On a successive property of strongly starlikeness for multivalent functions

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Abstract

For f analytic in the unit disk $\mathbb{D}$, of the form $f(z) = z^p + \dots$, we consider some consequences of strongly starlikeness of $f^{(p-1)}(z)/p!$.

Keywords Starlike · Strongly starlike · Multivalent

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1 Introduction

We denote by $\mathcal{H}$ the class of functions $f(z)$ which are holomorphic in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Denote by $\mathcal{A}_p$, $p \in \mathbb{N} = \{1, 2, \dots\}$, the class of functions $f(z) \in \mathcal{H}$ given by

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad (z \in \mathbb{D}). \quad (1.1)$$

Lemma 1.1 [2, Theorem 5] *If $f(z) \in \mathcal{A}_p$, then for all $z \in \mathbb{D}$, we have*

$$\Re \left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} > 0 \Rightarrow \forall k \in \{1, \dots, p-1\} : \Re \left\{ \frac{zf^{(k)}(z)}{f^{(k-1)}(z)} \right\} > 0. \quad (1.2)$$

In this paper we consider a generalization of the above result. In Lemma 1.1 we have assumed that $zf^{(p)}(z)/f^{(p-1)}(z)$ lies in the right half-plane while in this paper we work with a sector. The problem we solve here is: for what values of α, β does an analytic function of the form (1.1) satisfy

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$$\left| \arg \left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} \right| < \frac{\pi\alpha}{2} \Rightarrow \forall k \in \{1, \dots, p-1\} : \left| \arg \left\{ \frac{zf^{(k)}(z)}{f^{(k-1)}(z)} \right\} \right| < \frac{\pi\beta}{2}?$$

Recall that if $f(z) \in \mathcal{A}_p$ and

$$\Re \left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} > 0 \quad (z \in \mathbb{D}),$$

then $f^{(p-1)}(z)/p!$ is univalent in $\mathbb{D}$ and $f^{(p-1)}(z)/p!$ is called a starlike function. If $f(z) \in \mathcal{A}_p$, $\gamma \in (0, 1]$, and

$$\arg \left| \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right| < \frac{\pi\gamma}{2}, \quad z \in \mathbb{D}, \quad (1.3)$$

then $f^{(p-1)}(z)/p!$ is called a strongly starlike function of order γ and such functions we consider in the paper. This class for the case $p = 1$ was introduced by Brannan and Kirwan [1]. Also, if $f(z) \in \mathcal{A}_p$ satisfies (1.3), then $f(z)$ is called p -valently strongly starlike function of order γ . For the proof of main result we need the following lemma.

Lemma 1.2 [3] *Let $q(z) = 1 + \sum_{n \geq m} c_n z^n$, $c_m \neq 0$ be analytic function in $|z| < 1$ with $q(0) = 1$, $q(z) \neq 0$. If there exists a point z_0 , $|z_0| < 1$, such that*

$$|\arg \{q(z)\}| < \frac{\pi\beta}{2} \quad \text{for } |z| < |z_0|$$

and

$$|\arg \{q(z_0)\}| = \frac{\pi\beta}{2}$$

for some $\beta > 0$, then we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = \frac{2ik \arg \{q(z_0)\}}{\pi},$$

for some $k \geq m(a + a^{-1})/2 \geq m$, where

$$\{q(z_0)\}^{1/\beta} = \pm ia, \quad \text{and } a > 0.$$

2 Main results

For given $0 < \beta_{s-1} \leq 1$ let us consider the number

$$\beta_s = \beta_{s-1} + \frac{2}{\pi} \tan^{-1} \frac{\beta_{s-1} n(\beta_{s-1}) \sin[\pi(1 - \beta_{s-1})/2]}{sm(\beta_{s-1}) + \beta_{s-1} n(\beta_{s-1}) \cos[\pi(1 - \beta_{s-1})/2]}, \quad s = 2, 3, \dots, p, \quad (2.1)$$

where

$$m(\beta_{s-1}) = (1 + \beta_{s-1})^{(1+\beta_{s-1})/2}, \quad \text{and } n(\beta_{s-1}) = (1 - \beta_{s-1})^{(1-\beta_{s-1})/2}. \quad (2.2)$$

Notice that if $0 < \beta_{s-1} \leq 1$, then $0 < \beta_s \leq 1$ too because from (2.1), (2.2), we have

$$\begin{aligned}\beta_s &= \beta_{s-1} + \frac{2}{\pi} \tan^{-1} \frac{\beta_{s-1}n(\beta_{s-1}) \sin[\pi(1 - \beta_{s-1})/2]}{sm(\beta_{s-1}) + \beta_{s-1}n(\beta_{s-1}) \cos[\pi(1 - \beta_{s-1})/2]}, \\ &\leq \beta_{s-1} + \frac{2}{\pi} \tan^{-1} \frac{\beta_{s-1}n(\beta_{s-1}) \sin[\pi(1 - \beta_{s-1})/2]}{\beta_{s-1}n(\beta_{s-1}) \cos[\pi(1 - \beta_{s-1})/2]}, \\ &= 1.\end{aligned}$$

Therefore, if we have a number $\beta_1 \in (0, 1]$, then from (2.1), we can find a sequence $\beta_p, \beta_{p-1}, \dots, \beta_2, \beta_1$, such that

$$0 < \beta_1 \leq \beta_2 \leq \dots \leq \beta_{p-1} \leq \beta_p \leq 1. \quad (2.3)$$

Theorem 2.1 Let $f(z) \in \mathcal{A}_p$, $p \geq 2$. For given $\beta_{p-1} \in (0, 1]$ there exists $\beta_p \in (0, 1]$ of the form (2.1) such that for all $z \in \mathbb{D}$, we have

$$\left| \arg \left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} \right| < \frac{\pi\beta_p}{2} \Rightarrow \left| \arg \left\{ \frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)} \right\} \right| < \frac{\pi\beta_{p-1}}{2}. \quad (2.4)$$

Proof Let us put

$$q_1(z) = \frac{zf^{(p-1)}(z)}{2f^{(p-2)}(z)}, \quad q_1(0) = 1.$$

Then it follows that

$$\frac{zq_1'(z)}{q_1(z)} = 1 + \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} - \frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}$$

and

$$2q_1(z) + \frac{zq_1'(z)}{q_1(z)} = 1 + \frac{zf^{(p)}(z)}{f^{(p-1)}(z)}$$

and so

$$\arg \{q_1(z)\} + \arg \left\{ 2 + \frac{zq_1'(z)}{q_1^2(z)} \right\} = \arg \left\{ 1 + \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\}.$$

If there exists a point $z_0 \in \mathbb{D}$ such that

$$\begin{aligned}|\arg \{q_1(z)\}| &< \pi\beta_{p-1}/2 \quad \text{for } |z| < |z_0|, \quad |\arg \{q_1(z_0)\}| = \pi\beta_{p-1}/2, \\ \{q_1(z_0)\}^{1/\beta_{p-1}} &= \pm ia, \quad \text{and } a > 0,\end{aligned}$$

then from Lemma 1.2, we have

$$\frac{z_0q_1'(z_0)}{q_1(z_0)} = \frac{2ik \arg \{q_1(z_0)\}}{\pi} \quad (2.5)$$

for some real k with $k \geq (a + a^{-1})/2 \geq 1$. For the case $\arg \{q_1(z_0)\} = \pi\beta_{p-1}/2$, we have

$$\begin{aligned}\arg \left\{ \frac{z_0f^{(p)}(z_0)}{f^{(p-1)}(z_0)} \right\} &\geq \arg \left\{ 1 + \frac{z_0f^{(p)}(z_0)}{f^{(p-1)}(z_0)} \right\} \\ &= \arg \{q_1(z_0)\} + \arg \left\{ 2 + \frac{z_0q_1'(z_0)}{q_1(z_0)} \frac{1}{q_1(z_0)} \right\} \\ &\geq \frac{\pi\beta_{p-1}}{2} + \arg \left\{ 2 + e^{i\pi(1-\beta_{p-1})/2} \frac{1}{(ia)^{\beta_{p-1}}} \right\},\end{aligned}$$

where $\{q_1(z_0)\}^{1/\beta_{p-1}} = ia$ and a is a positive real number. Applying Lemma 1.2 we obtain

$$\begin{aligned} & \arg \left\{ \frac{z_0 f^{(p)}(z_0)}{f^{(p-1)}(z_0)} \right\} \\ & \geq \frac{\pi\beta_{p-1}}{2} + \arg \left\{ e^{i\pi(1-\beta_{p-1})/2} \left(\left(\frac{1+\beta_{p-1}}{1-\beta_{p-1}} \right)^{(1-\beta_{p-1})/2} \right. \right. \\ & \quad \left. \left. + \left(\frac{1+\beta_{p-1}}{1-\beta_{p-1}} \right)^{-(1+\beta_{p-1})/2} \right) + 2 \right\} \\ & = \frac{\pi\beta_{p-1}}{2} + \tan^{-1} \frac{\frac{\beta_{p-1}}{1-\beta_{p-1}} \left(\frac{1-\beta_{p-1}}{1+\beta_{p-1}} \right)^{(1+\beta_{p-1})/2} \sin \frac{\pi(1-\beta_{p-1})}{2}}{2 + \frac{\beta_{p-1}}{1-\beta_{p-1}} \left(\frac{1-\beta_{p-1}}{1+\beta_{p-1}} \right)^{(1+\beta_{p-1})/2} \cos \frac{\pi(1-\beta_{p-1})}{2}} \\ & = \frac{\pi\beta_{p-1}}{2} + \tan^{-1} \frac{\beta_{p-1}n(\beta_{p-1}) \sin \frac{\pi(1-\beta_{p-1})}{2}}{2m(\beta_{p-1}) + \beta_{p-1}n(\beta_{p-1}) \cos \frac{\pi(1-\beta_{p-1})}{2}}. \end{aligned}$$

From (2.1), we can see that

$$\arg \left\{ \frac{z_0 f^{(p)}(z_0)}{f^{(p-1)}(z_0)} \right\} \geq \frac{\pi\beta_p}{2}. \quad (2.6)$$

This contradicts hypothesis in (2.4).

For the case $\arg\{q_1(z_0)\} = -\pi\beta_{p-1}/2$, applying the same method as the above, gives

$$\arg \left\{ \frac{z_0 f^{(p)}(z_0)}{f^{(p-1)}(z_0)} \right\} \leq -\frac{\pi\beta_p}{2}. \quad (2.7)$$

This also contradicts hypothesis in (2.4) and therefore, we have

$$|\arg\{q_1(z)\}| < \pi\beta_{p-1}/2 \quad \text{for } |z| < |1|.$$

This completes the proof. $\square$

Let us go to next step and define the function

$$q_2(z) = \frac{zf^{(p-2)}(z)}{3f^{(p-3)}(z)}, \quad q_2(0) = 1$$

and applying the same method as the above, we have the following theorem.

Theorem 2.2 *Let $f(z) \in \mathcal{A}_p$, $p \geq 2$, $0 < \beta_2 \leq 1$ and suppose that*

$$\left| \arg \left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} \right| < \frac{\pi\beta_p}{2}, \quad z \in \mathbb{D}. \quad (2.8)$$

Then we have

$$\left| \arg \left\{ \frac{zf^{(p-2)}(z)}{f^{(p-3)}(z)} \right\} \right| < \frac{\pi\beta_{p-2}}{2}, \quad z \in \mathbb{D}. \quad (2.9)$$

where β_{p-2} we obtain from β_{p-1} using formula (2.1). Furthermore,

$$\left| \arg \left\{ \frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)} \right\} \right| < \frac{\pi\beta_{p-1}}{2}, \quad z \in \mathbb{D}, \quad (2.10)$$

and where β_{p-1} we obtain from β_p using formula (2.1) too.

Applying the same step as the above and under the hypothesis of Theorem 2.1, we have the following theorem

Theorem 2.3 Let $f(z) \in \mathcal{A}_p$, $p \geq 2$. For given $\beta_1 \in (0, 1]$ there exist $\beta_k \in (0, 1]$, $k = 2, \dots, p$, of the form (2.1) such that for all $z \in \mathbb{D}$, we have

$$\left| \arg \left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} \right| < \frac{\pi\beta_p}{2} \Rightarrow \forall k \in \{1, \dots, p-1\} : \left| \arg \left\{ \frac{zf^{(k)}(z)}{f^{(k-1)}(z)} \right\} \right| < \frac{\pi\beta_k}{2}.$$

Furthermore

$$0 < \beta_1 \leq \beta_2 \leq \dots \leq \beta_{p-1} \leq \beta_p \leq 1.$$

It is easy to see that Theorem 2.3 holds for the case $\beta_p = \beta_{p-1} = \dots = \beta_1 = 1$ and then Theorem 2.3 becomes Lemma 1.1 and in this sense Theorem 2.3 improves Lemma 1.1.

Corollary 2.4 Let $f(z) \in \mathcal{A}_p$, $p \geq 2$. If $\beta_1 \in (0, 1]$ and $\beta_k \in (0, 1]$, $k = 2, \dots, p$ are of the form (2.1), then for all $k = 1, \dots, p-1$ and for all $z \in \mathbb{D}$, we have

$$\left| \arg \left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} \right| < \frac{\pi\beta_k}{2} \Rightarrow \left| \arg \left\{ \frac{zf^{(k)}(z)}{f^{(k-1)}(z)} \right\} \right| < \frac{\pi\beta_k}{2}.$$

Proof For given $\beta_1 \in (0, 1]$ there exist $\beta_k \in (0, 1]$, $k = 2, \dots, p$, of the form (2.1) such that

$$\left| \arg \left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} \right| < \frac{\pi\beta_p}{2} \Rightarrow \forall k \in \{1, \dots, p-1\} : \left| \arg \left\{ \frac{zf^{(k)}(z)}{f^{(k-1)}(z)} \right\} \right| < \frac{\pi\beta_k}{2}.$$

where

$$0 < \beta_1 \leq \beta_2 \leq \dots \leq \beta_{p-1} \leq \beta_p \leq 1. \quad (2.11)$$

Therefore, from (2.11), we have

$$\begin{aligned} \left| \arg \left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} \right| < \frac{\pi\beta_k}{2} &\Rightarrow \left| \arg \left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} \right| < \frac{\pi\beta_p}{2} \\ \Rightarrow \left| \arg \left\{ \frac{zf^{(k)}(z)}{f^{(k-1)}(z)} \right\} \right| &< \frac{\pi\beta_k}{2}. \end{aligned}$$

□

Corollary 2.5 If $f(z) \in \mathcal{A}_p$, $p \geq 2$, then for all $\gamma \in (0, 1]$ and for all $k \in \{1, \dots, p\}$ and for all $s \in \{k, \dots, p-1\}$, and for all $z \in \mathbb{D}$, we have

$$\left| \arg \left\{ \frac{zf^{(s)}(z)}{f^{(s-1)}(z)} \right\} \right| < \frac{\pi\gamma}{2} \Rightarrow \left| \arg \left\{ \frac{zf^{(k)}(z)}{f^{(k-1)}(z)} \right\} \right| < \frac{\pi\gamma}{2}.$$

From the properties of the sequence (2.1) we have following corollary.

Corollary 2.6 Let $f(z) \in \mathcal{A}_p$, $p \geq 2$, $0 < \beta_p \leq 1$ and suppose that

$$\left| \arg \left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} \right| < \frac{\pi\beta_p}{2}, \quad z \in \mathbb{D}.$$

Then we have

$$\begin{aligned} \sup_{z \in \mathbb{D}} \left| \arg \left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} \right| &> \sup_{z \in \mathbb{D}} \left| \arg \left\{ \frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)} \right\} \right| > \cdots > \sup_{z \in \mathbb{D}} \left| \arg \left\{ \frac{zf^{(k)}(z)}{f^{(k-1)}(z)} \right\} \right| \\ &> \sup_{z \in \mathbb{D}} \left| \arg \left\{ \frac{zf^{(k-1)}(z)}{f^{(k-2)}(z)} \right\} \right| > \cdots > \sup_{z \in \mathbb{D}} \left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right|. \end{aligned}$$

Theorem 2.7 Let $\beta = \alpha + (2/\pi) \tan^{-1} \alpha$ and $f(z) \in \mathcal{A}_p$, $p \geq 2$. Suppose also that

$$\left| \arg \left\{ f^{(p)}(z) \right\} \right| < \frac{\pi\beta}{2}, \quad z \in \mathbb{D}. \quad (2.12)$$

Then we have

$$\left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\pi\beta_1}{2}, \quad z \in \mathbb{D}, \quad (2.13)$$

where β_1 is described in (2.1) with $\beta_p = \alpha + \beta$.

Proof If

$$\left| \arg \left\{ \frac{f^{(p-1)}(z)}{z} \right\} \right| < \frac{\pi\alpha}{2} \quad (2.14)$$

in $|z| < |z_0|$ and

$$\arg \left\{ \frac{f^{(p-1)}(z_0)}{z_0} \right\} = \frac{\pi\alpha}{2} \quad \text{or} \quad \arg \left\{ \frac{f^{(p-1)}(z_0)}{z_0} \right\} = -\frac{\pi\alpha}{2}, \quad (2.15)$$

then for the first case in (2.15), from Lemma 1.2, we have

$$\frac{z_0 f^{(p)}(z_0)}{f^{(p-1)}(z_0)} - 1 = ik\alpha$$

for some $k \geq 1$. This gives

$$\begin{aligned} \arg \left\{ f^{(p)}(z_0) \right\} &= \arg \left\{ \frac{f^{(p-1)}(z_0)}{z_0} (ik\alpha + 1) \right\} \\ &= \arg \left\{ \frac{f^{(p-1)}(z_0)}{z_0} \right\} + \arg \{ ik\alpha + 1 \} \\ &\geq \frac{\pi}{2} \{ \alpha + (2/\pi) \tan^{-1} \alpha \} = -\frac{\pi\beta}{2}. \end{aligned}$$

This contradicts hypothesis (2.12). In the second case in (2.15), applying the same method as in the first case, we obtain

$$\arg \left\{ f^{(p)}(z_0) \right\} \leq -\frac{\pi}{2} \{ \alpha + (2/\pi) \tan^{-1} \alpha \} = \frac{\pi\beta}{2}.$$

This also contradicts hypothesis (2.12). So (2.14) holds in the whole unit disc $\mathbb{D}$. From (2.12) and (2.14), we have

$$\begin{aligned} \left| \arg \left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} \right| &= \left| \arg \left\{ f^{(p)}(z) \frac{z}{f^{(p-1)}(z)} \right\} \right| \\ &\leq \left| \arg \left\{ f^{(p)}(z) \right\} \right| + \left| \arg \left\{ \frac{f^{(p-1)}(z)}{z} \right\} \right| \\ &< \frac{\pi(\alpha + \beta)}{2}, \quad z \in \mathbb{D}. \end{aligned}$$

Applying Theorem 2.3, we obtain (2.13). $\square$

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