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Superfluous ideals of *N*-groups

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Abstract
We consider a right nearring N and a module over N (known as, N-group). For an arbitrary ideal (or N-subgroup) Ω of an N-group G, we define the notions Ω-superfluous, strictly Ω-superfluous, g-superfluous ideals of G. We give suitable examples to distinguish between these classes and the existing classes studied in Bhavanari (Proc Japan Acad 61-A:23–25, 1985; Indian J Pure Appl Math 22:633–636, 1991; J Austral Math Soc 57:170–178, 1994), and prove some properties. For a zero-symmetric nearring with 1, we consider a module over a matrix nearring and obtain one-one correspondence between the superfluous ideals of an N-group (over itself) and those of $M_n(N)$-group N^n, where $M_n(N)$ is the matrix nearring over N. Furthermore, we define a graph of superfluous ideals of a nearring and prove some properties with necessary examples.

Keywords N-groups · Superfluous ideal · Supplement · Matrix nearring

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1 Introduction and Preliminaries

The notion of finite Goldie dimension (denoted by FGD) of a module was defined by Goldie [14] wherein, the key notions for the study of FGD are essential submodules, uniform sub-

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modules and complement of a submodule (see, [4, 5]). The dualization of this concept namely, finite spanning dimension (denoted by FSD) in modules over rings was defined by Fleury [13] with the notions such as superfluous submodules, hollow submodules and supplements. Later, these concepts were studied in [1, 2, 7, 18]. The idea of FSD was generalized to module over nearrings (known as N-groups) in [6, 11, 16, 20]. They introduced the notions such as superfluous ideal, hollow ideal and FSD in N-groups and proved the corresponding structure theorems. The motivation of this paper arises from a natural question that what if one substitutes an arbitrary ideal Ω in place of an N-group G, so that it generalises the existing study of these notions. The classes of N-groups with these new notions are different from the classes of N-groups studied in [6, 8, 11, 19, 20]. Eventually, in this paper we define Ω-superfluous ideal of an N-group G, where Ω is an ideal of G and obtain some connections to matrix nearrings, and some combinatorial aspects. In section 2, we define g-superfluous and g-supplement ideals of N-groups as a generalization of respective notions defined by [20]. We have constructed examples where G is non-abelian.
In Sect. 3, we introduce superfluous ideals of N-groups and prove some important properties and provide necessary examples. In Sect. 4, we consider the notion strictly superfluous in terms of N-subgroups and gave examples which indicate that the classes of N-groups with superfluous and strictly superfluous are different. Matrix nearrings over arbitrary nearrings were defined in [15] and studied in [23]. In Sect. 5, we introduce the superfluous and g-superfluous ideals in $M_n(N)$-group N^n, and establish a one-one correspondence between superfluous and g-superfluous ideals of N (over itself) and those of $M_n(N)$-group N^n. In Sect. 6, we introduce superfluous ideal graphs of nearrings and prove some properties with examples.

A (right) nearring is a set N together with two binary operations "$+$" and "$\cdot$" such that $(N, +)$ is a group, $(N, \cdot)$ is a semigroup and right distributive law holds. In general, for some $n \in N$, $n \cdot 0 \neq 0$, and so we call N is zero-symmetric if $n \cdot 0 = 0$ for all $n \in N$. A normal subgroup I of a nearring N is called an ideal of N (denoted by $I \trianglelefteq N$) if $IN \subseteq I$ and $a(b+i)-ab \in I$ for all $a, b \in N$ and $i \in I$. An additive group G is said to be an N-group if there exists a map $N \times G \to G$ defined by $(n, g) \longmapsto ng$ satisfying $(n+n_1)g = ng + n_1 g$ and $(n \cdot n_1)g = n(n_1 g)$ for all $n, n_1 \in N$ and $g \in G$. Throughout, we use G for an N-group. A subgroup H of G is said to be an N-subgroup (denoted as, $H \leq_N G$) of G if $NH \subseteq H$; and a normal subgroup I of G is called an ideal (denoted as, $I \trianglelefteq_N G$) of G if $n(g+i) - ng \in I$, for all $n \in N$, $g \in G$ and $i \in I$. An ideal S of G is said to be superfluous in G if $S + K = G$ and K is an ideal of G, imply $K = G$, and G is called hollow if every proper ideal of G is superfluous in G. For any two N-subgroups H and K of G, K is said to be a supplement for H if $H + K = G$ and $H + K^{'} \neq G$ for any proper ideal $K^{'}$ of K. For any ideals I, J, K of N (or of G), if $K \subseteq I$, then $I \cap (J + K) = (I \cap J) + K$. We use $I \oplus J$ to denote the direct sum of ideals I and J of G. We refer to Pilz [17] and Bhavanari and Kuncham [10] for fundamental literature in nearrings.

We consider simple and finite graphs, whose vertex set is V and edge set is E. A vertex v of a graph is called a universal vertex if degree of $v = |V| - 1$. If there exists a path between every two vertices of a graph, then the graph is connected otherwise the graph is disconnected. A graph whose vertex set is empty is called a null graph and a graph having atleast one vertex and empty edge set is called an empty graph.

Table 1 Multiplication $\star$ on N

$\star$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	2	2
3	0	0	0	0	0	0	2	2
4	0	0	0	0	0	0	4	4
5	0	0	0	0	0	0	4	4
6	0	0	0	0	0	0	6	6
7	0	0	0	0	0	0	6	6

2 Generalized supplements

The notion of superfluous submodule of module over a ring was studied by [3, 7]. We define generalized superfluous (briefly, g-superfluous) ideal of an N-group G as follows.

Definition 2.1 An ideal K of G is called g-superfluous if $G = K + T$ and $T \leq_e G$, then $T = G$. We denote this by $K \ll_{gs} G$.

Remark 2.2 Every superfluous ideal of G is g-superfluous.

Example 2.3 Consider the nearring $N = (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, +, \cdot)$ with the notation given in page no. 420, (N) of [17]. That is, $(0, 0, 0) = 0$, $(0, 0, 1) = 1$ $(0, 1, 0) = 2$ $(0, 1, 1) = 3$, $(1, 0, 0) = 4$, $(1, 0, 1) = 5$, $(1, 1, 0) = 6$, and $(1, 1, 1) = 7$. The multiplication table is given below. Let $G = N$ (Table 1).
The proper ideals of G are $I_1 = \{0, 2, 4, 6\}$ and $I_2 = \{0, 1\}$. It can be seen that I_1 and I_2 are g-superfluous but not superfluous in G as $I_1 + I_2 = G$

Proposition 2.4 *Let K, J be ideals of G such that $K \subseteq J$. If $K \ll_{gs} J$, then $K \ll_{gs} L$ for any ideal L of G with $J \subseteq L$.*

Proof Let $T \leq_e L$ such that $K+T = L$. We prove $T = L$. Clearly $T \subseteq L$. Since $K+T = L$, we have $(K + T) \cap J = L \cap J$. Since $K \subseteq J$ and modular law, we have $K + (T \cap J) = J$. Since $T \cap J \leq_e J$ and $K \ll_{gs} J$, we have $(T \cap J) = J$ and so $T \subseteq J$. Since $K \subseteq J \subseteq T$, we get $L = K + T \subseteq J + T = T$. Therefore $L = T$ and hence $K \ll_{gs} L$. □

Proposition 2.5 *Let K_1, K_2, G_1, G_2 be ideals of G such that $K_1 \subseteq G_1$ and $K_2 \subseteq G_2$. If $K_1 \ll_{gs} G_1$ and $K_2 \ll_{gs} G_2$, then $K_1 + K_2 \ll_{gs} G_1 + G_2$.*

Proof Since $K_i \ll_{gs} G_i$, by Proposition 2.4, we have $K_i \ll_{gs} G_1 + G_2$ for $i = \{1, 2\}$. Let $T \leq_e G_1 + G_2$ be such that $K_1 + K_2 + T = G_1 + G_2$. Since $T \leq_e G_1 + G_2$, we have $T + K_2 \leq_e G_1 + G_2$. Now $K_1 + (K_2 + T) = G_1 + G_2$ and $K_1 \ll_{gs} G_1 + G_2$, imply that $K_2 + T = G_1 + G_2$. Again, since $K_2 \ll_{gs} G_1 + G_2$ and $T \leq_e G_1 + G_2$ implies $T = G_1 + G_2$. □

Note 2.6 *Let X, K be ideals of G such that $K \subseteq X$. If $\frac{X}{K} \leq_e \frac{G}{K}$, then $X \leq_e G$.*

Proposition 2.7 *Let U, V and K be ideals of G. If $U \ll_{gs} V$, then $\frac{U+K}{K} \ll_{gs} \frac{V+K}{K}$.*

Proof Let $\frac{T}{K} \leq_e \frac{V+K}{K}$ be such that $\frac{U+K}{K} + \frac{T}{K} = \frac{V+K}{K}$. Then $U + K + T = V + K$. Since $K \subseteq T$, we get $U + T = V + K$. Now $U \ll_{gs} V$ implies that $U \ll_{gs} V + K$, and $\frac{T}{K} \leq_e \frac{V+K}{K}$ implies $T \leq_e V + K$. Since $U \ll_{gs} V + K$, $T \leq_e V + K$ and $U + T = V + K$, we get $T = V + K$, which implies $\frac{T}{K} = \frac{V+K}{K}$. Therefore, $\frac{U}{K} \ll_{gs} \frac{V+K}{K}$. □

Proposition 2.8 *Let J, K, L be ideals of G such that $K \subseteq J$.*

1. *If $J \ll_{gs} G$, then $K \ll_{gs} G$ and $\frac{J}{K} \ll_{gs} \frac{G}{K}$.*
2. *$J + L \ll_{gs} G$ if and only if $J \ll_{gs} G$ and $L \ll_{gs} G$.*

Proof 1. Suppose $J \ll_{gs} G$. To prove $K \ll_{gs} G$, let $T \leq_e G$ such that $K + T = G$. Since $K \subseteq J$, we get $J + T = G$. Since $J \ll_{gs} G$, we have $T = G$, shows that $K \ll_{gs} G$. Next we prove $\frac{J}{K} \ll_{gs} \frac{G}{K}$. Let $\frac{X}{K} \leq_e \frac{G}{K}$ such that $\frac{J}{K} + \frac{X}{K} = \frac{G}{K}$. Then $\frac{J+X}{K} = \frac{G}{K}$, implies that $J + X = G$. Since $X \leq_e G$, we get $X = G$. Therefore $\frac{X}{K} = \frac{G}{K}$.
2. Suppose $J + L \ll_{gs} G$. To prove $J \ll_{gs} G$ and $L \ll_{gs} G$, let $T \leq_e G$ such that $J + T = G$. Then $(J + L) + T = G$. Since $J + L \ll_{gs} G$, we have $T = G$. In a similar way, we get $L \ll_{gs} G$.
   Conversely, suppose that $J \ll_{gs} G$ and $L \ll_{gs} G$. To prove $J + L \ll_{gs} G$, let $T \leq_e G$ such that $(J + L) + T = G$. This means, $J + (L + T) = G$. Since $L + T \leq_e G$ and $J \ll_{gs} G$, it follows that $L + T = G$. Again since $L \ll_{gs} G$ and $T \leq_e G$, we get $T = G$. Therefore $J + L \ll_{gs} G$.

□

Definition 2.9 Let P and Q be N-subgroups of G. Q is said to be a g-supplement of P if $G = P + Q$ and $G = P + T$ with $T \leq_e Q$ implies that $T = Q$.
An N-group G is called g-supplemented if every ideal of G has a g-supplement.

Remark 2.10 Every supplemented N-group is g-supplemented.

Example 2.11 Consider the nearring given in (K(139), page 418 of [17]). Let $N = (D_8, +, \star)$, the dihedral group of order 8 and $G = N$.

The subgroups of $(D_8, +)$ are $H_1 = \langle e \rangle$, $H_2 = \langle s \rangle$, $H_3 = \langle sr^2 \rangle$, $H_4 = \langle r^2 \rangle$, $H_5 = \langle sr^3 \rangle$, $H_6 = \langle sr \rangle$, $H_7 = \langle \{s, r^2\} \rangle$, $H_8 = \langle r \rangle$ and $H_9 = \langle \{r^2, sr^3\} \rangle$. The N-subgroups are H_1, H_2, H_3, H_4, H_5, H_7 and H_9, and ideals of D_8 are H_4, H_7 and H_9. The ideals of H_9(when it is considered as an N-group) are H_4 and H_5 which are not essential in H_9. Observe that H_9 is not a supplement of H_7 as there exists an ideal H_5 of H_9 such that $H_7 + H_5 = D_8$. Furthermore, all ideals of H_9 are not essential, we do not have any essential ideal I of H_9 such that $H_7 + I = D_8$. Therefore H_9 is a g-supplement of H_7 (Table 2).

Lemma 2.12 *Let P, Q be ideals of G. Then Q is a g-supplement of P if and only if $G = P + Q$ and $P \cap Q \ll_{gs} Q$.*

Proof Suppose Q is a g-supplement of P in G. Then $P + Q = G$ and $P + Q' \neq G$ for any essential ideal Q' of Q. We prove $P \cap Q \ll_{gs} Q$. Let $T \leq_e Q$ such that $(P \cap Q) + T = Q$. Then $G = P + Q = P + (P \cap Q) + T = P + T$, as $(P \cap Q) \subseteq P$. Now $G = P + T$ where $T \leq_e Q$. Since Q is a g-supplement of P, we get $T = Q$. Therefore $P \cap Q \ll_{gs} Q$.

Conversely, suppose that $G = P + Q$ and $P \cap Q \ll_{gs} Q$. To show Q is a g-supplement of P in G, let $G = P + T$ for some essential ideal T of Q. Now, since $T \subseteq Q$, by modular law we get $Q = Q \cap G = Q \cap (P + T) = (Q \cap P) + T$. Since $(P \cap Q) \ll_{gs} Q$ and $T \leq_e Q$, we get $T = Q$. Therefore Q is a g-supplement of P in G. □

Table 2 Multiplication $\star$ on N

$\star$	e	r	r^2	r^3	s	sr^3	sr^2	sr
e	e	e	e	e	e	e	e	e
r	e	r	r^2	r^3	s	sr^3	sr^2	sr
r^2	e	r^2	e	r^2	e	e	e	e
r^3	e	r^3	r^2	r	s	sr^3	sr^2	sr
s	e	s	r^2	sr^2	s	e	sr^2	r^2
sr^3	e	sr^3	e	sr^3	e	sr^3	e	sr^3
sr^2	e	sr^2	r^2	s	s	e	sr^2	r^2
sr	e	sr	e	sr	e	sr^3	e	sr^3

Proposition 2.13 *[12]*

1. *Let G be an N-group and let I, J be the ideals of G with $G = I \oplus J$. Then $a+b = b+a$ for all $a \in I$ and $b \in J$.*
2. *If $N = N_0$, $n \in N$, $a \in I$, $b \in J$ and the sum $I \oplus J$ is direct in G, then $n(a+b) = na+nb$.*
3. *Let $N = N_0$ and $I \trianglelefteq_N G$ be a direct summand. Then each ideal of I is an ideal of G.*

Lemma 2.14 *Let A, B and C be ideals of G. Then*

$$A \cap (B + C) \trianglelefteq_N B \cap (A + C) + C \cap (A + B).$$

Proof We have $A \cap (B + C) \trianglelefteq_N G$. Let $p \in A \cap (B + C)$. Then $p \in A$ and $p \in B + C$, which implies $p = b + c$ for some $b \in B$ and $c \in C$. Now, $b = p - c \in A + C$ and $c = -b + p \in B + A = A + B$ and hence $p = b + c \in B \cap (A + C) + C \cap (A + B)$. Therefore $A \cap (B + C) \trianglelefteq_N G$, which is contained in $B \cap (A + C) + C \cap (A + B)$ and hence $A \cap (B + C) \trianglelefteq_N B \cap (A + C) + C \cap (A + B)$. □

Lemma 2.15 *Let N be zero-symmetric and G_1, U be ideals of G and G_1 be g-supplemented and a direct summand of G. If $G_1 + U$ has a g-supplement in G, then U has a g-supplement in G.*

Proof Let X be a g-supplement of $G_1 + U$ in G. Then by Lemma 2.12, $G_1 + U + X = G$ and $(G_1 + U) \cap X \ll_{gs} X$. Since G_1 is g-supplemented, $(U + X) \cap G_1$ has a g-supplement Y in G_1. That is, $G_1 \cap (U + X) + Y = G_1$ and $G_1 \cap (U + X) \cap Y \ll_{gs} Y$, by Lemma 2.12. Since G_1 is a direct summand, $Y \trianglelefteq_N G$. This yield,

$$G = G_1 \cap (U + X) + Y + (U + X) = U + X + Y$$

and

$$\begin{aligned} U \cap (X + Y) &\trianglelefteq_N X \cap (U + Y) + Y \cap (U + X) \\ &\trianglelefteq_N X \cap (G_1 + U) + Y \cap G_1 \cap (U + X) \\ &\ll_{gs} X + Y. \end{aligned}$$

Hence, $X + Y$ is a g-supplement of U in G. □

Proposition 2.16 *Let G be an N-group. Let K, U and V be ideals of G such that $K \subseteq U$. Let V be a g-supplement of U in G. Then $\frac{V+K}{K}$ is a g-supplement of $\frac{U}{K}$.*

Proof Since V is a g-supplement of U in G, we have $G = U + V$ and $U \cap V \ll_{gs} V$ which implies $\frac{U \cap V+K}{K} \ll_{gs} \frac{V+K}{K}$. Now $\frac{G}{K} = \frac{U+V}{K} = \frac{U}{K} + \frac{V+K}{K}$. Also $\frac{U}{K} \cap \frac{V+K}{K} = \frac{U \cap (V+K)}{K} = \frac{U \cap V+K}{K} \ll_{gs} \frac{V+K}{K}$. Therefore, $\frac{V+K}{K}$ is a g-supplement of $\frac{U}{K}$. □

3 Superfluous ideals

Definition 3.1 Let $\Omega \trianglelefteq_N G$. An ideal (or N-subgroup) H of G is said to be Ω-superfluous in G if $\Omega \nsubseteq H$ and for any ideal L of G, $\Omega \subseteq L + H$ implies $\Omega \subseteq L$. We denote it by $H \ll_\Omega G$.

Note 3.2 *If $\Omega = G$, then Ω-superfluous coincides with the notion of superfluous defined by [20]. In this case, we denote $H \ll G$ whenever an ideal H is superfluous in G. Trivially, the ideal* (0) *is superfluous in G.*

Example 3.3 Let $N = \mathbb{Z}$, the set of integers and $G = (\mathbb{Z}_{24}, +_{24})$. Then G is an N-group. Let $\Omega = 8\mathbb{Z}_{24}$. Then $6\mathbb{Z}_{24}$, $3\mathbb{Z}_{24}$, $12\mathbb{Z}_{24}$ are Ω-superfluous, whereas $3\mathbb{Z}_{24}$ is not superfluous in G, since $3\mathbb{Z}_{24} + 2\mathbb{Z}_{24} = \mathbb{Z}_{24}$ but $2\mathbb{Z}_{24} \neq \mathbb{Z}_{24}$.

Example 3.4 Let $N = \mathbb{Z}$ and $G = \mathbb{Z}_{12}$. Then G is an N-group. Let $\Omega = 4\mathbb{Z}_{12}$. Then $6\mathbb{Z}_{12}$, $3\mathbb{Z}_{12}$ are Ω-superfluous, whereas $3\mathbb{Z}_{12}$ is not superfluous in G, since $3\mathbb{Z}_{12} + 2\mathbb{Z}_{12} = \mathbb{Z}_{12}$ but $2\mathbb{Z}_{12} \neq \mathbb{Z}_{12}$.

Example 3.5 Let $N = \begin{pmatrix} 0 & \mathbb{Z}_{q^m} \\ 0 & 0 \end{pmatrix}$ and $G = N$. Then the ideals and N-subgroups are $H_i = \{\begin{pmatrix} 0 & q^i\mathbb{Z}_{q^m} \\ 0 & 0 \end{pmatrix} : 0 \le i \le m\}$. Let $\Omega = H_k$. Then $H_j \ll_\Omega G$ for all $j \le k$.

Example 3.6 Consider the N-group given in the Example 2.11.

1. $H_7 \ll_{H_9} G$, $H_4 \ll_{H_9} G$, $H_9 \ll_{H_7} G$, $H_4 \ll_{H_7} G$.
2. The N-subgroups H_2, H_3, H_4 and H_7 are Ω-superfluous in G with $\Omega = H_9$.
3. The N-subgroups H_4, H_5, H_9 are Ω-superfluous in G with $\Omega = H_7$.

Proposition 3.7 *Let Ω be an ideal and X be an ideal (or N-subgroup) of G. If X is Ω-superfluous in G, then $X \cap Y$ is Ω-superfluous in G for any ideal (or N-subgroup) Y of G.*

Proof Suppose X is Ω-superfluous in G. Let $Y \trianglelefteq_N G$. Since $\Omega \nsubseteq X$, we have $\Omega \nsubseteq X \cap Y$. On a contrary, suppose $X \cap Y$ is not superfluous in G. Then there exists a proper ideal K of G such that $\Omega \nsubseteq K$ and $\Omega \subseteq (X \cap Y) + K$. Now, since $X \cap Y \subseteq X$ we get $\Omega \subseteq X + K$, a contradiction as $X \ll_\Omega G$. Therefore $X \cap Y \ll_\Omega G$. □

Proposition 3.8 *Let Ω, K be ideals of G. If $K \ll_\Omega G$, then $K \cap \Omega \ll G$.*

Proof Let $K \ll_\Omega G$. To prove $K \cap \Omega \ll G$, let $L \trianglelefteq_N G$ be such that $(K \cap \Omega) + L = G$. Now $\Omega \subseteq (K \cap \Omega) + L \subseteq K + L$. Now since $K \ll_\Omega G$, we have that $\Omega \subseteq L$. Also since $K \cap \Omega \subseteq \Omega \subseteq L$, it follows that $L = (K \cap \Omega) + L = G$. Therefore, $K \cap \Omega \ll G$. □

Proposition 3.9 *Let N be zero-symmetric and $\Omega \trianglelefteq_N G$, which is a direct summand, and let $P \trianglelefteq_N G$ contained in Ω. Then $P \ll_\Omega G$ if and only if $P \ll \Omega$.*

Proof Suppose $P \ll_\Omega G$. Since $P \trianglelefteq_N G$ and $P \subseteq \Omega$, we have $P \trianglelefteq_N \Omega$. To prove $P \ll \Omega$, let $L \trianglelefteq_N \Omega$ be such that $P + L = \Omega$. Since Ω is a direct summand, by Proposition 2.13(3), $L \trianglelefteq_N G$. Now $\Omega \subseteq P + L$ and $P \ll_\Omega G$, we get $\Omega \subseteq L$. Since $L \subseteq \Omega$, it follows that $L = \Omega$.

Conversely, suppose that $P \ll \Omega$. Let $L \trianglelefteq_N G$ be such that $\Omega \subseteq P + L$. Now $\Omega = (P + L) \cap \Omega = P + (L \cap \Omega)$, by modular law, and since $P \ll \Omega$, it follows that $\Omega = L \cap \Omega$. Hence $\Omega \subseteq L$. □

Proposition 3.10 *Let $K \trianglelefteq_N G$ and let P and Ω be ideals of G which are contained in K. If $P \ll_\Omega K$, then $P \ll_\Omega G$.*

Proof Suppose that $P \ll_\Omega K$. To prove $P \ll_\Omega G$, let $L \trianglelefteq_N G$ be such that $\Omega \subseteq P + L$. Since $\Omega \subseteq K$ and by modular law, we have $\Omega \subseteq (P + L) \cap K = P + (L \cap K)$. Since $L \cap K \trianglelefteq_N K$ and $P \ll_\Omega K$, we have $\Omega \subseteq (L \cap K)$, which implies $\Omega \subseteq L$. Hence $P \ll_\Omega G$. □

Remark 3.11 It can be easily seen that the Propositions 3.7, 3.8, 3.9 and 3.10 hold for N-subgroups also.

Remark 3.12 The following proposition holds for ideals of G but not for N-subgroups, as sum of two N-subgroups need not be an N-subgroup.

Proposition 3.13 *Let N_1, N_2, Ω be ideals of G. Then $N_1 \ll_\Omega G$ and $N_2 \ll_\Omega G$ if and only if $N_1 + N_2 \ll_\Omega G$.*

Proof Suppose that $N_1 \ll_\Omega G$ and $N_2 \ll_\Omega G$. Let $L \trianglelefteq_N G$ be such that $\Omega \subseteq (N_1 + N_2) + L = N_1 + (N_2 + L)$. Since $N_1 \ll_\Omega G$, we have $\Omega \subseteq N_2 + L$, and again since $N_2 \ll_\Omega G$, we get $\Omega \subseteq L$.

Conversely, suppose $N_1 + N_2 \ll_\Omega G$. Let $L \trianglelefteq_N G$ be such that $\Omega \subseteq N_1 + L \subseteq (N_1 + N_2) + L$. Now since $N_1 + N_2 \ll_\Omega G$, we get $\Omega \subseteq L$. Similar assertion proves $N_2 \ll_\Omega G$. □

Note 3.14 *Let N be zero-symmetric and $K_1 \trianglelefteq_N G_1 \trianglelefteq_N G$ and $K_2 \trianglelefteq_N G_2 \trianglelefteq_N G$, $\Omega \trianglelefteq_N G$ such that $G_1 \oplus G_2 = G$. Then $K_1 \ll_\Omega G_1$ and $K_2 \ll_\Omega G_2$ if and only if $K_1 + K_2 \ll_\Omega G_1 + G_2$.*

Proposition 3.15 *Let Ω, K, P be ideals of G such that $K \subset \Omega$, $K \subseteq P$ and $\Omega \nsubseteq P$. Then $P \ll_\Omega G$ if and only if $K \ll_\Omega G$ and $\frac{P}{K} \ll_{\frac{\Omega}{K}} \frac{G}{K}$.*

Proof Suppose $P \ll_\Omega G$. To prove $K \ll_\Omega G$, let $L \trianglelefteq_N G$ such that $\Omega \subseteq K + L$. Since $K \subseteq P$, we get $\Omega \subseteq P + L$. Since $P \ll_\Omega G$, we have $\Omega \subseteq L$, and thus $K \ll_\Omega G$.
Now to prove $\frac{P}{K} \ll_{\frac{\Omega}{K}} \frac{G}{K}$, let $\frac{L}{K} \trianglelefteq_N \frac{G}{K}$, where $K \subseteq L \trianglelefteq_N G$ such that, $\frac{\Omega}{K} \subseteq \frac{P}{K} + \frac{L}{K} = \frac{(P+L)}{K}$. Then $\Omega \subseteq P + L$. Since $P \ll_\Omega G$, we get $\Omega \subseteq L$, which implies that $\frac{\Omega}{K} \subseteq \frac{L}{K}$. Hence, $\frac{P}{K} \ll_{\frac{\Omega}{K}} \frac{G}{K}$.

Conversely, suppose that $K \ll_\Omega G$ and $\frac{P}{K} \ll_{\frac{\Omega}{K}} \frac{G}{K}$. To prove $P \ll_\Omega G$, let $L \trianglelefteq_N G$ such that $\Omega \subseteq P + L$. Then $\frac{\Omega}{K} \subseteq \frac{(P+L)}{K} = \frac{P}{K} + \frac{L+K}{K}$. Since $\frac{P}{K} \ll_{\frac{\Omega}{K}} \frac{G}{K}$, it follows that $\frac{\Omega}{K} \subseteq \frac{L+K}{K}$, which implies $\Omega \subseteq L + K$. Since $K \ll_\Omega G$, we get $\Omega \subseteq L$. Hence, $P \ll_\Omega G$. □

Proposition 3.16 *Let $\{\Omega_i\}_{i \in I}$ be a family of ideals of G and $K \trianglelefteq_N G$. If for each $i \in I$, $K \ll_{\Omega_i} G$, then $K \ll_{\sum\limits_{i \in I} \Omega_i} G$.*

Proof Suppose $K \ll_{\Omega_i} G$ for each $i \in I$ and $\sum\limits_{i \in I} \Omega_i \subseteq K + L$ where $L \trianglelefteq_N G$. Then since $\Omega_i \subseteq \sum \Omega_i \subseteq K + L$ for each $i \in I$ and $K \ll_{\Omega_i} G$, we have $\Omega_i \subseteq L$, which shows that $\sum \Omega_i \subseteq L$. Hence, $K \ll_{\sum\limits_{i \in I} \Omega_i} G$. □

Corollary 3.17 *Let K_1 and K_2 be ideals of G such that $K_1 \ll_{K_2} G$ and $K_2 \ll_{K_1} G$. Then $K_1 \cap K_2 \ll_{K_1+K_2} G$.*

Proof First we show that $K_1 \cap K_2 \ll_{K_1} G$. For this, let $K_1 \subseteq (K_1 \cap K_2) + X$, where X is an ideal of G. Now $K_1 \subseteq K_2 + X$ and since $K_2 \ll_{K_1} G$ we get $K_1 \subseteq X$. Therefore, $K_1 \cap K_2 \ll_{K_1} G$. In a similar way, we get $K_1 \cap K_2 \ll_{K_2} G$. Hence, by Proposition 3.16, it follows that $K_1 \cap K_2 \ll_{K_1+K_2} G$. □

The converse of the Corollary 3.17 need not be true, as shown in the following example.

Example 3.18 Consider the N-group $\mathbb{Z}_{48}$ over $\mathbb{Z}$. Let $K_1 = 8\mathbb{Z}_{48}$ and $K_2 = 6\mathbb{Z}_{48}$. Then $8\mathbb{Z}_{48} \cap 6\mathbb{Z}_{48} \ll_{8\mathbb{Z}_{48}+6\mathbb{Z}_{48}} \mathbb{Z}_{48}$, whereas $8\mathbb{Z}_{48} \ll_{6\mathbb{Z}_{48}} \mathbb{Z}_{48}$ and $6\mathbb{Z}_{48}$ is not $8\mathbb{Z}_{48}$-superfluous in $\mathbb{Z}_{48}$.

Proposition 3.19 *Let K and Ω be ideals of G such that $\Omega \nsubseteq K$. Let G' be an N-group and $f : G \to G'$ be an epimorphism with $f(\Omega) \nsubseteq f(K)$. If $K \ll_{\Omega} G$, then $f(K) \ll_{f(\Omega)} G'$. The converse holds if f is injective.*

Proof Suppose that $K \ll_{\Omega} G$. Since f is an epimorphism, we have $f(K) \trianglelefteq_N G'$ by Theorem 1.30 of [17]. Let $X \trianglelefteq_N G'$ be such that $f(\Omega) \subseteq f(K) + X$. Then $\Omega \subseteq K + f^{-1}(X)$. Since $f^{-1}(X) \trianglelefteq_N G$ and $K \ll_{\Omega} G$, it follows that $\Omega \subseteq f^{-1}(X)$. Hence $f(\Omega) \subseteq X$.
Conversely, suppose that f is injective and $f(K) \ll_{f(\Omega)} G'$. Let $X \trianglelefteq_N G$ be such that $\Omega \subseteq K + X$. Then $f(\Omega) \subseteq f(K + X) = f(K) + f(X)$. Since $f(K) \ll_{f(\Omega)} G'$, we have $f(\Omega) \subseteq f(X)$. Therefore, $f^{-1}(f(\Omega)) \subseteq f^{-1}(f(X))$. Now by 2.17 of [17], $\Omega + ker\ f \subseteq X + ker\ f$. As f is injective, we get $\Omega \subseteq X$. □

Remark 3.20 Unlike in module over rings, the condition f is a homomorphism is not sufficient, as a homomorphic image of an ideal need not be an ideal. So we consider f to be an epimorphism. The following example justifies the condition f is a homomorphism is not sufficient.

Example 3.21 Consider the nearring given in the Example 3.6 and the ideals $H_9 = \{e, r^2, sr^3, sr\}$ and $H_7 = \{e, r^2, s, sr^2\}$ of G. Let f be an N-endomorphism of G defined by

$$f(g) = g \cdot sr \quad \text{for all} \quad g \in G.$$

Then $f(H_9) = \{e, sr^3\}$ and $f(H_7) = \{e, r^2\}$. It can be seen that $H_7 \ll_{H_9} G$, but $f(H_7)$ is not $f(H_9)$ superfluous in G, since $f(H_9) \ntrianglelefteq_N G$.

Definition 3.22 Let $\Omega \trianglelefteq_N G$. G is said to be Ω-hollow if every proper ideal of G which does not contain in Ω is Ω-superfluous in G.

Remark 3.23 1. Every hollow N-group is Ω-hollow with $\Omega = G$.
2. Ω-hollow need not be hollow and we justify this in the following example.

Example 3.24 Consider the Example 3.6 in which H_4, H_7 are H_9-superfluous in G and H_4,H_9 are H_7-superfluous in G. Hence it is H_7-hollow as well as H_9-hollow. However, G is not hollow, since H_7 is not superfluous in G as $H_7 + H_9 = G$ but $H_9 \neq G$.

Definition 3.25 Let N be zero-symmetric, and let Ω, H be ideals of G such that $\Omega \nsubseteq H$. An N-subgroup K of G is said to be an Ω-supplement of H if $\Omega \subseteq H + K$ and $\Omega \nsubseteq H + K'$ for any ideal K' of K.

Table 3 Multiplication ⋆ on N

$\star$	e	r	r^2	r^3	s	sr^3	sr^2	sr
e	e	e	e	e	e	e	e	e
r	e	e	e	e	e	r^2	e	e
r^2	e	e	e	e	e	e	e	e
r^3	e	e	e	e	e	r^2	e	e
s	e	e	e	e	e	e	e	e
sr^3	e	e	e	e	e	r^2	e	e
sr^2	e	e	e	e	e	e	e	e
sr	e	e	e	e	e	r^2	e	e

Example 3.26 Consider the Example 2.11.

Let $\Omega = H_7$. Here H_2 is an Ω-supplement of H_4, but H_2 is not a supplement of H_4 as $H_2 + H_4 \neq G$.

Example 3.27 $N = D_8$ with the multiplication given in the Table 3. Let $G = N$.
The ideals of G are $I_1 = G$, $I_2 = \{e, r^2, r^3, r\}$, $I_3 = \{e, sr^3, r^2, sr\}$, $I_4 = \{e, sr^2, s, r^2\}$, $I_5 = \{e, r^2\}$ and $I_6 = \{e\}$, and N-subgroups are $I_1, I_2, I_3, I_4, I_5, X_1 = \{e, s\}, X_2 = \{e, sr^2\}$, $X_3 = \{e, sr\}$. Let $\Omega = I_4$. Here I_3 is an Ω-supplement of I_2, X_1 is an Ω-supplement of I_2, I_3 and I_5. Further, X_1 is not a supplement of I_5 as $I_5 + X_1 \neq G$.

Note 3.28 *If $\Omega = G$, then Ω-supplement coincides with the supplement as defined by [20].*

4 Strictly superfluous ideals

In case of N-groups, we have substructures namely N-subgroups and ideals, whereas in modules over rings, these concepts coincide. So we consider the notion strictly superfluous in terms of N-subgroups. We provide explicit examples which indicate that the classes superfluous and strictly superfluous are different.

Definition 4.1 An ideal H of G is called strictly superfluous in G (denoted by $H \ll^s G$) if K is any N-subgroup of G such that $H + K = G$, then $K = G$.

Definition 4.2 Let G be an N-group and $\Omega \leq_N G$. An ideal H of G is said to be strictly Ω-superfluous in G if for any N-subgroup L of G, $\Omega \subseteq L + H$ implies $\Omega \subseteq L$. We denote this by $H \ll^s_\Omega G$.

Example 4.3 Let $N = \left(\begin{pmatrix} \mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}, +, \cdot \right)$ where $\mathbb{Z}_4$ is the set of residue classes modulo 4 and $G = N$.
N-subgroups of G are

$$H_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, H_2 = \begin{pmatrix} 2\mathbb{Z}_4 & 0 \\ 0 & 0 \end{pmatrix},$$
$$H_3 = \begin{pmatrix} 0 & 0 \\ 0 & 2\mathbb{Z}_4 \end{pmatrix}, H_4 = \begin{pmatrix} 0 & 2\mathbb{Z}_4 \\ 0 & 0 \end{pmatrix}, H_5 = \begin{pmatrix} 0 & 2\mathbb{Z}_4 \\ 0 & 2\mathbb{Z}_4 \end{pmatrix}, H_6 = \begin{pmatrix} 0 & 2\mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix},$$

$$H_7 = \begin{pmatrix} 2\mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & 0 \end{pmatrix}, H_8 = \begin{pmatrix} 2\mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & 2\mathbb{Z}_4 \end{pmatrix}, H_9 = \begin{pmatrix} 2\mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}, H_{10} = \begin{pmatrix} \mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & 0 \end{pmatrix},$$
$$H_{11} = \begin{pmatrix} \mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & 2\mathbb{Z}_4 \end{pmatrix}, H_{12} = \begin{pmatrix} 2\mathbb{Z}_4 & 0 \\ 0 & 2\mathbb{Z}_4 \end{pmatrix}, H_{13} = \begin{pmatrix} \mathbb{Z}_4 & 0 \\ 0 & 0 \end{pmatrix}, H_{14} = \begin{pmatrix} \mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}.$$

Ideals are $H_1, H_2, H_3, H_4, H_5, H_6, H_7, H_8, H_9, H_{10}, H_{11}, H_{12}, H_{14}$. Let $\Omega = H_3$. Then $H_{10} \ll^s_\Omega G$ but not strictly superfluous in G since $H_{10} + H_{11} = G$ and $H_{11} \neq G$. H_7 is not strictly H_{12}-superfluous in G as there exists H_5 such that $H_{12} \nsubseteq H_5$ and $H_{12} \subseteq H_7 + H_5$.

Example 4.4 Consider the N-group given in Example 3.6. Then $H_4 \ll^s_{H_5} G$, $H_7 \ll^s_{H_5} G$. Here H_7 is not strictly superfluous, since $H_7 + H_5 = G$ but $H_5 \neq G$. Also $H_7 \ll_{H_9} G$ but H_7 is not strictly H_9-superfluous since there exists $H_5 \leq_N G$ such that $H_9 \nsubseteq H_5$ but $H_9 \subseteq H_7 + H_5$.

Proposition 4.5 *Let $\Omega \leq_N G$, $K \trianglelefteq_N G$. If $K \ll^s_\Omega G$, then $K \cap \Omega \ll^s G$.*

Proof Let $K \ll^s_\Omega G$. To prove $K \cap \Omega \ll^s G$, let $L \leq_N G$ be such that $(K \cap \Omega) + L = G$. Now $\Omega \subseteq (K \cap \Omega) + L \subseteq K + L$. Since $K \ll^s_\Omega G$, we have that $\Omega \subseteq L$. Also since $K \cap \Omega \subseteq \Omega \subseteq L$, it follows that $L = (K \cap \Omega) + L = G$. Therefore, $K \cap \Omega \ll^s G$. □

Proposition 4.6 *Let $P \trianglelefteq_N G$ and K, Ω be N-subgroups of G such that P and Ω are contained in K. Then $P \ll^s_\Omega K$ implies $P \ll^s_\Omega G$.*

Proof Suppose that $P \ll^s_\Omega K$. To prove $P \ll^s_\Omega G$, let $L \leq_N G$ be such that $\Omega \subseteq P + L$. Since $\Omega \subseteq K$ and by modular law, we get $\Omega \subseteq (P + L) \cap K = P + (L \cap K)$. Since $L \cap K \leq_N K$ and $P \ll^s_\Omega K$, we conclude that $\Omega \subseteq L \cap K \subseteq L$. Therefore $P \ll^s_\Omega G$. □

The other implication follows when $K = \Omega$.

Proposition 4.7 *Let $P \trianglelefteq_N G$ and $\Omega \leq_N G$ such that $P \subset \Omega$. Then $P \ll^s_\Omega G$ if and only if $P \ll^s \Omega$.*

Proof Suppose $P \ll^s_\Omega G$. To prove $P \ll^s \Omega$, let $L \leq_N \Omega$ such that $P + L = \Omega$. Now $\Omega \subseteq P + L$ and $P \ll^s_\Omega G$, we get $\Omega \subseteq L$. Since $L \subseteq \Omega$, it follows that $L = \Omega$. □

Proposition 4.8 *Let N_1, N_2 be ideals of G. Let $\Omega \leq_N G$ such that $\Omega \nsubseteq N_1$, $\Omega \nsubseteq N_2$. Then $N_1 \ll^s_\Omega G$ and $N_2 \ll^s_\Omega G$ if and only if $N_1 + N_2 \ll^s_\Omega G$.*

Proof The proof is similar to the proof of Proposition 3.13. □

Proposition 4.9 *Let N be zero-symmetric and $\Omega \leq_N G$. Let K, P be ideals of G such that $K \subseteq P$, $K \subset \Omega$ and $\Omega \nsubseteq P$. Then $P \ll^s_\Omega G$ if and only if $K \ll^s_\Omega G$ and $\frac{P}{K} \ll^s_{\frac{\Omega}{K}} \frac{G}{K}$.*

Proof The proof is similar to the proof of Proposition 3.15. □

In Proposition 4.10 and 4.11, we assume N to be zero-symmetric, so that every ideal can also be considered as an N-group.

Proposition 4.10 *Let N be zero-symmetric, Ω be an N-subgroup of G, and $\{\Theta_j\}_{j \in J}$ be a family of ideals of G. If $K \trianglelefteq_N G$ such that $K \ll^s_\Omega G$ and $K \ll^s_{\Theta_j} G$ for all $j \in J$, then $K \ll^s_{\Omega + \sum_j \Theta_j} G$.*

Proof Let $K \ll^s_\Omega G$ and $K \ll^s_{\Theta_j} G$ for all $i \in I$ $j \in J$. Let $L \leq_N G$ be such that $\Omega + \sum_j \Theta_j \subseteq K + L$. Now $\Omega \subseteq \Omega + \sum_j \Theta_j \subseteq K + L$. Since $K \ll^s_\Omega G$, we get $\Omega \subseteq L$. Now $\Theta_j \subseteq \Omega + \sum_j \Theta_j \subseteq K + L$. Since $K \ll^s_{\Theta_j} G$, we get $\Theta_j \subseteq L$. Therefore $\Omega + \sum_j \Theta_j \subseteq L$. Hence $K \ll^s_{\Omega + \sum_j \Theta_j} G$. □

Proposition 4.11 *Let N be zero-symmetric and K_1, K_2 be ideals of G. If $K_1 \ll^s_{K_2} G$ and $K_2 \ll^s_{K_1} G$, then $K_1 \cap K_2 \ll^s_{K_1+K_2} G$.*

Proof Suppose $K_1 \ll^s_{K_2} G$ and $K_2 \ll^s_{K_1} G$. First we show that $K_1 \cap K_2 \ll^s_{K_1} G$. For this, let $K_1 \subseteq (K_1 \cap K_2) + X$, where $X \leq_N G$. Then $K_1 \subseteq K_2 + X$ and since $K_2 \ll^s_{K_1} G$ we get $K_1 \subseteq X$. Therefore $K_1 \cap K_2 \ll^s_{K_1} G$. In a similar way, we get $K_1 \cap K_2 \ll^s_{K_2} G$. Hence, by Proposition 4.10, $K_1 \cap K_2 \ll^s_{K_1+K_2} G$. □

Definition 4.12 Let G_1 and G_2 be N-groups and $\Omega \leq_N G$. An N-epimorphism $f : G_1 \to G_2$ is called strictly Ω-superfluous if $ker\ f \ll^s_\Omega G_1$.

Lemma 4.13 *Let $K \trianglelefteq_N G$ and $\Omega \leq_N G$ be such that $\Omega \nsubseteq K$. Then $K \ll^s_\Omega G$ if and only if the natural map $f : G \to \frac{G}{K}$ is strictly Ω-superfluous.*

Proof Since $ker\ f = \{g \in G : f(g) = 0 \in \frac{G}{K}\} = K$, the proof is clear. □

Lemma 4.14 *Let $K \trianglelefteq_N G$ and $\Omega \leq_N G$ be such that $\Omega \nsubseteq K$. Then $K \ll^s_\Omega G$ if and only if for every N-group G_1 and N-homomorphism $h : G_1 \to G$ with $\Omega \subseteq K + Im\ h$, $\Omega \subseteq Im\ h$.*

Proof Suppose $K \ll^s_\Omega G$. Let G_1 be an N-group and $h : G_1 \to G$ be an N-homomorphism with $\Omega \subseteq K + Im\ h$. Since $Im\ h$ is an N-subgroup of G and $K \ll^s_\Omega G$, we have $\Omega \subseteq Im\ h$. Conversely, suppose that $\Omega \subseteq K + X$ where $X \leq_N G$. Let $i : X \to G$ be an inclusion map. Clearly i is an N-homomorphism, and so by hypothesis, we can conclude that $\Omega \subseteq X$. Therefore, $K \ll^s_\Omega G$. □

Lemma 4.15 *Let Ω be an N-subgroup and K be an ideal of G. Let G' be an N-group and $f : G \to G'$ be an N-epimorphism such that $f(\Omega) \nsubseteq f(K)$. If $K \ll^s_\Omega G$, then $f(K) \ll^s_{f(\Omega)} G$. The converse holds if f is injective.*

Proof Suppose $K \ll^s_\Omega G$. Since f is an epimorphism, we have $f(K) \trianglelefteq_N G'$. Let $X \leq_N G'$ be such that $f(\Omega) \subseteq f(K) + X$. Then $\Omega \subseteq K + f^{-1}(X)$. Since $f^{-1}(X) \leq_N G$ and $K \ll^s_\Omega G$, we have $\Omega \subseteq f^{-1}(X)$. Hence $f(\Omega) \subseteq X$.
Conversely, suppose that $f(K) \ll^s_{f(\Omega)} G'$. Let $X \leq_N G$ be such that $\Omega \subseteq K + X$. Then $f(\Omega) \subseteq f(K + X) = f(K) + f(X)$. Since $f(K) \ll^s_{f(\Omega)} G'$, we have $f(\Omega) \subseteq f(X)$. Therefore, $f^{-1}(f(\Omega)) \subseteq f^{-1}(f(X))$ which implies $\Omega + ker\ f \subseteq X + ker\ f$. Since f is injective, we get $\Omega \subseteq X$. □

Example 4.16 Consider the Example 3.21. Then it can be seen that $H_9 \ll^s_{H_7} G$, but $f(H_9)$ is not strictly $f(H_7)$ superfluous in G, since $f(H_9)$ is not an ideal of G.

Definition 4.17 Let N be zero-symmetric nearring. Let $\Omega \leq_N G$ and $H \trianglelefteq_N G$ be such that $\Omega \nsubseteq H$. An N-subgroup K of G is said to be a strictly Ω-supplement of H if $\Omega \subseteq H + K$ and $\Omega \nsubseteq H + K'$ for any ideal K' of K.

The following remark is a straightforward observation.

Remark 4.18 1. If N is zero-symmetric and $\Omega = G$, then every strictly Ω-supplement is a supplement (defined by [20]).
2. Let N be zero-symmetric. Let $H \trianglelefteq_N G$ be such that $\Omega \nsubseteq H$. Then every Ω-supplement of H is a strictly Ω-supplement of H.
3. If N is zero-symmetric and $\Omega = G$, then every strictly Ω-supplement is a supplement.

5 Superfluous ideals of $M_n(N)$-group N^n

For a zero-symmetric right nearring N with 1, let N^n be the direct sum of n copies of $(N, +)$. The elements of N^n are column vectors and written as $(r_1, \cdots, r_n)$. The symbols i_j and π_j respectively, denote the i^{th} coordinate injective and j^{th} coordinate projective maps.
For an element $a \in N$, $i_i(a) = (0, \cdots, \underbrace{a}_{i^{th}}, \cdots, 0)$, and $\pi_j(a_1, \cdots, a_n) = a_j$, for any $(a_1, \cdots, a_n) \in N^n$. The nearring of $n \times n$ matrices over N, denoted by $M_n(N)$, is defined to be the subnearring of $M(N^n)$, generated by the set of functions $\{f_{ij}^a : N^n \to N^n \mid a \in N, 1 \leq i, j \leq n\}$ where $f_{ij}^a(k_1, \cdots, k_n) := (l_1, l_2, \cdots, l_n)$ with $l_i = ak_j$ and $l_p = 0$ if $p \neq i$. Clearly, $f_{ij}^a = i_i f^a \pi_j$, where $f^a(x) = ax$, for all $a, x \in N$. If N happens to be a ring, then f_{ij}^a corresponds to the $n \times n$-matrix with a in position (i, j) and zeros elsewhere.

Notation 5.1 ([9], Notation 1.1)
For any ideal $\mathcal{A}$ of $M_n(N)$-group N^n, we write

$$\mathcal{A}_{**} = \{a \in N : a = \pi_j A, \textit{ for some } A \in \mathcal{A}, 1 \leq j \leq n\}, \textit{ an ideal of } {}_N N.$$

We denote $M_n(N)$ for a matrix nearring, N^n for an $M_n(N)$-group N^n. We refer to Meldrum & Van der Walt [15] for preliminary results on matrix nearrings.
From [10], for any $s \in G$, the ideal generated by s is denoted by $\langle s \rangle$ and defined as, $\langle s \rangle = \bigcup_{i=1}^{\infty} U_{i+1}$, where $U_{i+1} = U_i^* \cup U_i^0 \cup U_i^+$ with $U_0 = \{s\}$, and $U_i^* = \{g+y-g : g \in G, y \in U_i\}$, $U_i^0 = \{p - q : p, q \in U_i\} \cup \{p + q : p, q \in U_i\}$, $U_i^+ = \{n(g + a) - ng : n \in N, g \in G, a \in U_i\}$.

Theorem 5.2 (Theorem 1.4 of [9]) *Suppose $A \subseteq N$.*

1. *If A^n is an ideal of ${}_{M_n(N)}N^n$, then $A = (A^n)_{\star\star}$.*
2. *If A is an ideal of ${}_N N$ if and only if A^n is an ideal of ${}_{M_n(N)}N^n$.*
3. *If A is an ideal of ${}_N N$, then $A = (A^n)_{\star\star}$.*

Lemma 5.3 (Lemma 1.5 of [9])

1. *If $\mathcal{I}$ is an ideal of ${}_{M_n(N)}N^n$, then $(\mathcal{I}_{\star\star})^n = \mathcal{I}$.*
2. *Every ideal $\mathcal{I}$ of ${}_{M_n(N)}N^n$ is of the form K^n for some ideal K of ${}_N N$.*

Note 5.4 (Note 1.7(iii) of [9]) *Let A be an ideal of $\frac{N}{N}$. Then $A \leq_e {}_N N$ if and only if $A^n \leq_e {}_{M_n(N)}N^n$.*

Theorem 5.5 (Theorem 1.9 [9]) *If $l \in N$, then $\langle l \rangle^n = \langle (l, 0, \cdots, 0) \rangle$.*

Lemma 5.6 *If I and J are ideals of N, then $(I + J)^n = I^n + J^n$.*

Proof Clearly, $I \subseteq I + J$ and $J \subseteq I + J$ which implies $I^n \subseteq (I + J)^n$ and $J^n \subseteq (I + J)^n$ and so $I^n + J^n \subseteq (I + J)^n$. To prove the other part, let $(x_1, x_2, \cdots, x_n) \in (I + J)^n$. Then $x_i \in I + J$ for every $1 \le i \le n$ which implies $x_i = a_i + b_i$, where $a_i \in I$ and $b_i \in J$. Now,

$$\begin{aligned}(x_1, x_2, \cdots, x_n) &= (a_1 + b_1, a_2 + b_2, \cdots, a_n + b_n)\\ &= (a_1, a_2, \cdots, a_n) + (b_1, b_2, \cdots, b_n)\\ &\in I^n + J^n\end{aligned}$$

Therefore, $(I + J)^n \subseteq I^n + J^n$. Hence, $(I + J)^n = I^n + J^n$. □

Lemma 5.7 *$I + J = G$ if and only if $(I + J)^n = G^n$ if and only if $I^n + J^n = G^n$.*

Definition 5.8 An ideal $\mathcal{A}$ of $M_n(N)$-group N^n is said to be superfluous if for any ideal $\mathcal{K}$ of N^n, $\mathcal{A} + \mathcal{K} = N^n$ implies $\mathcal{K} = N^n$.

Lemma 5.9 *Let B be an ideal of ${}_NN$. If $B \ll {}_NN$, then $B^n \ll {}_{M_n(N)}N^n$.*

Proof Let $\mathcal{A} \trianglelefteq {}_{M_n(N)}N^n$ such that $B^n + \mathcal{A} = N^n$. To show $\mathcal{A} = N^n$. Since $\mathcal{A} \trianglelefteq {}_{M_n(N)}N^n$, by Lemma 5.3, we have $\mathcal{A} = (\mathcal{A}_{\star\star})^n$, which implies $B^n + (\mathcal{A}_{\star\star})^n = N^n$. Now using Lemma 5.6, we get $(B + \mathcal{A}_{\star\star})^n = N^n$. Therefore, by Lemma 5.7, $B + \mathcal{A}_{\star\star} = N$. Since, $B \ll {}_NN$, we get $\mathcal{A}_{\star\star} = N$. Hence, $\mathcal{A} = (\mathcal{A}_{\star\star})^n = N^n$. □

Lemma 5.10 *If $\mathcal{A} \ll {}_{M_n(N)}N^n$, then $\mathcal{A}_{\star\star} \ll {}_NN$.*

Proof Let $B \trianglelefteq {}_NN$ such that $\mathcal{A}_{\star\star} + B = N$. By Lemma 5.7, we have $(\mathcal{A}_{\star\star} + B)^n = N^n$. By Lemma 5.6, we have $(\mathcal{A}_{\star\star})^n + B^n = N^n$ which implies $\mathcal{A} + B^n = N^n$. Since $B^n \trianglelefteq {}_{M_n(N)}N^n$ and $\mathcal{A} \ll {}_{M_n(N)}N^n$, we have $B^n = N^n$. Let $n \in N$. Then $(n, 0, \cdots, 0) \in N^n = B^n$. Therefore, $n \in (B^n)_{\star\star} = B$ (by Theorem 5.2(3)). Therefore, $B = N$. □

Theorem 5.11 *There is a one-one correspondence between the set of superfluous ideals of ${}_NN$ and those of $M_n(N)$-group N^n.*

Proof Let $P = \{A \trianglelefteq {}_NN : A \ll {}_NN\}$. $Q = \{\mathcal{A} \trianglelefteq {}_{M_n(N)}N^n : \mathcal{A} \ll {}_{M_n(N)}N^n\}$. Define $\Phi : P \to Q$ by $\Phi(A) = A^n$. Then by Lemma 5.9, $A^n \ll {}_{M_n(N)}N^n$. Define $\Psi : Q \to P$ by $\Psi(\mathcal{A}) = \mathcal{A}_{\star\star}$. By Lemma 5.10, $\mathcal{A}_{\star\star} \ll {}_NN$. Now $(\Psi \circ \Phi)(A) = \Psi(\Phi(A)) = \Psi(A^n) = (A^n)_{\star\star} = A$. Therefore, $(\Psi \circ \Phi) = Id_P$. Also, $(\Phi \circ \Psi)(\mathcal{A}) = \Phi(\Psi(\mathcal{A})) = \Phi(\mathcal{A}_{\star\star}) = (\mathcal{A}_{\star\star})^n = \mathcal{A}$, and hence $(\Phi \circ \Psi) = Id_Q$. □

Definition 5.12 An ideal $\mathcal{K}$ of $M_n(N)$-group N^n is said to be g-superfluous if for any ideal $\mathcal{A}$ of N^n, $\mathcal{K} + \mathcal{A} = N^n$ and $\mathcal{A} \le_e N^n$ implies $\mathcal{K} = N^n$.

Lemma 5.13 *Let I be an ideal of ${}_NN$. If $I \ll_{gs} {}_NN$, then $I^n \ll_{gs} {}_{M_n(N)}N^n$.*

Proof Let $I \ll_{gs} {}_NN$. To show $I^n \ll_{gs} {}_{M_n(N)}N^n$, let $\mathcal{K}$ be an ideal of ${}_{M_n(N)}N^n$ such that $I^n + \mathcal{K} = {}_{M_n(N)}N^n$ and $\mathcal{K} \le_e {}_{M_n(N)}N^n$. Since $\mathcal{K} \trianglelefteq {}_{M_n(N)}N^n$, by Lemma 5.3(2), we have $\mathcal{K} = A^n$ for some ideal A of ${}_NN$. Since $\mathcal{K} = A^n \le_e {}_{M_n(N)}N^n$, by Note 5.4, we have $A \le_e {}_NN$. Now, $I^n + \mathcal{K} = I^n + A^n = (I + A)^n = N^n$ which implies $I + A = N$. Since, $I \ll_{gs} {}_NN$, we get $A = N$. Therefore, $\mathcal{K} = A^n = N^n$. Hence, $I^n \ll_{gs} {}_{M_n(N)}N^n$. □

Lemma 5.14 *If $\mathcal{A} \ll_{gs} {}_{M_n(N)}N^n$, then $\mathcal{A}_{\star\star} \ll_{gs} {}_NN$.*

Proof Let $\mathcal{A} \ll_{gs\ M_n(N)} N^n$. To show $\mathcal{A}_{\star\star} \ll_{gs\ N} N$, let $B \leq_{e\ N} N$ such that $\mathcal{A}_{\star\star} + B =_N N$. Since $B \leq_{e\ N} N$, by Note 5.4, we have $B^n \leq_{e\ M_n(N)} N^n$. Now, $\mathcal{A}_{\star\star} + B = N$ implies $(\mathcal{A}_{\star\star} + B)^n = N^n$. By Lemma 5.6, we get $(\mathcal{A}_{\star\star})^n + B^n = N^n$. Therefore, $\mathcal{A} + B^n = N^n$. Since $\mathcal{A} \ll_{gs} N^n$, we get $B^n = N^n$. Now, by Theorem 5.2(3), we get $B = (B^n)_{\star\star} = (N^n)_{\star\star} = N$. Therefore, $mathcal A_{\star\star} \ll_{gs\ N} N$. □

Theorem 5.15 *There is a one-one correspondence between the set of g-superfluous ideals of* $\frac{N}{N}$ *and those of* $M_n(N)$*-group* N^n.

Proof Let $P = \{A \trianglelefteq {}_N N : A \ll_{gs\ N} N\}$. $Q = \{\mathcal{A} \trianglelefteq {}_{M_n(N)} N^n : \mathcal{A} \ll_{gs\ M_n(N)} N^n\}$. Define $\Phi : P \to Q$ by $\Phi(A) = A^n$. Then by Lemma 5.13, $A^n \ll_{gs\ M_n(N)} N^n$. Define $\Psi : Q \to P$ by $\Psi(\mathcal{A}) = \mathcal{A}_{\star\star}$. By Lemma 5.14, $\mathcal{A}_{\star\star} \ll_{gs\ N} N$. Now $(\Psi \circ \Phi)(A) = \Psi(\Phi(A)) = \Psi(A^n) = (A^n)_{\star\star} = A$. $(\Phi \circ \Psi)(\mathcal{A}) = \Phi(\Psi(\mathcal{A})) = \Phi(\mathcal{A}_{\star\star}) = (\mathcal{A}_{\star\star})^n = \mathcal{A}$. Therefore, $(\Psi \circ \Phi) = Id_P$ and $(\Phi \circ \Psi) = Id_Q$. □

Definition 5.16 An element $s \in G$ is called hollow if $\langle s \rangle$ is a hollow ideal of G. In this case we call s as an h-element of G.

Example 5.17
1. Let $N = (\mathbb{Z}_{12}, +_{\mathbb{Z}_{12}}, \cdot_{\mathbb{Z}_{12}})$ and $G = N$. Then $\langle 3 \rangle$ is hollow. Therefore, 3 is a hollow element.
2. Let $N = (D_8, +, \cdot)$ given in Example 2.11 and $G = N$. Then $\langle r^2 \rangle$ is hollow. Therefore, r^2 is a hollow element.

Proposition 5.18 *s is a hollow element of* ${}_N N$ *if and only if* $(s, 0, 0, \cdots, 0)$ *is a hollow element in* $M_n(N)$*-group* N^n.

Proof Suppose s is a hollow element then $\langle s \rangle$ is a hollow ideal. To show $\langle (s, 0, \cdots, 0) \rangle$ is a hollow ideal of $M_n(N)$-group N^n, let $\mathcal{I}$, $\mathcal{J}$ be ideals of $\langle (s, 0, \cdots, 0) \rangle$ such that $\mathcal{I} + \mathcal{J} = \langle (s, 0, \cdots, 0) \rangle$. Then by Lemma 5.3(1), we have $\mathcal{I} = (\mathcal{I}_{\star\star})^n$, $\mathcal{J} = (\mathcal{J}_{\star\star})^n$, which implies $(\mathcal{I}_{\star\star})^n + (\mathcal{J}_{\star\star})^n = \langle (s, 0, \cdots, 0) \rangle$. Using Lemma 5.6 and by Theorem 5.5, we get $(\mathcal{I}_{\star\star} + \mathcal{J}_{\star\star})^n = \langle (s, 0, \cdots, 0) \rangle = \langle s \rangle^n$ and so $\mathcal{I}_{\star\star} + \mathcal{J}_{\star\star} = \langle s \rangle$. Since, $\langle s \rangle$ is hollow, we get either $\mathcal{I}_{\star\star} = \langle s \rangle$ or $\mathcal{J}_{\star\star} = \langle s \rangle$.
Therefore,

$$\mathcal{I} = (\mathcal{I}_{\star\star})^n = \langle s \rangle^n = \langle (s, 0, \cdots, 0) \rangle$$

or

$$\mathcal{J} = (\mathcal{J}_{\star\star})^n = \langle s \rangle^n = \langle (s, 0, \cdots, 0) \rangle.$$

Conversely, suppose $(s, 0, \cdots, 0)$ is hollow in N^n. Then $\langle (s, 0, \cdots, 0) \rangle$ is a hollow ideal of $\frac{N^n}{M_n(N)}$, which implies $\langle s \rangle^n$ is a hollow ideal of $\frac{N^n}{M_n(N)}$. To show $\langle s \rangle$ is hollow in N, let I and J be two ideals of N contained in $\langle s \rangle$ such that $I + J = \langle s \rangle$. Now, $(I + J)^n = \langle s \rangle^n$. Therefore $I^n + J^n = \langle s \rangle^n$. Since $\langle s \rangle^n$ is hollow, we have $I^n = \langle s \rangle^n$ or $J^n = \langle s \rangle^n$, and hence, $I = \langle s \rangle$ or $J = \langle s \rangle$. □

Definition 5.19 $X = \{x_1, x_2, \cdots, x_n\} \subseteq G$ is said to be a spanning set for G if $\sum_{x_i \in X} \langle x_i \rangle = G$. If $\{x_i : 1 \leq i \leq n\}$ is a spanning set in G, then we say the elements x_i, $1 \leq i \leq n$ are spanning elements in G.

Theorem 5.20 $\{x_i : 1 \leq i \leq n\}$ *is a spanning set in* ${}_N N$ *if and only if* $\{(x_i, 0, \cdots, 0) : 1 \leq i \leq n\}$ *is a spanning set in* $M_n(N)$*-group* N^n.

Proof Suppose $\{x_i : 1 \leq i \leq n\}$ is a spanning set in ${}_N N$. Then

$$\begin{aligned}\sum_{1\leq i\leq n} \langle x_i \rangle = N &\Leftrightarrow \langle x_1 \rangle + \langle x_2 \rangle + \cdots + \langle x_n \rangle = N \\ &\Leftrightarrow (\langle x_1 \rangle + \langle x_2 \rangle + \cdots + \langle x_n \rangle)^n = N^n \\ &\Leftrightarrow \langle x_1 \rangle^n + \langle x_2 \rangle^n + \cdots + \langle x_n \rangle^n = N^n \\ &\Leftrightarrow \langle (x_1, \cdots, 0) \rangle + \langle (x_2, 0, \cdots, 0) \rangle + \cdots + \langle (x_n, 0, \cdots, 0) \rangle = N^n \\ &\Leftrightarrow \sum_{1\leq i\leq n} \langle (x_i, 0, \cdots, 0) \rangle = N^n.\end{aligned}$$

Therefore $\{(x_i, 0, \cdots, 0) : 1 \leq i \leq n\}$ is a spanning set in $M_n(N)$-group N^n. □

Definition 5.21 A subset X of G is said to be a h-spanning set if every element of X is a h-element and X is a spanning set.

Theorem 5.22 *Suppose $x_1, x_2, \cdots, x_n \in N$. Then $\{x_i : 1 \leq i \leq n\}$ is a h-spanning set in N if and only if $\{(x_i, 0, \cdots, 0) : 1 \leq i \leq n\}$ is a h-spanning set in $M_n(N)$-group N^n.*

Proof $\{x_i : 1 \leq i \leq n\}$ is a h-spanning set.

$$\begin{aligned}&\Leftrightarrow x_i,\ 1 \leq i \leq n \text{ are } h\text{-elements and } \sum_{1\leq i\leq n} \langle x_i \rangle = N \\ &\Leftrightarrow (x_i, 0, \cdots, 0),\ 1 \leq i \leq n \text{ are } h\text{-elements in } {}_{M_n(N)}N^n \\ &\quad \text{and } \sum_{1\leq i\leq n} \langle (x_i, 0, \cdots, 0) \rangle = N^n.\end{aligned}$$

Therefore $\{(x_i, 0, \cdots, 0) : 1 \leq i \leq n\}$ is a h-spanning set in $M_n(N)$-group N^n. □

6 Superfluous ideal graph of a nearring

The authors [22] studied graphs with respect to superfluous elements in a lattice, and in [21] the authors studied the graphs with respect to the dual aspects such as essential elements, complements, etc. Lattice aspects of modules over rings are well-known due to [3, 7]. In this section, we define the superfluous ideal graph of a nearring and study some of its properties.

Definition 6.1 Let N be a nearring. An ideal I of N is said to be superfluous if for any ideal J of N, $I + J = N$ implies $J = N$.

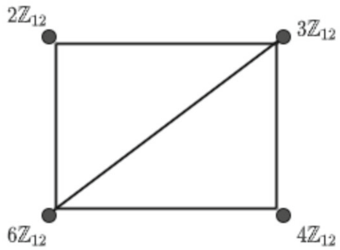
Definition 6.2 The superfluous ideal graph of N, denoted by $S_N(G)$, is a graph having set of all non-zero proper ideals of N as vertices and two vertices I and J are adjacent if $I \cap J \ll N$.

Example 6.3 1. If N is simple, then $S_N(G)$ is a null graph.

2. Suppose N is a finitely generated nearring which contains only one non-zero maximal ideal, then every proper ideal of N is superfluous. The vertices of $S_N(G)$ are the non-zero proper ideals of N. Since every proper ideal of N is superfluous in N, we have $I \cap J \ll G$ for all proper ideals I, J of N. Therefore $S_N(G)$ is a complete graph.
   For example, let $N = (Z_{p^n}, +_{p^n}, \cdot)$ where p is prime. Then possible ideals are of the form $\langle p^i \rangle,\ i \in \{0, 1, \cdots, n-1\}$. If N is simple, then $S_N(G)$ is a null graph. If N is not simple, then N has only one non-zero maximal ideal of the form $\langle p^k \rangle$ for some $0 \leq k \leq n-1$.

Fig. 1 $S_{\mathbb{Z}_6}(G)$

Fig. 2 $S_{\mathbb{Z}_{12}}(G)$



Hence, N is a local nearring. In this case we get a complete graph.
Consider $R = (Z_{p^n}, +_{p^n}, \cdot_{p^n})$ where addition and multiplication are modulo p^n. Then R is a ring. In this case, the superfluous ideal graph is a complete graph with $(n-1)$ vertices.

Example 6.4 Let $N = (\mathbb{Z}_6, +_{\mathbb{Z}_6}, \cdot_{\mathbb{Z}_6})$. Then $V(S_{\mathbb{Z}_6}(G)) = \{2\mathbb{Z}_6, 3\mathbb{Z}_6\}$. Now $2\mathbb{Z}_6 \cap 3\mathbb{Z}_6 = (0) \ll N$. The graph $S_{\mathbb{Z}_6}(G)$ is shown in Fig. 1.

Example 6.5 Let $N = (\mathbb{Z}_{12}, +_{\mathbb{Z}_{12}}, \cdot_{\mathbb{Z}_{12}})$. Non-zero proper ideals of N are $2\mathbb{Z}_{12}, 3\mathbb{Z}_{12}, 4\mathbb{Z}_{12}, 6\mathbb{Z}_{12}$ and $6\mathbb{Z}_{12}$ is superfluous in $\mathbb{Z}_{12}$. Then the corresponding superfluous ideal graph is given in Fig. 2.

Example 6.6 Let $N = (\mathbb{Z}_2 \times \mathbb{Z}_2, +_{\mathbb{Z}_2}, \cdot_{\mathbb{Z}_2})$ where addition and multiplication are carried out component-wise. All non-zero proper ideals are of N are $(0) \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times (0)$ and $((0) \times \mathbb{Z}_2) \cap (\mathbb{Z}_2 \times (0)) = (0) \ll N$. Therefore, the superfluous ideal graph is given in Fig. 3.

Example 6.7 Let $N = (\mathbb{Z}_4 \times \mathbb{Z}_2, +, \cdot)$ where addition and multiplication are carried out component-wise with the first component modulo 4 and the second component modulo 2. Then the nontrivial ideals are $I_1 = \{(0,0), (1,0), (2,0), (3,0)\}$, $I_2 = \{(0,0), (2,0), (0,1), (2,1)\}$, $I_3 = \{(0,0), (0,1)\}$, $I_4 = \{(0,0), (2,0)\}$ and I_4 is a superfluous ideal. The corresponding superfluous ideal graph is given in Fig. 4.

Example 6.8 Consider the nearring given in the Example 2.11. The ideals of N are H_4, H_9, H_7 and it can be seen that H_4 is superfluous in N. We have $H_9 \cap H_7 = H_4$, $H_9 \cap H_4 = H_4$ and $H_7 \cap H_4 = H_4$. Hence, we get a complete graph given in Fig. 5.

Proposition 6.9 *Every non-zero superfluous ideal of N is a universal vertex in $S_N(G)$.*

Proof Let X be a non-zero superfluous ideal of N. To prove $XY \in E$ for every $Y \in V$. Let $Y \in V$. By Lemma 3.7, $X \cap Y \ll N$ which implies $XY \in E$. Since Y is arbitrary, X is a universal vertex. □

The converse of the Proposition 6.9 need not be true. We justify this in the following example.

Example 6.10 In Example 6.5, $N = (\mathbb{Z}_{12}, +_{\mathbb{Z}_{12}}, \cdot_{\mathbb{Z}_{12}})$. Then $6\mathbb{Z}_{12}$ is a non-zero superfluous ideal which is a universal vertex in the corresponding superfluous ideal graph given in Fig. 2. The vertex $3\mathbb{Z}_{12}$ is universal but it is not superfluous, as $3\mathbb{Z}_{12} + 2\mathbb{Z}_{12} = \mathbb{Z}_{12}$, and $2\mathbb{Z}_{12} \neq \mathbb{Z}_{12}$.

Fig. 3 $S_{\mathbb{Z}_2 \times \mathbb{Z}_2}(G)$

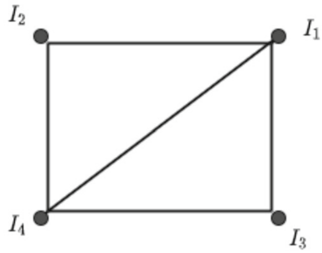


Fig. 4 $S_{\mathbb{Z}_4 \times \mathbb{Z}_2}(G)$

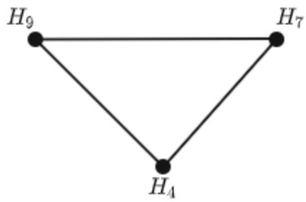


Fig. 5 Superfluous ideal graph of the nearring in Example 2.11

Proposition 6.11 *The subgraph* $S_N[min(N)]$ *induced by* $min(N)$ *is a clique, where* $min(N)$ *is the set of minimal ideals of* N.

Proof Case 1: Suppose N has exactly one minimal ideal. Then we get a clique K_1.
Case 2: Suppose N has more than one minimal ideal. Let M_1, M_2 be two arbitrary minimal ideals of N. We prove that $M_1M_2 \in E(S_N(G))$. Since, M_1 and M_2 both are minimal $M_1 \cap M_2 = (0)$, which is a superfluous ideal of N, which implies $M_1M_2 \in E(S_N(G))$. Since M_1 and M_2 are arbitrary, we conclude that there exists an edge between any two minimal ideals. Therefore $S_N[min(N)]$ is a clique (Fig. 3). □

Proposition 6.12 $S_N(G)$ *is an empty graph if and only if N has exactly one non-zero proper ideal.*

Proof If N has exactly one non-zero proper ideal then $S_N(G) = K_1$. Conversely, suppose $S_N(G)$ is an empty graph. We prove that N has exactly one non-zero proper ideal. First we prove that N has exactly one minimal ideal. Suppose on a contrary, N has two minimal ideals M_1 and M_2. Then by Proposition 6.11, M_1 and M_2 are adjacent in $S_N(G)$, a contradiction since $S_N(G)$ is an empty graph. Therefore N has a unique minimal ideal say, M. So every non-zero ideal of N different from M contains M. Therefore M is superfluous. We claim that M is the only unique proper ideal of N. On a contrary, suppose $I \neq M$ be a non-zero proper ideal of N. Then $M \subseteq I$, $M \cap I = M$, which is superfluous in N, and we get $MI \in E(S_N(G))$, a contradiction, since $S_N(G)$ is an empty graph. Therefore M is the unique non-zero proper ideal of N (Fig. 4). □

Definition 6.13 Let I be an ideal of N. The dual annihilator of I, denoted as $ann_d(I)$ is the intersection of all ideals J of N such that $I + J = N$. That is, $ann_d(I) = \bigcap_{J \trianglelefteq_N N,\ I+J=N} J$ (Fig. 5).

Example 6.14 1. In the nearring given in Example 2.11, the ideals of N are H_7, H_9, H_5 and $\{e\}$. Therefore $ann_d(H_7) = \cap\{H_9, N\} = H_9$.

2. In the nearring N given in Example 3.27, the ideals of N are N, I_2, I_3, I_4, I_5 and $\{e\}$. We have $I_2 + I_3 = N$ and $I_2 + I_4 = N$. Therefore $ann_d(I_2) = \bigcap\{I_3, I_4\} = I_5$.

Proposition 1 *Let I be any arbitrary ideal of N. Then $I \cap (ann_d(I)) \ll N$.*

Proof Let $K \trianglelefteq N$ such that $I \cap (ann_d(I)) + K = N$. Since $I \cap (ann_d(I)) \subseteq I$, we have $I + K = N$, which implies $ann_d(I) \subseteq K$ and so $I \cap (ann_d(I)) \subseteq K$. Now $K = K + I \cap (ann_d(I)) = N$. Therefore, $I \cap (ann_d(I)) \ll N$. □

7 Conclusion

We have defined the notions superfluous, strictly superfluous (with respect to an ideal Ω), generalised superfluous, generalised suppements in N-groups. We have proved some properties and exhibited examples which are different from the existing notions. We have defined graph on superfluous ideals of a nearring, and gave some properties. The concepts can be extended to study various finite spanning dimension aspects and related chain conditions in N-groups and those of $M_n(N)$-group N^n.

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