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A new iterative method for approximating common fixed points of two non-self mappings in a CAT(0) space

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Abstract
In this paper, we introduce a new iterative process for approximating common fixed points of two non-self mappings in the setting of CAT(0) spaces. Then we establish Δ-convergence and strong convergence results for two nonexpansive non-self mappings under appropriate conditions. Moreover, we establish strong convergence theorems for approximating common fixed points of two Lipschitz quasi-nonexpansive non-self mappings under some additional conditions. Our results improve, complement and unify most of the results in the literature.

Keywords Fixed points · Iterative methods · Nonexpansive mappings · Non-self mappings

Mathematics Subject Classification 47H09 · 47H10 · 47J25

1 Introduction

Let X be a CAT(0) space and K be a nonempty subset of X. A point x in K is called a *fixed point* of the mapping T if $x = Tx$. The set of all fixed points of the mapping T is denoted by $F(T)$, i.e $F(T) = \{x \in K : x = Tx\}$. A mapping $T : K \to X$ is said to be

(a) *L-Lipschitz* if there exists $L > 0$ such that

$$d(Tx, Ty) \leq Ld(x, y), \text{ for all } x, y \in K.$$

(b) *nonexpansive* mapping if $d(Tx, Ty) \leq d(x, y)$, for all $x, y \in K$.
(c) *quasi-nonexpansive* mapping if $F(T) \neq \emptyset$ and such that

$$d(Tx, Tp) \leq d(x, p), \forall x \in K, \forall p \in F(T).$$

We observe that the class of Lipschitz mappings includes the class of nonexpansive mappings. In fact, a nonexpansive mapping is 1-Lipschiz mapping. One can also easily observe that a nonexpansive mapping with nonempty fixed point set is quasi-nonexpansive mapping. However, a quasi-nonexpansive mapping need not be nonexpansive (see, e.g., [24]).

Interests to study fixed points of nonlinear operators stems mainly from its application in diverse fields of sciences such as differential equations, optimization, control theory,

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economics, graph theory, biology and computer sciences among others. Consequently, the existence of fixed points and their approximations for nonexpansive mappings and their generalizations have been studied by several authors (see, e.g. [1, 8, 11, 13, 17, 19–23, 30, 36] and the references therein).

Most of the iterative processes in the literature are well defined only for self mappings. For approximating fixed points of non-self mappings, researchers have been using metric projection or sunny nonexpansive retraction mappings (see [23, 25]). However, the process of calculating the projection or sunny nonexpansive mapping can be a resource consuming task and may require an approximating iterative scheme by itself. In the attempt to get rid of these difficulties, Colao and Marino [5] have successfully introduced and studied the following iterative process for non-self mapping without the use of any auxiliary operator.

$$\begin{cases} x_0 \in C, \\ \alpha_0 = \max\{\frac{1}{2}, h(x_0)\}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ \alpha_{n+1} = \max\{\alpha_n, h(x_{n+1})\}, n \geq 0, \end{cases} \tag{1}$$

where $h(x) := \inf\{\lambda \geq 0 : \lambda x + (1-\lambda)Tx \in C\}, \forall x \in C \subseteq H$. They established weak and strong convergence results of the algorithm for approximating fixed points of nonexpansive non-self mappings.

This method of Colao and Marino [5] have been used by many authors to construct and study iterative processes for approximating fixed points of non-self mappings (see, e.g., [6, 7, 12, 29, 32–34]). In particular, Tufa and Zegeye [33] introduced an iterative scheme for approximating fixed points of nonexpansive non-self mappings in the setting of CAT(0) spaces (see Corollary 3.5 of [33]). They established strong and Δ-convergence results of the scheme.

On the other hand, iterative schemes for approximating common fixed points of two self mappings have been extensively studied (see, e.g., [14–16, 27, 28, 37]) as approximating common fixed points of two mappings is applicable in minimization problems (see, for instance, [26]. In [37], Yao and Chen introduced and studied the following iteration process for common fixed points of two self mappings:

$$x_1 \in C, x_{n+1} = \alpha_n x_n + \beta_n Tx_n + \gamma_n Sx_n, n \in \mathbb{N},$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are in [0, 1] and $\alpha_n + \beta_n + \gamma_n = 1$. They obtained weak and strong convergence results of the proposed algorithm to a common fixed point of two asymptotically nonexpansive mappings in a uniformly convex Banach space. Many authors have been using nonexpansive retraction mappings to construct iterative methods for approximating common fixed points of two non-self mappings (see, for instance, [35, 38]). Recently, Tufa [31] obtained some weak and strong convergence results for two quasi-non-expansive non-self mappings using a new iterative scheme which does not involve any auxiliary operators such as projection and nonexpansive retraction mappings at the expense of lowering the space to a real Hilbert space.

Motivated by the above results, our purpose in this paper is to construct and study new iterative methods for approximating common fixed points of two non-self mappings in a complete CAT(0) space. We obtain Δ-convergence and strong convergence results of the proposed method for nonexpansive non-self mappings under appropriate conditions. Moreover, we establish strong convergence results for approximating common fixed points of two L-Lipschitz quasi-nonexpansive non-self mappings under some additional conditions. Our results extend and generalize many of the results in the literature.

2 Preliminaries

Let (X, d) be a metric space and $x, y \in X$. A geodesic path joining x to y is a map $r : [0, l] \subset \mathbb{R} \to X$ such that $r(0) = x, r(l) = y$ and $d(r(t), r(t_0)) = |t - t_0|$ for all $t, t_0 \in [0, l]$. The image of r is called a *geodesic segment* joining x and y. When it is unique this geodesic segment is denoted by $[x, y]$. This means that $z \in [x, y]$ if and only if there exists $t \in [0, 1]$ such that $d(x, z) = (1-t)d(x, y)$ and $d(y, z) = td(x, y)$. In this case, we write $z = tx \oplus (1-t)y$.

The metric space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic and it is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A uniquely geodesic space (X, d) is said to be an $\mathbb{R}$-*tree,* if $x, y, z \in X$ with $[x, y] \cap [y, z] = \{y\}$ implies $[x, z] = [x, y] \cup [y, z]$. Hereafter, we denote a geodesic space (X, d) simply by X.

A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic space X consists of three points x_1, x_2, x_3 of X and three geodesic segments joining each pair of vertices. A comparison triangle of a geodesic triangle $\triangle(x_1, x_2, x_3)$ is the triangle $\bar{\triangle}(x_1, x_2, x_3) := \triangle(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in the Euclidean space $\mathbb{R}^2$ such that $d(x_i, x_j) = d_{\mathbb{R}^2}(\bar{x_i}, \bar{x_j})$ for all $i, j = 1, 2, 3$.

A geodesic space X is said to be a *CAT(0) space* if all geodesic triangles satisfy the following comparison axiom:

$$d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y}), \quad \forall x, y \in \triangle, \bar{x}, \bar{y} \in \bar{\triangle}, \tag{2}$$

where $\triangle$ is a geodesic triangle in X and $\bar{\triangle}$ is its comparison triangle in $\mathbb{R}^2$. It is well known that a CAT(0) space X is uniquely geodesic. Pre-Hilbert spaces, $\mathbb{R}-$trees and Euclidean buildings are examples of $CAT(0)$ spaces. For details we refer the readers to standard texts such as [2, 3].

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$. The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known [9] that in a CAT(0) space X, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\} \subseteq X$ is said to be $\triangle$-*convergent* to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. From the uniqueness of asymptotic center, it follows that the CAT(0) space X satisfies Opial's property, i.e., for a sequence $\{x_n\}$ in X such that x_n $\triangle$-converges to x and given $y \in X$ with $y \neq x$, one has

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y).$$

A subset K of a CAT(0) space X is said to be convex if K includes every geodesic segment joining any two of its points. A convex set K is said to be *strictly convex* if for $x, y \in \partial K$ and $t \in (0, 1)$, we have $tx \oplus (1-t)y \in \mathring{K}$, where ∂K and $\mathring{K}$ denotes boundary and interior of K respectively.

A mapping $T : K \to X$ is said to be *inward* on K if $Tx \in I_K(x)$, where $I_K(x) := \{w \in X : w = x \ \ or \ \ y = (1 - \frac{1}{\lambda})x \oplus \frac{1}{\lambda}w,$ for some $y \in K, \lambda \geq 1\}$ for all $x \in K$. If for a sequence $\{x_n\}$ in K such that x_n $\triangle$-converges to p and $d(x_n, Tx_n) \to 0$ implies $p = Tp$, then the mapping $I - T$ is called demiclosed at zero.

We may use the following lemmas in the sequel.

Lemma 1 [36] *Let X be a complete CAT(0) space and $\{x, x_1, x_2, \ldots, x_n\} \subseteq X$. If $\{\lambda_1, \lambda_2, \cdots, \lambda_n\} \subseteq [0, 1]$ with $\sum_{i=1}^{n} \lambda_i = 1$, then for each $i, j \in \{1, 2, \ldots, n\}$, we have*

$$d^2\left(\bigoplus_{i=1}^{n} \lambda_i x_i, x\right) \leq \lambda_1 d^2(x_1, x) + \lambda_2 d^2(x_2, x) + \cdots + \lambda_n d^2(x_n, x) - \lambda_i \lambda_j d(x_i, x_j) d^2(x_i, x_j).$$

Lemma 2 [18] *Every bounded sequence in a complete CAT(0) space always has a $\triangle-$convergent subsequence.*

Lemma 3 [4] *Let X be a CAT(0) space. Then for each $x, y, z \in X$ and $\lambda \in [0, 1]$, one has*

$$d(\lambda x \oplus (1-\lambda)y, \mu x \oplus (1-\mu)y) \leq |\lambda - \mu| d(x, y).$$

Lemma 4 [10] *Let X be a CAT(0) space. Then the following inequalities hold true for all $x, y, z \in X$ and $\lambda \in [0, 1]$.*

$$d(\lambda x \oplus (1-\lambda)y, z) \leq \lambda d(x, z) + (1-\lambda) d(y, z).$$

The following lemmas are consequences of Lemmas 3.1, 3.2 and 3.3 of Tufa and Zegeye [33], respectively.

Lemma 5 [33] *Let K be a nonempty subset of a CAT(0) space X. If a mapping $T : K \to X$ is $L-$Lipschitz quasi-nonexpansive, then $F(T)$ is closed and convex.*

Lemma 6 [33] *Let K be a nonempty, closed and convex subset of a complete CAT(0) space X. If a mapping $T : K \to X$ is nonexpansive, then $I - T$ is demiclosed at zero.*

Lemma 7 [33] *Let K be a nonempty, closed and convex subset of a CAT(0) space X and $T : K \to X$ be a mapping. Define $h : K \to \mathbb{R}$ by*

$$h(x) = \inf\{\lambda \geq 0 : (1-\lambda)x \oplus \lambda T x \in K\}.$$

Then for any $x \in K$ the following hold:

(1) *$h(x) \in [0, 1]$ and $h(x) = 0$ if and only if $Tx \in K$.*
(2) *If $\beta \in [0, h(x)]$, then $(1-\beta)x \oplus \beta T x \in K$.*
(3) *If T is inward mapping, then $h(x) < 1$.*
(4) *If $Tx \notin K$, then $(1-h(x))x \oplus h(x) T x \in \partial K$.*

3 Results and discussions

In this section, we construct an algorithm which involves two non-self mappings in the framework of CAT(0) spaces. Then we establish convergence results to a common fixed point of the mappings. We start with the following lemma.

Lemma 8 *Let K be a nonempty, closed and convex subset of a CAT(0) space X and $T, S : K \to X$ be two non-self mappings. Given $\theta \in [0, 1]$, define $f_\theta : K \to [0, \infty]$ by*

$$f_\theta(x) = \inf\{\alpha \geq 0 : \alpha\big[\theta x \oplus (1-\theta) T x\big] \oplus (1-\alpha) S x \in K\}.$$

Then for any $x \in K$ the following hold:

(1) *If $\theta x \oplus (1-\theta) T x \in K$, then $f_\theta(x) \in [0, 1]$ and $f_\theta(x) = 0$ iff $Sx \in K$.*
(2) *If $\theta x \oplus (1-\theta) T x \in K$, then $\beta(\theta x + (1-\theta) T x\big) \oplus (1-\beta) S x \in K$ for any $\beta \in [f_\theta(x), 1]$.*

(3) *If T and S are inward mappings, then $f_\theta(x) < 1$.*
(4) *If $\theta x \oplus (1-\theta)Tx \in K$ and $Sx \notin K$, then*

$$f_\theta(x)\big[\theta x \oplus (1-\theta)Tx\big] \oplus \big(1 - f_\theta(x)\big)Sx \in \partial K.$$

Proof The proofs of (1) and (2) are obvious. Thus, we need to prove only (3) and (4).
(3) Let T and S be inward mappings and $x \in K$. Then for some $c_1, c_2 \geq 1$, we have

$$u_1 := \frac{1}{c_1}Tx \oplus \left(1 - \frac{1}{c_1}\right)x \in K \text{ and } u_2 := \frac{1}{c_2}Sx \oplus \left(1 - \frac{1}{c_2}\right)x \in K.$$

Then, we have

$$\begin{aligned}\frac{1}{2}u_1 \oplus \frac{1}{2}u_2 &= \frac{1}{2c_1}Tx \oplus \frac{1}{2}\left(1 - \frac{1}{c_1}\right)x \oplus \frac{1}{2c_2}Sx \oplus \frac{1}{2}\left(1 - \frac{1}{c_2}\right)x \\ &= \left(1 - \frac{1}{2c_2}\right)\left[\frac{c_2}{c_1(2c_2-1)}Tx \oplus \left(1 - \frac{c_2}{c_1(2c_2-1)}\right)x\right] \oplus \frac{1}{2c_2}Sx. \quad (3)\end{aligned}$$

Then since K is convex, we can conclude that

$$\left(1 - \frac{1}{2c_2}\right)\left[\frac{c_2}{c_1(2c_2-1)}Tx \oplus \left(1 - \frac{c_2}{c_1(2c_2-1)}\right)x\right] \oplus \frac{1}{2c_2}Sx \in K.$$

On the other hand, one can easily verify that $\theta := \frac{c_2}{c_1(2c_2-1)} \in (0, 1]$. Hence,

$$f_\theta(x) \leq 1 - \frac{1}{2c_2} < 1.$$

(4) Assume that $\theta x \oplus (1-\theta)Tx \in K$ and $Sx \notin K$. Then $f_\theta(x) \in [0, 1]$ by (1). Let $\{w_n\} \subseteq (0, f_\theta(x))$ be a real sequence such that $w_n \to f_\theta(x)$. Then from the definition of f_θ, it follows that

$$z_n := w_n(\theta x \oplus (1-\theta)Tx) \oplus (1 - w_n)Sx \notin K.$$

Thus, by Lemma 3, we have

$$d(z_n, f_\theta(x)(\theta x \oplus (1-\theta)Tx) \oplus (1 - f_\theta(x))Sx) \leq |w_n - f_\theta(x)| d(\theta x \oplus (1-\theta)Tx, Sx)).$$

Then, since $w_n \to f_\theta(x)$, it follows that

$$z_n \to f_\theta(x)(\theta x \oplus (1-\theta)Tx) \oplus (1 - f_\theta(x))Sx \in K$$

and since $z_n = w_n\big(\theta x \oplus (1-\theta)Tx\big) \oplus (1 - w_n)Sx \notin K$, for all $n \geq 1$, we obtain that

$$f_\theta(x)\big(\theta x \oplus (1-\theta)Tx\big) \oplus \big(1 - f_\theta(x)\big)Sx \in \partial K.$$

The proof is complete. □

Note that if $\theta = 1$ or T is an identity map, then Lemma 8 reduces to Lemma 7.

We now construct an iterative scheme which involves two non-self mappings. Let K be a nonempty, closed and convex subset of a CAT(0) space X and $S, T : K \to X$ be two non-self mappings. Given $x_1 \in K$, let

$$h(x_1) = \inf\{\theta \geq 0 : \theta x_1 \oplus (1-\theta)Tx_1 \in K\}.$$

Take $\theta_1 := \max\{\frac{1}{2}, h(x_1)\}$. Then $\theta_1 x_1 \oplus (1-\theta_1)Tx_1 \in K$.

Now, let

$$f_{\theta_1}(x_1) := \inf\{\alpha \geq 0 : \alpha[\theta_1 x_1 \oplus (1-\theta_1)Tx_1] \oplus (1-\alpha)Sx_1 \in K\}.$$

Put $\alpha_1 = \max\{\frac{1}{2}, f_{\theta_1}(x_1)\}$. Then by Lemma 8

$$x_2 := \alpha_1[\theta_1 x_1 \oplus (1-\theta_1)Tx_1] \oplus (1-\alpha_1)Sx_1 \in K.$$

If we continue the process in this fashion, by the principle of mathematical induction, we obtain a sequence $\{x_n\}$ defined recursively as follows:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = \alpha_n[\theta_n x_n \oplus (1-\theta_n)Tx_n] \oplus (1-\alpha_n)Sx_n \\ \theta_{n+1} = \max\{\theta_n, h(x_{n+1})\}, \\ \alpha_{n+1} = \max\{\alpha_n, f_{\theta_{n+1}}(x_{n+1})\}, \end{cases} \tag{4}$$

where $h : K \to \mathbb{R}$ and $f : K \to [0, \infty]$ are defined, respectively, by $h(x) = \inf\{\lambda \geq 0 : \lambda x \oplus (1-\lambda)Tx \in K\}$ and $f_\theta(x) = \inf\{\alpha \geq 0 : \alpha[\theta x \oplus (1-\theta)Tx] \oplus (1-\alpha)Sx \in K\}$, for given $\theta \in [0, 1]$.

We observe that Algorithm (4) reduces to Mann iterative scheme when T or S is an identity map. Now, we prove the following results.

Theorem 9 *Let K be a nonempty, closed and convex subset of a complete CAT(0) space X and $S, T : K \to X$ be two nonexpansive inward mappings with $\mathcal{F} = F(T) \cap F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined in (4).*

(i) *If there exists $b \in (0, 1)$ such that $\theta_n, \alpha_n \leq b, \forall n \geq 1$, then $\{x_n\}$ Δ-converges to a point in $\mathcal{F}$.*
(ii) *If K is strictly convex, $\sum_{n=1}^{\infty}(1-\theta_n) < \infty$ and $\sum_{n=1}^{\infty}(1-\alpha_n) < \infty$, then $\{x_n\}$ strongly converges to a point in $\mathcal{F}$.*

Proof Let $p \in \mathcal{F}$. Since T and S are nonexpansive, by Lemma 4, we have

$$\begin{aligned} d(x_{n+1}, p) &= d(\alpha_n[\theta_n x_n \oplus (1-\theta_n)Tx_n] \oplus (1-\alpha_n)Sx_n, p) \\ &\leq \alpha_n d(\theta_n x_n \oplus (1-\theta_n)Tx_n, p) + (1-\alpha_n)d(Sx_n, p) \\ &\leq \alpha_n\theta_n d(x_n, p) + \alpha_n(1-\theta_n))d(Tx_n, p) + (1-\alpha_n)d(Sx_n, p) \\ &\leq \alpha_n\theta_n d(x_n, p) + \alpha_n(1-\theta_n)d(x_n, p) + (1-\alpha_n)d(x_n, p) \\ &= d(x_n, p). \end{aligned} \tag{5}$$

Then the sequensce $\{d(x_n, p)\}$ is decreasing which implies that

$$\lim_{n\to\infty} d(x_n, p) \text{ exists.}$$

This in turn implies that $\{x_n\}$ is bounded.

(i) Suppose there exists $b \in (0, 1)$ such that $\theta_n, \alpha_n \leq b, \forall n \geq 1$. Then by Lemma 1, we have

$$\begin{aligned} d^2(x_{n+1}, p) &= d^2(\alpha_n\theta_n x_n \oplus \alpha_n(1-\theta_n)Tx_n \oplus (1-\alpha_n)Sx_n, p) \\ &\leq \alpha_n\theta_n d^2(x_n, p) + \alpha_n(1-\theta_n)d^2(Tx_n, p) + (1-\alpha_n)d^2(Sx_n, p) \\ &\quad -\alpha_n^2\theta_n(1-\theta_n)d^2(Tx_n, x_n) \\ &\leq \alpha_n\theta_n d^2(x_n, p) + \alpha_n(1-\theta_n)d^2(x_n, p) + (1-\alpha_n)d^2(x_n, p) \\ &\quad -\alpha_n^2\theta_n(1-\theta_n)d^2(Tx_n, x_n) \\ &= d^2(x_n, p) - \alpha_n^2\theta_n(1-\theta_n)d^2(Tx_n, x_n). \end{aligned}$$

This yields

$$\alpha_n^2\theta_n(1-\theta_n)d^2(Tx_n,x_n)\le d^2(x_n,p)-d^2(x_{n+1},p).$$

Then since $\frac{1}{2}\le\theta_n,\alpha_n\le b$, it follows that

$$\sum_{n=1}^{\infty}\frac{1}{8}(1-b)d^2(Tx_n,x_n)\le\sum_{n=1}^{\infty}\alpha_n^2\theta_n(1-\theta_n)d^2(Tx_n,x_n)<\infty.$$

This implies that

$$\lim_{n\to\infty}d(x_n,Tx_n)=0.$$

Similarly, one can easily show that

$$\lim_{n\to\infty}d(x_n,Sx_n)=0.$$

On the other hand, since $\{x_n\}$ is bounded, Lemma 2 implies that the set of all $\triangle$-cluster points of $\{x_n\}$ is nonempty, that is

$$w(x_n):=\{x\in X: x_{n_i}\ \ \triangle\text{-converges to } x \text{ for some subsequence } \{x_{n_i}\}\text{of } \{x_n\}\}\ne\emptyset.$$

Thus, if $x\in w(x_n)$, then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that x_{n_i} Δ-converges to x as $n\to\infty$. Then since $I-T$ and $I-S$ are demiclosed at zero (see Lemma 6), we have

$$x\in\mathcal{F}=F(T)\cap F(S).$$

Hence, $w(x_n)\subseteq\mathcal{F}$. To show uniqueness, suppose $x,y\in w(x_n)$ such that $x\ne y$. Then there exist subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of the sequence $\{x_n\}$ such that x_{n_i} Δ-converges to x and x_{n_j} Δ-converges to y as $i,j\to\infty$. Then since $\lim_{n\to\infty}d(x_n,x)$ exists for all $x\in\mathcal{F}$ and CAT(0) space satisfies the Opial's property, we have

$$\begin{aligned}\lim_{n\to\infty}d(x_n,x)&=\lim_{i\to\infty}d(x_{n_i},x)<\lim_{i\to\infty}d(x_{n_i},y)\\&=\lim_{n\to\infty}d(x_n,y)=\lim_{j\to\infty}d(x_{n_j},y)\\&<\lim_{j\to\infty}d(x_{n_j},y)=\lim_{n\to\infty}d(x_n,x).\end{aligned}$$

But this is a contradiction and hence $x=y$. Thus, every subsequence of $\{x_n\}$ Δ-converges to x and hence $\{x_n\}$ Δ-converges to $x\in\mathcal{F}$.

(ii) Suppose that K is strictly convex, $\sum_{n=1}^{\infty}(1-\theta_n)<\infty$ and $\sum_{n=1}^{\infty}(1-\alpha_n)<\infty$.

Then

$$\sum_{n=0}^{\infty}\theta_n(1-\alpha_n)<\infty.$$

Moreover, from (4) and Lemma 4, we have

$$d(x_n,x_{n+1})\le\alpha_n(1-\theta_n)d(x_n,Tx_n)+(1-\alpha_nd(x_n,Sx_n).$$

Then since $\{x_n\}$, $\{Tx_n\}$ and $\{Sx_n\}$ are bounded, we obtain that

$$\sum_{n=0}^{\infty}d(x_n,x_{n+1})<\infty.$$

Thus, $\{x_n\}$ is strongly Cauchy sequence and hence $x_n \to x^* \in K$ as $n \to \infty$.

On the other hand, since $\lim_{n\to\infty} \theta_n = 1$ and $\theta_n = \max\{\theta_{n-1}, h(x_n)\}$, we can choose a subsequence $\{x_{n_i}\}$ such that $\{h(x_{n_i})\}$ is non-decreasing and $\lim_{i\to\infty} h(x_{n_i}) = 1$. In particular, for any $\gamma < 1$,

$$t_{n_i} := \gamma x_{n_i} \oplus (1-\gamma)Tx_{n_i} \notin K, \text{ eventually holds.}$$

Now, choose $\gamma_1, \gamma_2 \in (h(x^*), 1)$ such that $\gamma_1 \neq \gamma_2$ and let

$$v_1 = \gamma_1 x^* \oplus (1-\gamma_1)Tx^* \text{ and } v_2 = \gamma_2 x^* \oplus (1-\gamma_2)Tx^*.$$

Then for all $\gamma \in [\gamma_1, \gamma_2]$, we have

$$v := \gamma x^* \oplus (1-\gamma)Tx^* \in K.$$

Since $x_{n_i} \to x^*$ and $Tx_{n_i} \to Tx^*$ as $i \to \infty$, it follows that $t_{n_i} \to v$ as $i \to \infty$ and hence $v \in \partial K$. Furthermore, since γ is arbitrary, it follows that $[v_1, v_2] \subseteq \partial K$. Then by strict convexity of K, we obtain that $v_1 = v_2$. Then $d(v_1, Tx^*) = d(v_2, Tx^*)$ which implies that

$$\gamma_1 d(x^*, Tx^*) = \gamma_2 d(x^*, Tx^*).$$

Then since $\gamma_1 \neq \gamma_2$, it follows that $x^* = Tx^*$ and hence, $x^* \in F(T)$. It remains to show that $x^* \in F(S)$. To this end, we observe that $y_n := \theta_n x_n \oplus (1-\theta_n)Tx_n \to x^*$. Then, since $\lim_{n\to\infty} \alpha_n = 1$ and $\alpha_n = \max\{\alpha_{n-1}, f_{\theta_n}(x_n)\}$, repeating the above arguments we obtain the required result. Indeed, we can choose a subsequence $\{x_{n_j}\}$ such that $\{f_{\theta_{n_j}}(x_{n_j})\}$ is non-decreasing and $\lim_{j\to\infty} f_{\theta_{n_j}}(x_{n_j}) = 1$. In particular, for any $\mu < 1$,

$$s_{n_j} := \mu y_{n_j} \oplus (1-\mu)Sx_{n_j} \notin K, \text{ eventually holds.}$$

Now, choose $\mu_1, \mu_2 \in (f_{\theta_{n_j}}(x^*), 1)$ such that $\mu_1 \neq \mu_2$ and let

$$u_1 = \mu_1 x^* \oplus (1-\mu_1)Sx^* \text{ and } u_2 = \mu_2 x^* \oplus (1-\mu_2)Sx^*.$$

Thus, for any $\mu \in [\mu_1, \mu_2]$, we have

$$u := \mu x^* \oplus (1-\mu)Sx^* \in K.$$

Since $y_{n_j} \to x^*$ and $Sx_{n_j} \to Sx^*$ as $j \to \infty$, it follows that $S_{n_j} \to u$ as $j \to \infty$ and hence $u \in \partial K$. Furthermore, since μ is arbitrary, it follows that $[u_1, u_2] \subseteq \partial K$. Then by strict convexity of K, it follows that $u_1 = u_2$ and $d(u_1, Sx^*) = d(u_2, Sx^*)$ which implies that $\mu_1 d(x^*, Sx^*) = \mu_2 d(x^*, Sx^*)$. Then since $\mu_1 \neq \mu_2$, we have that $x^* = Sx^*$ and hence $x^* \in F(S)$. Therefore, $\{x_n\}$ strongly converges to $x^* \in \mathcal{F}$. □

Next, we prove strong convergence results using the condition (I). For this, we first give the definition of condition (I) for a pair of mappings. Recall that a mapping $T : K \to X$ is said to satisfy condition (I) if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$, for all $r \in (0,\infty)$ such that $d(x, Tx) \geq f(d(x, F(T)))$, for all $x \in K$, where $d(x, F(T)) = \inf\{d(x, p) : p \in F(T)\}$. Analogous definition for a pair of mappings is given below.

A pair of mappings S and T denoted by $\{S, T\}$ is said to satisfy condition (I) if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$, for all $r \in (0,\infty)$ such that

$$d(x, Tx) \geq f(d(x, \mathcal{F})) \text{ or } d(x, Sx) \geq f(d(x, \mathcal{F})), \forall x \in K,$$

where $d(x, \mathcal{F}) = \inf\{d(x, p) : p \in \mathcal{F} = F(T) \cap F(S)\}$.

Now, we consider the following Algorithm:

$$\begin{cases} x_1 \in K, \\ \theta_1 = \max\left\{\frac{1}{2}, h(x_1)\right\}, \\ \alpha_1 = \max\{\theta_1, f_{\theta_1}(x_1)\}, \\ x_{n+1} = \alpha_n[\theta_n x_n \oplus (1-\theta_n)Tx_n] \oplus (1-\alpha_n)Sx_n, \\ \theta_{n+1} \in \big[\max\{\theta_n, h(x_{n+1})\}, 1\big), \\ \alpha_{n+1} \in \big[\max\{\alpha_n, \ \theta_n, \ f_{\theta_{n+1}}(x_{n+1})\}, 1\big), \end{cases} \tag{6}$$

where $h : K \to \mathbb{R}$ and $f : K \to [0, \infty]$ are defined, respectively, by $h(x) = \inf\{\lambda \geq 0 : \lambda x \oplus (1-\lambda)Tx \in K\}$ and $f_\theta(x) = \inf\{\alpha \geq 0 : \alpha[\theta x \oplus (1-\theta)Tx] \oplus (1-\alpha)Sx_n \in K\}$, for a given $\theta \in [0, 1]$.

Theorem 10 *Let K be a nonempty, closed convex subset of a complete CAT(0) space X and $S, T : K \to X$ be two L-Lipschitz quasi-nonexpansive inward mappings. Let $\{x_n\}$ be a sequence as defined in (6) such that $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. If the pair $\{S, T\}$ satisfies the condition(I) and $\mathcal{F} = F(T) \cap F(S) \neq \emptyset$, then $\{x_n\}$ converges strongly to a point in $\mathcal{F}$.*

Proof From the method of the proof of Theorem 9, it is easy to see that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in \mathcal{F}$. In addition, by Lemma 1, we have

$$\begin{aligned} d^2(x_{n+1}, p) &= d^2(\alpha_n\theta_n x_n \oplus \alpha_n(1-\theta_n)Tx_n \oplus (1-\alpha_n)Sx_n, p) \\ &\leq \alpha_n\theta_n d^2(x_n, p) + \alpha_n(1-\theta_n)d^2(Tx_n, p) + (1-\alpha_n)d^2(Sx_n, p) \\ &\quad -\alpha_n^2\theta_n(1-\theta_n)d^2(Tx_n, x_n) \\ &\leq \alpha_n\theta_n d^2(x_n, p) + \alpha_n(1-\theta_n)d^2(x_n, p) + (1-\alpha_n d^2(x_n, p) \\ &\quad -\alpha_n^2\theta_n(1-\theta_n)d^2(Tx_n, x_n) \\ &= d^2(x_n, p) - \alpha_n^2\theta_n(1-\theta_n)d^2(Tx_n, x_n). \end{aligned}$$

This implies that

$$\alpha_n^2\theta_n(1-\theta_n)d^2(x_n, Tx_n) \leq d^2(x_n, p) - d^2(x_{n+1}, p),$$

which in turn implies that

$$\sum_{n=1}^{\infty} \alpha_n^2\theta_n(1-\theta_n)d^2(Tx_n, x_n) < \infty. \tag{7}$$

Also, from Lemma 1, we have:

$$\begin{aligned} d^2(x_{n+1}, p) &= d^2(\alpha_n\theta_n x_n \oplus \alpha_n(1-\theta_n)Tx_n \oplus (1-\alpha_n)Sx_n, p) \\ &\leq \alpha_n\theta_n d^2(x_n, p) + \alpha_n(1-\theta_n)d^2(Tx_n, p) + (1-\alpha_n)d^2(Sx_n, p) \\ &\quad -\alpha_n\theta_n(1-\alpha_n)d^2(Sx_n, x_n) \\ &\leq \alpha_n\theta_n d^2(x_n, p) + \alpha_n(1-\theta_n)d^2(x_n, p) + (1-\alpha_n)d^2(x_n, p) \\ &\quad -\alpha_n\theta_n(1-\alpha_n)d^2(Sx_n, x_n) \\ &= d^2(x_n, p) - \alpha_n\theta_n(1-\alpha_n)d^2(Sx_n, x_n). \end{aligned}$$

Hence, we obtain

$$\sum_{n=1}^{\infty} \alpha_n\theta_n(1-\alpha_n)d^2(Sx_n, x_n) < \infty. \tag{8}$$

Furthermore, since $\sum_{n=1}^{\infty}(1-\alpha_n)=\infty$ and $\alpha_n \geq \theta_n \geq \frac{1}{2}$ for each n, it follows that

$$\sum_{n=1}^{\infty}\alpha_n^2\theta_n(1-\theta_n)=\infty=\sum_{n=1}^{\infty}\alpha_n\theta_n(1-\alpha_n).$$

Thus, from (7) and (8), we get

$$\liminf_{n\to\infty} d(x_n, Tx_n)=0=\liminf_{n\to\infty} d(x_n, Sx_n).$$

Then, since $\{S, T\}$ satisfies the Condition (I), we have $\liminf_{n\to\infty} f(d(x_n, \mathcal{F}))=0$ for some increasing function $f:[0,\infty)\to[0,\infty)$ with $f(0)=0$, $f(r)>0$ for $r\in(0,\infty)$. This gives $\liminf_{n\to\infty} d(x_n, \mathcal{F})=0$. Moreover, since $d(x_{n+1}, p)\leq d(x_n, p)$, taking infimum over all $p\in\mathcal{F}$, we have

$$d(x_{n+1}, \mathcal{F})\leq d(x_n, \mathcal{F}).$$

Then, the sequence $\{d(x_n, \mathcal{F})\}$ is decreasing and hence $\lim_{n\to\infty} d(x_n, \mathcal{F})=0$.

Now, for arbitrary $p\in\mathcal{F}$ and any $n, m\geq 1$, we have

$$d(x_{n+m}, x_n)\leq d(x_{n+m}, p)+d(x_n, p)\leq 2d(x_n, p),$$

which implies that

$$d(x_{n+m}, x_n)\leq 2d(x_n, \mathcal{F}).$$

Then $\{x_n\}$ is a Cauchy sequence and hence $x_n\to x^*\in K$. Thus, we have

$$d(x^*, \mathcal{F})\leq d(x_n, x^*)+d(x_n, \mathcal{F})\to 0.$$

Then, it follows from Lemma 5 that $x^*\in\mathcal{F}$. This completes the proof. □

If, in Theorem 10, T and S are nonexpansive and $\mathcal{F}=F(T)\cap F(S)\neq\emptyset$, then T and S are 1-Lipschitz quasi-nonexpansive and so we have the following corollary.

Corollary 11 *Let K be a nonempty, closed convex subset of a complete CAT(0) space X and $S, T: K\to X$ be two nonexpansive inward mappings. Let $\{x_n\}$ be a sequence as defined in (6) such that $\sum_{n=1}^{\infty}(1-\alpha_n)=\infty$. If the pair $\{S, T\}$ satisfies the condition (I) and $\mathcal{F}=F(T)\cap F(S)\neq\emptyset$, then $\{x_n\}$ converges to a common fixed point of S and T.*

A mapping $T: K\to X$ is called hemicompact if, for any sequence $\{x_n\}$ in K such that $d(x_n, Tx_n)\to 0$, there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j}\to p\in K$. We note that if K is compact, then every mapping $T: K\to X$ is hemicompact.

Now, we prove the following theorem.

Theorem 12 *Let K be a nonempty, closed and convex subset of a complete CAT(0) space X and $S, T: K\to X$ be two L-Lipschitz quasi-nonexpansive inward mappings. Let $\{x_n\}$ be a sequence as defined in (6) such that $\sum_{n=1}^{\infty}(1-\alpha_n)=\infty$. Assume that $\mathcal{F}=F(T)\cap F(S)\neq\emptyset$ and T or S is hemicompact. Then $\{x_n\}$ converges strongly to a point in $\mathcal{F}$.*

Proof From the proof of Theorem 10, we have

$$\liminf_{n\to\infty} d(x_n, Tx_n)=0=\liminf_{n\to\infty} d(x_n, Sx_n).$$

Then, there exists a subsequence $\{x_m\}$ of $\{x_n\}$ such that $\lim_{m\to\infty} d(x_m, Tx_m)=0$. Without loss of generality, assume that T is hemicompact. Then there is a subsequence $\{x_{m_k}\}$ of $\{x_m\}$ such that $x_{m_k}\to x^*\in K$ as $k\to\infty$. Then the continuity of T implies

$$\lim_{k\to\infty} d(x_{m_k}, Tx_{m_k})=d(x^*, Tx^*)=0. \tag{9}$$

Moreover, we have:

$$\begin{aligned} d(x^*, Sx^*) &= \lim_{k\to\infty} d(x_{m_k}, Sx^*) \\ &= \liminf_{k\to\infty} d(x_{m_k}, Sx^*) \\ &\le \liminf_{k\to\infty} [d(x_{m_k}, Sx_{m_k}) + d(Sx_{m_k}, Sx^*)] \\ &\le \liminf_{k\to\infty} [d(x_{m_k}, Sx_{m_k}) + Ld(x_{m_k}, x^*)] = 0. \end{aligned} \tag{10}$$

From (9) and (10), we obtain $x^* \in \mathcal{F}$. By employing the method used to obtain (5), we can see that $\lim_{n\to\infty} d(x_n, x^*)$ exists. Thus,

$$\lim_{n\to\infty} d(x_n, x^*) = \lim_{k\to\infty} d(x_{m_k}, x^*) = 0.$$

Hence, $\{x_n\}$ converges strongly to $x^* \in \mathcal{F}$. □

If, in Theorem 12, we assume that K is Compact, then T and S are hemicompact and hence we have the following corollary.

Corollary 13 *Let K be a nonempty, compact and convex subset of a CAT(0) space X and $S, T : K \to X$ be two L-Lipschitz quasi-nonexpansive inward mappings with $F(T) \cap F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined in (6) such that $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. Then $\{x_n\}$ converges strongly to a common fixed point of T and S.*

Remark 1 In this paper, a new iterative method for finding a common fixed point of a pair of non-self mappings is studied in the setting of CAT(0) spaces. strong convergence and Δ-convergence results of the scheme to a common fixed point of two nonexpansive non-self mappings are obtained under mild conditions. Moreover, strong convergence results for a pair of quasi-nonexpansive non-self mappings are established under some additional conditions. Our results extend and generalize many of the results in the literature. For instance, our results extends the results of Yao and Chem [37] in the sense that our results are valid for non-self mappings in a CAT(0) space more general than Hilbert spaces.

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