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A bound on the genus of a curve with Cartier operator of small rank

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Abstract
Ekedahl showed that the genus of a curve in characteristic $p > 0$ with zero Cartier operator is bounded by $p(p-1)/2$. We show the bound $p + p(p-1)/2$ in case the rank of the Cartier operator is 1, improving a result of Re.

Keywords Cartier operator · Non-hyperelliptic curves · Effective divisors

Mathematics Subject Classification 14C20 · 14H05 · 14H55 · 14F10

1 Introduction

In [1] Ekedahl gave a bound for the genus g of an irreducible smooth complete curve over an algebraically closed field of characteristic $p > 0$ with zero Cartier operator: $g \leq p(p-1)/2$. This bound is sharp and was generalized by Re to curves with Cartier operator of given rank [2]. He showed for hyperelliptic curves whose Cartier operator has rank m the bound $g < mp + (p+1)/2$, and for non-hyperelliptic curves

$$g \leq mp + (m+1)p(p-1)/2. \tag{1}$$

He also showed that if the Cartier operator $\mathcal{C}$ is nilpotent and $\mathcal{C}^r = 0$, then

$$g \leq p^r(p^r - 1)/2.$$

In this paper we give a strengthening of the result (1) of Re. One can find other related results in [3,4] and [5].

Theorem 1.1 *Let X be an irreducible smooth complete curve of genus g over an algebraically closed field of characteristic $p > 0$. If the rank of the Cartier operator of X equals 1, we have $g \leq p(p+1)/2$.*

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This is sharp for example for $p = 2$, see [6, Lemma 4.8]. In the case of higher rank we have the following result.

Theorem 1.2 *Let X be an irreducible smooth complete curve of genus g over an algebraically closed field of characteristic $p > 0$. If the rank of the Cartier operator of X equals 2, and if X possesses a point R such that linear system $|pR|$ is base point free, then $g \leq p(p+3)/2$, while if X does not have such a point, one has the bound $g \leq p(4p+1)/3$.*

2 The Cartier operator and linear systems

From now on, by a curve we mean an irreducible smooth complete curve over an algebraically closed field k of characteristic $p > 0$. For a curve X with function field $k(X)$, Cartier [7] defined an operator on rational differential forms with the following properties:

(1) $\mathcal{C}(\omega_1 + \omega_2) = \mathcal{C}(\omega_1) + \mathcal{C}(\omega_2)$,
(2) $\mathcal{C}(f^p \omega) = f\,\mathcal{C}(\omega)$,
(3) $\mathcal{C}(\mathrm{d}f) = 0$,
(4) $\mathcal{C}(\mathrm{d}f/f) = \mathrm{d}f/f$,

where $f \in k(X)$ is non-zero. Moreover, recall that if x is a separating variable of $k(X)$, any $f \in k(X)$ can be written as

$$f = f_0^p + \cdots + f_{p-1}^p x^{p-1}, \text{ with } f_i \in k(X). \tag{2}$$

For a rational differential form $\omega = f\,\mathrm{d}x$ with f as in (2), we have $\mathcal{C}(\omega) = f_{p-1}\,\mathrm{d}x$. In particular $\mathcal{C}^n(f^i\,\mathrm{d}f) = f^{(i+1)/p^n - 1}\,\mathrm{d}f$ if $p^n | i+1$, and $\mathcal{C}^n(f^i\,\mathrm{d}f) = 0$ otherwise. Furthermore, for distinct points Q_1, Q_2 on X, if there is a rational differential form ω that $\mathrm{ord}_{Q_1}(\omega) \geq p$ and $\mathrm{ord}_{Q_2}(\omega) = p-1$, then by property 2) above we have $\mathrm{ord}_{Q_1}(\mathcal{C}(\omega)) \geq 1$ and $\mathrm{ord}_{Q_2}(\mathcal{C}(\omega)) = 0$.

This operator $\mathcal{C}$ induces a map $\mathcal{C}: H^0(X, \Omega_X^1) \to H^0(X, \Omega_X^1)$ which is σ^{-1}-linear, that is, it satisfies properties (1) and (2) above, with σ denoting the Frobenius automorphism of k. We are interested in the relation between the rank of the Cartier operator, defined as $\dim_k \mathcal{C}(H^0(X, \Omega_X^1))$, and the genus g.

Re showed that there is a relation between the rank of Cartier operator and the geometry of linear systems on a curve. We will list some results that we will use and refer for the proof to Re's paper [2]. In the following, X denotes a non-hyperelliptic curve and for D a divisor on X, we will denote by $H^i(D)$ the vector space $H^i(X, \mathcal{O}_X(D))$.

We will say that a statement holds for a general effective divisor of degree n on X if the statement is true for divisors in a nonempty open set of effective divisors of degree n on X. By a general point we mean a general effective divisor of degree 1. We start with a few results of Re.

Proposition 2.1 [2, Prop. 2.2.2] *Let X be a non-hyperelliptic curve with* $\mathrm{rank}(\mathcal{C}) = m$. *Then for a general effective divisor $D = Q_1 + \cdots + Q_{m+1}$ on X with* $\deg D = m+1$, *one has*

$$h^0(pD) = 1 + h^0(pD - Q_{m+1}).$$

This implies for a general divisor D with $\deg D > \mathrm{rank}(\mathcal{C})$, that the linear system $|pD|$ is base point free. As a corollary, we have the following.

Corollary 2.2 [2, Prop. 2.2.3] *If X is a non-hyperelliptic curve with zero Cartier operator, then $h^0(pQ) \geq 2$ for any point Q on X.*

The following lemma gives a way of estimating dimensions of linear systems.

Lemma 2.3 [2, Lemma 2.3.1] *Assume that Q_1 and Q_2 are general points on a non-hyperelliptic curve X and that D is a divisor. Then we have*

$$h^0(pD + pQ_1 + pQ_2) - h^0(pD + pQ_1) \geq h^0(pD + pQ_1) - h^0(pD).$$

We now give a generalization of a result of Re.

Proposition 2.4 *Let D, E, F be effective divisors on a non-hyperelliptic curve X such that*

(1) *$|F|$ is base point free;*
(2) *$D > 0$ and $\mathrm{Supp}(D) \cap \mathrm{Supp}(E) = \emptyset$;*
(3) *There are points $Q_1, \ldots, Q_{m+1} \in \mathrm{Supp}(D)$ and a divisor $F_1 \in |F|$ such that $\mathrm{ord}_{Q_i}(F_1) = 1$ for $1 \leq i \leq m+1$ and $\mathrm{Supp}(D) \cap \mathrm{Supp}(F_1) = \{Q_1, \ldots, Q_{m+1}\}$;*
(4) *For these points Q_i one has $h^0(E + \sum_{i=1}^{m+1} Q_i) = h^0(E)$;*
(5) *Q_i is not a base point of $|D+E+F|$ for $i = 1, \ldots, m+1$ and there exist $s_1, \ldots, s_{m+1} \in H^0(D + E + F)$ such that*

$$\mathrm{ord}_{Q_i}(s_i) = 0, \ \ \mathrm{ord}_{Q_i}(s_j) \geq p, \ \ i \neq j, \ i, j = 1, \ldots, m+1.$$

Then we have

$$h^0(D + E + F) - h^0(E + F) \geq h^0(D + E) - h^0(E) + m + 1.$$

Proof Let $s_{F_1} \in H^0(F)$ with divisor F_1 and $s_D \in H^0(D)$ with divisor D. We have a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(E) & \xrightarrow{\cdot s_D} & H^0(D+E) & \longrightarrow & H^0(\mathcal{O}_D) \\ & & \downarrow{\scriptstyle \cdot s_{F_1}} & & \downarrow{\scriptstyle \cdot s_{F_1}} & & \downarrow{\scriptstyle \cdot s_{F_1}|_D} \\ 0 & \longrightarrow & H^0(E+F) & \xrightarrow{\cdot s_D} & H^0(D+E+F) & \longrightarrow & H^0(\mathcal{O}_D). \end{array}$$

Claim 2.5 *Multiplication by s_{F_1} induces an injective map*

$$H^0(D + E)/s_D \cdot H^0(E) \xrightarrow{s_{F_1}} H^0(D + E + F)/s_D \cdot H^0(E + F).$$

This Claim follows if

$$s_{F_1} \cdot H^0(D + E) \cap s_D \cdot H^0(E + F) = s_{F_1} \cdot s_D \cdot H^0(E).$$

Because of assumptions (2) and (3), the left hand side of this equation is equal to $s_D \cdot s'_{F_1} \cdot H^0(E + \sum_{i=1}^{m+1} Q_i)$, where $s'_{F_1} = s_{F_1}/s_0$ for a section $s_0 \in H^0(\sum_{i=1}^{m+1} Q_i)$ with $\mathrm{div}(s_0) = \sum_{i=1}^{m+1} Q_i$. Then (4) implies $H^0(E + \sum_{i=1}^{m+1} Q_i) = s_0 \cdot H^0(E)$. The Claim follows.

By (5), there exist $s_1, \ldots, s_{m+1}$ such that for all i, j with $i \neq j$ we have $\mathrm{ord}_{Q_i}(s_i) = 0$ and $\mathrm{ord}_{Q_i}(s_j) \geq p$. Now we will show that $s_1, \ldots, s_{m+1}$ generate an $m+1$-dimensional subspace of $H^0(D+E+F)/s_D \cdot H^0(E+F)$ with zero intersection with $\mathrm{Im}(s_{F_1})$. First we will prove the zero intersection part. Assume there exist $c_1, \ldots, c_{m+1} \in k$ such that $\xi = \sum_{i=1}^{m+1} c_i s_i$ lies in $\mathrm{Im}(s_{F_1})$. That means $\xi = s_{F_1} \cdot r + s_D \cdot t$ with some $r \in H^0(D + E)$ and $t \in H^0(E + F)$. If $c_1 \neq 0$ then we obtain $\mathrm{ord}_{Q_1}(\xi) = 0$. However, because $\mathrm{ord}_{Q_1}(F_1) = \mathrm{ord}_{Q_1}(s_{F_1}) = 1$ and $\mathrm{ord}_{Q_1}(s_D) = \mathrm{ord}_{Q_1}(D) \geq 1$, we have $0 = \mathrm{ord}_{Q_1}(\xi) = \mathrm{ord}_{Q_1}(s_{F_1} \cdot r + s_D \cdot t) \geq 1$, a contradiction if $c_1 \neq 0$. Similarly, we can show $c_2 = \cdots = c_{m+1} = 0$. Then for any non-zero element ξ in $< s_1, \ldots, s_{m+1} >$ one has $\xi \notin \mathrm{Im}(s_{F_1})$.

Now for the linear independence of $s_1, \ldots, s_{m+1}$, if $\xi = \sum_{i=1}^{m+1} c_i s_i$ lies in $s_D \cdot H^0(E+F)$, then $\xi = s_D \cdot t$ with $t \in H^0(E+F)$ and we can apply the same argument on the orders of ξ at Q_i as above with $r = 0$. Then we find $c_i = 0$ for $i = 1, \ldots, m+1$. So $s_1, \ldots, s_{m+1}$ are linearly independent in $H^0(D+E+F)/s_D \cdot H^0(E+F)$.

By the injectivity Claim 2.5 above we then have

$$h^0(D+E+F) - h^0(D+E) \geq h^0(E+F) - h^0(E) + m + 1.$$

□

3 Proofs of the Theorems 1.1 and 1.2

Before giving the proofs of theorems, we need several lemmas on the relation between the rank of the Cartier operator and geometrical properties of linear systems on a curve.

Lemma 3.1 *Let X be a non-hyperelliptic curve with* $\mathrm{rank}(\mathcal{C}) = m \geq 1$. *Then there exists points $Q_1, \ldots, Q_m$ on X such that with $D = \sum_{i=1}^m Q_i$ we have*

$$h^0(pD) = 1 + h^0(pD - Q_m).$$

Proof Suppose that $\omega_1, \ldots, \omega_m$ are differentials that generate $\mathrm{Im}(\mathcal{C})$. Assume the lemma is not true, that is, for any m-tuple $\alpha = (Q_1, \ldots, Q_m)$, we have with $D = \sum_{i=1}^m Q_i$ that $h^0(pD) = h^0(pD - Q_m)$. Then by Serre duality and Riemann–Roch, there exists a $\omega_D \in H^0(X, \Omega_X^1)$ that

$$\mathrm{ord}_{Q_i}(\omega_D) \geq p, \ 1 \leq i \leq m-1, \ \mathrm{ord}_{Q_m}(\omega_D) = p - 1. \tag{3}$$

Let $\eta := \mathcal{C}(\omega_D) = \sum_{i=1}^m \lambda_i \omega_i$ with $\lambda_i \in k$. Then one has

$$\mathrm{ord}_{Q_i}(\eta) \geq 1, \ 1 \leq i \leq m-1, \ \mathrm{ord}_{Q_m}(\eta) = 0. \tag{4}$$

Suppose now that $\omega_1, \ldots, \omega_m$ have a common base point R. Then define $Q_m = R$ and choose general points $Q_1, \ldots, Q_{m-1}$ such that $Q_1, \ldots, Q_{m-1}, R$ form m distinct points. Then with $D = \sum_{i=1}^{m-1} Q_i + R$ we have $h^0(pD) = h^0(pD - R)$, hence there exists a ω_D satisfying (3). Then $\eta = \mathcal{C}(\omega_D)$ satisfies (4) and we have $0 = \mathrm{ord}_{Q_m}(\eta) = \mathrm{ord}_{Q_m}(\sum_{i=1}^m \lambda_i \omega_i) \geq 1$, a contradiction.

So we may assume that $\omega_1, \ldots, \omega_m$ have no common base point. Choose a point Q_1 such that ω_1 does not vanish at Q_1, but $\omega_2, \ldots, \omega_m$ vanish at Q_1. More generally, assume furthermore that we have $Q_1, \ldots, Q_n$ such that $\mathrm{ord}_{Q_i}(\omega_i) = 0$ and $\mathrm{ord}_{Q_i}(\omega_j) > 0$ for $i = 1, \ldots, n$ and $i < j \leq m$.

If $\omega_{n+1}, \ldots, \omega_m$ have a base point R different from Q_i for $i = 1, \ldots, n$, then we choose $Q_{n+1}, \ldots, Q_{m-1}$ general distinct points, $Q_m = R$ and let $\alpha = (Q_1, \ldots, Q_m)$. By assumption $h^0(pD) = h^0(pD - Q_m)$ for $D = \sum_{i=1}^m Q_i$, and we find a differential form ω_D satisfying (3) and therefore a form $\eta = \mathcal{C}(\omega_D)$ satisfying (4), again a contradiction.

So we may assume that $\omega_{n+1}, \ldots, \omega_m$ do not have common base points except $Q_1, \ldots, Q_n$. Choose now a point Q_{n+1} different from $Q_1, \ldots, Q_n$ such that ω_{n+1} does not vanish at Q_{n+1}, but $\omega_{n+2}, \ldots, \omega_m$ all vanish at Q_{n+1}. By induction on n, we find points $Q_1, \ldots, Q_{m-1}$ with $\mathrm{ord}_{Q_i}(\omega_i) = 0$ and $\mathrm{ord}_{Q_i}(\omega_j) \geq 1$ for $j > i$ and $j = 2, \ldots, m$.

Now if ω_m has a zero distinct from Q_i for $i = 1, \ldots, m-1$, say Q_m, we let $\alpha = (Q_1, \ldots, Q_m)$ and $D = \sum_{i=1}^m Q_i$. The assumption $h^0(pD) = h^0(pD - Q_m)$ gives us a differential form ω_D and $\eta = \mathcal{C}(\omega_D) = \sum_{i=1}^{m} \lambda_i \omega_i$. By (4) we have $0 = \mathrm{ord}_{Q_m}(\eta) = \mathrm{ord}_Q(\lambda_m \omega_m) \geq 1$, a contradiction. So ω_m has no zeros outside $Q_1, \ldots, Q_{m-1}$.

Now $\deg(\omega_m) = 2g-2 \geq m$ for $g \geq 2$, so ω_m vanishes at one Q_i with multiplicity larger than one, say Q_{m-1}. Then with $D = \sum_{i=1}^{m-2} Q_i + 2Q_{m-1}$ we have $h^0(pD) = h^0(pD - Q_{m-1})$, giving us a differential form ω_D, and $\eta = \mathcal{C}(\omega_D) = \sum_{i=1}^{m} \lambda_i \omega_i$. Then we have $\mathrm{ord}_{Q_i}(\eta) \geq 1$ for $i = 1, \ldots, m-2$ and $\mathrm{ord}_{Q_{m-1}}(\eta) = 1$. However, by the induction assumption

$$\mathrm{ord}_{Q_i}(\omega_i) = 0, \mathrm{ord}_{Q_i}(\omega_j) \geq 1, \quad 1 \leq i < j \leq m-1,$$
$$\mathrm{ord}_{Q_l}(\omega_m) \geq 1, \mathrm{ord}_{Q_{m-1}}(\omega_m) \geq 2, \quad l = 1, 2, \ldots, m-2.$$

So we must have $\lambda_i = 0$ for $i = 1, \ldots, m-1$ and $\mathrm{ord}_{Q_{m-1}}(\eta) \geq 2$, and we therefore find $h^0(pD) = 1 + h^0(pD - Q_m)$. ◻

By putting $m = 1$ in the Lemma 3.1 above, we have the following.

Corollary 3.2 *Let X be a non-hyperelliptic curve. If the Cartier operator has* $\mathrm{rank}(\mathcal{C}) = 1$, *there exists a point R of X such that*

$$h^0(pR) = 1 + h^0((p-1)R).$$

Combining Lemma 3.1 above and Proposition 2.4, we have the following result. We denote the canonical divisor (class) by K_X.

Corollary 3.3 *Let X be a non-hyperelliptic curve with* $\mathrm{rank}(\mathcal{C}) = 1$ *and let* T_n *be a general effective divisor of degree n. Put* $E = pT_n$ *and let R be a point of X with* $h^0(pR) = 1 + h^0((p-1)R)$. *Then the following holds.*

(i) *If* $h^0(K_X - E) \leq 1$, *one has for general points* Q_1, Q_2

$$h^0\left(E + pR + \sum_{i=1}^{2} pQ_i\right) - h^0\left(E + \sum_{i=1}^{2} pQ_i\right) = p.$$

(ii) *If* $h^0(K_X - E) \geq 2$, *one has for general points* Q_1, Q_2

$$h^0\left(E + pR + \sum_{i=1}^{2} pQ_i\right) - h^0\left(E + \sum_{i=1}^{2} pQ_i\right) \geq 2 + h^0(E + pR) - h^0(E).$$

Proof Note that the existence of R is provided by Corollary 3.2 above.

(i) If $h^0(K_X - E) = 0$, i.e. E is non-special, Riemann–Roch implies statement i). If $h^0(K_X - E) = 1$, we choose Q_1 a non-base point of $|K_X - E|$, then $h^0(K_X - E - Q_1) = 0$, hence $h^0(K_X - E - pQ_1) = 0$. Therefore $h^0(K_X - E - \sum_{i=1}^{2} pQ_i) = h^0(K_X - E - \sum_{i=1}^{2} pQ_i - pR) = 0$ and by Riemann–Roch we have $h^0(E + \sum_{i=1}^{2} pQ_i + pR) - h^0(E + \sum_{i=1}^{2} pQ_i) = p$.

(ii) If $h^0(K_X - E) \geq 2$, we write $D = pQ_1 + pQ_2$, $E = pT_n$ and $F = pR$ and we proceed to verify the conditions (1)−(5) of Proposition 2.4 in this case. Conditions (1) and (2) are easy consequences of the generality assumptions of Q_1, Q_2 and R. For condition (3), if the linear system $|pR|$ induces a separable map to projective space, then we can choose Q_1 and Q_2 to be points where the map is smooth and find an effective divisor F_1 such that $\mathrm{ord}_{Q_1}(F_1) = \mathrm{ord}_{Q_2}(F_1) = 1$. If, on the contrary, the map induced by $|pR|$ is inseparable, then $\dim |R| \geq 1$, which is not true for curves of genus larger than zero.

Condition (4) is satisfied once we choose Q_1 to be a non-base point of $|K_X - E|$ and Q_2 a non-base point of $|K_X - E - Q_1|$, since $h^0(K_X - E) \geq 2$. Then we have $h^0(K_X - E - \sum_{i=1}^{2} Q_i) = h^0(K_X - E) - 2$.

Condition (5) holds as $|E + pR + pQ_1 + pQ_2|$ is base point free by Proposition 2.1 if Q_1 and Q_2 are general. Furthermore by Proposition 2.1, we have $h^0(E + pQ_i) = 1 + h^0(E + (p-1)Q_i)$ for $i = 1, 2$. Then we obtain s_1 and s_2 in $H^0(E + pR + pQ_1 + pQ_2) = H^0(D + E + F)$ such that for all i, j we have $\mathrm{ord}_{Q_i}(s_i) = 0$ and $\mathrm{ord}_{Q_i}(s_j) \geq p$ for $j \neq i$.

Then we conclude by Proposition 2.4 above. □

Now we can state some numerical consequences of Corollary 3.3.

Corollary 3.4 *Let X be a non-hyperelliptic curve with* $\mathrm{rank}(\mathcal{C}) = 1$. *Denote by D_n a general divisor of degree n. Then for any integer $n \geq 1$, one has*

(i) $p \geq h^0(pD_{2n}) - h^0(pD_{2n-1}) \geq \min(2n-1, p)$.
(ii) *For* $1 \leq n \leq \lceil (p+1)/2 \rceil$, *one has* $p \geq h^0(pD_{2n-1}) - h^0(pD_{2n-2}) \geq 2n-2$.
(iii) pD_p *is non-special, i.e.* $h^0(K_X - pD_p) = 0$.
(iv) *For* $1 \leq n \leq [(p+1)/2]$, *one has* $h^0(pD_{2n}) - h^0(pD_{2n-2}) \geq 4n-3$.
(v) *For* $1 \leq n \leq [(p+1)/2]$, *one has*

$$h^0(K_X - pD_{2n-2}) - h^0(K_X - pD_{2n}) \leq 2p - 4n + 3.$$

(vi) $h^0(K_X - pD_{p-1}) \leq 1$ *for* $p \geq 3$.

Proof (i) For $n \in \mathbb{Z}_{>0}$, one can always has $p \geq h^0(pD_{2n}) - h^0(pD_{2n-1})$. We will prove the second inequality in (i) by induction on n.

In the case $n = 1$, by Proposition 2.1, for general points Q_1, Q_2 one has

$$h^0(pQ_1 + pQ_2) = 1 + h^0(pQ_1 + (p-1)Q_2),$$

and thus with $D_2 = Q_1 + Q_2$ and $D_1 = Q_1$, we see $h^0(pD_2) \geq 1 + h^0(pD_1)$. Now we do induction and assume $h^0(pD_{2n-2}) - h^0(pD_{2n-3}) \geq 2n-3$. We apply Corollary 3.3 with $E = pD_{2n-3}$ for $n \geq 2$. If $h^0(K_X - E) \leq 2$, then we have $h^0(pD_{2n}) - h^0(pD_{2n-1}) = p$. Otherwise, Corollary 3.3 implies

$$h^0(pD_{2n}) - h^0(pD_{2n-1}) \geq 2 + h^0(pD_{2n-2}) - h^0(pD_{2n-3}) \geq 2n-1.$$

and we are done.

(ii) The case $n = 1$ is trivial. Assuming the assertion for $n-1$, we will prove

$$h^0(pD_{2n-1}) - h^0(pD_{2n-2}) \geq 1 + h^0(pD_{2n-2}) - h^0(pD_{2n-3}), \tag{5}$$

and by (i) the right hand side is at least $2n-2$, which suffices for (ii). To prove the inequality (5), take $D = pQ_1$, $E = pD_{2n-3}$ and $F = pR$ with the point R satisfying $h^0(pR) = 1 + h^0((p-1)R)$ (see Corollary 3.2) and Q_1 a general point. We are going to verify the conditions (1)–(5) of Proposition 2.4 in the case of $m = 0$. Conditions (1) and (2) are obvious by the property of R and generality of Q_1. For condition (3), the map induced by $|pR|$ is separable for curves of genus $g > 0$. We can choose Q_1 to be a point where the map is smooth.

For condition (4) we can choose Q_1 to be a non-base point of $|K_X - E|$ as it is non-empty. For condition (5), as $E = pD_{2n-3}$ with $n \geq 2$, we have for any point Q in $\mathrm{Supp}(E)$, $|pQ + pQ_1|$ is base point free due to Proposition 2.1. Then $|D + E + F|$ is base point free and by Proposition 2.4 we have

$$h^0(pD_{2n-1}) - h^0(pD_{2n-2}) \geq 1 + h^0(pD_{2n-2}) - h^0(pD_{2n-3}) \geq 2n-2.$$

(iii) For p odd, we let $n = (p+1)/2$ and apply (i) get $h^0(pD_{p+1}) - h^0(pD_p) \geq p$. For $p = 2$, we let $n = 2$ and apply (ii) we also get $h^0(pD_{p+1}) - h^0(pD_p) \geq p$. So we have $h^0(K_X - pD_p) = h^0(K_X - pD_{p+1})$. In other words, for a general point Q, we see that pQ lies in the base locus of $|K_X - pD_p|$. This can only happen when $h^0(K_X - pD_p) = 0$. Property (iv) follows by combining (i) and (ii). Property (v) follows by (iv) and Riemann–Roch. For property (vi), by (ii) and (iii), it is known that

$$h^0(pD_p) - h^0(pD_{p-1}) \geq p-1,\ h^0(K_X - pD_p) = 0.$$

We have

$$\begin{aligned} h^0(K_X - pD_{p-1}) &= h^0(pD_{p-1}) - 1 + g - p(p-1) \\ &\leq h^0(pD_p) - 1 + g - p(p-1) - (p-1) \\ &= h^0(K_X - pD_p) + 1 = 1. \end{aligned}$$

□

Using the inequalities in Corollary 3.4, we can easily prove Theorem 1.1.

Proof (Proof of Theorem 1.1) We estimate $g = h^0(K_X)$ by

$$\begin{aligned} h^0(K_X) &= \sum_{n=1}^{\lceil (p-1)/2 \rceil} (h^0(K_X - pD_{2n-2}) - h^0(K_X - pD_{2n})) + h^0(K_X - pD_{2\lceil (p-1)/2 \rceil}) \\ &\leq \sum_{n=1}^{\lceil (p-1)/2 \rceil} (2p - 4n + 3) + h^0(K_X - pD_{2\lceil (p-1)/2 \rceil}) = p(p+1)/2. \end{aligned}$$

□

Our approach to the case $\operatorname{rank}(\mathcal{C}) = 2$ is similar, but there are differences due to the existence of special linear systems. We now give the analogue of Corollary 3.3.

Corollary 3.5 *Let X be a non-hyperelliptic curve with* $\operatorname{rank}(\mathcal{C}) = 2$*, and let T_n be a general effective divisor of degree n and put $E = pT_n$. Let Q_1, Q_2, Q_3 be general points of X and put $D = E + \sum_{i=1}^{3} pQ_i$.*
(1) *Assume there exists R of X such that $h^0(pR) = 1 + h^0((p-1)R)$.*
(a) *If $h^0(K_X - E) \leq 2$, then one has $h^0(D + pR) - h^0(D) = p$.*
(b) *If $h^0(K_X - E) \geq 3$, then one has*

$$h^0(D + pR) - h^0(D) \geq h^0(E + pR) - h^0(E) + 3.$$

(2) *If there does not exist such a point R, we choose points R_1, R_2 satisfying $h^0(\sum_{i=1}^{2} pR_i) = 1 + h^0(\sum_{i=1}^{2} pR_i - R_2)$ and let* $\deg E \geq 2p$.
(a) *If $h^0(K_X - E) \leq 2$, then one has $h^0(D + \sum_{j=1}^{2} pR_j) - h^0(D) = 2p$.*
(b) *If $h^0(K_X - E) \geq 3$, then one has*

$$h^0\left(D + \sum_{i=1}^{2} pR_j\right) - h^0(D) \geq h^0\left(E + \sum_{j=1}^{2} pR_j\right) - h^0(E) + 3.$$

Note that in (2) we can choose such R_1 and R_2 by Lemma 3.1. The proof is similar to the proof of Corollary 3.3. But we point out that in the proof of part (2), instead of using the separable map induced by $|pR|$ in part (ii) of the proof of Corollary 3.3, we consider the map induced by $|pR_1 + pR_2|$ with points R_1 and R_2. This map is separable, otherwise $\dim |R_1 + R_2| \geq 1$, contradicting that X is non-hyperelliptic.

The following two corollaries are the analogues of Corollary 3.4.

Corollary 3.6 *Let X be a non-hyperelliptic curve with* $\mathrm{rank}(\mathcal{C}) = 2$. *Denote by D_n a general divisor of degree n. If there exists a point R of X that $|pR|$ is base point free, then for any integer $n \geq 1$, one has*

(i) $p \geq h^0(pD_{3n}) - h^0(pD_{3n-1}) \geq \min(3n-2, p)$.
(ii) *For* $1 \leq n \leq \lceil (p+2)/3 \rceil$, *one has*

$$2p \geq h^0(pD_{3n-1}) - h^0(pD_{3n-3}) \geq \max(6n-7, 0).$$

(iii) pD_{p+1} *is non-special, i.e.* $h^0(K_X - pD_{p+1}) = 0$.
(iv) *For* $1 \leq n \leq [(p+2)/3]$, *one has* $h^0(pD_{3n}) - h^0(pD_{3n-3}) \geq 9n - 9$.
(v) *For* $1 \leq n \leq [(p+2)/3]$, *one has*

$$h^0(K_X - pD_{3n-3}) - h^0(K_X - pD_{3n}) \leq 3p - 9n + 9.$$

(vi) $h^0(K_X - pD_{3\lceil (p-1)/3 \rceil}) \leq 3$.

Corollary 3.7 *Let X be a non-hyperelliptic curve with* $\mathrm{rank}(\mathcal{C}) = 2$. *Denote by D_n a general divisor of degree n. If X does not possess a point R such that $|pR|$ is base point free, then for any integer $n \geq 1$, one has*

(i) $2p \geq h^0(pD_{3n}) - h^0(pD_{3n-2}) \geq \min(3n-2, 2p)$.
(ii) *For* $2 \leq n \leq \lceil (2p+2)/3 \rceil$, *one has*

$$2p \geq h^0(pD_{3n-2}) - h^0(pD_{3n-3}) \geq 1.$$

(iii) pD_{2p} *is non-special, i.e.* $h^0(K_X - pD_{2p}) = 0$.
(iv) *For* $2 \leq n \leq [(2p+2)/3]$, *one has* $h^0(pD_{3n}) - h^0(pD_{3n-3}) \geq 3n - 1$. *For* $n = 1$, *one has* $h^0(pD_3) - h^0(pD_0) \geq 1$.
(v) *For* $2 \leq n \leq [(2p+2)/3]$, *one has*

$$h^0(K_X - pD_{3n-3}) - h^0(K_X - pD_{3n}) \leq 3p - 3n + 1.$$

For $n = 1$, *one has* $h^0(K_X) - h^0(K_X - pD_3) \leq 3p - 1$.
(vi) $h^0(K_X - pD_{3\lceil (2p-1)/3 \rceil}) \leq p - 1$.

The proofs of two corollaries above are similar to the proof of Corollary 3.4 and therefore we omit these. The corollaries above now readily imply the proof of theorem in the case of $\mathrm{rank}(\mathcal{C}) = 2$.

Proof (Proof of Theorem 1.2) (1) If $|pR|$ is base point free, then by Corollary 3.6 we have

$$\begin{aligned} h^0(K_X) &\leq \sum_{n=1}^{\lceil (p-1)/3 \rceil} (h^0(K_X - pD_{3n-3}) - h^0(K_X - pD_{3n})) + h^0(K_X - pD_{3\lceil (p-1)/3 \rceil}) \\ &\leq \sum_{n=2}^{\lceil (p-1)/3 \rceil} (3p - 9n + 9) + 1 + 3 = p(p+3)/2. \end{aligned}$$

(2) Otherwise, by Corollary 3.7 we have

$$h^0(K_X) \leq \sum_{n=1}^{\lceil (2p-1)/3 \rceil} (h^0(K_X - pD_{3n-3}) - h^0(K_X - pD_{3n})) + h^0(K_X - pD_{3\lceil (2p-1)/3 \rceil})$$

$$\leq \sum_{n=2}^{\lceil (2p-1)/3 \rceil} (3p - 3n + 1) + 3p - 1 + p - 1 = p(4p + 1)/3.$$

□

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