Groups satisfying the minimal condition on subgroups which are not transitively normal

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Abstract

A subgroup X of a group G is called *transitively normal* if X is normal in any subgroup Y of G such that $X \leq Y$ and X is subnormal in Y . Thus all subgroups of a group G are transitively normal if and only if normality is a transitive relation in every subgroup of G (i.e. G is a $\bar{T}$ -group). It is proved that a group G with no infinite simple sections satisfies the minimal condition on subgroups that are not transitively normal if and only if either G is Černikov or a $\bar{T}$ -group.

Keywords T -group · $\bar{T}$ -group · Transitively normal subgroup

Mathematics Subject Classification 20E15

1 Introduction

A group G is said to be a T -group (or to have the T -property) if normality in G is a transitive relation, i.e. if all subnormal subgroups of G are normal. The structure of soluble T -groups was described by W. Gaschütz [4] in the finite case and by D.J.S. Robinson [10]

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for arbitrary groups. In particular, it was proved that all soluble groups with the T -property are metabelian and locally supersoluble, and that a finitely generated soluble T -group either is finite or abelian. Obviously, any simple group has the T -property, so that the class of T -groups is not subgroup closed; a group is called a $\overline{T}$ -group if all its subgroups have the T -property. It is known that every finite soluble T -group is a $\overline{T}$ -group and that all finite groups with the $\overline{T}$ -property are soluble.

A subgroup X of a group G is said to be *transitively normal* if X is normal in any subgroup Y of G such that $X \leq Y$ and X is subnormal in Y (see [8]). Obviously, an ascendant subgroup which is transitively normal is normal and all self-normalizing subgroups of an arbitrary group are transitively normal. Transitively normal subgroups may be considered as the most natural generalization of the so-called pronormal subgroups: recall here that a subgroup X of a group G is *pronormal* if for each element g of G the subgroups X and X^g are conjugate in $\langle X, X^g \rangle$. It is straightforward to show that pronormal subgroups are transitively normal, while there exists a group of order $2 \cdot 3 \cdot 7^2$ containing a subgroup of order 21 which is transitively normal but not pronormal (see for instance [3] and [7], where transitively normal subgroups were considered under different denominations).

Of course, a group G has the $\overline{T}$ -property if and only if all its subgroups are transitively normal; since there exist soluble $\overline{T}$ -groups containing non-pronormal subgroups, it follows that even a soluble group with only transitively normal subgroups may contain subgroups which are not pronormal. The aim of this paper is to give a further contribution to this topic of the theory by studying groups which have many transitively normal subgroups. More precisely, we investigate here the class $\overline{T}_0$ consisting of all groups satisfying *min* – *ntn*, the minimal condition on subgroups which are not transitively normal. In order to avoid *Tarski groups* (i.e. infinite simple groups whose proper non-trivial subgroups have prime order) and similar pathological groups belonging to the class $\overline{T}_0$, in our main result the attention is restricted to a suitable class of generalized soluble groups.

Theorem *Let G be a group satisfying the minimal condition on subgroups which are not transitively normal. If G has no infinite simple sections, then either G is Černikov or all subgroups of G are transitively normal. In particular, if G is not periodic, then it is abelian.*

The proof of the theorem will be treated separately in the periodic and in the non-periodic case.

Our notation is mostly standard and can be found in [11]. Notice that the symbol T_0 was used in [13], with a completely different meaning, to denote the class of groups G whose Frattini factor group $G/\Phi(G)$ has the T -property.

2 Some preliminaries

If X is any transitively normal subgroup of a group G , it is clear that X is transitively normal in every subgroup Y of G such that $X \leq Y$; moreover, if N is a normal subgroup of G , a subgroup X/N of G/N is transitively normal if and only if X is transitively normal in G . It follows that the class $\overline{T}_0$ is closed with respect to subgroups and homomorphic images.

Lemma 2.1 *Let G be a group and let $(X_i)_{i \in I}$ be a collection of transitively normal subgroups of G . If $[X_i, X_j] \leq X_i \cap X_j$ for all $i, j \in I$, then also the subgroup $X = \langle X_i \mid i \in I \rangle$ is transitively normal in G .*

Proof Let Y be any subgroup of G such that $X \leq Y$ and X is subnormal in Y . For each $i \in I$, the subgroup X_i is obviously normal in X and so subnormal in Y . Thus every X_i is normal in Y , so that X itself is normal in Y . Therefore X is a transitively normal subgroup of G . $\square$

Corollary 2.2 *Let G be a group, and let A be an abelian subgroup which is generated by transitively normal subgroups of G . Then A itself is transitively normal in G .*

Our next easy result shows in particular that if all cyclic subgroups of a locally nilpotent group G are transitively normal, then G is a Dedekind group.

Lemma 2.3 *Let G be a locally nilpotent group, and let X be a finitely generated transitively normal subgroup of G . Then X is normal in G .*

Proof If g is any element of G , the subgroup $\langle X, g \rangle$ is nilpotent, so that in particular X is subnormal in $\langle X, g \rangle$. Then $X^g = X$, and hence X is normal in G . $\square$

Let G be a finitely generated T -group which is not soluble. Since G/G'' is either finite or abelian, the subgroup G'' contains a maximal G -invariant subgroup M and G''/M is a simple non-abelian group. In particular, if G has the $\bar{T}$ -property, the section G''/M must be infinite. As a consequence, we have the following elementary result.

Lemma 2.4 *Let G be a $\bar{T}$ -group with no infinite simple sections. Then G is metabelian.*

Of course, every $\bar{T}_0$ -group satisfies the minimal condition on subnormal non-normal subgroups and our next two lemmas describe the behaviour of chief factors and derived series in groups with such property.

Lemma 2.5 *Let G be a group with no infinite simple sections. If G satisfies the minimal condition on subnormal non-normal subgroups, then every chief factor of G is finite.*

Proof Let X/Y be a chief factor of G and let Z be any normal subgroup of X properly containing Y . Then Z is a subnormal non-normal subgroup of G and so X/Y satisfies the minimal condition on normal subgroups. It follows that X/Y is the direct product of finitely many simple subgroups (see [11] Part 1, Corollary 2 to Lemma 5.23) and hence it is finite. $\square$

Lemma 2.6 *Let G be a group satisfying the minimal condition on subnormal non-normal subgroups. Then the derived series of G stops after finitely many steps.*

Proof Assume for a contradiction that $G^{(n)} \neq G^{(n+1)}$ for each non-negative integer n . Then $G^{(n)}/G^{(n+3)}$ is a soluble group of derived length 3 and so it contains a subnormal non-normal subgroup $X_n/G^{(n+3)}$. Then

$$X_0 > X_3 > \cdots > X_{3n} > \cdots$$

is an infinite descending chain of subnormal non-normal subgroups of G , contradicting the hypothesis. $\square$

Lemma 2.7 *Let G be a $\overline{T}_0$ -group. Then the Fitting subgroup F of G is hypercentral.*

Proof Obviously, all nilpotent normal subgroups of G satisfying the minimal condition on subgroups are contained in the hypercentre of F , so that without loss of generality we may suppose that G contains at least one nilpotent normal subgroup K which is not Černikov. Let N be any nilpotent normal subgroup of G . As the product KN is likewise nilpotent, it satisfies the minimal condition on non-normal subgroups and hence it is a Dedekind group by a result of Černikov (see [2] or [9]). It follows that in this case F itself is a Dedekind group and so the statement is proved. $\square$

Since every finitely generated non-trivial group contains a maximal normal subgroups, it is clear that all groups with no infinite simple sections are *locally graded*, i.e. all their finitely generated non-trivial subgroups contain proper subgroups of finite index.

Lemma 2.8 *Let G be a finitely generated locally graded $\overline{T}_0$ -group. Then G contains a subgroup X of finite index whose finite homomorphic images have the $\overline{T}$ -property. Moreover, X/X'' is a T -group and $X'' = X^{(3)}$.*

Proof It follows from the $\overline{T}_0$ -property that G contains a subgroup X of finite index such that all proper subgroups of finite index of X are transitively normal in G , and hence all finite homomorphic images of X are $\overline{T}$ -groups. Clearly, $X/X^{(3)}$ is a finitely generated soluble group whose finite homomorphic images have the T -property, so that also $X/X^{(3)}$ is a T -group (see [12], Theorem 2), and hence it is metabelian. Therefore $X'' = X^{(3)}$ and X/X'' has the T -property. $\square$

Recall that a group class $\mathfrak{X}$ is said to be *countably recognizable* if, whenever all countable subgroups of a group G belong to $\mathfrak{X}$, then G itself is an $\mathfrak{X}$ -group. Countably recognizable group classes have been introduced by Baer [1] more than fifty years ago and recently many relevant group classes have been proved to be detectable by the behaviour of countable groups (see for instance [5] and [6]). In particular, Baer proved that hyperabelian groups form a countably recognizable class; for our purposes we need to show that also the class of hyperabelian-by-finite groups is countably recognizable.

Theorem 2.9 *The class of groups containing a hyperabelian subgroup of finite index is countably recognizable.*

Proof Let G be an uncountable group whose countable subgroups have a hyperabelian subgroup of finite index and assume that G has no abelian non-trivial normal subgroups. Since the class of hyper-(abelian or finite) groups is countably recognizable (see [5], Lemma 4.3), G contains a finite non-trivial normal subgroup X_1 and

$$X_1 \cap C_G(X_1) = \zeta(X_1) = \{1\}.$$

Obviously, the index $|G : C_G(X_1)|$ is finite and so $C_G(X_1)$ contains a finite non-trivial G -invariant subgroup X_2 . Again we have

$$\langle X_1, X_2 \rangle \cap C_G(\langle X_1, X_2 \rangle) = \{1\}$$

and $|G : C_G(\langle X_1, X_2 \rangle)|$ is finite, so that the iteration of the argument allows us to construct a collection $(X_n)_{n \in \mathbb{N}}$ of finite non-trivial normal subgroups of G such that

$$X = \langle X_n \mid n \in \mathbb{N} \rangle = \text{Dr}_{n \in \mathbb{N}} X_n.$$

As the normal subgroup X of G is countably infinite, it has an abelian non-trivial normal subgroup A . Then the normal closure $A^G \leq X$ is hyperabelian and hence it contains an abelian non-trivial G -invariant subgroup. This contradiction shows that every uncountable homomorphic image of G has an abelian non-trivial normal subgroup. On the other hand, if G/N is a countably infinite homomorphic image of G , we have $G = NL$ for a suitable countable subgroup L of G , so that G/N has a hyperabelian subgroup of finite index and hence also an abelian non-trivial normal subgroup. Therefore all infinite homomorphic images of G have an abelian non-trivial normal subgroup and so G contains a hyperabelian subgroup of finite index and the statement is proved. $\square$

3 Periodic $\overline{T}_0$ -groups

The first result of this section shows that $\overline{T}_0$ -groups with no infinite simple sections are close to be hyperabelian.

Lemma 3.1 *Let G be a $\overline{T}_0$ -group with no infinite simple sections. Then G contains a hyperabelian subgroup of finite index.*

Proof As the class $\overline{T}_0$ is subgroup closed, by Theorem 2.9 it is enough to prove the statement when G is countable. Assume for a contradiction that G is infinite but has no abelian non-trivial normal subgroups. Let $\mathfrak{L}$ be any chain of infinite normal subgroups of G such that

$$L = \bigcap_{X \in \mathfrak{L}} X$$

is finite, and put

$$G \setminus L = \{x_n \mid n \in \mathbb{N}\}.$$

Since $\mathfrak{L}$ cannot have a smallest member, there are elements $X_1, \dots, X_n, \dots$ of $\mathfrak{L}$ such that

$$X_n \cap \{x_1, \dots, x_n\} = \emptyset \quad \text{and} \quad X_n > X_{n+1}$$

for all n , and clearly

$$\bigcap_{n \in \mathbb{N}} X_n = L.$$

It follows now from the $\overline{T}_0$ -property that there exists a positive integer m such that for every $n \geq m$ the group X_n/X_{n+1} has the $\overline{T}$ -property and hence is metabelian by Lemma 2.4. Then X_m/L is soluble by Lemma 2.6 and so X_m is soluble-by-finite. This contradiction shows that the intersection of any chain of infinite normal subgroups of G is infinite and so an application of Zorn's Lemma yields that G contains an infinite normal subgroup M whose proper

G -invariant subgroups are finite. Since all chief factors of G are finite by Lemma 2.5, it follows that M is generated by its proper G -invariant subgroups. Let N be any proper G -invariant subgroup of M . Since N is finite, the centralizer $C_M(N)$ has finite index in M and so $N \leq \zeta(M)$ because M has no proper G -invariant subgroups of finite index. Therefore M is abelian and this last contradiction completes the proof. $\square$

We are now ready to prove the main theorem in the periodic case.

Theorem 3.2 *Let G be a periodic $\overline{T}_0$ -group with no infinite simple sections. Then G is either Černikov or a $\overline{T}$ -group.*

Proof By Lemma 3.1 the group G contains a hyperabelian normal subgroup K of finite index. Let F be the Fitting subgroup of K and assume first that F is Černikov. Then also $K/C_K(F)$ is a Černikov group (see [11] Part 1, Theorem 3.29) and hence G is Černikov because $C_K(F) \leq F$.

Suppose now that F is not a Černikov group. Then F does not satisfy the minimal condition on abelian subgroups and so it contains a subgroup of the form

$$A = \operatorname{Dr}_{n \in \mathbb{N}} A_n,$$

where each A_n has prime order. For every positive integer k , put

$$B_k = \operatorname{Dr}_{n \geq k} A_n,$$

so that

$$B_1 > B_2 > \cdots > B_n > B_{n+1} > \cdots$$

and there exists a positive integer m such that B_n is transitively normal in G for all $n \geq m$. Since F is hypercentral by Lemma 2.7, all its subgroups are ascendant in G and so B_n is normal in G for every $n \geq m$.

Consider an arbitrary finite subgroup E of G and let X be any subnormal subgroup of E . Then the product XB_n is a subnormal subgroup of EB_n for each $n \geq m$. Moreover, there is an integer $r \geq m$ such that XB_n is transitively normal in G , and so normal in EB_n , for each $n \geq r$. On the other hand, we may obviously choose an integer $s \geq r$ such that $E \cap B_s = \{1\}$ and so $X = XB_s \cap E$ is normal in E . Therefore all finite subgroups of G have the T -property and hence G is a $\overline{T}$ -group. $\square$

4 Non-periodic $\overline{T}_0$ -groups

As we mentioned in the introduction, a finitely generated soluble T -group is either finite or abelian and hence it follows from Lemma 2.4 that non-periodic $\overline{T}$ -groups with no infinite simple sections are abelian. Thus, in order to complete the proof of our main theorem, we need to show that such a conclusion holds in the case of groups with the $\overline{T}_0$ -property.

Lemma 4.1 *Let G be a $\overline{T}_0$ -group, and let A be an abelian non-periodic subgroup of G . Then A is transitively normal in G .*

Proof Let a be any element of infinite order of A , and let p and q be different prime numbers. As G satisfies $\min - ntn$, there exist positive integers h and k such that both subgroups $\langle a^{p^h} \rangle$ and $\langle a^{q^k} \rangle$ are transitively normal in G . Then $\langle a \rangle = \langle a^{p^h}, a^{q^k} \rangle$ is likewise transitively normal in G by Corollary 2.2. Since A is generated by its elements of infinite order, a further application of Corollary 2.2 yields that also A is transitively normal in G . $\square$

Lemma 4.2 *Let G be a non-periodic locally nilpotent $\overline{T}_0$ -group. Then G is abelian.*

Proof By Lemma 4.1 every infinite cyclic subgroup of G is transitively normal, and so even normal in G by Lemma 2.3. As G is locally nilpotent, it follows that all elements of infinite order of G belong to the centre, and hence G is abelian because it is generated by elements of infinite order. $\square$

The behaviour of the Hirsch-Plotkin radical plays an important role in the study of non-periodic $\overline{T}_0$ -groups, as the following lemma shows.

Lemma 4.3 *Let G be a $\overline{T}_0$ -group whose Hirsch-Plotkin radical H is not periodic. Then H is contained in the centre $\zeta(G)$ of G .*

Proof Assume for a contradiction that the statement is false, so that there exists an element g of G such that $[H, g] \neq \{1\}$. Then $\langle g, H \rangle$ is a counterexample to the statement and hence we may suppose without loss of generality that $G = \langle g, H \rangle$. The subgroup H is abelian by Lemma 4.2, so that in particular G is soluble and $C_G(H) = H$. Of course, all subgroups of H are subnormal in G and it follows from Lemma 4.1 that every infinite cyclic subgroup of H is normal in G . Let a be an element of infinite order of H . If b is any element of finite order of H , the subgroup $\langle b \rangle$ is obviously characteristic in $\langle a, b \rangle$ and $\langle a, b \rangle$ is normal in G by Lemma 4.1, and so $\langle b \rangle$ is normal in G . Therefore all subgroups of H are normal in G , so that G/H has order 2 and $x^g = x^{-1}$ for all $x \in H$. Since $g^2 \in H$, we have $g^4 = 1$ and hence $\langle g \rangle \cap \langle a \rangle = \{1\}$. It follows that

$$\langle g, a \rangle > \langle g, a^2 \rangle > \langle g, a^4 \rangle > \cdots > \langle g, a^{2^n} \rangle > \langle g, a^{2^{n+1}} \rangle > \cdots$$

is a strictly descending chain of subgroups, and so there exists a positive integer k such that $\langle g, a^{2^n} \rangle$ is a transitively normal subgroup of G for each $n \geq k$. On the other hand, every $\langle g, a \rangle / \langle a^{2^n} \rangle$ is a finite 2-group, so that $\langle g, a^{2^n} \rangle$ is subnormal in $\langle g, a \rangle$ and hence $\langle g, a^{2^n} \rangle$ is normal in $\langle g, a \rangle$ for all $n \geq k$. Therefore also

$$\langle g \rangle = \bigcap_{n \geq k} \langle g, a^{2^n} \rangle$$

is a normal subgroup of $\langle g, a \rangle$, whence $[g, a] = 1$. This contradiction proves the statement. $\square$

Our final result completes the proof of the main theorem.

Theorem 4.4 *Let G be a non-periodic $\overline{T}_0$ -group which has no infinite simple sections. Then G is abelian.*

Proof Assume for a contradiction that the statement is false. Clearly, we may suppose that G is finitely generated, so that there exists in G an infinite descending chain of normal subgroups

$$G = G_0 > G_1 > G_2 > \cdots > G_n > G_{n+1} > \cdots$$

such that the index $|G : G_n|$ is finite for each non-negative integer n . It follows from the $\bar{T}_0$ -property that there is a positive integer k such that each subgroup of G_n/G_{n+1} is transitively normal in G/G_{n+1} for all $n \geq k$. In particular, G_n/G_{n+1} is a finite $\bar{T}$ -group and so it is soluble. Thus G_k/G_n is a finite soluble group for each $n \geq k$.

As G_k is finitely generated, Lemma 2.8 yields that it contains a subgroup X of finite index whose finite homomorphic images are $\bar{T}$ -groups, X/X'' has the T -property and $X'' = X^{(3)}$. Then XG_n/G_n is metabelian and hence $X'' \leq G_n$ for all $n \geq k$. Since X is finitely generated, the infinite metabelian $\bar{T}$ -group X/X'' cannot be periodic, so that it is abelian and $X' = X''$ is the last term of the derived series of X . Assume that $X' \neq \{1\}$. Clearly, X' is the normal closure in X of a finite subset and hence it contains a maximal proper X -invariant subgroup Y . Then X'/Y is a minimal normal subgroup of X/Y , and the $\bar{T}_0$ -property yields that X'/Y satisfies the minimal condition on normal subgroups, so that it is the direct product of finitely many simple groups (see [11] Part 1, p.150). On the other hand, G has no infinite simple sections, so that X'/Y is finite and hence X/Y is soluble-by-finite and then even soluble. This contradiction shows that X is abelian. Then G is abelian-by-finite, and it follows from Lemma 4.3 that $G/\zeta(G)$ is finite.

Among all finitely generated counterexamples choose one G for which the index $|G : \zeta(G)|$ is smallest possible. Then every proper subgroup of G containing $\zeta(G)$ is abelian, so that all proper subgroups of the finite group $G/\zeta(G)$ are abelian; in particular, $G/\zeta(G)$ is soluble and hence G itself is a soluble group. On the other hand, $\zeta(G)$ coincides with the Hirsch-Plotkin radical of G by Lemma 4.3, and so $G = C_G(\zeta(G)) = \zeta(G)$, which is of course impossible. This contradiction completes the proof. $\square$

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