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On groups of finite prüfer rank III

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Abstract
If G is a group of finite Prüfer rank we prove that G has a characteristic subgroup K of finite index such that every finite image of its derived subgroup K′ is nilpotent, so every finite image of K is nilpotent-by-abelian.

Keywords Groups of finite Prüfer rank

A group G has finite (Prüfer) rank if there is an integer r such that each finitely generated subgroup of G can be generated by at most r elements, the least such r being the rank rk(G) of G. Soluble groups of finite rank are well understood, as are groups of finite rank in the very much larger group class generated by the abelian groups and the finite groups under closure by the local and the ascending series operators (symbolically the class $<$L, P′$>$(**AF**), see [1]). Results about groups of finite rank in general seem to be rare. The following is the main result of this paper.

THEOREM 1 Let G be a group of finite rank. Then G has a characteristic subgroup K of finite index such that every finite image of its derived subgroup K′ is nilpotent and every finite image of K is nilpotent-by-abelian.

If G is a group of finite rank, it follows from [2] that G has a characteristic subgroup H of finite index such that every finite image of H is soluble and we proved in [3] that for every finite set π of primes there is a characteristic subgroup H = H(π) of G of finite index such that every finite π-image of H is nilpotent. (The special cases of the latter where G is also soluble or finitely generated are proved in [4].)

In Theorem 1 we cannot in general choose K with every finite image of K nilpotent. For example, suppose G is a polycyclic group that is not nilpotent-by-finite. Then G has finite

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rank and any subgroup of G with all its finite images nilpotent is itself nilpotent (theorem of Hirsch, e.g. [5] 2.8). (The simplest example of such a group is probably the group G=<x>A, where A is a free abelian group of rank 2 and x is the 2 by 2 companion matrix of the polynomial $x^2 - 2X - 1$.) However for a polycyclic group G we can always choose K so that K´ is actually nilpotent ([5] 4.5).

Say that a group G has finite residual rank at most the integer r if every finite image of G has finite rank at most r, the least such r being the residual rank of G. In [2] we were able to prove the main results of [3] and [2] under the weaker assumption of finite residual rank. However we cannot weaken finite rank in Theorem 1 to just finite residual rank, as the following example shows.

(I am very grateful to the referee for bringing to my attention the book [6] by Lubotzky and Segal and informing me that it contains a discussion of finite residual rank, but under the name ‘finite upper rank’, and that some of the material I quote here from [3] and [2] can also be found in [6].)

Let S be a finite non-abelian simple group, A an infinite cyclic group and G the wreath product of S by A. If B denotes the base group of G, then B is the direct product of the S^a as a ranges over A and hence the only normal subgroups of G in B are B and <1>. It follows that if K is a normal subgroup of G of finite index, then B≤K and hence K´ ≤ B=B´ ≤ K´. Further G/K is cyclic, so G has residual rank 1. If N is a normal subgroup of K´ = B of index at least 2, since B=B´, so K´/N is not nilpotent. Clearly for every positive integer m there exists such an N with $(K' : N) = |S|^m$.

The following is a soluble example. Let G denote the wreath product of the dihedral group of order 6 by the additive group Q of the rational numbers. Then G is a soluble group of derived length 3 and residual rank 1 and for every normal subgroup H of G of finite index, H´ has a finite non-nilpotent image. (Clearly there is no such metabelian example.)

For let B denote the base group of G, so G=QB, and consider some normal subgroup H of G of finite index. Since Q is divisible, Q≤H and since Q acts on the finite group B/(H∩B), Q centralizes it; that is, [B, Q]≤H∩B. In particular, [B´, Q]≤H∩B´. Now B´ is isomorphic as Q-module in the obvious way to the group algebra F_3Q of Q over the field F_3 of 3 elements with the image of H∩B´ in F_3Q containing the augmentation ideal of Q. Since H∩B´ is normal in B, this image will contain

$$(2x-1) - (x-1) = x \text{ for every } x \in Q.$$

Consequently H∩B´ = B´.

Now B/B´ is isomorphic as Q-module to the group algebra F_2Q of Q over the field F_2 of 2 elements. Since B/(H∩B) is Q-central, the image of (H∩B)/B´ in F_2Q contains the augmentation ideal of Q and hence (B : H∩B)≤2. Thus we have only two possibilities for H, namely H=G or H=Q(H∩B) with H∩B actually the inverse image of the augmentation ideal. In particular G/H has order 1 or 2 and G has residual rank 1. Trivially G is soluble of derived length 3.

Suppose every finite image of H´ is nilpotent. Since H´ ≤ B it follows that every finite image of H´ is abelian. But clearly B is residually finite, so H´ is abelian. If x ϵ H´ ≤ B has order 2, there exists some a ϵ B of order 3 with $a^x = a^{-1}$. But H´ is normal in G, so H´ contains $x^a = xa^2$. But then the abelian group H´ contains both the non-commuting elements x and a. Consequently no such x exists and hence H´ ≤ B´. Therefore G/B´ is abelian-by-finite.

Also $G/B' = QB/B' = HB/B'$, H/B' and B/B' are abelian and hence (Fitting's Lemma) G/B' is nilpotent. Hence the augmentation ideal of Q in F_2Q is nilpotent. But if $x \in Q \backslash \langle 1 \rangle$, then in F_2Q, $(1-x)^m \neq 0$ for every positive integer m. This final contradiction shows that H has a non-nilpotent finite image.

However we can at least prove the following, from which Theorem 1 is immediate.

THEOREM 2 Let G be a group with finite residual rank. Then G has a characteristic subgroup K of finite index such that for every normal subgroup N of G that has finite index in K' the group K'/N is nilpotent and every finite image of K is nilpotent-by-abelian. Suppose further that the derived subgroup of every characteristic subgroup of G of finite index also has finite residual rank. Then G has a characteristic subgroup K of finite index such that every finite image of its derived subgroup K' is nilpotent.

It is possible even for a periodic metabelian group to have finite residual rank while each of its characteristic subgroups of finite index has its derived subgroup of infinite residual rank, as the following example shows.

For some prime p let F be an infinite locally finite field of characteristic p. Set $A = F^+$ the additive group of F and $B = F^*$ the multiplicative group of F. Then A is a B-module in the obvious way. Let $G = BA$, the split extension of A by B. Then A is irreducible as B-module, B has finite rank 1, every normal subgroup of G of finite index contains A and hence G has residual rank at most 1. Also $G' = A$ is an elementary abelian p-group of infinite rank and in particular G' does not have finite residual rank.

If F is the algebraic closure of GF(p), then B is divisible and G is the only subgroup of G of finite index. In general, if K is a normal subgroup of G of finite index, then $A \leq K$. For if E denotes the subfield of F generated by $K \cap B$, then $(B : K \cap B)$ is finite, $\dim_E F$ is finite, E is irreducible as $K \cap B$-module, A is a direct sum of finitely many copies of E as $K \cap B$-module, $A \leq K$ and hence K has finite residual rank, while $K' = A$ does not have finite residual rank.

The following theorem is the critical step in our proof of Theorems 1 and 2.

THEOREM 3 Let G be a group of finite residual rank r. Then there exists a positive integer $e = e(r)$, depending only on r, and a characteristic subgroup H of G of finite index such that if N is a normal subgroup of H of finite index, then there exists normal subgroups X and Y of H such that $N \leq X \leq Y \leq H$ with X/N nilpotent, Y/X abelian and H/Y nilpotent of finite exponent dividing e and derived length at most 2r.

Properties of linear groups are at the heart of our proofs below, so perhaps one should look more generally at linear groups of finite residual rank. By an old theorem of Platonov (e.g. [7] 10.9) linear groups of finite rank are soluble-by-finite (even abelian-by-finite in positive characteristic). The same cannot hold for linear groups of finite residual rank, since infinite simple linear groups trivially have 0 residual rank. If G is a finitely generated subgroup of GL(n, F), then $G \leq GL(n, R)$ for some finitely generated subring of the field F. Thus the following corollary of Theorem 2 covers all finitely generated linear groups.

COROLLARY If R is a finitely generated integral domain and if G is any subgroup of GL(n, R) of finite residual rank, then G is soluble-by-finite.

Fitt(G) denotes the Fitting subgroup of the group G.

LEMMA 1 Let G be a finite soluble group of rank at most the integer r. Then there is a positive integer $e=e(r)$, depending only on r, such that G/Fitt(G) has derived length at most 2r and is abelian-by-(of exponent dividing e).

Proof If $r\leq 1$ the claims are vacuous. There exists a nilpotent normal subgroup B of G of class at most 2 such that $Z=C_G(B)\leq B$ (e.g. see [8] 1.A.8). Set.

$$L = \cap_p (C_G(Z/Z^p)\cap(B/B^pZ)),$$

the intersection being over the set of primes p dividing |B|.

There is a positive integer e, depending only on r, such that every soluble linear group of degree r has derived length at most 2r (see [9] or 6.2 A of [10]) and is unipotent-by-abelian-by-(finite of order dividing e), e.g. see [7] 3.6. Now, since G has rank at most r, G/L is residually linear of degree r. Set $H = (G^e)'L$. Then G/L has derived length at most 2r and H/L acts unipotently (i.e. nilpotently) on each Z/Z^p and B/B^pZ. Hence H acts nilpotently on Z and B/Z and hence also on B. Consequently $H/C_H(B)$ is nilpotent (e.g. [5] 1.17). But $C_H(B)\leq C_G(B)\leq B$ and H acts nilpotently on B. Therefore H is nilpotent. Finally $L\leq H\leq \text{Fitt}(G)$ and the lemma follows.

*Proof of Theorem*3. With $e=e(r)$ as in Lemma 1, let π denote the (finite) set of prime divisors of e. By the Theorem, the Proposition and Lemma 2 of [2] there is a characteristic subgroup H of G of finite index such that every finite image of H is soluble and every finite π-image of H is nilpotent. Now apply Lemma 1 to each finite image of H.

LEMMA 2 Let G be a soluble, residually finite group with finite exponent e, finite residual rank r and derived length d. Then G is finite of order dividing e^{dr}.

Proof If G is finite, clearly |G| divides e^{dr}. In general choose a normal subgroup N of G of finite index with (G : N) maximal; note that always $(G : N)\leq e^{dr}$. Then every normal subgroup of G of finite index contains N. But G is residually finite. Consequently $N=\langle 1\rangle$, G is finite and |G| divides e^{dr}.

The following lemma is a special case of [2] Lemma 2, where we have chosen both G and H equal to L.

LEMMA 3 Let N be a normal subgroup of finite index in the group L. If L has finite residual rank and $M = \cap_\varphi N\varphi$, where φ ranges over Aut(L), then L/M is finite.

*Proof of Theorem*2. Choose $e=e(r)$ and $H\leq G$ as in Theorem 3 and set K equal to the intersection of all the normal subgroups N of H of finite index with H/N of exponent dividing e and derived length at most $d=2r$. Then K is a characteristic subgroup of H (and hence of G) such that (H : K) divides e^{dr} by Lemma 2.

Now suppose N is a normal subgroup of H with N of finite index in K′. Then $C=C_H(K'/N)$ is a normal subgroup of H of finite index. Hence by the choice of H and the definition of K we have that K′C/C is nilpotent. But then $K'/(C\cap K')$ is nilpotent and $(C\cap K')/N=\zeta_1(K'/N)$.

Consequently K'/N is nilpotent. If N is now a normal subgroup of K of finite index, set $N_H = \cap_{x\epsilon H} N^x$. Then $N_H \cap K'$ is normal in H and of finite index in K′, so by the above $K'/(N_H \cap K')$ is nilpotent. Consequently K/N_H and hence K/N are nilpotent-by-abelian.

Finally assume that K′ has finite residual rank and let N denote a normal subgroup of K′ of finite index. With L=K′ and M as in Lemma 3, clearly $M \leq N_H = \cap_{x\epsilon H} N^x$, so K'/N_H is finite. By the previous paragraph K'/N_H is nilpotent. Consequently K′/N is nilpotent and the proof of Part b) is complete.

Proof of the Corollary. Let K be the normal subgroup of G given by Theorem 2. Then G/K′ is abelian-by-finite and if whenever N is a normal subgroup of G of finite index in K′, we have K′ /N nilpotent. But then K′ is nilpotent by [7] 4.16 (or if you prefer by [7] 10.5). Consequently G is soluble-by-finite.

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