A normal criterion concerning zero numbers

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Abstract

Let $n \geq 4$ be a positive integer, $\mathcal{F}$ be a family of meromorphic functions in D and let $a(z) (\neq 0)$, $b(z)$ be two holomorphic functions in D . If, for any function $f \in \mathcal{F}$, (1) $f(z) \neq \infty$ when $a(z) = 0$, (2) $f'(z) - a(z)f^n(z) - b(z)$ has at most one zero in D , then $\mathcal{F}$ is normal in D .

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1 Introduction and main results

Let D be a domain in $\mathbb{C}$ and $\mathcal{F}$ be a family of meromorphic functions in D . A family $\mathcal{F}$ is said to be normal in D , in the sense of Montel, if each sequence $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ that converges spherically locally uniformly in D to a meromorphic function or to the constant ∞ .

The following well-known normal conjecture was proposed by Hayman in 1967.

Theorem A [1] *Let $n \geq 5$ be a positive integer, $\mathcal{F}$ be a family of meromorphic functions in D and let $a(\neq 0)$, b be two finite complex numbers. If, for any function $f \in \mathcal{F}$, $f'(z) - af^n(z) \neq b$ in D , then $\mathcal{F}$ is normal in D .*

The conjecture was proved by Li [2], Li [3], Langley [4] (for $n \geq 5$), Pang [5] (for $n = 4$), Chen and Fang [6], Bergweiler and Eremenko [7] (for $n = 3$).

Up to now, this result has undergone various extensions and improvements (see [8–12]).

The following generalization of Theorem A was proved by Yang et al. [13].

Theorem B *Let $n \geq 4$ be a positive integer, $\mathcal{F}$ be a family of meromorphic functions in D and let $a(z) (\neq 0)$, $b(z)$ be two holomorphic functions in D . If, for any function $f \in \mathcal{F}$, (1) $f(z) \neq \infty$ when $a(z) = 0$, (2) $f'(z) - a(z)f^n(z) \neq b(z)$ in D , then $\mathcal{F}$ is normal in D .*

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It is natural to ask what can be said if $f'(z) - a(z)f^n(z) = b(z)$ has solutions in Theorem B.

In this article, we study this problem and prove the following result.

Theorem 1.1 *Let $n \geq 4$ be a positive integer, $\mathcal{F}$ be a family of meromorphic functions in D and let $a(z) (\not\equiv 0)$, $b(z)$ be two holomorphic functions in D . If, for any function $f \in \mathcal{F}$, (1) $f(z) \neq \infty$ when $a(z) = 0$, (2) $f'(z) - a(z)f^n(z) - b(z)$ has at most one zero in D , then $\mathcal{F}$ is normal in D .*

The following examples show that both $n \geq 4$ and the conditions (1), (2) in Theorem 1.1 are necessary and sharp.

Example 1.1 Let $\Delta = \{z : |z| < 1\}$ and $a(z) = -2z^3$, $b(z) = 0$. Let $\mathcal{F} = \{f_j(z)\}$, where

$$f_j(z) = \frac{1}{z^2 - \frac{1}{j}}, z \in \Delta, j = 1, 2, \dots$$

Then $f'_j(z) - a(z)f_j^3(z) - b(z) = \frac{\frac{2}{j}z}{(z^2 - \frac{1}{j})^3}$, which has a zero in Δ . Clearly, $f \neq \infty$ when $a(z) = 0$, however $\mathcal{F}$ is not normal at 0.

Example 1.2 Let $\Delta = \{z : |z| < 1\}$, $a(z) = -z$, $b(z) = 0$. Let $\mathcal{F} = \{f_j(z)\}$, where

$$f_j(z) = \frac{1}{j+z}, z \in \Delta, j = 1, 2, \dots$$

Then $f'_j(z) - a(z)f_j^4(z) - b(z) = \frac{-z^2 + (1-2j)z - j^2}{(j+z)^4}$, which has exactly two distinct zeros in Δ . Clearly, $f \neq \infty$ when $a(z) = 0$, however $\mathcal{F}$ is not normal at 0.

Example 1.3 Let $\Delta = \{z : |z| < 1\}$, $a(z) = -z$, $b(z) = 0$. Let $\mathcal{F} = \{f_j(z)\}$, where

$$f_j(z) = \frac{1}{jz}, z \in \Delta, j = 2, 3, \dots$$

Then $f'_j(z) - a(z)f_j^4(z) - b(z) = \frac{1-j^3z}{j^4z^3}$, which has a zero in Δ . But $\mathcal{F}$ is not normal at 0.

The next example shows that Theorem 1.1 is not valid if $a(z)$ is a meromorphic function in D .

Example 1.4 Let $\Delta = \{z : |z| < 1\}$, $a(z) = z^{-4}$, $b(z) = 0$. Let $\mathcal{F} = \{f_j(z)\}$, where

$$f_j(z) = jz, z \in \Delta, j = 2, 3, \dots$$

Then $f'_j(z) - a(z)f_j^4(z) - b(z) = j - j^4$, which has no zero in Δ . But $\mathcal{F}$ is not normal at 0.

Remark 1.1 Clearly Theorem 1.1 improves and extends Theorem B.

2 Some lemmas

Lemma 2.1 [14] *Let k be a positive integer and let $\mathcal{F}$ be a family of functions meromorphic in the unit disc Δ , all of whose zeros have multiplicity at least k . If $\mathcal{F}$ is not normal in any neighbourhood of $z_0 \in \Delta$, then for each α , $0 \leq \alpha < k$ there exist a sequence of complex numbers $z_n, z_n \rightarrow z_0, z_0 \in \Delta$, a sequence of positive numbers $\rho_n \rightarrow 0$, and a sequence of functions $f_n \in \mathcal{F}$ such that $g_n(\xi) = \rho_n^{-\alpha} f_n(z_n + \rho_n \xi) \rightarrow g(\xi)$ spherically uniformly on compact subsets of $\mathbb{C}$, where g is a non-constant meromorphic function, all of whose zeros have multiplicity at least k . Moreover, $g(\xi)$ has order at most 2.*

We remark that one can take $-k < \alpha \leq 0$ in the above lemma if all poles of each $f_n \in \mathcal{F}$ have multiplicity at least k (see [15]).

Lemma 2.2 [16, Theorem 4, p. 381] *Let $f(z)$ be a transcendental meromorphic function with finite order, all of whose zeros are of multiplicity at least 2, and let $P(z) (\not\equiv 0)$ be a polynomial, then $f'(z) - P(z)$ has infinitely many zeros.*

Lemma 2.3 [17, Lemma 6] *Let k be a positive integer, $f(z)$ be a transcendental meromorphic function with finite order, all of whose zeros have multiplicity at least $k + 1$, then $f^{(k)} - 1$ has infinitely many zeros.*

Lemma 2.4 [18, Lemma 5] *Let k be a positive integer, $f(z)$ be a nonconstant rational function, all of whose zeros have multiplicity at least $k + 2$, and all of whose poles have multiplicity at least 3, then $f^{(k)}(z) - 1$ has at least two distinct zeros.*

Lemma 2.5 [12, Lemma 3] *Let $n \geq 4$ be a positive integer, $a \neq 0$ be a finite complex number and let $f(z)$ be a nonconstant meromorphic function, then $f'(z) - af^n(z)$ has at least two distinct zeros.*

Lemma 2.6 [19, Lemma 7(iv), p. 261] *Let k, l be two positive integers, $Q(z)$ be a rational function, all of whose zeros have multiplicity at least $k + 2$ and all of whose poles have multiplicity at least 2 with the possible exception of $z = 0$, then $Q^{(k)}(z) = z^l$ has a solution in $\mathbb{C}$.*

Lemma 2.7 [20, Lemma 11] *Let k, m be two positive integers, $f(z)$ be a rational function. If $f(z) \neq 0$ for $z \in \mathbb{C}$, and $f^{(k)}(z) \neq z^m$ for $z \neq z_0$, where $z_0 \in \mathbb{C}$, then $f(z)$ is a constant.*

Lemma 2.8 *Let $n \geq 4$ be a positive integer, $\mathcal{F} = \{f_j\}$ be a family of meromorphic functions in a domain D , and let $a_j(z), b_j(z)$ be two sequences of analytic functions in D such that $a_j(z) \rightarrow a(z) \neq 0, b_j(z) \rightarrow b(z)$. If $f_j'(z) - a_j(z)f_j^n(z) - b_j(z)$ has at most one zero, then $\mathcal{F}$ is normal in D .*

Proof Suppose that $\mathcal{F}$ is not normal at $z_0 \in D$. By Lemma 2.1, there exists $z_j \rightarrow z_0, \rho_j \rightarrow 0^+$, and $f_j \in \mathcal{F}$ such that

$$g_j(\xi) = \rho_j^{\frac{1}{n-1}} f_j(z_j + \rho_j \xi) \rightarrow g(\xi)$$

locally uniformly on compact subsets of $\mathbb{C}$, where $g(\xi)$ is a non-constant meromorphic function in $\mathbb{C}$.

For each $\xi \in \mathbb{C} / \{g^{-1}(\infty)\}$, we have

$$g'_j(\xi) - a_j(z_j + \rho_j \xi) g''_j(\xi) - \rho_j^{\frac{n}{n-1}} b_j(z_j + \rho_j \xi) \longrightarrow g'(\xi) - a(z_0) g''(\xi).$$

Obviously, $g'(\xi) - a(z_0) g''(\xi) \neq 0$.

Suppose that $g'(\xi) - a(z_0) g''(\xi) \equiv 0$, then $g(\xi)$ must be an entire function. Hence, since $n \geq 3$ and $a(z_0) \neq 0$, we have

$$nT(r, g) = T(r, g^n) = T(r, \frac{g'}{a(z_0)}) = T(r, g') + \log^+ \frac{1}{|a(z_0)|} \leq T(r, g) + S(r, g).$$

It follows that we have $T(r, g) = S(r, g)$. But this is impossible since $g(\xi)$ is a non-constant meromorphic function.

Claim: $g'(\xi) - a(z_0) g''(\xi)$ has at most one zero.

Suppose this is not the case, and $g'(\xi) - a(z_0) g''(\xi)$ has two distinct zeros ξ_1 , and ξ_2 . We choose a positive number δ small enough such that $D_1 \cap D_2 = \emptyset$ and $g'(\xi) - a(z_0) g''(\xi)$ has no other zeros in $D_1 \cup D_2$ except for ξ_1 and ξ_2 , where $D_1 = \{\xi : |\xi - \xi_1| < \delta\}$ and $D_2 = \{\xi : |\xi - \xi_2| < \delta\}$.

By Hurwitz's theorem, for sufficiently large j , there exist points $\xi_{1,j} \rightarrow \xi_1$ and $\xi_{2,j} \rightarrow \xi_2$ such that

$$f'_j(\xi_{1,j}) - a_j(z_j + \rho_j \xi_{1,j}) f''_j(z_j + \rho_j \xi_{1,j}) - b_j(z_j + \rho_j \xi_{1,j}) = 0$$

and

$$f'_j(\xi_{2,j}) - a_j(z_j + \rho_j \xi_{2,j}) f''_j(z_j + \rho_j \xi_{2,j}) - b_j(z_j + \rho_j \xi_{2,j}) = 0$$

Since $f'_j(z) - a_j(z) f''_j(z) - b_j(z)$ has at most one zero in D , then

$$z_j + \rho_j \xi_{1,j} = z_j + \rho_j \xi_{2,j},$$

this is

$$\xi_{1,j} = \xi_{2,j} = \frac{z_0 - z_j}{\rho_j},$$

which contradicts the fact $D_1 \cap D_2 = \emptyset$. The claim is proved.

From Lemma 2.5, we have $g'(\xi) - a(z_0) g''(\xi)$ has at least two distinct zeros, this contradicts Claim which says that $g'(\xi) - a(z_0) g''(\xi)$ has at most one zero. Therefore $\mathcal{F}$ is normal in D . $\square$

3 Proof of theorems

Proof of Theorem 1.1 Suppose that $\mathcal{F}$ is not normal at z_0 . From Lemma 2.8, we obtain $a(z_0) = 0$. Without loss of generality, we assume that $z_0 = 0$ and $a(z) = z^m \phi(z)$, where $m \in \mathbb{N}$, and ϕ is holomorphic with $\phi(0) \neq 0$. We can say $\phi \neq 0$ in $\Delta(0, \delta_0) = \{z : |z| < \delta_0\}$ with the normalization $\phi(0) = 1$.

Since $f(z) \neq \infty$ when $a(z) = 0$ and $f'(z) - a(z) f''(z) - b(z)$ has at most one zero in Δ , then $\frac{f'(z)}{a(z) f''(z)} - \frac{b(z)}{a(z) f''(z)} - 1$ has at most one zero in Δ . Consider the family as follows

$$\mathcal{G} = \{g(z) = \frac{1}{a(z)f^{n-1}(z)}, f \in \mathcal{F}\}, \quad (3.1)$$

where all zeros and poles of $g(z)$ have multiplicity at least $n-1$ (≥ 3), except possibly the pole at 0, which has order at least m . Hence $\mathcal{G}$ is not normal at $z_0 = 0$ in Δ . Then, by Lemma 2.1, there exists $z_j \rightarrow 0$, $g_j \in \mathcal{G}$ and $\rho_j \rightarrow 0^+$ such that

$$G_j(\xi) = \frac{g_j(z_j + \rho_j \xi)}{\rho_j} \rightarrow G(\xi) \quad (3.2)$$

locally uniformly on compact subsets of $\mathbb{C}$, where $G(\xi)$ is a non-constant meromorphic functions in $\mathbb{C}$, and all zeros and poles of $G(\xi)$ have multiplicity at least $n-1$ (≥ 3), except possibly a pole, which has order at least m .

Next we consider two cases according to whether the sequence $\frac{z_n}{\rho_n}$ is bounded or unbounded.

Case 1. First assume that the sequence $\frac{z_n}{\rho_n}$ is unbounded. Then, there exists a subsequence, which we continue to call $\frac{z_n}{\rho_n}$, such that $\frac{z_n}{\rho_n} \rightarrow \infty$.

By simple calculation, from (3.1) and (3.2), we have

$$\begin{aligned} G'_j(\xi) &= \left[\frac{g_j(z_j + \rho_j \xi)}{\rho_j} \right]' = \left[\frac{1}{\rho_j a(z_j + \rho_j \xi) f_j^{n-1}(z_j + \rho_j \xi)} \right]' \\ &= - \frac{a'(z_j + \rho_j \xi)}{a^2(z_j + \rho_j \xi) f_j^{n-1}(z_j + \rho_j \xi)} - \frac{(n-1) f'_j(z_j + \rho_j \xi)}{a(z_j + \rho_j \xi) f_j^n(z_j + \rho_j \xi)} \\ &= - \frac{a'(z_j + \rho_j \xi)}{a^2(z_j + \rho_j \xi) f_j^{n-1}(z_j + \rho_j \xi)} - \frac{(n-1) b(z_j + \rho_j \xi)}{a(z_j + \rho_j \xi) f_j^n(z_j + \rho_j \xi)} \\ &\quad - (n-1) \left[\frac{f'_j(z_j + \rho_j \xi)}{a(z_j + \rho_j \xi) f_j^n(z_j + \rho_j \xi)} - \frac{b(z_j + \rho_j \xi)}{a(z_j + \rho_j \xi) f_j^n(z_j + \rho_j \xi)} \right]. \end{aligned}$$

Noting that

$$\begin{aligned}
& \frac{a'(z_j + \rho_j \xi)}{a^2(z_j + \rho_j \xi) f_j^{n-1}(z_j + \rho_j \xi)} \\
&= \frac{m(z_j + \rho_j \xi)^{m-1} \rho_j \phi(z_j + \rho_j \xi) + (z_j + \rho_j \xi)^m \rho_j \phi'(z_j + \rho_j \xi)}{(z_j + \rho_j \xi)^{2m} \phi^2(z_j + \rho_j \xi)} \\
& \quad g_j(z_j + \rho_j \xi)(z_j + \rho_j \xi)^m \phi(z_j + \rho_j \xi) \\
&= \frac{m \rho_j}{z_j + \rho_j \xi} g_j(z_j + \rho_j \xi) + \frac{\rho_j \phi'(z_j + \rho_j \xi)}{\phi(z_j + \rho_j \xi)} g_j(z_j + \rho_j \xi) \rightarrow 0, \text{ and} \\
& \left[\frac{b(z_j + \rho_j \xi)}{a(z_j + \rho_j \xi) f_j^n(z_j + \rho_j \xi)} \right]^{n-1} \\
&= \rho_j^n a(z_j + \rho_j \xi) b^{n-1}(z_j + \rho_j \xi) \left[\frac{1}{\rho_j a(z_j + \rho_j \xi) f_j^{n-1}(z_j + \rho_j \xi)} \right]^n \\
&= \rho_j^n a(z_j + \rho_j \xi) b^{n-1}(z_j + \rho_j \xi) G_j^n(\xi) \rightarrow 0
\end{aligned}$$

uniformly on compact subsets of $\mathbb{C}$ disjoint from the poles of G . we deduce that

$$-(n-1) \left[\frac{f_j'(z_j + \rho_j \xi)}{a(z_j + \rho_j \xi) f_j^n(z_j + \rho_j \xi)} - \frac{b(z_j + \rho_j \xi)}{a(z_j + \rho_j \xi) f_j^n(z_j + \rho_j \xi)} \right] \rightarrow G'(\xi)$$

uniformly on compact subsets of $\mathbb{C}$ disjoint from the poles of G .

If $G'(\xi) \equiv -(n-1) = 1-n$, then $G(\xi) = (1-n)\xi + A$, where A is a constant, which contradicts the fact that all zeros of $G(\xi)$ have multiplicity at least $n-1$. Hence, we have $G'(\xi) \not\equiv -(n-1)$.

Using an argument similar to Claim in Lemma 2.8, we can obtain $G'(\xi) + (n-1)$ has at most one zero.

On the other hand, by Lemmas 2.3 and 2.4, we have $G'(\xi) + (n-1)$ has at least two distinct zeros. Hence $G(\xi)$ is a constant, a contradiction.

Case 2. Now we consider the case that $\frac{z_m}{\rho_n}$ is bounded. Then, there is a subsequence, which we continue to call $\frac{z_m}{\rho_n}$, such that $\frac{z_m}{\rho_n} \rightarrow \alpha \in \mathbb{C}$.

It follows from (3.2) that

$$\frac{g_j(\rho_j \xi)}{\rho_j} = \frac{g_j(z_j + \rho_j(\xi - \frac{z_j}{\rho_j}))}{\rho_j} = G_j(\xi - \frac{z_j}{\rho_j}) \rightarrow G(\xi - \alpha) = \hat{G}(\xi) \quad (3.3)$$

spherically uniformly on compact subsets of $\mathbb{C}$. Clearly, all zeros and all poles of $\hat{G}(\xi)$ have multiplicity at least $n-1$, and $\xi = 0$ is a pole of $\hat{G}(\xi)$ with multiplicity at least m .

Set

$$H_j(\xi) = \frac{1}{\rho_j^{m+1} f_j^{n-1}(\rho_j \xi)}.$$

It follows from (3.2) and (3.3) that

$$H_j(\xi) = a(\rho_j \xi) \frac{g_j(z_j + \rho_j(\xi - \frac{z_j}{\rho_j}))}{\rho_j} \rightarrow H(\xi) = \xi^m G(\xi - \alpha) = \xi^m \hat{G}(\xi) \quad (3.4)$$

spherically locally uniformly on $\mathbb{C} \setminus \{0\}$, or on $\mathbb{C}$ if $G(-\alpha) \neq \infty$. Obviously, all zeros and all poles of $H(\xi)$ have multiplicity at least $n-1$, and $H(0) \neq 0$ since $f(\xi) \neq \infty$ when $a(\xi) = 0$.

Noting that

$$\begin{aligned} H'_j(\xi) + (n-1) \frac{b(\rho_j \xi)}{\rho_j^m f_j^n(\rho_j \xi)} &= \left[\frac{1}{\rho_j^{m+1} f_j^{n-1}(\rho_j \xi)} \right]' + (n-1) \frac{b(\rho_j \xi)}{\rho_j^m f_j^n(\rho_j \xi)} \\ &= -(n-1) \xi^m \phi(\rho_j \xi) \left[\frac{f'_j(\rho_j \xi)}{a(\rho_j \xi) f_j^n(\rho_j \xi)} - \frac{b(\rho_j \xi)}{a(\rho_j \xi) f_j^n(\rho_j \xi)} \right]. \end{aligned}$$

On the other hand,

$$\begin{aligned} \left[\frac{b(\rho_j \xi)}{\rho_j^m f_j^n(\rho_j \xi)} \right]^{n-1} &= \frac{\rho_j^n b^{n-1}(\rho_j \xi) a^n(\rho_j \xi)}{\rho_j^{m(n-1)}} \frac{1}{\rho_j^n a^n(\rho_j \xi) f_j^{n(n-1)}(\rho_j \xi)} \\ &= \frac{\rho_j^n b^{n-1}(\rho_j \xi) a^n(\rho_j \xi)}{\rho_j^{m(n-1)}} G_j^n(\rho_j \xi) = \frac{\rho_j^n b^{n-1}(\rho_j \xi) a^n(\rho_j \xi)}{\rho_j^{m(n-1)}} G_j^n(z_j + \rho_j(\xi - \frac{z_j}{\rho_j})) \\ &= \frac{\rho_j^{n(m+1)} b^{n-1}(\rho_j \xi) \xi^{nm} \phi^n(\rho_j \xi)}{\rho_j^{m(n-1)}} G_j^n(z_j + \rho_j(\xi - \frac{z_j}{\rho_j})) \\ &= \rho_j^{m+n} b^{n-1}(\rho_j \xi) \xi^{nm} \phi^n(\rho_j \xi) G_j^n(z_j + \rho_j(\xi - \frac{z_j}{\rho_j})) \rightarrow 0 \end{aligned}$$

uniformly on compact subsets of $\mathbb{C}$ disjoint from the poles of G . Hence

$$-(n-1) \xi^m \phi(\rho_j \xi) \left[\frac{f'_j(\rho_j \xi)}{a(\rho_j \xi) f_j^n(\rho_j \xi)} - \frac{b(\rho_j \xi)}{a(\rho_j \xi) f_j^n(\rho_j \xi)} \right] \rightarrow H'(\xi)$$

uniformly on compact subsets of $\mathbb{C}$ disjoint from the poles of G .

If $H'(\xi) \equiv -(n-1) \xi^m$, then $H(\xi) = \frac{(1-n)\xi^{m+1}}{m+1} + B$, where B is a constant. Thus, $H(\xi)$ has at least one zero in $\mathbb{C}$. Let ξ_0 be a zero of $H(\xi)$, then by the fact that ξ_0 has multiplicity at least $n-1$ (≥ 3), we get $H'(\xi_0) = 0$ and hence $\xi_0 = 0$ by $H'(\xi) \equiv -(n-1) \xi^m$. It follows that $H(\xi) = c \xi^l$, $c \in \mathbb{C}$ and $l \in \mathbb{N}$. Thus, $\frac{(1-n)\xi^{m+1}}{m+1} + B = c \xi^l$. By comparing the degrees and coefficients, we see that $B = 0$. Thus, $H(\xi) = \frac{(1-n)\xi^{m+1}}{m+1}$. It follows that $H(0) = 0$, which contradicts that $H(0) \neq 0$.

Using an argument similar to Claim in Lemma 2.8, we can obtain $H'(\xi) + (n-1) \xi^m$ has at most one zero.

Next we consider two subcases according to whether $H'(\xi) + (n-1) \xi^m$ has zero or not.

Case 2.1 $H'(\xi) \neq (n-1) \xi^m$. It now follows from Lemmas 2.2 and 2.6 that Case 2.1 cannot occur.

Case 2.2 $H'(\xi) = (n-1) \xi^m$. Then $H'(\xi) \neq (n-1) \xi^m$ for $\xi \neq \xi_0$, where $\xi_0 \in \mathbb{C}$.

Case 2.2.1 $\xi_0 = 0$. Thus $H'(0) = 0$. It follows that $H(0) = 0$, which contradicts that $H(0) \neq 0$.

Case 2.2.2 $\xi_0 \neq 0$.

If $H(\xi_0) = 0$, then $H'(\xi_0) = 0$ since all zeros of $H(\xi)$ have multiplicity at least $n-1$. Thus $\xi_0 = 0$, which contradicts that $\xi_0 \neq 0$.

If $H(\xi) \neq 0, \xi \in \mathbb{C}$. Combined with the fact that $H'(\xi) \neq (n-1)\xi^m$ for $\xi \neq \xi_0$, where $\xi_0 \in \mathbb{C}$. Hence by Lemma 2.7 $H(\xi)$ is a constant, a contradiction.

If $H(\xi) = 0$, and $H(\xi_0) \neq 0, \xi \in \mathbb{C} \setminus \{\xi_0\}$. Combined with the fact that $H'(\xi) \neq (n-1)\xi^m$ for $\xi \in \mathbb{C} \setminus \{\xi_0\}$. It now follows from Lemmas 2.2 and 2.6 that this case cannot occur. Hence we show that $\mathcal{G}$ is normal at $z_0 = 0$.

Next, we show that $\mathcal{F}$ is normal at $z_0 = 0$. Since $\mathcal{G}$ is normal at $z_0 = 0$, let $g_j \rightarrow g$ in a neighborhood of 0, then there exist $\Delta_\delta = \{z : |z| < \delta\}$ and a subsequence of $\{g_j\}$ such that $\{g_j\}$ converges uniformly to a meromorphic function or ∞ . Noting $g(0) = \infty$, we can find a ε with $0 < \varepsilon < \delta$ and $M > 0$ such that $|g(z)| > M, z \in \Delta_\varepsilon$. So, for sufficiently large j , we get $|g_j(z)| \geq \frac{M}{2}$, hence $|a(z)f_j^{n-1}(z)| \leq \frac{2}{M}$. Therefore $f_j(z) \neq \infty$ for sufficiently large j and $z \in \Delta_\varepsilon$.

Hence $f_j(z)$ is analytic in Δ_ε . Choosing ε small enough that $|a(z)| \geq \frac{|z|^m}{M}$, it follows that, for sufficiently large j , we have

$$|f_j(z)| = \left| \frac{1}{a(z)g_j(z)} \right|^{\frac{1}{n-1}} \leq \left(\frac{2^{m+1}}{\varepsilon^m} \right)^{\frac{1}{n-1}}, \quad |z| = \frac{\varepsilon}{2}.$$

By the Maximum Principle and Montel's theorem, $\mathcal{F}$ is normal at $z_0 = 0$. The complete proof of Theorem 1.1 is given. $\square$

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References

1. Hayman, W.K.: Research Problems of Function Theory. Athlone Press of Univ of London, London (1967)
2. Li, S.Y.: The normality criteria of a class of the functions. J. Fujian Norm. Univ. **2**, 156–158 (1984)
3. Li, X.J.: Proof of Hayman's conjecture on normal families. Sci. Sin. **28**, 596–603 (1985)
4. Langley, J.K.: On normal families and a result of Drasin. Proc. R. Soc. Edinb. **98A**, 385–393 (1984)
5. Pang, X.C.: Bloch's principle and normal criterion. Sci. Sin. Ser. A **11**, 1153–1159 (1988)
6. Chen, H.H., Fang, M.L.: On the value distribution of $f^n f'$. Sci. China Ser. A **38**, 789–798 (1995)
7. Bergweiler, W., Eremenko, A.: On the singularities of the inverse to a meromorphic function of finite order. Rev. Mat. Iberoam **11**, 355–373 (1995)
8. Xu, Y.: Normal families of meromorphic functions. J. Math. **21**(4), 381–386 (2001)
9. Zhang, Q.C.: Normal families of meromorphic functions concerning shared values. J. Math. Anal. Appl. **338**, 545–551 (2008)
10. Yuan, W.J., Li, Z.R., Lin, J.M.: Some normality criteria of function families concerning shared values. Math. Methods Appl. Sci. **35**(17), 2095–2100 (2012)
11. Qiu, H.L., Liu, D., Fang, M.L.: Hayman's question concerning normal families and shared values. Sci. Sin. Math. **42**(6), 603–610 (2012). ((Chinese))
12. Deng, B.M., Qiu, H.L., Fang, M.L.: Hayman's question on normal families concerning zero numbers, complex variables and elliptic equations. Int. J. (2013). <https://doi.org/10.1080/17476933.2012.750307>
13. Yang, J.H., Yang, Q., Pang, X.C.: A normal criterion concerning omitted holomorphic function. Acta Math. Sin. Engl. Ser. **35**(12), 1972–1978 (2019)
14. Pang, X.C., Zalcman, L.: Normal families and shared values. Bull. Lond. Math. Soc. **32**, 325–331 (2000)
15. Zalcman, L.: Normal families: new perspectives. Bull. Am. Math. Soc. **35**, 215–230 (1998)
16. Fang, M.L.: Picard values and normality criterion. Bull. Korean Math. Soc. **38**(2), 379–387 (2001)
17. Wang, Y.F., Fang, M.L.: Picard values and normal families of meromorphic functions with multiple zeros. Acta Math. Sci. **14**(5), 17–26 (1998)

18. Chang, J.M.: Normality of meromorphic functions whose derivatives have 1-points. Arch. Math. **94**, 555–564 (2010)
19. Pang, X.C., Yang, D.G., Zalcman, L.: Normal families of meromorphic functions whose derivatives omit a function. Comput. Methods Funct. Theory **2**(1), 257–265 (2002)
20. Chang, J.M.: Normality of meromorphic functions and uniformly discrete exceptional sets. Comput. Methods Funct. Theory **13**(1), 47–63 (2013)

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