A note on the asymptotics for incomplete Betafunctions

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Abstract

We determine the asymptotic behaviour of certain incomplete Betafunctions.

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1 Introduction and results

For integers $k, \ell \geq 1$ define

$$P_{k,\ell} = \frac{(k+\ell)!}{(k-1)!\ell!} \int_{k/(k+\ell)}^1 t^{k-1}(1-t)^\ell dt.$$

By repeated partial integration we obtain the representation

$$P_{k,\ell} = \left(\frac{\ell}{\ell+k}\right)^{k+\ell} \sum_{v=0}^{k-1} \binom{k+\ell}{v} \left(\frac{k}{\ell}\right)^v.$$

Vietoris[5] showed that $P_{k,\ell} \leq \frac{1}{2}$. Alzer and Kwong [1] proved that $P_{k,\ell} \geq \frac{1}{4}$ holds for all k, ℓ . Interest in bounds for $P_{k,\ell}$ stems from application to statistics, see [4].

In this note we consider the asymptotic behaviour of $P_{k,\ell}$. We prove the following.

Theorem 1 *We have*

$$P_{k,\ell} = e^{-k} \sum_{v=0}^{k-1} \frac{k^v}{v!} + \mathcal{O}\left(\frac{k^2}{\ell}\right), \quad (1)$$

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$$P_{k,\ell} = 1 - e^{-\ell} \sum_{v=0}^{\ell} \frac{\ell^v}{v!} + \mathcal{O}\left(\frac{\ell^2}{k}\right) \quad (2)$$

and

$$P_{k,\ell} = \frac{1}{2} + \mathcal{O}\left(\frac{1}{\sqrt{\min(k, \ell)}}\right). \quad (3)$$

Note that the first two estimates are good if one of the parameters k, ℓ is rather small, whereas the third one gives information in the general case.

Comparing (1) and (2) with Vietoris' result that $P_{k,\ell} \leq \frac{1}{2}$ we see that

$$\sum_{v=0}^{\ell-1} \frac{\ell^v}{v!} < \frac{1}{2} e^{\ell} < \sum_{v=0}^{\ell} \frac{\ell^v}{v!}.$$

Note that equality is impossible as e is transcendental. A more precise version of this inequality has been asked by Ramanujan (Question 294) and was answered by Karamata [2]. For a detailed discussion of this result, Uhlmann's inequalities [4] and Vietoris bound we refer the reader to the historical notes by Vietoris [6].

From Theorem 1 we deduce the following.

Corollary 1 *The only accumulation points of the set $\{P_{k,\ell} : k, \ell \in \mathbb{N}\}$ are $\frac{1}{2}$ and the real numbers $e^{-k} \sum_{v=1}^{k-1} \frac{k^v}{v!}$ and $1 - e^{-\ell} \sum_{v=1}^{\ell} \frac{\ell^v}{v!}$.*

Proof Suppose that $(k_n), (\ell_n)$ are integer sequences such that the pairs (k_n, ℓ_n) are all distinct and that the sequence (P_{k_n, ℓ_n}) converges. If both k_n, ℓ_n tend to infinity, then by (3) we have that P_{k_n, ℓ_n} tends to $\frac{1}{2}$. If one of these sequences does not tend to infinity, then we can pass to a subsequence and assume that k_n or ℓ_n is constant. Then our claim follows from (1) or (2). $\square$

Together with Vietoris bound we obtain the following.

Corollary 2 *We have*

$$\lim_{\ell \rightarrow \infty} P_{k,\ell} \geq \frac{1}{2} - \frac{1}{\sqrt{2\pi k}}.$$

Proof We have

$$\lim_{\ell \rightarrow \infty} (P_{k,\ell} + P_{\ell,k}) = 1 - \frac{k^k}{e^k k!},$$

using Stirling's formula in the form $k! = \left(\frac{k}{e}\right)^k \sqrt{2\pi k} e^{\theta/12k}$ with $0 \leq \theta \leq 1$, and $P_{\ell,k} \leq \frac{1}{2}$, we conclude

$$\lim_{\ell \rightarrow \infty} P_{k,\ell} \geq \frac{1}{2} - \frac{k^k}{e^k k!} \geq \frac{1}{2} - \frac{1}{\sqrt{2\pi k}}.$$

$\square$

2 Preliminary estimates

For the proof of (3) we use another representation of $P_{k,\ell}$, which is due to Raab [3].

Theorem 2 We have $P_{k,\ell} = U_{k,\ell} V_{k,\ell}$, where

$$U_{k,\ell} = \exp \int_0^\infty \frac{1}{t} \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) (e^{-(k+\ell)t} - e^{-kt} - e^{-\ell t}) dt,$$

$$V_{k,\ell} = \frac{\sqrt{\ell}}{2\pi} \sum_{v=1}^{\infty} \frac{c_v(k/\ell)}{\sqrt{v(v+\ell)}},$$

and

$$c_v(x) = \exp \int_0^\infty \frac{1}{t} \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) (e^{-v(1+x)t} - e^{-vxt} - e^{-vt}) dt.$$

We first compute the occurring integrals.

Lemma 1 We have for $x \geq 1$

$$\int_0^\infty \frac{1}{t} \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) e^{-xt} dt = \frac{1}{12x} + \mathcal{O}\left(\frac{1}{x^2}\right)$$

and

$$\int_0^\infty \frac{1}{t} \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) e^{-xt} dt = \frac{1}{2} \log x + C + \mathcal{O}(x)$$

for $0 < x \leq 1$, where C is some constant.

Proof The series expansion of e^x yields for $t \rightarrow 0$

$$\frac{1}{t} \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) = \frac{1}{t^2} \left(\frac{1}{1 + t/2 + t^2/6 + \mathcal{O}(t^3)} - 1 + \frac{t}{2} \right) = 1/12 + \mathcal{O}(t).$$

For $t \rightarrow \infty$ this expression tends to 0, in particular, it is bounded for all positive t . Hence for $x \rightarrow \infty$ the integral in question becomes

$$\frac{1}{12} \int_0^\infty e^{-xt} dt + \mathcal{O}\left(\int_0^\infty t e^{-xt} dt\right) = \frac{1}{12x} + \mathcal{O}\left(\frac{1}{x^2}\right).$$

For $x \rightarrow 0$ we use $e^{-xt} = 1 - xt + \mathcal{O}(x^2 t^2)$ and obtain

$$\int_0^1 \frac{1}{t} \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) e^{-xt} dt = \int_0^1 \frac{1}{t} \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) dt + \mathcal{O}(x) = C_1 + \mathcal{O}(x)$$

and

$$\int_1^\infty \frac{1}{t(e^t - 1)} e^{-xt} dt = \int_1^\infty \frac{1}{t(e^t - 1)} dt - x \int_1^\infty \frac{1}{e^t - 1} dt + \mathcal{O}(x^2).$$

As a function of x , the integral $\int_1^\infty \frac{e^{-xt}}{t^2} dt$ defines a function that is differentiable from the right in 0 and has bounded second derivative in $(0, \infty)$, hence, for $x \geq 0$ this integral is $1 + \mathcal{O}(x)$.

Finally

$$\begin{aligned} \int_1^\infty \frac{e^{-xt}}{2t} dt &= \int_x^\infty \frac{e^{-s}}{2s} ds = \\ &= \int_x^1 \frac{dt}{2t} + \int_0^1 \frac{e^{-t} - 1}{2t} dt + \int_1^\infty \frac{e^{-t}}{2t} dt + \int_0^x \frac{1 - e^{-t}}{2t} dt \\ &= -\frac{1}{2} \log x + C_3 + \mathcal{O}(x). \end{aligned}$$

Combining these estimates our claim follows. $\square$

From this we obtain

Lemma 2 *We have*

$$U_{k,\ell} = 1 - \frac{1}{12} \left(\frac{1}{k} + \frac{1}{\ell} - \frac{1}{k+\ell} \right) + \mathcal{O} \left(\frac{1}{\min(k, \ell)^2} \right).$$

Proof From Lemma 1 we obtain

$$U_{k,\ell} = \exp \left(\frac{1}{12} \left(\frac{1}{k+\ell} - \frac{1}{k} - \frac{1}{\ell} \right) + \mathcal{O} \left(\frac{1}{\min(k, \ell)^2} \right) \right),$$

inserting the Taylor series for exp and using the fact that $k, \ell \geq 1$ our claim follows. $\square$

Next we compute $c_v(x)$.

Lemma 3 *If $v x \geq 1$, then*

$$c_v(x) = 1 + \frac{1}{12v(1+x)} - \frac{1}{12vx} + \frac{1}{12v} + \mathcal{O} \left(\frac{1}{v^2 \min(1, x^2)} \right).$$

If $v x \leq 1$, then

$$c_v(x) = K \sqrt{vx} + \mathcal{O} \left(\sqrt{\frac{x}{v}} + (vx)^{3/2} \right)$$

for some constant K .

Proof If $v x > 1$, then we apply Lemma 1 to obtain

$$\begin{aligned} c_v(x) &= \exp \left(\frac{1}{12v(x+1)} - \frac{1}{12vx} + \frac{1}{12v} + \mathcal{O} \left(\frac{1}{v^2 \min(1, x^2)} \right) \right) \\ &= 1 + \frac{1}{12v(x+1)} - \frac{1}{12vx} + \frac{1}{12v} + \mathcal{O} \left(\frac{1}{v^2 \min(1, x^2)} \right). \end{aligned}$$

If $vx < 1$, then $v(x+1) \geq 1$, and we obtain

$$\begin{aligned} c_v(x) &= \exp \left(\frac{1}{12v(x+1)} + \frac{1}{2} \log(vx) + C + \frac{1}{12v} + \mathcal{O} \left(\frac{1}{v^2} + vx \right) \right) \\ &= K \sqrt{vx} + \mathcal{O} \left(\sqrt{\frac{x}{v}} + (vx)^{3/2} \right). \end{aligned}$$

□

We now compute $V_{k,\ell}$.

Lemma 4 *We have*

$$V_{k,\ell} = \frac{1}{2} + \mathcal{O} \left(\frac{1}{\sqrt{k}} + \frac{1}{\sqrt{\ell}} \right)$$

Proof We have

$$\sum_{v \leq \ell/k} \frac{c_v(k/\ell)}{\sqrt{v(v+\ell)}} \ll \sum_{v \leq \ell/k} \frac{\sqrt{vk/\ell}}{\sqrt{v(v+\ell)}} \ll \frac{1}{\sqrt{\ell k}}$$

thus,

$$\begin{aligned} V_{k,\ell} &= \frac{\sqrt{\ell}}{2\pi} \sum_{v > \ell/k} \frac{1}{\sqrt{v(v+\ell)}} \left(1 + \frac{1}{12v \frac{k+\ell}{\ell}} - \frac{1}{12v \frac{k}{\ell}} + \frac{1}{12v} + \mathcal{O} \left(\frac{1}{v^2 \min(1, k^2/\ell^2)} \right) \right) \\ &\quad + \mathcal{O} \left(\frac{1}{\sqrt{k}} \right) \\ &= \frac{\sqrt{\ell}}{2\pi} \sum_{v > \ell/k} \frac{1}{\sqrt{v(v+\ell)}} \left(1 - \frac{1}{12v \frac{k}{\ell}} + \mathcal{O} \left(\frac{1}{v^2 \min(1, k^2/\ell^2)} \right) \right) \\ &\quad + \mathcal{O} \left(\frac{1}{\sqrt{k}} + \sum_{v > \ell/k} \frac{1}{v^{3/2} \sqrt{\ell}} \right) \\ &= \frac{\sqrt{\ell}}{2\pi} \sum_{v > \ell/k} \frac{1}{\sqrt{v(v+\ell)}} \left(1 - \frac{1}{12v \frac{k}{\ell}} + \mathcal{O} \left(\frac{1}{v^2 \min(1, k^2/\ell^2)} \right) \right) \\ &\quad + \mathcal{O} \left(\frac{1}{\sqrt{k}} + \frac{1}{\sqrt{\ell}} \right) \end{aligned}$$

If $k > \ell$ we obtain

$$\begin{aligned}
V_{k,\ell} &= \frac{\sqrt{\ell}}{2\pi} \sum_{v=1}^{\infty} \frac{1}{\sqrt{v(v+\ell)}} + \mathcal{O}\left(\frac{1}{\sqrt{k}} + \frac{1}{\sqrt{\ell}} + \sqrt{\ell} \sum_{v=1}^{\infty} \frac{1}{v^{3/2}(v+\ell)}\right) \\
&= \frac{1}{2\pi} \int_0^{\infty} \frac{dt}{\sqrt{t(t+1)}} + \mathcal{O}\left(\frac{1}{\sqrt{k}} + \frac{1}{\sqrt{\ell}}\right) \\
&= \frac{1}{2} + \mathcal{O}\left(\frac{1}{\sqrt{\ell}}\right).
\end{aligned}$$

We have

$$\begin{aligned}
\frac{\sqrt{\ell}}{2\pi} \sum_{v>\ell} \frac{1}{\sqrt{v(v+\ell)}} \left(1 - \frac{1}{12v_{\ell}^k} + \mathcal{O}\left(\frac{1}{v^2 \min(1, k^2/\ell^2)}\right)\right) &= \frac{\sqrt{\ell}}{2\pi} \sum_{v>\ell} \frac{1}{\sqrt{v(v+\ell)}} \\
&+ \mathcal{O}\left(\frac{1}{\sqrt{k}}\right),
\end{aligned}$$

and for $\ell/k \leq N \leq \ell$

$$\sum_{N \leq v \leq 2N} \frac{1}{v^{3/2}(v+\ell)} \leq \frac{1}{\sqrt{N}\ell}$$

as well as

$$\sum_{N \leq v \leq 2N} \frac{1}{v^{5/2}(v+\ell)} \leq \frac{1}{N^{3/2}\ell}$$

and therefore

$$\sqrt{\ell} \sum_{v \geq \ell/k} \frac{1}{\sqrt{v(v+\ell)}} \cdot \frac{1}{v_{\ell}^k} = \frac{\ell^{3/2}}{k} \sum_{v \geq \ell/k} \frac{1}{v^{3/2}(v+\ell)} = \mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$$

and

$$\sqrt{\ell} \sum_{v \geq \ell/k} \frac{1}{\sqrt{v(v+\ell)}} \cdot \frac{1}{v^2 \min(k^2/\ell^2)} = \frac{\ell^{5/2}}{k^2} \sum_{v \geq \ell/k} \frac{1}{v^{5/2}(v+\ell)} = \mathcal{O}\left(\frac{1}{\sqrt{k}}\right).$$

Using these estimates we obtain

$$\begin{aligned}
\frac{\sqrt{\ell}}{2\pi} \sum_{\ell/k \leq v \leq \ell} \frac{1}{\sqrt{v(v+\ell)}} \left(1 - \frac{1}{12v_{\ell}^k} + \mathcal{O}\left(\frac{1}{v^2 \min(1, k^2/\ell^2)}\right)\right) &= \\
\frac{\sqrt{\ell}}{2\pi} \sum_{\ell/k \leq v \leq \ell} \frac{1}{\sqrt{v(v+\ell)}} + \mathcal{O}\left(\frac{1}{\sqrt{k}}\right).
\end{aligned}$$

For $k \leq \ell$ we obtain

$$\begin{aligned}
V_{k,\ell} &= \frac{\sqrt{\ell}}{2\pi} \sum_{v \geq \ell/k} \frac{1}{\sqrt{v(v+\ell)}} + \mathcal{O}\left(\frac{1}{\sqrt{k}}\right) \\
&= \frac{1}{2\pi} \int_{1/k}^{\infty} \frac{dt}{\sqrt{t(t+1)}} + \mathcal{O}\left(\frac{1}{\sqrt{k}}\right) \\
&= \frac{1}{2\pi} \int_0^{\infty} \frac{dt}{\sqrt{t(t+1)}} + \mathcal{O}\left(\frac{1}{\sqrt{k}}\right) \\
&= \frac{1}{2} + \mathcal{O}\left(\frac{1}{\sqrt{k}}\right),
\end{aligned}$$

and the proof is complete. $\square$

3 Proof of Theorem 1

We first prove (1). Note that this inequality is only interesting if the error term is $o(1)$, in particular we may assume that $k^2 < \ell$. Under this assumption we have

$$\begin{aligned}
P_{k,\ell} &= \left(\frac{\ell}{\ell+k}\right)^{k+\ell} \sum_{v=0}^{k-1} \binom{k+\ell}{v} \left(\frac{k}{\ell}\right)^v \\
&= \sum_{v=0}^{k-1} \left(\frac{\ell}{\ell+k}\right)^{k+\ell-v} \frac{k^v}{v!} \exp\left(\sum_{\mu=1}^v \log \frac{k+\ell-\mu}{k+\ell}\right) \\
&= \sum_{v=0}^{k-1} \left(\frac{\ell}{\ell+k}\right)^{k+\ell-v} \frac{k^v}{v!} \exp\left(-\sum_{\mu=1}^v \frac{\mu}{k+\ell} + \mathcal{O}\left(\frac{\mu^2}{(k+\ell)^2}\right)\right) \\
&= \sum_{v=0}^{k-1} \left(\frac{\ell}{\ell+k}\right)^{k+\ell-v} \frac{k^v}{v!} \exp\left(\mathcal{O}\left(\frac{v^2}{\ell}\right)\right) \\
&= \sum_{v=0}^{k-1} \left(\frac{\ell}{\ell+k}\right)^{k+\ell} \frac{k^v}{v!} \exp\left(\mathcal{O}\left(\frac{k^2}{\ell}\right)\right) \\
&= \exp\left((k+\ell) \log \frac{\ell}{\ell+k}\right) \sum_{v=0}^{k-1} \frac{k^v}{v!} \exp\left(\mathcal{O}\left(\frac{k^2}{\ell}\right)\right) \\
&= \exp\left(-k + \mathcal{O}\left(\frac{k^2}{\ell}\right)\right) \sum_{v=0}^{k-1} \frac{k^v}{v!} \\
&= e^{-k} \sum_{v=0}^{k-1} \frac{k^v}{v!} + \mathcal{O}\left(\frac{k^2}{\ell}\right).
\end{aligned}$$

The proof of (2) is quite similar. Analogously to the previous case we may assume that $\ell^2 < k$.

$$\begin{aligned}
P_{k,\ell} &= \left(\frac{\ell}{\ell+k} \right)^{k+\ell} \left(\left(\frac{\ell+k}{\ell} \right)^{k+\ell} - \sum_{v=k}^{k+\ell} \binom{k+\ell}{v} \left(\frac{k}{\ell} \right)^v \right) \\
&= 1 - \left(\frac{\ell}{\ell+k} \right)^{k+\ell} \sum_{\mu=0}^{\ell} \binom{k+\ell}{\mu} \left(\frac{k}{\ell} \right)^{k+\ell-\mu} \\
&= 1 - \left(\frac{k}{\ell+k} \right)^{k+\ell} \sum_{\mu=0}^{\ell} \binom{k+\ell}{\mu} \left(\frac{\ell}{k} \right)^{\mu} \\
&= 1 - \exp \left(-\ell + \mathcal{O} \left(\frac{\ell^2}{k} \right) \right) \sum_{\mu=0}^{\ell} \frac{\ell^{\mu}}{\mu!} \left(\frac{k+\ell}{k} \right)^{\mu} \\
&= 1 - e^{-\ell} \sum_{\mu=0}^{\ell} \frac{\ell^{\mu}}{\mu!}.
\end{aligned}$$

Finally (3) follows from Theorem 2, Lemma 2 and Lemma 4.

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