max\{\alpha_n, h_{\mathcal{E},T_{\mu_{n+1}}}(x_{n+1})\},\end{cases}$$

which depends on the initial point x_1. Now, we define a mapping $S_{n,x_1} : \mathcal{E} \to \mathcal{X}$ by

$$S_{n,x_1} := \alpha_n I + (1-\alpha_n)T_n \text{ for all } n \in \mathbb{N}.$$

Then using Lemma 2.11 we can say S_{n,x_1} is nonexpansive. Moreover, $x_{n+1} = S_{n,x_1}(x_n)$ and $\cap_{n=1}^{\infty} F(T_n) \subseteq \cap_{n=1}^{\infty} F(S_{n,x_1})$. Let $V_{n,m} : \mathcal{E} \to \mathcal{X}$ be the mapping defined as

$$V_{n,m} = S_{n+m-1,x_1} R_{\mathcal{E}} S_{n+m-2,x_1} R_{\mathcal{E}} ... S_{n+1,x_1} R_{\mathcal{E}} S_{n,x_1} R_{\mathcal{E}}.$$

Where $R_{\mathcal{E}} : \mathcal{X} \to \mathcal{E}$ is the nonexpansive retraction. Since a retraction does not move point into $\mathcal{E}$, it can be seen that $x_{n+m} = V_{n,m}(x_n)$ and $\cap_{n=1}^{\infty} F(T_n) \subseteq \cap_{n=1}^{\infty} F(V_{n,m})$. Moreover,

$$\|V_{n,m}(x) - V_{n,m}(y)\| \le \|x - y\| \text{ for all } x, y \in \mathcal{E}.$$

Set

$$\xi_{n,m}(\beta) := \|\beta V_{n,m}(x_n) + (1-\beta)p - V_{n,m}\{\beta x_n + (1-\beta)p\}\|.$$

Now, by Lemma 2.14, ∃ a strictly increasing convex continuous function $\tau : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\tau(0) = 0$ such that

$$\begin{aligned}\tau\left(\xi_{n,m}(\beta)\right) &= \tau\left(\|\beta V_{n,m}(x_n) + (1-\beta)p - V_{n,m}\{\beta x_n + (1-\beta)p\}\|\right)\\ &\le \|x_n - p\| - \|V_{n,m}(x_n) - V_{n,m}(p)\|\\ &= \|x_n - p\| - \|x_{n+m} - p\|.\end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - p\|$ exists and hence last difference is zero. Therefore $\lim_{n,m\to\infty} \tau\left(\xi_{n,m}(\beta)\right) = 0$ and $\lim_{n,m\to\infty} \xi_{n,m}(\beta) = 0$. Now, we have

$$\begin{aligned}\zeta_{n+m}(\beta) &= \|\beta x_{n+m} + (1-\beta)p - q\|\\ &= \|\beta V_{n,m}(x_n) + (1-\beta)p - q\|\\ &\le \xi_{n,m}(\beta) + \|V_{n,m}\{\beta x_n + (1-\beta)p\} - q\|\\ &\le \xi_{n,m}(\beta) + \|\beta x_n + (1-\beta)p - q\|\\ &\le \xi_{n,m}(\beta) + \zeta_n(\beta).\end{aligned}$$

Hence

$$\begin{aligned}\limsup_{n\to\infty} \zeta_n(\beta) &\le \lim_{n\to\infty} \xi_{n,m}(\beta) + \liminf_{n\to\infty} \zeta_n(\beta)\\ &\le \liminf_{n\to\infty} \zeta_n(\beta).\end{aligned}$$

That is, there exists $\lim_{n\to\infty} \|\beta x_n + (1-\beta)p - q\|$ $\forall \beta \in (0, 1)$. Hence the proof is complete.

Lemma 3.4 *Suppose $\mathcal{E}$ be a convex closed and nonempty subset of a given q-USBS $\mathcal{X}$. Suppose $\{T_n\}$ be a family of k_n-strict pseudocontractive weakly inward mappings from $\mathcal{E}$ into $\mathcal{X}$ with $\emptyset \neq \cap_{n=1}^{\infty} F(T_n)$. Assume that $\{x_n\}$ is a sequence given by (3.1) then $\omega_w(x_n)$ is a singleton set.*

Proof Since $\{x_n\}$ is a bounded sequence in $\mathcal{X}$, $\lim_{n\to\infty} \|\beta x_n + (1-\beta)p - q\|$ exists $\forall \beta \in [0, 1]$, $p, q \in \cap_{n=1}^{\infty} F(T_n)$ and the norm defined on $\mathcal{X}$ is Fréchet differentiable, by (2.4), we have

$$\begin{aligned}\frac{1}{2}\|p-q\|^2 + \beta\langle x_n - p, j(p-q)\rangle &\le \frac{1}{2}\|\beta x_n + (1-\beta)p - q\|^2\\ &\le \frac{1}{2}\|p-q\|^2 + \beta\langle x_n - p, j(p-q)\rangle\\ &\quad + g(\beta\|x_n - p\|).\end{aligned}$$

Since the middle term admits limit, $\lim_{n\to\infty}\|x_n - p\|$ exists. Thus

$$\begin{aligned}\frac{1}{2}\|p-q\|^2+\beta\limsup_{n\to\infty}\langle x_n-p, j(p-q)\rangle &\le \frac{1}{2}\lim_{n\to\infty}\|\beta x_n+(1-\beta)p-q\|^2\\ &\le \frac{1}{2}\|p-q\|^2+\beta\liminf_{n\to\infty}\langle x_n-p, j(p-q)\rangle\\ &\quad +o(\beta).\end{aligned}$$

Therefore, we have

$$\limsup_{n\to\infty}\langle x_n-p, j(p-q)\rangle \le \liminf_{n\to\infty}\langle x_n-p, j(p-q)\rangle + \frac{o(\beta)}{\beta},$$

if $\beta \to 0$, $\lim_{n\to\infty}\langle x_n-p, j(p-q)\rangle$ exists $\forall p, q \in \cap_{n=1}^{\infty} F(T_n)$. Now, we prove that the set $\omega_w(x_n)$ is singleton. For this, assume $\tilde{p}, \tilde{q} \in \omega_w(x_n)$ and there exist two subsequences $\{x_{n_i}\}$, $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \tilde{p}$ and $x_{n_j} \rightharpoonup \tilde{q}$, thus

$$\begin{aligned}\langle \tilde{p}-p, j(p-q)\rangle &= \lim_{i\to\infty}\langle x_{n_i}-p, j(p-q)\rangle\\ &= \lim_{j\to\infty}\langle x_{n_j}-p, j(p-q)\rangle\\ &= \langle \tilde{q}-p, j(p-q)\rangle.\end{aligned}$$

That is,

$$\langle \tilde{p}-\tilde{q}, j(p-q)\rangle = 0$$

for all $p, q \in \cap_{n=1}^{\infty} F(T_n)$. Hence $\tilde{p}=\tilde{q}$ and $\omega_w(x_n)$ is a singleton set.

Theorem 3.5 *Suppose $\mathcal{E}$ be a convex closed and nonempty subset of a given q-USBS $\mathcal{X}$. Suppose $\mathcal{E}$ is a nonexpansive retract of $\mathcal{X}$. Suppose $\{T_n\}$ be a family of k_n-strict pseudocontractive weakly inward mappings from $\mathcal{E}$ into $\mathcal{X}$ with $\emptyset \ne \cap_{n=1}^{\infty} F(T_n)$. If $\{T_{\mu_n}\}$ satisfies NST-condition, then the sequence $\{x_n\}$ defined by (3.1) converges weakly to a point in $\cap_{n=1}^{\infty} F(T_n)$, provided $\{\alpha_n\} \subseteq [a, b] \subset (0, 1)$.*

Proof In view of Lemma 3.3, the limit $\lim_{n\to\infty}\|\beta x_n + (1-\beta)p - q\|$ exists $\forall\beta \in [0, 1]$, $p, q \in \cap_{n=1}^{\infty} F(T_n)$. Using Lemma 3.1 sequence $\{x_n\}$ is bounded, so $\exists$ a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to some $x^* \in \omega_w\{x_n\} \subset \mathcal{E}$. By Lemma 3.4, the set $\omega_w\{x_n\}$ is singleton. Thus $\{x_n\}$ converges weakly to $x^* \in \omega_w\{x_n\}$. Since $\{\alpha_n\} \subseteq [a, b] \subset (0, 1)$, by Lemma 3.2, we get $\sum_{n=1}^{\infty}\psi(\|x_n - T_{\mu_n}(x_n)\|) < \infty$. This implies that $\lim_{n\to\infty}\psi(\|x_n - T_{\mu_n}(x_n)\|) = 0$, $\lim_{n\to\infty}\|x_n - T_{\mu_n}(x_n)\| = 0$. By the assumption that $\{T_{\mu_n}\}$ satisfies NST-condition, hence we have $\omega_w\{x_n\} \subset \cap_{n=1}^{\infty} F(T_{\mu_n})$ and hence $\omega_w\{x_n\} \subset \cap_{n=1}^{\infty} F(T_n)$.

Theorem 3.6 *Suppose $\mathcal{E}$ be a convex closed and nonempty subset of a given q-USBS $\mathcal{X}$. Suppose $\mathcal{E}$ is a nonexpansive retract of $\mathcal{X}$, $\{T_n\}$ be a family of k_n-strict pseudocontractive weakly inward mappings from $\mathcal{E}$ into $\mathcal{X}$ with $\emptyset \ne \cap_{n=1}^{\infty} F(T_n)$. If $\{T_{\mu_n}\}$ satisfies AKTT-condition* (I)*, mapping T_μ is defined by* (2.6) *and $F(T_\mu) = \cap_{n=1}^{\infty} F(T_{\mu_n})$, then sequence $\{x_n\}$ given by* (3.1) *converges weakly to a point in $\cap_{n=1}^{\infty} F(T_n)$, provided $\{\alpha_n\} \subseteq [0, 1]$ with $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$.*

Proof First, we prove that $\lim_{n\to\infty} \|x_n - T(x_n)\|$ exists. Since $\{T_{\mu_n}\}$ satisfies AKTT-condition (I) and sequence $\{x_n\}$ is bounded, we have

$$\sum_{n=1}^{\infty} \sup\{\|T_{\mu_{n+1}}(x) - T_{\mu_n}(x)\| : x \in \{x_n\}\} < \infty. \tag{3.4}$$

Moreover, by (3.1) and the triangle inequality, we get

$$\begin{aligned}
\|x_{n+1} - T_{\mu_{n+1}}(x_{n+1})\| &= \|\alpha_n x_n + (1-\alpha_n)T_{\mu_n}(x_n) - T_{\mu_{n+1}}(x_{n+1})\| \\
&\leq \alpha_n\|x_n - T_{\mu_n}(x_n)\| + \|T_{\mu_n}(x_n) - T_{\mu_{n+1}}(x_{n+1})\| \\
&\leq \alpha_n\|x_n - T_{\mu_n}(x_n)\| + \|T_{\mu_n}(x_n) - T_{\mu_n}(x_{n+1})\| \\
&\quad + \|T_{\mu_n}(x_{n+1}) - T_{\mu_{n+1}}(x_{n+1})\| \\
&\leq \alpha_n\|x_n - T_{\mu_n}(x_n)\| + \|x_n - x_{n+1}\| + \|T_{\mu_n}(x_{n+1}) - T_{\mu_{n+1}}(x_{n+1})\| \\
&= \alpha_n\|x_n - T_{\mu_n}(x_n)\| + (1-\alpha_n)\|x_n - T_{\mu_n}(x_n)\| \\
&\quad + \|T_{\mu_n}(x_{n+1}) - T_{\mu_{n+1}}(x_{n+1})\| \\
&= \|x_n - T_{\mu_n}(x_n)\| + \|T_{\mu_n}(x_{n+1}) - T_{\mu_{n+1}}(x_{n+1})\| \\
&\leq \|x_n - T_{\mu_n}(x_n)\| + \sup\{\|T_{\mu_n}(x) - T_{\mu_{n+1}}(x)\| : x \in \{x_n\}\}.
\end{aligned}$$

Now, using (3.4) and the Lemma (2.3), $\lim_{n\to\infty} \|x_n - T_{\mu_n}(x_n)\|$ exists. Since $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$ and using Lemma 3.2, we have $\liminf_{n\to\infty} \psi(\|x_n - T_{\mu_n}(x_n)\|) = 0$. This implies that $\liminf_{n\to\infty} \|x_n - T_{\mu_n}(x_n)\| = 0$. Hence $\lim_{n\to\infty} \|x_n - T_{\mu_n}(x_n)\| = 0$. Now, we prove $\omega_w\{x_n\} \subset \cap_{n=1}^{\infty} F(T_n)$. Let $p \in \omega_w\{x_n\}$, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ in such a way that $x_{n_k} \rightharpoonup p$. The mapping T_μ is nonexpansive and $I - T_\mu$ is demiclosed at zero. From Lemma 2.19, we have

$$\lim_{k\to\infty} \sup\{\left\|T_\mu(x) - T_{\mu_{n_k}}(x)\right\| : x \in \{x_n\}\} = 0. \tag{3.5}$$

From (3.5) and the triangle inequality, we get

$$\begin{aligned}
\|x_{n_k} - T_\mu(x_{n_k})\| &\leq \|x_{n_k} - T_{\mu_{n_k}}(x_{n_k})\| + \left\|T_{\mu_{n_k}}(x_{n_k}) - T_\mu(x_{n_k})\right\| \\
&\leq \|x_{n_k} - T_{\mu_{n_k}}(x_{n_k})\| + \sup\{\left\|T_{\mu_{n_k}}(x) - T_\mu(x)\right\| : x \in \{x_n\}\}
\end{aligned}$$

making $k \to \infty$, $\|x_{n_k} - T_\mu(x_{n_k})\| \to 0$. Using the demiclosedness of mapping $I - T_\mu$, $p \in F(T_\mu)$ and $\omega_w\{x_n\} \subset F(T_\mu) = \cap_{n=1}^{\infty} F(T_{\mu_n})$. Hence $\{T_{\mu_n}\}$ satisfies NST-condition. Now the proof directly follows from Theorem 3.5.

Theorem 3.7 *Suppose $\mathcal{E}$ be a convex closed and nonempty subset of a given q-USBS $\mathcal{X}$ which has Opial condition. Suppose $\{T_n\}$ be a family of k_n-strict pseudocontractive inward mappings from $\mathcal{E}$ into $\mathcal{X}$ with $\emptyset \neq \cap_{n=1}^{\infty} F(T_n)$. If $\{T_{\mu_n}\}$ satisfies NST-condition, then sequence $\{x_n\}$ given by (3.1) converges weakly to a point in $\cap_{n=1}^{\infty} F(T_n)$, provided $\{\alpha_n\} \subseteq [a, b] \subset (0, 1)$ with $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$.*

Proof Since $\{\alpha_n\} \subseteq [a, b] \subset (0, 1)$, using Lemma 3.2, we get $\sum_{n=1}^{\infty} \psi(\|x_n - T_{\mu_n}(x_n)\|) < \infty$. This implies that $\lim_{n\to\infty} \psi(\|x_n - T_{\mu_n}(x_n)\|) = 0$ and $\lim_{n\to\infty} \|x_n - T_{\mu_n}(x_n)\| = 0$. By the assumption that $\{T_{\mu_n}\}$ satisfies the NST-condition, we have $\omega_w\{x_n\} \subset \cap_{n=1}^{\infty} F(T_{\mu_n})$. In view of Lemma 3.1, sequence $\{x_n\}$ is bounded. Then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to some $x^* \in \omega_w\{x_n\} \subset \mathcal{E}$. Thus, $x^* \in \cap_{n=1}^{\infty} F(T_{\mu_n}) = \cap_{n=1}^{\infty} F(T_n)$. To prove weak convergence of the sequence $\{x_n\}$ to a point in $\cap_{n=1}^{\infty} F(T_n)$, it is suffice to prove that $\omega_w\{x_n\}$ is singleton. Arguing by contradiction, let $\tilde{p}, \tilde{q} \in \omega_w\{x_n\}$, $\{x_{n_k}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \tilde{p}$ and $x_{n_j} \rightharpoonup \tilde{q}$, respectively with $\tilde{p} \neq \tilde{q}$. Since $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in \cap_{n=1}^{\infty} F(T_n)$, from the Opial condition, we have

$$\begin{aligned}\lim_{n\to\infty} \|x_n - \tilde{p}\| &= \lim_{k\to\infty} \|x_{n_k} - \tilde{p}\| \\ &< \lim_{k\to\infty} \|x_{n_k} - \tilde{q}\| = \lim_{j\to\infty} \|x_{n_j} - \tilde{q}\| \\ &< \lim_{j\to\infty} \|x_{n_j} - \tilde{p}\| = \lim_{n\to\infty} \|x_n - \tilde{p}\|,\end{aligned}$$

a contradiction. This completes the proof.

Theorem 3.8 *Suppose $\mathcal{E}$ be a convex closed and nonempty subset of a given q-USBS $\mathcal{X}$ which has Opial condition. Suppose $\{T_n\}$ be a family of k_n-strict pseudocontractive weakly inward mappings from $\mathcal{E}$ into $\mathcal{X}$ with $\emptyset \neq \cap_{n=1}^{\infty} F(T_n)$. If $\{T_{\mu_n}\}$ satisfies AKTT-condition* (I)*, mapping T_μ is defined by* (2.6) *and $F(T_\mu) = \cap_{n=1}^{\infty} F(T_{\mu_n})$, then the sequence $\{x_n\}$ given by* (3.1) *converges weakly to a point in $\cap_{n=1}^{\infty} F(T_n)$, provided $\{\alpha_n\} \subseteq [0,1]$ with $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$.*

Proof Since the sequence of mapping $\{T_{\mu_n}\}$ satisfies AKTT-condition (I) along with $F(T_\mu) = \cap_{n=1}^{\infty} F(T_{\mu_n})$, following the proof of Theorem 3.6, it can be shown that $\lim_{n\to\infty} \|x_n - T_{\mu_n}(x_n)\| = 0$ and $\{T_{\mu_n}\}$ satisfies NST-condition. Following largely the similar argument for application of Opial condition as in proof of Theorem 3.7, we can conclude that the sequence $\{x_n\}$ converges weakly to $p \in \cap_{n=1}^{\infty} F(T_n)$.

4 Strong convergence results

Theorem 4.1 *Suppose $\mathcal{E}$ be a convex closed and nonempty subset of a given q-USBS $\mathcal{X}$. Suppose $\{T_n\}$ be a family of k_n-strict pseudocontractive inward mappings from $\mathcal{E}$ into $\mathcal{X}$ with $\emptyset \neq \cap_{n=1}^{\infty} F(T_n)$. If $\{T_{\mu_n}\}$ satisfies AKTT-condition (I), mapping T_μ is defined by* (2.6) *and $\cap_{n=1}^{\infty} F(T_{\mu_n}) = F(T_\mu)$, then sequence $\{x_n\}$ given by* (3.1) *converges strongly to a point in $\cap_{n=1}^{\infty} F(T_n)$, provided $\sum_{n=1}^{\infty}(1-\alpha_n) < \infty$.*

Proof By Lemma 3.1, $\forall p \in \cap_{n=1}^{\infty} F(T_n)$, $\lim_{n\to\infty} \|x_n - p\|$ exists. Since

$$\|x_n - x_{n+1}\| = (1-\alpha_n)\|x_n - T_{\mu_n}(x_n)\|,$$

using boundedness of the sequences $\{x_n\}$, $\{T_{\mu_n}(x)\}$,

$$\sum_{n=1}^{\infty} \|x_n - x_{n+1}\| < \infty.$$

That is, $\{x_n\}$ is a strongly Cauchy sequence. Therefore, $\exists\, x^* \in \mathcal{E}$ such that

$$\lim_{n\to\infty} x_n = x^*.$$

Now, it suffices to show that $x^* \in \cap_{n=1}^{\infty} F(T_n)$.

Since the family $\{T_{\mu_n}\}$ is weakly inward $\forall n \in \mathbb{N}$, $T_{\mu_n}(x) \in \overline{I_{\mathcal{E}}(x)}$ $\forall n \in \mathbb{N}$. Hence

$$T_\mu(x) = \lim_{n\to\infty} T_{\mu_n}(x) \in \overline{I_{\mathcal{E}}(x)},$$

and T_μ is weakly inward mapping. Using Lemma 2.24, we know that $h_{\mathcal{E},T_\mu}(x) < 1$ $\forall x \in \mathcal{E}$. Lemma 2.23 (Z1) implies that for all $\delta \in (h_{\mathcal{E},T_\mu}(x^*), 1)$

$$\delta x^* + (1-\delta)T_\mu(x^*) \in \mathcal{E}. \tag{4.1}$$

On the other hand, $\sum_{n=1}^{\infty}(1-\alpha_n) < \infty$ ensures that $\lim_{n\to\infty} \alpha_n = 1$ where $\alpha_n = \max\{\alpha_{n-1}, h_n(x_n)\}$. So we can choose a subsequence $\{x_{n_j}\}$ with the property that $\left\{h_{\mathcal{E},T_{\mu_{n_j}}}(x_{n_j})\right\}$ is nondecreasing and $h_{\mathcal{E},T_{\mu_{n_j}}}(x_{n_j}) \to 1$. In particular, Lemma 2.23 (Z2) ensures that for any fixed $\delta < 1$,

$$\delta x_{n_j} + (1-\delta)T_{\mu_{n_j}}(x_{n_j}) \notin \mathcal{E} \text{ for sufficiently large } j. \tag{4.2}$$

Take two positive real numbers $\delta_1, \delta_2 \in (h_{\mathcal{E},T_\mu}(x^*), 1)$ with $\delta_1 \neq \delta_2$ and set $\rho_1 = \delta_1 x^* + (1-\delta_1)T_\mu(x^*)$ and $\rho_2 = \delta_2 x^* + (1-\delta_2)T_\mu(x^*)$. Now, for any $\delta \in [\delta_1, \delta_2]$ by (4.1), we get

$$\rho = \delta x^* + (1-\delta)T_\mu(x^*) \in \mathcal{E}. \tag{4.3}$$

Now, we prove that $T_{\mu_n}(x_n) \to T_\mu(x^*)$ as $n \to \infty$. For this, take $\mathcal{C} = \mathcal{B}_r(x^*) \cap \mathcal{E}$, $\forall r > 0$, then $\mathcal{C}$ is a bounded subset of $\mathcal{E}$. Using triangle inequality, we have

$$\begin{aligned}\|T_{\mu_n}(x_n) - T_\mu(x^*)\| &\le \|T_{\mu_n}(x_n) - T_{\mu_n}(x^*)\| + \|T_{\mu_n}(x^*) - T_\mu(x^*)\| \\ &\le \|x_n - x^*\| + \sup\{\|T_{\mu_n}(x) - T_\mu(x)\| : x \in \mathcal{C}\}.\end{aligned}$$

Since $x_n \to x^*$ as $n \to \infty$ and from Lemma 2.19, we have $T_{\mu_n}(x_n) \to T_\mu(x^*)$ as $n \to \infty$. By (4.2), $\delta x_{n_j} + (1-\delta)T_{\mu_{n_j}}(x_{n_j}) \notin \mathcal{E}$. Since $x_n \to x^*$ and $T_{\mu_n}(x_n) \to T_\mu(x^*)$ as $n \to \infty$, we get

$$\lim_{j\to\infty} \delta x_{n_j} + (1-\delta)T_{\mu_{n_j}}(x_{n_j}) = \rho, \text{ and } \rho \in \partial\mathcal{E}.$$

Since ρ is any arbitrary point in segment $[\rho_1, \rho_2]$, the entire segment $[\rho_1, \rho_2] \subset \partial\mathcal{E}$. The strict convexity of $\mathcal{E}$ implies that $\rho_1 = \rho_2$, that is,

$$\delta_1 x^* + (1-\delta_1)T_\mu(x^*) = \delta_2 x^* + (1-\delta_2)T_\mu(x^*),$$

hence $T_\mu(x^*) = x^*$, so $x^* \in F(T_\mu)$. Since $\cap_{n=1}^{\infty} F(T_n) = \cap_{n=1}^{\infty} F(T_{\mu_n}) = F(T)$, therefore, $x^* \in \cap_{n=1}^{\infty} F(T_n)$.

Corollary 4.2 *Suppose $\mathcal{E}$ be a convex closed and nonempty subset of a given q-USBS $\mathcal{X}$, $\{T_n\}$ a family of weakly inward nonexpansive mappings from $\mathcal{E}$ into $\mathcal{X}$ with $\emptyset \neq \cap_{n=1}^{\infty} F(T_n)$. If $\{T_{\mu_n}\}$ satisfies AKTT-condition (I) and mapping T_μ defined by (2.6), $\cap_{n=1}^{\infty} F(T_{\mu_n}) = F(T_\mu)$, then sequence $\{x_n\}$ given by (3.1) converges strongly to a point in $\cap_{n=1}^{\infty} F(T_n)$, provided $\sum_{n=1}^{\infty}(1-\alpha_n) < \infty$.*

Theorem 4.3 *Suppose $\mathcal{E}$ be a convex closed and nonempty subset of a given q-USBS $\mathcal{X}$. Suppose $\{T_n\}$ be a family of k_n-strict pseudocontractive weakly inward mappings from $\mathcal{E}$ into $\mathcal{X}$ with $\emptyset \neq \cap_{n=1}^{\infty} F(T_n)$. If $\{T_{\mu_n}\}$ satisfies AKTT*-condition with $\mathfrak{T}_\mu$, and $\mathfrak{T}_\mu$ is a family of closed mappings from $\mathcal{E}$ into $\mathcal{X}$ with $\cap_{n=1}^{\infty} F(T_{\mu_n}) = F(\mathfrak{T}_\mu)$. If there exists a mapping $\bar{T}_\mu \in \mathfrak{T}_\mu$ such that $\bar{T}_\mu$ is semicompact, then sequence $\{x_n\}$ given by (3.1) converges strongly to a point in $\cap_{n=1}^{\infty} F(T_n)$, provided $\{\alpha_n\} \subseteq [0, 1]$ with $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$.*

Proof From Theorem 3.6, we have $\lim_{n\to\infty} \|x_n - T_{\mu_n}(x_n)\| = 0$, and since family of mappings $\{T_{\mu_n}\}$ satisfies AKTT*-condition, we have

$$\lim_{n\to\infty} \|x_n - T_\mu(x_n)\| = 0 \ \ \forall\, T_\mu \in \mathfrak{T}_\mu. \tag{4.4}$$

In particular,

$$\lim_{n\to\infty} \|x_n - \bar{T}_\mu(x_n)\| = 0.$$

Since mapping $\bar{T}_\mu \in \mathfrak{T}_\mu$ is semicompact, we can find a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p \in \mathcal{E}$ as $j \to \infty$. From (4.4)

$$\lim_{j\to\infty} \|x_{n_j} - T_\mu(x_{n_j})\| = 0 \ \forall\, T_\mu \in \mathfrak{T}_\mu.$$

Using triangle inequality,

$$\|T_\mu(x_{n_j}) - p\| \le \|T_\mu(x_{n_j}) - x_{n_j}\| + \|x_{n_j} - p\|.$$

Thus $\|T_\mu(x_{n_j}) - p\| \to 0$ as $j \to \infty$. Since each mapping $T_\mu \in \mathfrak{T}_\mu$ is closed, it confirms that $T_\mu(p) = p\ \forall\, T_\mu \in \mathfrak{T}_\mu$, hence $p \in F(\mathfrak{T}_\mu)$. Using Lemma 3.1 and $\cap_{n=1}^{\infty} F(T_{\mu_n}) = F(\mathfrak{T}_\mu)$, it can be easily seen that the sequence $\{x_n\}$ converges strongly to $p \in \cap_{n=1}^{\infty} F(T_{\mu_n})$ and hence $p \in \cap_{n=1}^{\infty} F(T_n)$.

Theorem 4.4 *Suppose $\mathcal{E}$ be a convex closed and nonempty subset of a given q-USBS $\mathcal{X}$. Suppose $\{T_n\}$ be a family of k_n-strict pseudocontractive weakly inward mappings from $\mathcal{E}$ into $\mathcal{X}$ with $\emptyset \neq \cap_{n=1}^{\infty} F(T_n)$. If the family of mappings $\{T_{\mu_n}\}$ satisfies AKTT*-condition with $\mathfrak{T}_\mu$, and $\mathfrak{T}_\mu$ is a family of closed mappings from $\mathcal{E}$ into $\mathcal{X}$ with $\cap_{n=1}^{\infty} F(T_{\mu_n}) = F(\mathfrak{T}_\mu)$. If $\mathfrak{T}_\mu$ is finite and satisfies condition (II), then sequence $\{x_n\}$ given by (3.1) converges strongly to a point in $\cap_{n=1}^{\infty} F(T_n)$, provided $\{\alpha_n\} \subseteq [0, 1]$ with $\sum_{n=1}^{\infty}(1 - \alpha_n) = \infty$.*

Proof Lemma 3.1 confirms that for each $p \in \cap_{n=1}^{\infty} F(T_n)$, $\lim_{n\to\infty} \|x_n - p\|$ exists. Now, we prove that the sequence $\{x_n\}$ is a Cauchy sequence. From (3.2) we get

$$\|x_{n+1} - p\| \le \|x_n - p\|.$$

Following this way, $\forall\, n, m \in \mathbb{N}$, we can write

$$\|x_{n+m} - p\| \le \|x_n - p\|. \tag{4.5}$$

From Theorem 3.6, we have $\lim_{n\to\infty} \|x_n - T_{\mu_n}(x_n)\| = 0$, and since family of mappings $\{T_{\mu_n}\}$ satisfies AKTT*-condition, we have

$$\lim_{n\to\infty} \|x_n - T_\mu(x_n)\| = 0 \ \forall\, T_\mu \in \mathfrak{T}_\mu. \tag{4.6}$$

Since $\mathfrak{T}_\mu$ satisfies condition (II), there exists a function f such that

$$f(d(x_n, F(\mathfrak{T}_\mu))) \le \max\{\|x_n - T_\mu(x_n)\| : T_\mu \in \mathfrak{T}_\mu\}.$$

From (4.6), we get

$$\lim_{n\to\infty} d(x_n, F(\mathfrak{T}_\mu)) = 0. \tag{4.7}$$

Thus for any given $\varepsilon > 0$, there exists a natural number n_0 such that

$$d(x_{n_0}, F(\mathfrak{T}_\mu)) < \frac{\varepsilon}{2}.$$

Since $F(\mathfrak{T}_\mu)$ is closed, from (4.7), there exists a point $x^* \in F(\mathfrak{T}_\mu))$ such that $\|x_{n_0} - x^*\| < \frac{\varepsilon}{2}$. From (4.5), we have for all $n \ge n_0$ and $m \ge 1$,

$$\begin{aligned}\|x_{n+m} - x_n\| &\le \|x_{n+m} - p\| + \|x_n - p\|\\ &\le 2\|x_n - p\|\\ &\le 2\|x_{n_0} - p\| < 2\frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Thus the sequence $\{x_n\}$ is a Cauchy sequence in $\mathcal{E}$. Since $\mathcal{X}$ is complete and $\mathcal{E}$ is a closed subset of $\mathcal{X}$, $\mathcal{E}$ is also complete. Therefore the sequence $\{x_n\}$ converges to some $\tilde{p} \in \mathcal{E}$. Since the mapping T_μ is closed, from (4.6), it can be easily follows that $T_\mu(\tilde{p}) = \tilde{p}\ \forall\ T_\mu \in \mathfrak{T}_\mu$ and $\tilde{p} \in F(\mathfrak{T}_\mu)$. Since $\cap_{n=1}^{\infty} F(T_{\mu_n}) = \cap_{n=1}^{\infty} F(T_n) = F(\mathfrak{T}_\mu)$, sequence $\{x_n\}$ converges strongly to $\tilde{p} \in \cap_{n=1}^{\infty} F(T_n)$.

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Declarations

Conflict of interest The authors declare that they have no conflict of interests.

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