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Generalization of prime ideals in $M_n(N)$-group N^n

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Abstract
The notion of a matrix nearring over an arbitrary nearring was introduced by (Meldrum and Walt Arch. Math. 47(4): 312–319, 1986). In this paper, we define the notions such as weakly τ-prime ($\tau = 0, c, 3, e$) ideals of an N-group G, which are the generalization of the classes of τ-prime ideals of G, and provide suitable examples to distinguish between the two classes. We extend the concept to obtain the one-one correspondence between weakly τ-prime ideals ($\tau = 0, c, 3, e$) of N-group (over itself) and those of $M_n(N)$-group N^n, where $M_n(N)$ is the matrix nearring over the nearring N. Further, we prove the correspondence between weakly 2-absorbing ideals of these classes.

Keywords Nearring · Matrix nearring · Prime ideal · Absorbing ideal

Mathematics Subject Classification 16Y30

1 Introduction

Nearrings are generalized rings where the addition need not be abelian and only one distributive property is allowed. Rings can be viewed as algebraic systems of 'linear' functions on groups, while nearrings describe the general non-linear case [2]. Matrix nearrings over arbitrary nearrings were introduced by Meldrum & Van der Walt [1],

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wherein the correspondence between the two-sided ideals in nearring N and those of matrix nearring $M_n(N)$ were obtained. We denote N for a zero-symmetric (right) nearring with 1, and G for an N-group (denoted as, ${}_N G$). Considerable developments in matrix nearrings over arbitrary nearrings were due to Meyer [3], Booth and Groenewald [4]. Meyer [3] proved that for a left ideal L of N, L^n is its corresponding ideal in $M_n(N)$-module N^n. Van der Walt [5] studied the relationship between primitive modules over a nearring N and those of the matrix nearring $M_n(N)$. Indeed, when G is a locally monogenic N-group, then the action of G^n over $M_n(N)$ was defined. However, for an N-group (over itself), N^n becomes an $M_n(N)$-group. The motivation to study the interrelations between the N-group over N and the $M_n(N)$-group N^n is an analogy to the concept of Morita equivalence. Two unital rings are Morita equivalent if they have equivalent categories of left modules. If A is a left R-module, then A^n, $n \in \mathbb{N}$ is a $M_n(R)$-module in a canonical way. The correspondence $A \to A^n$ preserves homomorphisms and conversely every $M_n(R)$-module can be obtained in this way. In this paper, we introduce the concepts weakly τ-prime ($\tau = 0, c, 3, e$) ideals of an N-group G and obtain the one-one correspondence between weakly τ-prime ideals of N-group (over itself) and those of $M_n(N)$-group N^n. As a generalization of a c-prime and a 3-prime ideal of N-group G, we define the notions weakly (c, 2)-absorbing ideal and weakly (3, 2)-absorbing ideal, respectively, and provide suitable examples. Further, we prove the one-one correspondence between (i, 2), $i \in \{c, 3\}$-absorbing ideals of N (over N) and those of $M_n(N)$-group N^n. Bhavanari et.al [6] proved the correspondence between the prime left ideals of N and that of $M_n(N)$. Juglal et.al [7] studied different prime N-ideals and prime relations between generalized matrix nearring and multiplication modules over a nearring. Further, Juglal and Groenewald [8] studied the class of strongly prime nearring modules and shown that it forms a τ-special class. In Bhavanari and Kuncham [9], the relation between the ideals of the N-group N and the ideals of $M_n(N)$-group N^n has been studied. Badawi and Darani [10] studied weakly 2-absorbing ideals as a generalization of prime ideals in commutative rings.

For standard notations and definitions in nearrings, we refer to Pilz [11], Bhavanari and Kuncham [2].

We denote $H \trianglelefteq_N G$ for an ideal H of G (if H is proper, we use $H \triangleleft_N G$).

From [2], for any $u \in G$, the ideal generated by u is denoted by $\langle u \rangle$ and defined as, $\langle u \rangle = \bigcup_{i=1}^{\infty} S_{i+1}$, where $S_{i+1} = S_i^* \cup S_i^0 \cup S_i^+$ with $S_0 = \{u\}$, and $S_i^* = \{g + y - g : g \in G, y \in S_i\}$, $S_i^0 = \{p - q : p, q \in S_i\} \cup \{p + q : p, q \in S_i\}$, $S_i^+ = \{n(g + a) - ng : n \in N, g \in G, a \in S_i\}$.

Definition 1.1 [7, 12] Let $P \triangleleft_N G$ such that $NG \nsubseteq P$. Then

1. P is *prime* (or *0-prime*) if for every ideal A of N and every ideal B of G, $AB \subseteq P$, then $AG \subseteq P$ or $B \subseteq P$,
2. P is 3-*prime* if for $n \in N$ and $g \in G$, $nNg \subseteq P$, then $nG \subseteq P$ or $g \in P$.
3. P is *completely prime* (denoted as, *c-prime*) if for $n \in N$ and $g \in G$, $ng \in P$, then $nG \subseteq P$ or $g \in P$.
4. P is *equiprime* (denoted as, *e-prime*) if $a \in N$ and $g_1, g_2 \in G$, $ang_1 - ang_2 \in P$, for all $n \in N$, then $aG \subseteq P$ or $g_1 - g_2 \in P$.

2 Weakly prime ideals of *N*-group

We define weakly ($\tau = 0, c, 3, e$)-prime ideal of an N-group G and provide examples to distinguish between the classes weakly τ-prime ideals and τ-prime ideals of G.

Definition 2.1 An ideal P of G with $NG \nsubseteq P$ is called a *weakly* 0-*prime* ideal (or simply *weakly prime* ideal) of G, if for any ideal A of N and ideal B of G, $(0) \neq AB \subseteq P$, then $AG \subseteq P$ or $B \subseteq P$.

Proposition 2.2 *The following conditions are equivalent for an ideal P of G.*

1. *P is a weakly prime ideal of G.*
2. *If $x \in N, y \in G, (0) \neq \langle x\rangle\langle y\rangle \subseteq P$, then $xG \subseteq P$ or $y \in P$.*
3. *If $x \in N, y \in G, (0) \neq x\langle y\rangle \subseteq P$, then $xG \subseteq P$ or $y \in P$.*
4. *If $x \in N, y \in G, (0) \neq \langle x\rangle(P + \langle y\rangle) \subseteq P$, then $xG \subseteq P$ or $y \in P$.*
5. *If A is an ideal of N, B an ideal of G and $(0) \neq A(P + B) \subseteq P$, then $AG \subseteq P$ or $B \subseteq P$.*

Proof (1)⇒(2): Suppose that P is a weakly prime ideal of G.

Let $(0) \neq \langle x\rangle\langle y\rangle \subseteq P$. Then by (1), $\langle x\rangle G \subseteq P$ or $\langle y\rangle \subseteq P$. Now $xG \subseteq \langle x\rangle G \subseteq P$ or $y \in \langle y\rangle \subseteq P$, implies $xG \subseteq P$ or $y \in P$.

(2)⇒(3) is obvious.

(3)⇒(4): Suppose that $(0) \neq \langle x\rangle(P + \langle y\rangle) \subseteq P$. Then there exists $(0) \neq x\langle y\rangle \subseteq \langle x\rangle(P + \langle y\rangle) \subseteq P$. This implies $(0) \neq x\langle y\rangle \subseteq P$. Now by (3), $xG \subseteq P$ or $y \in P$.

(4)⇒(5): Suppose that $(0) \neq A(P + B) \subseteq P$. In a contrary way, suppose that $AG \nsubseteq P$ or $B \nsubseteq P$. Then there exist $xg \in AG\backslash P$ and $y \in B\backslash P$, for some $x \in A$, $g \in G$, which implies $xg \notin P$ or $y \notin P$. Then by (4), $(0) \neq \langle x\rangle(P + \langle y\rangle) \nsubseteq P$.

But $\langle x\rangle(P + \langle y\rangle) \subseteq A(P + B) \subseteq P$, a contradiction. Therefore, $AG \subseteq P$ or $B \subseteq P$.

(5)⇒(1): Let A be an ideal of N and B an ideal of G such that $(0) \neq AB \subseteq P$. In a contrary way, $AG \nsubseteq P$ and $B \nsubseteq P$. Then by (5), $(0) \neq A(P + B) \nsubseteq P$, which implies there exist $x \in A$ and $y \in B$ such that $0 \neq xy \in A(P + B) \nsubseteq P$. This implies, $0 \neq xy \in P$, a contradiction. Therefore, $AG \subseteq P$ or $B \subseteq P$. □

Definition 2.3 An ideal I of G with $NG \nsubseteq I$, is called

1. *weakly c-prime* if $n \in N$, $g \in G$ and $0 \neq ng \in I$, then $nG \subseteq I$ or $g \in I$.
2. *weakly* 3-*prime* if $n \in N$, $g \in G$ and $(0) \neq nNg \subseteq I$, then $nG \subseteq I$ or $g \in I$.
3. *weakly e-prime* if $a \in N$, $g_1, g_2 \in G$ and $0 \neq ang_1 - ang_2 \in I$, for all $n \in N$, then $aG \subseteq I$ or $g_1 - g_2 \in I$.

Proposition 2.4 *Let be an ideal of G such that $NG \nsubseteq P$. Then P is*

1. *weakly c-prime implies weakly* 3-*prime*;
2. *weakly* 3-*prime implies weakly prime*;
3. *weakly e-prime implies weakly* 3-*prime*.

Proof (1) Suppose P is a weakly c-prime ideal. Let $a \in N$ and $g \in G$ such that $(0) \neq aNg \subseteq P$. To show $aG \subseteq P$ or $g \in P$. If $g \in P$, it is clear.

Case-(i) Let $g \notin P$. If $(0) = Ng$, then since $1 \in N$, $0 = g$, a contradiction. Now assume that, $(0) \neq Ng \subseteq P$. Now, $0 \neq ag \in P$, for all $a \in N$. Since P is weakly c-prime and $g \notin P$, we get $aG \subseteq P$.

Case-(ii) Let $g \notin P$ and $(0) \neq Ng \nsubseteq P$. Then, there exists $n \in N$ such that $0 \neq ng \in P$, but $0 \neq ang \in P$. Since P is weakly c-prime and $g \notin P$, we get $aG \subseteq P$. Therefore, P is weakly 3-prime.

(2) Suppose P is weakly 3-prime. Let A be an ideal of N and B be an ideal of G such that $(0) \neq AB \subseteq P$. To show $AG \subseteq P$ or $B \subseteq P$. If $B \subseteq P$, it is clear. If $B \nsubseteq P$, there exists $x \in B \setminus P$, and since $1 \in N$, for every $a \in A$, we have $(0) \neq aNx \subseteq ANB \subseteq AB$ (by Pilz [11], Proposition 1.34) $\subseteq P$. Since P is weakly 3-prime and $x \notin P$, it follows that $aG \subseteq P$, for all $a \in A$. Therefore, $AG \subseteq P$, shows that P is weakly prime.

(3) Suppose P is weakly e-prime. Let $a \in N$ and $g \in G$ such that $(0) \neq aNg \subseteq P$. To show $aG \subseteq P$ or $g \in P$. If $g \in P$, it is clear. So, let $g \notin P$. Since $(0) \neq aNg \subseteq P$, $0 \neq ang \in P$, for all $n \in N$. Now $0 \neq ang = ang - an0_G \in P$, as N is zero-symmetric. Since P is weakly e-prime and $g = g - 0_G \notin P$, we get $aG \subseteq P$. Therefore, P is weakly 3-prime. □

Example 2.5

1. Consider $N = (\mathbb{Z}, +, \cdot)$ and $G = ((\mathbb{Z}_4 \times \mathbb{Z}_2), +)$. Then G is an N-group. The ideal $\langle(0,1)\rangle$ is weakly prime, but not prime, as $2\mathbb{Z} \cdot \langle(2,1)\rangle = \langle(0,0)\rangle \subseteq \langle(0,1)\rangle$, but $\langle(2,1)\rangle \nsubseteq \langle(0,1)\rangle$ and $2\mathbb{Z} \cdot G \nsubseteq \langle(0,1)\rangle$.
2. Let $p \neq q$, be distinct prime. Consider $N = (\mathbb{Z}, +, \cdot)$ and $G = (\mathbb{Z}_{pq}, +_{pq})$. Then G is an N-group. Here, (0) is weakly prime ideal but not prime.
3. Let $N = \{0, a, b, c\}$ ([11], Table E(8)), $G = N$. Then G is an N-group over itself.

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

·	0	a	b	c
0	0	0	0	0
a	0	0	0	a
b	0	a	b	b
c	0	a	b	c

The ideals of G are $I_1 = \{0, a\}, I_2 = \{0, b\}, I_3 = \{0, c\}$. An easy observation yield, $I_1I_2 = \{0\}$, $I_2I_1 = I_1$, $I_2I_3 = I_2$, $I_3I_2 = I_2$, $I_3I_1 = I_1$, $I_1I_3 = I_1$.

(a) I_1 is weakly prime as well as prime.
(b) I_2 is neither weakly prime nor prime, since $(0) \neq I_2I_3 = I_2$, but $I_2G \nsubseteq I_2, I_3 \nsubseteq I_2$.

(c) I_3 is weakly prime but not prime, since $I_1I_2 = \{0\} \subseteq I_3$, and $I_1G \nsubseteq I_3, I_2 \nsubseteq I_3$.
(d) I_2 is weakly c-prime, but not c-prime, since $a.b = 0 \in I_3$, and $aG \nsubseteq I_3$, $b \notin I_3$.
(e) I_3 is weakly 3-prime, but not 3-prime, since $aNb = 0 \in I_3$, and $aG \nsubseteq I_3$, $b \notin I_3$.
(f) I_2 is weakly 3-prime, but not 3-prime, since $aN = 0 \in I_3$, and $aG \nsubseteq I_3$, $a \notin I_3$.
(g) I_2 is weakly e-prime, but not e-prime, since $0 = ana - anb \in I_2$, and $aG \nsubseteq I_2$, $a - b \notin I_2$.

4. Let $(N, +)$ be any group with at least 3 distinct non-zero elements, say a, g_1, g_2. Define multiplication on N by

$$ab = \begin{cases} a & \text{if } b \neq 0, \\ 0 & \text{if } b = 0. \end{cases}$$

Then $(N, +, \cdot)$ is a zero-symmetric nearring. Here, $\{0\}$ is weakly e-prime ideal but not e-prime, since $0 \notin \{a, g_1, g_2\} \subseteq N$ such that $g_1 \neq g_2$ and $aN \nsubseteq \{0\}$. However, for each element $n \in N$,

$$ang_1 = \begin{cases} an & \text{if } g_1 \neq 0, \\ 0 & \text{if } g_1 = 0. \end{cases}, \quad ang_2 = \begin{cases} an & \text{if } g_2 \neq 0, \\ 0 & \text{if } g_2 = 0. \end{cases}$$

Thus, $ang_1 - ang_2 = 0$.

3 Weakly prime ideals of $M_n(N)$-group N^n

Now we give some basic definitions about the matrix nearring from [1]. For a zero-symmetric right nearring N with identity 1, let N^n will denote the direct sum of n copies of $(N, +)$. The elements of N^n are column vectors and written as $\left(r_1, \dots, r_n\right)$. The symbols i_j and π_j respectively, denote the i^{th} coordinate injective and j^{th} coordinate projective maps.

For an element $a \in N$, $i_i(a) = (0, \dots, \underbrace{a}_{i^{th}}, \dots, 0)$, and $\pi_j(a_1, \dots, a_n) = a_j$, for any $(a_1, \dots, a_n) \in N^n$. The nearring of $n \times n$ matrices over N, denoted by $M_n(N)$, is defined to be the subnearring of $M(N^n)$, generated by the set of functions $\{f_{ij}^r : N^n \to N^n \mid r \in N, 1 \leq i, j \leq n\}$ where $f_{ij}^r\left(r_1, \dots, r_n\right) := \left(s_1, s_2, \dots, s_n\right)$ with $s_i = rr_j$ and $s_k = 0$ if $k \neq i$. Clearly, $f_{ij}^r = i_i f^r \pi_j$, where $f^r : N \to N$, $r \in N$ by $f^r(x) = rx$, for all $x \in N$. If N happens to be a ring, then f_{ij}^r corresponds to the $n \times n$-matrix with r in position (i, j) and zeros elsewhere. Following the notation from ([9], Notation 1.1), for any ideal $\mathcal{I}$ of $M_n(N)$-group N^n, we write

$$\mathcal{I}_{**} = \{a \in N : a = \pi_j A, \text{ for some } A \in \mathcal{I}, 1 \leq j \leq n\},$$

where π_j is the j^{th} projection map from N^n to N.

Throughout, we denote $M_n(N)$ for a matrix nearring, N^n for an $M_n(N)$-group N^n; 0, $\bar{\mathbf{0}}$ and $\bar{0}$ respectively denote the additive identities of N, $M_n(N)$ and N^n. As usual, I^n denotes the direct sum of n copies of an ideal I of N. Furthermore, we refer to Meldrum & Van der Walt [1] for detailed literature on matrix nearrings. Now, we introduce the notions weakly τ-prime ($\tau = 0, c, 3, e$) ideals of $M_n(N)$-group N^n and prove the correspondence between the weakly τ-prime ideals ($\tau = 0, c, 3, e$) of N-group (over itself) and those of $M_n(N)$-group N^n.

The following Lemma 3.1 and Lemma 3.3 from [9] are useful, and so, for completeness we brief the proofs.

Lemma 3.1 *For any $t \in N$, let $i_i(t) = (0, \dots, t, \dots, 0)$ with t in the i^{th} position. Then $\langle i_i(t)\rangle = \langle (t, 0, \dots, 0)\rangle$.*

Proof Suppose t is an element of N. Then $(t, 0, \dots, 0) \in N^n$. Now $(t, 0, \dots, 0) = f^1_{1i}(i_i(t)) \in \langle i_i(t)\rangle$. Therefore, $\langle (t, 0, \dots, 0)\rangle \subseteq \langle i_i(t)\rangle$. On the other hand, $i_i(t) = f^1_{1i}(t, 0, \dots, 0) \in \langle (t, 0, \dots, 0)\rangle$. Therefore, $\langle i_i(t)\rangle \subseteq \langle (t, 0, \dots, 0)\rangle$. □

Lemma 3.2 *If s, t are elements of N and $s \in \langle t\rangle$, then*

$$(s, 0, \dots, 0) \in \langle (t, 0, \dots, 0)\rangle.$$

Lemma 3.3 *If t is an element of N, then $\langle t\rangle^n = \langle (t, 0, \dots, 0)\rangle$.*

Proof Since $(t, 0, \dots, 0) \in \langle t\rangle^n$, we have $\langle (t, 0, \dots, 0)\rangle \subseteq \langle t\rangle^n$. Take $(s_1, s_2, \dots, s_n) \in \langle t\rangle^n$. This implies $s_i \in \langle t\rangle^n$, for $1 \le i \le n$. By Lemma 3.2, $(s_i, 0, \dots, 0) \in \langle (t, 0, \dots, 0)\rangle$ for all i. Now, from ([11], Proposition 1.34), it follows that

$$\begin{aligned}(s_1, s_2, \dots, s_n) &= f^1_{11}(s_1, 0, \dots, 0) + f^1_{21}(s_2, 0, \dots, 0)\\ &\quad + \dots + f^1_{n1}(s_n, 0, \dots, 0)\\ &\in \langle (t, 0, \dots, 0)\rangle.\end{aligned}$$

□

Definition 3.4 An ideal $\mathcal{I}$ of N^n with $M_n(N)N^n \nsubseteq \mathcal{I}$ is said to be a *weakly prime* ideal if for any ideal $\mathcal{A}$ of $M_n(N)$, and an ideal B of N^n with $(\bar{0}) \nsubseteq \mathcal{A}B \subseteq \mathcal{I}$, then either $\mathcal{A}N^n \subseteq \mathcal{I}$ or $B \subseteq \mathcal{I}$.

Proposition 3.5 *If I is a weakly prime ideal in ${}_NN$, then I^n is a weakly prime ideal in $M_n(N)$-group N^n.*

Proof Let I be weakly prime in ${}_NN$. By ([1], Proposition 4.1), I^n is an ideal of N^n. To prove, I^n is weakly prime in N^n, let $\mathcal{A}$ an ideal of $M_n(N)$ and K be an ideal of N^n such that $\mathcal{A}N^n \nsubseteq I^n$ and $K \nsubseteq I^n$. By ([9], Lemma 1.5), $K = B^n$ for some ideal B of ${}_NN$. Then there exist $A \in \mathcal{A}$, $\rho \in N^n$ and $(b_1, b_2, \dots, b_n) \in B^n$ such that $A\rho \notin I^n$ and $(b_1, b_2, \dots, b_n) \notin I^n$. Suppose $w(A) = 1$, say $A = f^a_{ij}$, $a \in N$, and $\rho = (x_1, x_2, \dots, x_n)$. Then $(0, \dots, ax_j, \dots, 0) = f^a_{ij}(x_1, x_2, \dots, x_n) \notin I^n$ and $(b_1, b_2, \dots, b_n) \notin I^n$, yield $ax_j \notin I$ and $b_k \notin I$, for some k. Since I is weakly prime, by Proposition 2.2, we have $\langle 0\rangle \neq a\langle b_k\rangle \nsubseteq I$. Then there exists $b \in \langle b_k\rangle$ such that $0 \neq ab \notin I$. Now,

$$\begin{aligned}(0, \dots, 0) &\neq (0, \dots, ab, \dots, 0)\\ &= f^{ab}_{ij}(1, \dots, 1)\\ &= f^a_{ij} f^b_{jj}(1, \dots, 1)\ (([14, \textit{Lemma}\ 3.1(3))]\\ &= f^a_{ij}(0, \dots, b, \dots, 0).\end{aligned}$$

Since $f_{ij}^a \in \mathcal{A}$, and $(0, \dots, b, \dots, 0) \in \langle b_k \rangle^n \subseteq \langle (0, \dots, b_k, \dots, 0) \rangle$ by Lemma 3.3 $\subseteq \langle (b_1, \dots, b_n) \rangle$, it follows that $(\bar{0}) \neq \mathcal{A}B^n \nsubseteq I^n$. We use the induction on the complexity of A. Assume the induction hypothesis: for all $A \in \mathcal{A}$ and $w(A) < n$, and now consider $w(A) = n$. Then $A = C + D$ or $A = CD$, where $w(C), w(D) < n$. Let $A = C + D$. Now, $(\bar{0}) \neq AB^n = (C + D)B^n = CB^n + DB^n \notin I^n + I^n = I^n$, by induction hypothesis. Therefore, $(\bar{0}) \neq \mathcal{A}B^n \nsubseteq I^n$. Let $A = CD$. Then $(\bar{0}) \neq AB^n = (CD)B^n = C(DB^n) \notin I^n$, by induction hypothesis. Therefore, $(\bar{0}) \neq \mathcal{A}B^n \nsubseteq I^n$. □

Proposition 3.6 *If $\mathcal{I}$ is a weakly prime ideal in $M_n(N)$-group N^n, then $\mathcal{I}_{**}$ is a weakly prime ideal in ${}_N N$.*

Proof Suppose that $\mathcal{I}$ is weakly prime N^n. By ([9], Lemma 1.3), $\mathcal{I}_{**}$ is an ideal of ${}_N N$. To prove $\mathcal{I}_{**}$ is weakly prime, let A and B be ideals of ${}_N N$ with $A \nsubseteq \mathcal{I}_{**}$ and $B \nsubseteq \mathcal{I}_{**}$. Then there exist $a \in A \backslash \mathcal{I}_{**}$ and $b \in B \backslash \mathcal{I}_{**}$. By ([9], Lemma 1.2), $(a, 0, \dots, 0) \notin \mathcal{I}$ and $(b, 0, \dots, 0) \notin \mathcal{I}$. That is, $f_{11}^a(1, \dots, 1) \notin \mathcal{I}$ and $(b, 0, \dots, 0) \notin \mathcal{I}$. Then, $f_{11}^a N^n \nsubseteq \mathcal{I}$ and $(b, 0, \dots, 0) \notin \mathcal{I}$. Since $\mathcal{I}$ is weakly prime, we have $(0) \neq f_{11}^a \langle (b, 0, \dots, 0) \rangle \nsubseteq \mathcal{I}$. Then there exists $\left(b', 0, \dots, 0\right) \in \langle (b, 0, \dots, 0) \rangle$ such that

$$\begin{aligned} \bar{0} \neq & f_{11}^a\left(b', 0, \dots, 0\right) \\ & = \left(ab', 0, \dots, 0\right) \\ & \notin \mathcal{I}. \end{aligned}$$

Hence, by ([9], Lemma 1.2), $0 \neq ab' \notin \mathcal{I}_{**}$, where $a \in A$ and $\left(b', 0, \dots, 0\right) \in \langle (b, 0, \dots, 0) \rangle = \langle b \rangle^n \subseteq B^n$, yields $b' \in B$. Therefore, $(0) \neq AB \nsubseteq \mathcal{I}_{**}$, and proves $\mathcal{I}_{**}$ is weakly prime in ${}_N N$. □

Definition 3.7 An ideal $\mathcal{I}$ of N^n with $M_n(N)N^n \nsubseteq \mathcal{I}$, is said to be *weakly c-prime* if for any $A \in M_n(N)$ and $\rho \in N^n$, $\bar{0} \neq A\rho \notin \mathcal{I}$, then $AN^n \subseteq \mathcal{I}$ or $\rho \in \mathcal{I}$.

Proposition 3.8 *If I is a weakly c-prime ideal of ${}_N N$, then I^n is weakly c-prime in $M_n(N)$-group N^n.*

Proof Let I be a weakly c-prime ideal of ${}_N N$. By ([1], Proposition 4.1), I^n is an ideal of N^n. To prove, I^n is weakly c-prime in N^n, let $A \in M_n(N)$ and $\left(b_1, \dots, b_n\right) \in N^n$ such that $AN^n \nsubseteq I^n$ and $\left(b_1, \dots, b_n\right) \notin I^n$. Let $w(A) = 1$, say $A = f_{ij}^a$, $a \in N$ such that $f_{ij}^a(x_1, x_2, \dots, x_n) \notin I^n$, for some $(x_1, x_2, \dots, x_n) \in N^n$, and $(b_1, b_2, \dots, b_n) \notin I^n$. This implies $(0, \dots, ax_j, \dots, 0) \notin I^n$ and $(b_1, b_2, \dots, b_n) \notin I^n$. Then $ax_j \notin I$ and $b_k \notin I$ for some k. Since I is weakly c-prime, $0 \neq ab_k \notin I$. Hence,

$$\begin{aligned} (0, \dots, 0, \dots, 0) \neq & (0, \dots, ab_k, \dots, 0) \\ & = f_{ij}^{ab_k}(1, \dots, 1) \\ & = f_{ij}^a f_{jk}^{b_k}(1, \dots, 1)([14, \mathit{Lemma}\ 3.1(3))] \\ & = f_{ij}^a(b_1, \dots, b_k, \dots, b_n) \notin I^n. \end{aligned}$$

Therefore, $\bar{0} \neq A(b_1, \dots, b_n) \notin I^n$. The rest of the proof follows by induction on the complexity of A as in the Proposition 3.5. □

Proposition 3.9 *If $\mathcal{I}$ is a weakly c-prime ideal of $M_n(N)$-group N^n, then $\mathcal{I}_{**}$ is a weakly c-prime ideal of ${}_NN$.*

Proof Suppose that $\mathcal{I}$ is a weakly c-prime ideal of N^n. In view of ([9], Lemma 1.3), it is enough to show the weakly primeness of $\mathcal{I}_{**}$. Let $a, b \in N$ such that $a \notin \mathcal{I}_{**}$ and $b \notin \mathcal{I}_{**}$. Then, by ([9], Lemma 1.2), $(a, 0, \dots, 0) \notin \mathcal{I}$ and $(b, 0, \dots, 0) \notin \mathcal{I}$. This implies $f_{11}^{a}(1, \dots, 1) \notin \mathcal{I}$ and $(b, 0, \dots, 0) \notin \mathcal{I}$. By taking $A = f_{11}^{a}$, we get $AN^n \nsubseteq \mathcal{I}$ and $(b, 0, \dots, 0) \notin \mathcal{I}$. Since $\mathcal{I}$ is a weakly c-prime ideal of N^n, we have $\bar{0} \neq A(b, 0, \dots, 0) \notin \mathcal{I}$. So $\bar{0} \neq f_{11}^{a}(b, 0, \dots, 0) = (ab, 0, \dots, 0) \notin \mathcal{I}$. Thus, by ([9], Lemma 1.2), $0 \neq ab \notin \mathcal{I}_{**}$. □

Definition 3.10 An ideal $\mathcal{I}$ of N^n with $M_n(N)N^n \nsubseteq \mathcal{I}$, is said to be *weakly* 3*-prime* if for any $A \in M_n(N)$ and $\rho \in N^n$ with $(\bar{0}) \neq AM_n(N)\rho \subseteq \mathcal{I}$, then $AN^n \subseteq \mathcal{I}$ or $\rho \in \mathcal{I}$.

Proposition 3.11 *If I is a weakly* 3*-prime ideal of ${}_NN$, then I^n is a weakly* 3*-prime ideal in $M_n(N)$-group N^n.*

Proof Let I be a weakly 3-prime ideal of ${}_NN$. By ([1], Proposition 4.1), I^n is an ideal of N^n . To show I^n is weakly 3-prime in N^n, let $A \in M_n(N)$ and $\left(b_1, \dots, b_n\right) \in N^n$ such that $AN^n \nsubseteq I^n$ and $\left(b_1, \dots, b_n\right) \notin I^n$. The proof is based on complexity of A. For this, suppose that $w(A) = 1$, say $A = f_{ij}^{a}$, $a \in N$ such that $f_{ij}^{a}(x_1, x_2, \dots, x_n) \notin I^n$, for some $(x_1, x_2, \dots, x_n) \in N^n$, implies $(0, \dots, ax_j, \dots, 0) \notin I^n$, and since $(b_1, b_2, \dots, b_n) \notin I^n$, we get $ax_j \notin I$ and $b_k \notin I$, for some k. Since I is weakly 3-prime, we have $(0) \neq aNb_k \nsubseteq I$. Then there exists $n_1 \in N$ such that $0 \neq an_1b_k \notin I$. Therefore,

$$
\begin{aligned}
(0, \dots, 0) \neq (0, \dots, \underbrace{an_1b_k}_{i^{th}}, \dots, 0) \\
= f_{ik}^{an_1}(b_1, \dots, b_k, \dots, b_n) \\
= f_{ij}^{a}f_{jk}^{n_1}(b_1, \dots, b_k, \dots, b_n) \notin I^n, \text{ as } f_{jk}^{n_1} \in M_n(N).
\end{aligned}
$$

Therefore, $(\bar{0}) \neq AM_n(N)(b_1, \dots, b_n) \nsubseteq I^n$. The rest of the proof follows by induction on the complexity of A as in the Proposition 3.5. □

Proposition 3.12 *If $\mathcal{I}$ is a weakly* 3*-prime ideal of $M_n(N)$-group N^n, then $\mathcal{I}_{**}$ is a weakly* 3*-prime ideal of ${}_NN$.*

Proof Suppose $\mathcal{I}$ is a weakly 3-prime ideal of N^n. By ([9], Lemma 1.3), $\mathcal{I}_{**}$ is an ideal of ${}_NN$. To show $\mathcal{I}_{**}$ is weakly 3-prime, let $a, b \in N$ such that $aN \nsubseteq \mathcal{I}_{**}$ and $b \notin \mathcal{I}_{**}$. Then, $ax_1 \notin \mathcal{I}_{**}$, for some $x_1 \in N$ and $b \notin \mathcal{I}_{**}$, and by ([9], Lemma 1.2), $\left(ax_1, 0, \dots, 0\right) \notin \mathcal{I}$ and $(b, 0, \dots, 0) \notin \mathcal{I}$, which implies that $f_{11}^{a}\left(x_1, \dots, x_n\right) \notin \mathcal{I}$ and $(b, 0, \dots, 0) \notin \mathcal{I}$. Take $A = f_{11}^{a}$. Then $AN^n \nsubseteq \mathcal{I}$ and $(b, 0, \dots, 0) \notin \mathcal{I}$. Since $\mathcal{I}$ is weakly 3-prime, $\bar{0} \neq AM_n(N)(b, 0, \dots, 0) \nsubseteq \mathcal{I}$. Then there exists $f_{ij}^{n_1} \in M_n(N)$, $n_1 \in N$ such that $(0, 0, \dots, 0) \neq f_{11}^{a}f_{ij}^{n_1}(b, 0, \dots, 0) \notin \mathcal{I}$. By ([1], Lemma 3.1 (3)), and N is zero-symmetric, we have the following.

If $1 \neq i = j$, then

$$
\begin{aligned}
f^{a}_{11} f^{n_1}_{ij}(b,0,\dots,0) &= f^{a0}_{1j}(b,0,\dots,0)\\
&= f^{0}_{1j}(b,0,\dots,0)\\
&= (0,0,\dots,0), \quad \text{a contradiction.}
\end{aligned}
$$

If $1 = i \neq j$, then

$$
\begin{aligned}
f^{a}_{11} f^{n_1}_{ij}(b,0,\dots,0) &= f^{an_1}_{1j}(b,0,\dots,0)\\
&= \left(an_1 0,0,\dots,0\right)\\
&= (0,0,\dots,0), \quad \text{a contradiction.}
\end{aligned}
$$

If $1 = i = j$, then

$$
\begin{aligned}
\left(an_1 b,0,\dots,0\right) &= f^{an_1}_{11}(b,0,\dots,0)\\
&= f^{a}_{11} f^{n_1}_{ij}(b,0,\dots,0)\\
&\notin \mathcal{I}.
\end{aligned}
$$

Therefore, by ([9], Lemma 1.2), $0 \neq an_1 b \notin \mathcal{I}_{**}$, and thus $(0) \neq aNb \nsubseteq \mathcal{I}_{**}$. □

Definition 3.13 An ideal $\mathcal{I}$ of N^n with $M_n(N)N^n \nsubseteq \mathcal{I}$, is said to be *weakly equiprime* (weakly e-prime) if $A \in M_n(N)$ and $\rho_1, \rho_2 \in N^n$ with $\bar{0} \neq AB\rho_1 - AB\rho_2 \in \mathcal{I}$, for all $B \in M_n(N)$, then $AN^n \subseteq \mathcal{I}$ or $\rho_1 - \rho_2 \in \mathcal{I}$.

Proposition 3.14 *If I is a weakly e-prime ideal of ${}_N N$, then I^n is a weakly e-prime ideal in $M_n(N)$-group N^n.*

Proof Let I be a weakly e-prime ideal of ${}_N N$. By ([1], Proposition 4.1), I^n is an ideal of N^n. To prove I^n is weakly e-prime, let $M_n(N)N^n \nsubseteq I^n$ and $A \in M_n(N)$, $\rho_1 = (x_1,\dots,x_n)$, $\rho_2 = (y_1,\dots,y_n) \in N^n$ such that $AN^n \nsubseteq I^n$ and $\rho_1 - \rho_2 \notin I^n$. Suppose that $w(A) = 1$, say $A = f^{a}_{ij}, a \in N$ such that $f^{a}_{ij}(z_1,\dots,z_n) \notin I^n$ for some $(z_1,\dots,z_n) \in N^n$ and $\rho_1 - \rho_2 = (x_1 - y_1,\dots,x_n - y_n) \notin I^n$. This means that,

$$
(0,\dots,\underbrace{az_j}_{i^{th}},\dots,0) \notin I^n \text{ and } (x_1 - y_1,\dots,x_n - y_n) \notin I^n,
$$

and yield $az_j \notin I$ and $x_k - y_k \notin I$, for some k. That is, $aN \nsubseteq I$ and $x_k - y_k \notin I$. Since I is weakly e-prime, we have $0 \neq an_1x_k - an_1y_k \notin I$, where $n_1 \in N$. Then,

$$
\begin{aligned}
(0,\dots,0) \quad &\neq (0,\dots,\underbrace{an_1x_k - an_1y_k}_{i^{th}},\dots,0)\\
\neq (0,\dots,\underbrace{an_1x_k}_{i^{th}},\dots,0) &- (0,\dots,\underbrace{an_1y_k}_{i^{th}},\dots,0)\\
&= f^{an_1}_{ik}\rho_1 - f^{an_1}_{ik}\rho_2\\
= f^{a}_{ij} f^{n_1}_{jk}\rho_1 - f^{a}_{ij} f^{n_1}_{jk} &\;\rho_2 ([14]),\ \text{Lemma 3.1(3))}\\
&\notin I^n.
\end{aligned}
$$

Hence, $\bar{0} \neq AB\rho_1 - AB\rho_2 \notin I^n$, with $A = f^{a}_{ij}$, $B = f^{n_1}_{jk}$. The rest of the proof follows by induction on the complexity of A as in the Proposition 3.5. □

Proposition 3.15 *If $\mathcal{I}$ is a weakly e-prime ideal of an $M_n(N)$-group N^n, then $\mathcal{I}_{**}$ is a weakly e-prime ideal of $_NN$.*

Proof Suppose that $\mathcal{I}$ is a weakly e-prime ideal of N^n. By ([9], Lemma 1.3), we have $\mathcal{I}_{**}$ is an ideal of $_NN$. To prove $\mathcal{I}_{**}$ is weakly e-prime , let $a, x, y \in N$ such that $aN \nsubseteq \mathcal{I}_{**}$ and $x - y \notin \mathcal{I}_{**}$. Then there exists $m_1 \in N$ such that $am_1 \notin \mathcal{I}_{**}$ and $x - y \notin \mathcal{I}_{**}$. By ([9], Lemma 1.2), we have $\left(am_1, 0, \ldots, 0\right) \notin \mathcal{I}$ and $(x - y, 0, \ldots, 0) \notin \mathcal{I}$. That is,

$$f_{11}^{a}\left(m_1, \ldots, m_n\right) = \left(am_1, 0, \ldots, 0\right) \notin \mathcal{I} \text{ and } (x, 0, \ldots, 0) - (y, 0, \ldots, 0) \notin \mathcal{I}.$$

Take $A = f_{11}^{a} \in M_n(N)$, and $\rho_1 = (x, 0, \ldots, 0)$, $\rho_2 = (y, 0, \ldots, 0)$. Then $AN^n \nsubseteq \mathcal{I}$ and $\rho_1 - \rho_2 \notin \mathcal{I}$. Since $\mathcal{I}$ is weakly e-prime, there exist $B \in M_n(N)$ such that $\bar{0} \neq AB\rho_1 - AB\rho_2 \notin \mathcal{I}$. Let $B = f_{ij}^{n'}$, $n' \in N$. Then,

$$\bar{0} \neq f_{11}^{a} f_{ij}^{n'} \rho_1 - f_{11}^{a} f_{ij}^{n'} \rho_2 \notin \mathcal{I}.$$

By ([1], Lemma 3.1(3)), and N is zero-symmetric, we have the following.
If $1 \neq i = j$, then

$$\begin{aligned} f_{11}^{a} f_{ij}^{n'} \rho_1 - f_{11}^{a} f_{ij}^{n'} \rho_2 &= f_{1j}^{a0} \rho_1 - f_{1j}^{a0} \rho_2 \\ &= f_{1j}^{0} \rho_1 - f_{1j}^{0} \rho_2 \\ &= (0, 0, \ldots, 0), \quad \text{a contradiction.} \end{aligned}$$

If $1 = i \neq j$, then

$$\begin{aligned} f_{11}^{a} f_{ij}^{n'} \rho_1 - f_{11}^{a} f_{ij}^{n'} \rho_2 &= f_{1j}^{an'} (x, 0, \ldots, 0)_1 - f_{1j}^{an'} (y, 0, \ldots, 0) \\ &= \left(an'0, 0, \ldots, 0\right) - \left(an'0, 0, \ldots, 0\right) \\ &= (0, 0, \ldots, 0), \quad \text{a contradiction.} \end{aligned}$$

Now, in case $1 = i = j$, by ([1], Lemma 3.1(3)), it follows that

$$\begin{aligned} \bar{0} &\neq f_{11}^{a} f_{ij}^{n'} \rho_1 - f_{11}^{a} f_{ij}^{n'} \rho_2 \\ &= f_{11}^{an'} \rho_2 - f_{11}^{an'} \rho_2 \\ &= \left(an'x, 0, \ldots, 0\right) - \left(an'y, 0, \ldots, 0\right) \\ &\notin \mathcal{I}. \end{aligned}$$

Therefore, $0 \neq an'x - an'y \notin \mathcal{I}_{**}$, proves $\mathcal{I}_{**}$ is weakly e-prime. □

We establish a one-one correspondence between the set of all weakly τ-prime ($\tau = 0, c, 3, e$), (resp. IFP, $(i, 2)$-absorbing ($i \in \{c, 3\}$)) ideals of $_NN$ and those of $M_n(N)$ -group N^n.

Theorem 3.16 *There is a one-one correspondence between the set of all weakly τ-prime ($\tau = 0, c, 3, e$) ideals of $_NN$ and those of $M_n(N)$-group N^n.*

Proof Let $\tau \in \{0, c, 3, e\}$. Write $\mathcal{P} = \{I \trianglelefteq_N N : \text{I is weakly } \tau\text{-prime}\}$ and $\mathcal{Q} = \{\mathcal{I} \trianglelefteq_{M_n(N)} N^n : \mathcal{I} \text{ is weakly } \tau\text{-prime}\}$. Define

$$\Phi : \mathcal{P} \to \mathcal{Q} \text{ by } \Phi(I) = I^n, \text{ and } \psi : \mathcal{Q} \to \mathcal{P} \text{ by } \psi(\mathcal{I}) = \mathcal{I}_{**}.$$

Then $\Phi(I)$ and $\psi(\mathcal{I})$ are weakly τ-prime ideals of N^n and ${}_NN$ respectively, as proved above. Now

$$\begin{aligned}(\Phi\circ\psi)(\mathcal{I}) &= \Phi(\mathcal{I}_{**})\\ &= \left(\mathcal{I}_{**}\right)^n\\ &= \mathcal{I} \text{ (([10, } \textit{Lemma } 1.5(i))],\end{aligned}$$

and

$$\begin{aligned}(\psi\circ\Phi)(I) &= \psi(I^n)\\ &= (I^n)_{**}\\ &= I \text{ (([10, } \textit{Lemma } 1.4(iii))].\end{aligned}$$

Therefore, $(\Phi\circ\psi) = id_{\mathcal{Q}}$, and $(\psi\circ\Phi) = id_{\mathcal{P}}$, proves that $\mathcal{P}$ and $\mathcal{Q}$ are isomorphic. □

Definition 3.17 An ideal I of G is said to have *weakly insertion of factors property* (denoted as, *weakly IFP*), if for any $a \in N$, $g \in G$ with $0 \neq ag \in I$, then $ang \in I$, for all $n \in N$.

Example 3.18 ([13], *Sonata* Table no. 134) Consider $N = ((\mathbb{Z}_4 \times \mathbb{Z}_2), +, \cdot)$ and $G = (N, +)$. Then G is an N-group. Then $\{0\}$ has weakly IFP, but not IFP, since, $7 \cdot 4 = 0$, but $7N4 = \{0, 2\} \nsubseteq \{0\}$.

Definition 3.19 An ideal $\mathcal{I}$ of N^n is said to have *weakly IFP*, if for any $A \in M_n(N)$ and $\rho \in N^n$ with $(0) \neq A\rho \in \mathcal{I}$, then $AB\rho \in \mathcal{I}$, for all $B \in M_n(N)$.

Lemma 3.20 *If I has weakly IFP in ${}_NN$, then I^n has weakly IFP in $M_n(N)$-group N^n.*

Proof Assume that I has weakly IFP, and let $AB\rho \notin I^n$, for some $A, B \in M_n(N)$, $\rho \in N^n$. Let $A = f^a_{ij}$, $B = f^b_{kj}$, $\rho = (x_1, \dots, x_n)$. Then

$f^a_{ij}f^b_{kj}(x_1, \dots, x_n) \notin I^n$. By ([1], Lemma 3.1(3)), and N is zero-symmetric, we have the following.

If $j \neq k$, then

$$\begin{aligned}f^a_{ij}f^b_{kj}(x_1, \dots, x_n) &= f^{a0}_{ij}(x_1, \dots, x_n)\\ &= f^0_{ij}(x_1, \dots, x_n)\\ &= (0, \dots, 0),\end{aligned}$$

a contradiction.

If $j = k$, then

$$\begin{aligned}(0, \dots, \underbrace{abx_j}_{i^{th}}, \dots, 0) &= f^{ab}_{ij}(x_1, \dots, x_n)\\ &= f^{ab}_{ij}f^b_{kj}(x_1, \dots, x_n)\\ &\notin I^n.\end{aligned}$$

This implies $abx_j \notin I$. Since I has weakly IFP, we have $0 \neq ax_j \notin I$. Therefore, $\bar{0} \neq (0, \ldots, ax_j, \ldots, 0) = f_{ij}^{a}(x_1, \ldots, x_n) \notin I^n$, yields $\bar{0} \neq A\rho \notin I^n$. □

Lemma 3.21 *If $\mathcal{I}$ has weakly IFP in $M_n(N)$-group N^n, then $\mathcal{I}_{**}$ has weakly IFP in ${}_NN$.*

Proof Assume that $\mathcal{I}$ has weakly IFP, and suppose $a, g, n' \in N$ such that $an'g \notin \mathcal{I}_{**}$. Then by ([9], Lemma 1.2), we have $(an'g, \ldots, 0) \notin \mathcal{I}$. Now $f_{11}^{a} f_{11}^{n'}(g, \ldots, 0) = f_{11}^{an'}(g, \ldots, 0) \notin \mathcal{I}$. Taking $A = f_{11}^{a}$, $B = f_{11}^{n'}$ and $\rho = (g, 0, \ldots, 0)$ we get $AB\rho \notin \mathcal{I}$. Since $\mathcal{I}$ has weakly IFP, we have $\bar{0} \neq A\rho \notin \mathcal{I}$. This implies, $\bar{0} \neq f_{11}^{a}(g, \ldots, 0) = (ag, \ldots, 0) \notin \mathcal{I}$, and get $0 \neq ag \notin \mathcal{I}_{**}$. □

From Lemma 3.20, Lemma 3.21, ([9], Lemma 1.5(i), 1.4(iii)), we get

Theorem 3.22 *There is a one-one correspondence between the set of all weakly IFP ideals of ${}_NN$ and those of $M_n(N)$-group N^n.*

Proof Let $\mathcal{P} = \{I \trianglelefteq_N N : I \text{ has weakly IFP}\}$ and
$\mathcal{Q} = \{\mathcal{I} \trianglelefteq_{M_n(N)} N^n : \mathcal{I} \text{ has weakly IFP}\}$. Define

$$\Phi : \mathcal{P} \to \mathcal{Q} \text{ by } \Phi(I) = I^n.$$

By Lemma 3.20, $\Phi(I)$ has IFP. Define

$$\psi : \mathcal{Q} \to \mathcal{P} \text{ by } \psi(\mathcal{I}) = \mathcal{I}_{**}.$$

By Lemma 3.21, $\psi(\mathcal{I})$ has IFP. Now $(\Phi \circ \psi)(\mathcal{I}) = \Phi(\mathcal{I}_{**}) = \left(\mathcal{I}_{**}\right)^n = \mathcal{I}$ ([9], Lemma 1.5(i)) and $(\psi \circ \Phi)(I) = \psi(I^n) = (I^n)_{**} = I$ ([9], Lemma 1.4(iii)). Therefore, $(\Phi \circ \psi) = id_{\mathcal{Q}}$, and $(\psi \circ \Phi) = id_{\mathcal{P}}$. Hence, $\mathcal{P}$ and $\mathcal{Q}$ are isomorphic. □

We introduce the notions weakly $(c, 2)$-absorbing ideal and weakly $(3, 2)$-absorbing ideal as generalizations of c-prime and 3-prime ideal of an N-group G, respectively. We establish a one-one correspondence between $(i, 2), i \in \{c, 3\}$-absorbing ideals of N (over N) and those of N^n. Tapatee et. al [14] introduced completely 2-absorbing (abbr. $(c, 2)$-absorbing) ideals in G and derived the properties such as homomorphic images, inverse images of $(c, 2)$-absorbing ideals.

Definition 3.23 Let $I \triangleleft_N G$ such that $NG \nsubseteq I$. Then

1. I is weakly $(c, 2)$-absorbing if for $a, b \in N$, $g \in G$, $0 \neq abg \in I$, then $abG \subseteq I$ or $ag \in I$ or $bg \in I$.
2. I is weakly $(3, 2)$-absorbing if for $a, b \in N$, $g \in G$, $(0) \neq abNg \subseteq I$, then $abG \subseteq I$ or $ag \in I$ or $bg \in I$.

Example 3.24 Consider $N = (A_4, +, \cdot)$ and $G = N$, given in ([11], Table O(1)). Then G is an N-group, where $+$ and $\cdot$ is defined as follows:

+	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11

+	0	1	2	3	4	5	6	7	8	9	10	11
1	1	0	3	2	7	6	5	4	10	11	8	9
2	2	3	0	1	5	4	7	6	11	10	9	8
3	3	2	1	0	6	7	4	5	9	8	11	10
4	4	5	6	7	8	9	10	11	0	1	2	3
5	5	4	7	6	11	10	9	8	2	3	0	1
6	6	7	4	5	9	8	11	10	3	2	1	0
7	7	6	5	4	10	11	8	9	1	0	3	2
8	8	9	10	11	0	1	2	3	4	5	6	7
9	9	8	11	10	3	2	1	0	6	7	4	5
10	10	11	8	9	1	0	3	2	7	6	5	4
11	11	10	9	8	2	3	0	1	5	4	7	6

·	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	4	4	4	6
5	0	0	0	0	0	0	0	0	4	4	4	6
6	0	0	0	0	0	0	0	0	4	4	4	6
7	0	0	0	0	0	0	0	0	4	4	4	6
8	0	0	0	0	0	0	0	0	8	8	8	11
9	0	0	0	0	0	0	0	0	8	8	8	11
10	0	0	0	0	0	0	0	0	8	8	8	11
11	0	0	0	0	0	0	0	0	8	8	8	11

The ideals are $I_1 = \{0\}$ and $I_2 = \{0, 1, 2, 3\}$. Then I_2 is not a weakly 3-prime ideal of G, since $4N5 = \{0\} \in I_2$, but $4G \nsubseteq I_2$ and $5 \notin I_2$ whereas I_2 is weakly (3, 2)-absorbing ideal of G.

Definition 3.25 Let $\mathcal{I} \lhd_{M_n(N)} N^n$ such that $M_n(N)N^n \nsubseteq \mathcal{I}$. Then

1. $\mathcal{I}$ is weakly (c, 2)-absorbing if for $A, B \in M_n(N)$ and $\rho \in N^n$, $\bar{0} \neq AB\rho \in \mathcal{I}$, then $ABN^n \subseteq \mathcal{I}$ or $A\rho \in \mathcal{I}$ or $B\rho \in \mathcal{I}$.
2. $\mathcal{I}$ is weakly (3, 2)-absorbing if for $A, B \in M_n(N)$ and $\rho \in N^n$, $(\bar{0}) \neq ABM_n(N)\rho \subseteq \mathcal{I}$, then $ABN^n \subseteq \mathcal{I}$ or $A\rho \in \mathcal{I}$ or $B\rho \in \mathcal{I}$.

Proposition 3.26 *If I is a weakly (c, 2)-absorbing ideal of ${}_NN$, then I^n is a weakly (c, 2)-absorbing ideal in $M_n(N)$-group N^n.*

Proof Let I be a weakly (c, 2)-absorbing ideal of ${}_NN$. By ([1], Proposition 4.1), I^n is an ideal of N^n. To show I^n is weakly (c, 2)-absorbing in N^n, let $A, B \in M_n(N)$ and $\rho \in N^n$ such that $\bar{0} \neq AB\rho \in I^n$. Suppose that $w(A) = w(B) = 1$, say $A = f_{11}^a,$, $B = f_{11}^b$, $a, b \in N$ and $\rho = (x_1, \dots, x_n)$. Then by ([1], Lemma 3.1 (3)), we have

$$\begin{aligned}\bar{0} \neq & f_{11}^{a} f_{11}^{b}(x_1, \ldots, x_n)\\ &= f_{11}^{ab}(x_1, \ldots, x_n)\\ &= (abx_1, \ldots, 0)\\ &\in I^n.\end{aligned}$$

This implies, $0 \neq abx_1 \in I$. Since I is weakly (c, 2)-absorbing in ${}_NN$, we have $abN \subseteq I$ or $ax_1 \in I$ or $bx_1 \in I$, and so $abx_i \in I$, for all i. Thus,

$$(abx_1, 0, \ldots, 0) \in I^n \text{ or } (ax_1, \ldots, 0) \in I^n \text{ or } (bx_1, \ldots, 0) \in I^n.$$

Now,

$$\begin{aligned}f_{11}^{a} f_{11}^{b}(x_1, x_2, \ldots, x_n) &= f_{11}^{ab}(x_1, x_2, \ldots, x_n)\\ &= (abx_1, 0, \ldots, 0)\\ &\in I^n, \text{ for all } x_i \in N\end{aligned}$$

or

$$f_{11}^{a}(x_1, x_2, \ldots, x_n) \in I^n \text{ or } f_{11}^{b}(x_1, x_2, \ldots, x_n) \in I^n,$$

implies that $ABN^n \subseteq I^n$ or $A\rho \in I^n$ or $B\rho \in I^n$. □

Proposition 3.27 *If $\mathcal{I}$ is a weakly* (c, 2)*-absorbing ideal of $M_n(N)$-group N^n, then $\mathcal{I}_{**}$ is a weakly* (c, 2)*-absorbing ideal of ${}_NN$.*

Proof Suppose that $\mathcal{I}$ is a weakly (c, 2)-absorbing ideal of N^n. By ([9], Lemma 1.3), $\mathcal{I}_{**}$ is an ideal of ${}_NN$. To prove $\mathcal{I}_{**}$ is a weakly (c, 2)-absorbing ideal of ${}_NN$, let $a, b, c \in N$ such that $0 \neq abc \in \mathcal{I}_{**}$. Then by ([9], Lemma 1.2), we have $0 \neq (abc, 0, \ldots, 0) \in \mathcal{I}$. This implies that,

$$\begin{aligned}\bar{0} \neq (abc, 0, \ldots, 0) &= f_{11}^{ab}(c, 0, \ldots, 0)\\ &= f_{11}^{a} f_{11}^{b}(c, 0, \ldots, 0)\\ &\in \mathcal{I},\end{aligned}$$

shows that $\bar{0} \neq AB\rho_1 \in \mathcal{I}$, where $A = f_{11}^{a}$, $B = f_{11}^{b}$, $\rho_1 = (c, 0, \ldots, 0)$. Since $\mathcal{I}$ is a weakly (3, 2)-absorbing ideal of N^n, we get $ABN^n \subseteq \mathcal{I}$ or $A\rho_1 \in \mathcal{I}$ or $B\rho_1 \in \mathcal{I}$. Hence, $f_{11}^{a} f_{11}^{b}(x_1, \ldots, x_n) \in \mathcal{I}$, for all $(x_1, \ldots, x_n) \in N^n$ or $f_{11}^{a}(c, \ldots, 0) \in \mathcal{I}$ or $f_{11}^{b}(c, \ldots, 0) \in \mathcal{I}$, implies that $(abx_1, \ldots, 0) \in \mathcal{I}$ or $(ac, \ldots, 0) \in \mathcal{I}$ or $(bc, \ldots, 0) \in \mathcal{I}$. Then, by ([9], Lemma 1.2), we have $abx_1 \in \mathcal{I}_{**}$, for all $x_1 \in N$ or $ac \in \mathcal{I}_{**}$ or $bc \in \mathcal{I}_{**}$, and so $abN \subseteq \mathcal{I}_{**}$ or $ac \in \mathcal{I}_{**}$ or $bc \in \mathcal{I}_{**}$. □

Proposition 3.28 *If I is a weakly* (3, 2)*-absorbing ideal of ${}_NN$, then I^n is a weakly* (3, 2)*-absorbing ideal of $M_n(N)$-group N^n.*

Proof Let I be a weakly (3, 2)-absorbing ideal of ${}_NN$. By ([1], Proposition 4.1), I^n is an ideal of N^n. To show I^n is weakly (3, 2)-absorbing in N^n, let $A, B \in M_n(N)$ and $\rho \in N^n$ such that $(\bar{0}) \neq ABM_n(N)\rho \subseteq I^n$. Then, $\bar{0} \neq ABC\rho \in I^n$, for every $C \in M_n(N)$. Suppose that $w(A) = w(B) = w(C) = 1$, say $A = f_{11}^{a}$,, $B = f_{11}^{b}$, $C = f_{11}^{c}$, $a, b, c \in N$ and $\rho = (x_1, \ldots, x_n)$. Then by ([1], Lemma 3.1 (3)), we have

$$\begin{aligned}\bar{0} \neq & f_{11}^a f_{11}^b f_{11}^c(x_1, \ldots, x_n) \\ &= f_{11}^{abc}(x_1, \ldots, x_n) \\ &= (abcx_1, \ldots, 0) \\ &\in I^n.\end{aligned}$$

This implies, $0 \neq abcx_1 \in I$, for all $c \in N$. Hence, $(0) \neq abNc_1 \subseteq I$. Since I is weakly (3, 2)-absorbing in ${}_NN$, we have $abN \subseteq I$ or $ax_1 \in I$ or $bx_1 \in I$, and so $abx_i \in I$, for all i. Thus,

$$(abx_1, 0, \ldots, 0) \in I^n \text{ or } (ax_1, \ldots, 0) \in I^n \text{ or } (bx_1, \ldots, 0) \in I^n.$$

Now,

$$\begin{aligned}f_{11}^a f_{11}^b(x_1, x_2, \ldots, x_n) &= f_{11}^{ab}(x_1, x_2, \ldots, x_n) \\ &= (abx_1, 0, \ldots, 0) \\ &\in I^n, \text{ for all} x_i \in N\end{aligned}$$

or

$$f_{11}^a(x_1, x_2, \ldots, x_n) \in I^n \text{ or} f_{11}^b(x_1, x_2, \ldots, x_n) \in I^n,$$

implies that $ABN^n \subseteq I^n$ or $A\rho \in I^n$ or $B\rho \in I^n$. □

Proposition 3.29 *If $\mathcal{I}$ is a weakly* (3, 2)*-absorbing ideal of $M_n(N)$-group N^n, then $\mathcal{I}_{**}$ is a weakly* (3, 2)*-absorbing ideal of ${}_NN$.*

Proof Suppose that $\mathcal{I}$ is a weakly (3, 2)-absorbing ideal of N^n. By ([9], Lemma 1.3), $\mathcal{I}_{**}$ is an ideal of ${}_NN$. To prove $\mathcal{I}_{**}$ is a weakly (3, 2)-absorbing ideal of ${}_NN$, let $a, b, x \in N$ such that $abN \nsubseteq \mathcal{I}_{**}$, $ax \notin \mathcal{I}_{**}$ and $bx \notin \mathcal{I}_{**}$. Then there exist $n \in N$ such that $abn \notin \mathcal{I}_{**}$, $ax \notin \mathcal{I}_{**}$ and $bx \notin \mathcal{I}_{**}$. Now by ([9], Lemma 1.2) $(abn, \ldots, 0) \notin \mathcal{I}$, $(ax, \ldots, 0) \notin \mathcal{I}$ and $(bx, \ldots, 0) \notin \mathcal{I}$. Hence, $f_{11}^{ab}(n, \ldots, 0) \notin \mathcal{I}$, $f_{11}^a(x, 0, \ldots, 0) \notin \mathcal{I}$ and $f_{11}^b(x, \ldots, 0) \notin \mathcal{I}$. Take $A = f_{11}^a$, $B = f_{11}^b$ and $\rho = (x, 0, \ldots, 0)$. Then $ABN^n \nsubseteq \mathcal{I}$, $A\rho \notin \mathcal{I}$ and $B\rho \notin \mathcal{I}$. Since $\mathcal{I}$ is weakly (3, 2)-absorbing in N^n, we have $(\bar{0}) \neq ABM_n(N)\rho \nsubseteq \mathcal{I}$. Then $(\bar{0}) \neq ABC\rho \notin \mathcal{I}$, for some $C \in M_n(N)$. Now,

$$\begin{aligned}ABC = f_{11}^a f_{11}^b C = & f_{11}^a f_{11}^b C f_{11}^e \\ &= f_{11}^a f_{11}^b (f_{11}^{r_1} + \ldots + f_{n1}^{r_n}), \text{ (([7, } \textit{Lemma}\, 2.3)] \\ &= f_{11}^a f_{11}^b f_{11}^{r_1} \text{ (([14, } \textit{Lemma}\, 3.1(5))] \\ &= f_{11}^{abr_1} \text{ (([14, } \textit{Lemma}\, 3.1(3))].\end{aligned}$$

Hence,

$$\begin{aligned}(\bar{0}) \neq ABC\rho &= f_{11}^{abr_1}\rho \\ &= f_{11}^{abr_1}(x, 0, \ldots, 0) \\ &= (abr_1x, 0, \ldots, 0) \\ &\mathcal{I}.\end{aligned}$$

Therefore, by ([9], Lemma 1.2), $0 \neq abr_1x \notin \mathcal{I}_{**}$, shows that, $(0) \neq abNx \nsubseteq \mathcal{I}_{**}$. □

The following is a one-one correspondence similar to Theorem 3.16.

Theorem 3.30 *There is a one-one correspondence between the set of all weakly* $(i, 2)$*-absorbing*, $i \in \{c, 3\}$ *ideals of* ${}_{N}N$ *and those of* $M_n(N)$*-group* N^n.

Proof Let $i \in \{c, 3\}$ and let $\mathcal{P} = \{I \trianglelefteq_N N : I \text{ is weakly } (i, 2)\text{-absorbing}\}$ and $\mathcal{Q} = \{\mathcal{I} \trianglelefteq_{M_n(N)} N^n : \mathcal{I} \text{ is weakly } (i, 2)\text{-absorbing}\}$. Define $\Phi : \mathcal{P} \to \mathcal{Q}$ by $\Phi(I) = I^n$. By Theorem 3.26 and 3.28, $\Phi(I)$ is weakly $(i, 2)$-absorbing. Define $\psi : \mathcal{Q} \to \mathcal{P}$ by $\psi(\mathcal{I}) = \mathcal{I}_{**}$. By Theorem 3.27 and 3.29, $\psi(\mathcal{I})$ is weakly $(i, 2)$-absorbing. Now $(\Phi \circ \psi)(\mathcal{I}) = \Phi(\mathcal{I}_{**}) = \left(\mathcal{I}_{**}\right)^n = \mathcal{I}$ ([9], Lemma 1.5(i)) and $(\psi \circ \Phi)(I) = \psi(I^n) = (I^n)_{**} = I$ ([9], Lemma 1.4(iii)). Therefore, $(\Phi \circ \psi) = id_{\mathcal{Q}}$, and $(\psi \circ \Phi) = id_{\mathcal{P}}$. Hence, $\mathcal{P}$ and $\mathcal{Q}$ are isomorphic. □

4 Conclusion

We have proved the one-one correspondence between the weakly τ-prime ideals $(\tau = 0, c, 3, e)$ of N-group N and those of $M_n(N)$-group N^n. Further, we have introduced the notions weakly $(c, 2)$-absorbing ideals and weakly $(3, 2)$-absorbing ideals as a generalization of c-prime and 3-prime ideals of N-group G, respectively; and finally obtained the one-one correspondence between $(i, 2)$, $i \in \{c, 3\}$-absorbing ideals of N (over N) and those of $M_n(N)$-group N^n. As a future research work, one can study the radical properties and hyperstructural aspects [15, 16] of weakly prime ideals in $M_n(N)$-group N^n.

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