Singular ideals of skew Hurwitz polynomial rings

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Abstract

For a ring R and an endomorphism α of R , we provide a full description of left and right singular ideals of the skew Hurwitz polynomial ring (hR, α) . We obtain that if α is an automorphism of R , then R is right (resp., left) nonsingular if and only if (hR, α) is right (resp., left) nonsingular. We give an example of a ring R and an endomorphism α of R such that the skew Hurwitz polynomial ring (hR, α) is left nonsingular, but not right nonsingular.

Keywords Singular ideal · Nonsingular ring · Skew Hurwitz polynomial ring

Mathematics Subject Classification 16D25 · 16D80

Rings of formal power series have been of interest and have had important applications in numerous areas, one of which has been differential algebra. In an earlier paper by Keigher [4], a variant of the ring of formal power series was considered, and some of its properties, particularly its categorical properties, were studied. In the papers [5,6], Keigher showed that the ring of Hurwitz series has many interesting applications in differential algebra.

Throughout this article, R denotes an associative ring with unity and $\alpha : R \rightarrow R$ is an endomorphism. The ring $(H(R), \alpha)$, or simply (HR, α) , of the skew Hurwitz series over a ring R is defined as follows: the elements of (HR, α) are functions $f : \mathbb{N} \rightarrow R$, where $\mathbb{N}$ is the set of all natural numbers. Let $\text{supp}(f)$ denote the support of $f \in (HR, \alpha)$, i.e. $\text{supp}(f) = \{i \in \mathbb{N} \mid 0 \neq f(i) \in R\}$, $\Pi(f)$ denote the minimal element in $\text{supp}(f)$ and $\Delta(f)$ denote the greatest element in $\text{supp}(f)$ if it exists. The operation of addition in (HR, α) is componentwise and the operation of multiplication for each $f, g \in (HR, \alpha)$ is defined by:

$$(fg)(n) = \sum_{k=0}^n \binom{n}{k} f(k) \alpha^k(g(n-k)) \quad \text{for all } n \in \mathbb{N},$$

where $\binom{n}{k}$ is the binomial coefficient.

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Define the mappings $h_n : \mathbb{N} \rightarrow R$, $n \geq 1$ via $h_n(n-1) = 1$ and $h_n(m) = 0$ for each $(n-1) \neq m \in \mathbb{N}$ and $h'_r : \mathbb{N} \rightarrow R$ via $h'_r(0) = r$ and $h'_r(n) = 0$ for each $0 \neq n \in \mathbb{N}$ and $r \in R$. It can be easily shown that (HR, α) is a ring with identity h_1 .

For a ring R , the subring R' of (HR, α) defined by $\{h'_r \mid r \in R\}$, we have $R \cong R'$. For any subset A of R define $A' = \{h'_r \mid r \in A\}$. If A is an ideal of R , then A' is an ideal of R' .

The ring (hR, α) of skew Hurwitz polynomials over a ring R is the subring of (HR, α) that consists elements of the form $f \in (HR, \alpha)$ with $\Delta(f) < \infty$ (see [1].)

A right ideal of R is (right) *essential* if it has nontrivial intersection with every non-zero right ideal. the *right singular ideal* $\mathcal{Z}_r(R)$ of R is defined to be the set of all elements of R whose right annihilators are essential right ideals. Equivalently

$$\mathcal{Z}_r(R) = \{x \in R \mid \forall y (\neq 0) \in R, \exists z \in R \text{ such that } yz \neq 0, xyz = 0\}.$$

The left singular ideal is defined similarly; it will be denoted by $\mathcal{Z}_\ell(R)$. $\mathcal{Z}_r(R)$ and $\mathcal{Z}_\ell(R)$ are two-sided ideals of R . The singular ideals were introduced by Johnson [3], and have since become important objects of study in ring theory, for example in obtaining criteria for the existence of classical quotient rings, and in investigating the structure of maximal quotient rings, see [2, 7].

A ring R is said to be a *right nonsingular* ring if $\mathcal{Z}_r(R) = 0$. Left nonsingular rings are defined similarly. Right nonsingular rings include all reduced rings, all group rings FG over formally real fields F , and all right semihereditary rings (in particular all von Neumann regular rings).

The aim of this note is to characterize the left and right singular ideal of the skew Hurwitz polynomial rings. It is not a surprise that the obtained characterization is not left-right symmetric. It shows up, in particular, that the right (resp., left) singular ideal of a skew Hurwitz polynomial ring is an extension of an ideal of the coefficient ring. If α is an automorphism of R , then $\mathcal{Z}_\ell(hR, \alpha) = (h(\mathcal{Z}_\ell(R)), \alpha)$ and $\mathcal{Z}_r(hR, \alpha) = (h(\mathcal{Z}_r(R)), \alpha)$. We deduce that, for a ring R with an automorphism α , R is right (resp., left) nonsingular if and only if (hR, α) is right (resp., left) nonsingular.

For a nonempty subset X of R , $r_R(X)$ and $\ell_R(X)$ denote the right and left annihilators of X in R respectively. For a nonzero polynomial $f \in (hR, \alpha)$, we define the length $le(f)$ equal to $\Delta(f) - \Pi(f)$ and set $le(0) = 0$. Moreover, we define the degree $deg(f)$ equal to $\Delta(f)$ and set $deg(0) = \infty$.

For an endomorphism α of R and each $i \geq 0$ let us define

$$T_i = \left\{ a \in R \mid \forall r (\neq 0) \in R, \forall m \geq 0, \exists b (\neq 0) \in r\alpha^m(R) \right. \\ \left. \text{such that } \binom{i+m}{i} a\alpha^i(b) = 0 \right\}.$$

When α is a monomorphism, then the above condition can be replaced by $r_R(a) \cap \binom{i+m}{i} \alpha^i(r) \alpha^{i+m}(R) \neq 0$. Notice also that always $T_0 \subseteq \mathcal{Z}_r(R)$. Keeping the above notation, we have the following.

Lemma 1 *If α is an automorphism of R , then for each $i \geq 0$, $\mathcal{Z}_\ell(R) = T_i$.*

Proof Let $a \in T_i$, $i \geq 0$ and $0 \neq r \in R$. Then there exists $c \in R$ such that $\binom{i}{i} a\alpha^i(\alpha^{-i}(r))\alpha^i(\alpha^{-i}(c)) = 0$. So $rca = 0$ and that $a \in \mathcal{Z}_r(R)$.

Now, assume that $a \in \mathcal{Z}_r(R)$, $0 \neq r \in R$ and $u \geq 0$. Then there exists $0 \neq \alpha^{i+u}(c) \in R$ such that $a\alpha^i(r)\alpha^{i+u}(c) = 0$, so $\binom{i+u}{i} a\alpha^i(r)\alpha^{i+u}(c) = 0$. Therefore $a \in T_i$. $\square$

Theorem 2 *For any endomorphism α of R , $\mathcal{Z}_r(hR, \alpha) = \sum_{i=0}^{\infty} T'_i h_{i+1}$.*

Proof Let $a \in T_i$, for some $i \geq 0$ and J be a nonzero right ideal of (hR, α) . Choose a polynomial $0 \neq g \in J$ such $h'_a h_{i+1} g$ is of minimal possible degree. We claim that $h'_a h_{i+1} g = 0$. For the contrary, assume that $h'_a h_{i+1} g \neq 0$, say $\Delta(h'_a h_{i+1} g) = i + n$, then

$$(h'_a h_{i+1} g)(i + n) = \sum_{k=0}^{i+n} \binom{i+n}{k} (h'_a h_{i+1})(k) \alpha^k(g(i+n-k)) = \binom{i+n}{i} a \alpha^i(g(n)) \neq 0.$$

Since $a \in T_i$, there exists $r \in R$ such that

$$g(n) \alpha^n(r) \neq 0 \text{ and } \binom{i+n}{i} a \alpha^i(g(n)) \alpha^{i+n}(r) = 0.$$

The polynomial $gh'_r \in J$ is nonzero, since $(gh'_r)(n) = g(n) \alpha^n(r) \neq 0$. Moreover $\deg(h'_a h_{i+1} g h'_r) < \deg(h'_a h_{i+1} g)$, since

$$\begin{aligned} (h_a h_{i+1} g h'_r)(i + n) &= \sum_{k=0}^{i+n} \binom{i+n}{k} (h'_a h_{i+1} g)(k) \alpha^k(h'_r(i+n-k)) \\ &= \binom{i+n}{i} a \alpha^i(g(n)) \alpha^{i+n}(r) = 0. \end{aligned}$$

This contradicts the choice of g and shows that $h'_a h_{i+1} g = 0$, i.e. $r(hR, \alpha)(h'_a h_{i+1}) \cap J \neq 0$. Therefore $T'_i h_{i+1} \subseteq \mathcal{Z}_r(hR, \alpha)$ and so $\sum_{i=0}^{\infty} T'_i h_{i+1} \subseteq \mathcal{Z}_r(hR, \alpha)$ follows.

Assume that $\mathcal{Z}_r(hR, \alpha) \setminus \sum_{i=0}^{\infty} T'_i h_{i+1} \neq \emptyset$. Now choose $f \in \mathcal{Z}_r(hR, \alpha) \setminus \sum_{i=0}^{\infty} T'_i h_{i+1}$ of minimal length, say $\ell(f) = n - m$ where $n \geq m \geq 0$. We claim that $f(m) \in T_m$. For doing so, let $0 \neq r \in R$ and $k \geq 0$. Consider

$$J = \left\{ \sum_{i=k}^{\infty} h'_{b_i} h_{i+1} \mid b_i \in r \alpha^k(R), b_i = 0 \text{ for all but finite number of } i \text{'s} \right\}.$$

It is easy to see that J is a right ideal of (hR, α) . Thus, since $f \in \mathcal{Z}_r(hR, \alpha)$, there exists $0 \neq g \in J$ such that $fg = 0$. Looking at the term of the smallest degree in this relation we obtain $0 = h'_{f(m)} h_{m+1} h'_b h_{t+1} = \binom{m+t}{m} f(m) \alpha^m(b)$ where $0 \neq b \in r \alpha^k(R)$ and $\Pi(g) = t$. This shows that $f(m) \in T_m$. Hence, by the first part, $h'_{f(m)} h_{m+1} \in \mathcal{Z}_r(hR, \alpha)$ and $f - h'_{f(m)} h_{m+1} \in \mathcal{Z}_r(hR, \alpha)$ is of shorter length than f . This yields $f \in \sum_{i=0}^{\infty} T'_i h_{i+1}$, contradicting the assumption, and shows that $\mathcal{Z}_r(hR, \alpha) = \sum_{i=0}^{\infty} T'_i h_{i+1}$. $\square$

The above theorem gives us the following.

Corollary 3 For any $i \geq 0$, T_i is a left ideal of R such that $T_i + \alpha(T_i) \subseteq T_{i+1}$.

As a consequence of Lemma 1 and Theorem 2, we obtain the following corollaries.

Corollary 4 For any automorphism α of R , $\mathcal{Z}_r(hR, \alpha) = (h(\mathcal{Z}_r(R)), \alpha)$.

Corollary 5 Let R be a ring and α be an automorphism of R . Then R is right nonsingular if and only if (hR, α) is right nonsingular.

For an endomorphism α of R , we defined

$$\begin{aligned} A = \{ a \in R \mid \forall r (\neq 0) \in R, \forall u \in \mathbb{Z}, \exists v \in \mathbb{Z} \\ \text{such that } \ell_R(\alpha^{u+v}(a)) \cap R \binom{u+v}{v} \alpha^v(r) \neq 0 \}. \end{aligned}$$

Lemma 6 If α is an automorphism of R , then $\mathcal{Z}_\ell(R) = A$.

Proof Let $a \in A$ and $0 \neq r \in R$. Then there exist $v \geq 0$ and $c \in R$ such that $\binom{v}{0}c\alpha^v(r)\alpha^v(a) = 0$. So $\alpha^v(\alpha^{-v}(c)ra) = 0$ and that $\alpha^{-v}(c)ra = 0$. Hence $a \in \mathcal{Z}_\ell(R)$.

Now, assume that $a \in \mathcal{Z}_\ell(R)$, $0 \neq r \in R$ and $u \geq 0$. Then there exists $0 \neq \alpha^{-u}(c) \in R$ such that $\alpha^{-u}(c)\alpha^{-u}(r)a = 0$, so $\binom{u}{0}cr\alpha^u(a) = 0$. Therefore $a \in A$. $\square$

Theorem 7 For any endomorphism α of R , $\mathcal{Z}_\ell(hR, \alpha) = (hA, \alpha)$.

Proof Let $a \in A$ and J be a nonzero left ideal of (hR, α) . Among nonzero polynomials f from J choose one, such that $le(fh'_a)$ is minimal. Assume that $le(fh'_a) > 0$, and $\Delta(fh'_a) = m$, then $(fh'_a)(m) = f(m)\alpha^m(a) \neq 0$. Since $a \in A$, there are $v \in \mathbb{Z}$ and $c \in R$ such that $0 \neq \binom{m+v}{v}c\alpha^v(f(m)) \in \ell_R(\alpha^{m+v}(a))$. Since J is a left ideal of (hR, α) , $g = h'_c h_{v+1}f \in J$. Note that

$$g(m+v) = \sum_{k=0}^{m+v} \binom{m+v}{k} (h'_c h_{v+1})(k) \alpha^k(f(m+v-k)) = \binom{m+v}{v} c \alpha^v(f(m)) \neq 0,$$

so $g \neq 0$. Moreover, because

$$(gh'_a)(m) = \sum_{k=0}^m \binom{m}{k} g(k) \alpha^k(h'_a(m-k)) = g(m) \alpha^m(a) = c \alpha^v(f(m)) \alpha^{m+v}(a) = 0,$$

$le(gh'_a) < le(fh'_a)$. This contradicts the choice of f and yields that $le(fh'_a) = 0$, i.e. $fh'_a = 0$. Hence $0 \neq f \in J \cap \ell_{(hR, \alpha)}(h'_a)$ and $A \subseteq \mathcal{Z}_\ell(hR, \alpha)$. Since for each $n \in \mathbb{N}$, $\mathcal{Z}_\ell(hR, \alpha)h_n \subseteq \mathcal{Z}_\ell(hR, \alpha)$, $(hA, \alpha) \subseteq \mathcal{Z}_\ell(hR, \alpha)$ follows.

Suppose that $\mathcal{Z}_\ell(hR, \alpha) \setminus (hA, \alpha) = \emptyset$, and take $f \in \mathcal{Z}_\ell(hR, \alpha) \setminus (hA, \alpha)$ of minimal length with $\Pi(f) = k$. Let $r \in R$ and $u \in \mathbb{Z}$. Consider

$$J = R'h'_r h_1 + R'h'_{\alpha(r)} h_2 + R'h'_{\alpha^2(r)} h_3 + \cdots.$$

It is easy to see that J is a nonzero left ideal of (hR, α) . Thus, since $h_{u+1}f \in \mathcal{Z}_\ell(hR, \alpha)$, there exists $0 \neq g \in J$ such that $gh_{u+1}f = 0$. Assume $\Pi(g) = v$, then we have $h'_{g(v)} h_{v+1} h_{u+1} h'_{f(k)} h_{k+1} = 0$, where $h'_{g(v)} \in R'h'_{\alpha^v(r)}$. Then we obtain

$$\binom{u+v+k}{u+v} \binom{u+v}{v} g(v) \alpha^{u+v}(f(k)) = 0,$$

where $\binom{u+v+k}{u+v} \binom{u+v}{v} g(v) \in R \binom{u+v}{v} \alpha^v(r)$. Then $f(k) \in A \subseteq \mathcal{Z}_\ell(hR, \alpha)$. Hence the polynomial $f - h'_{f(k)} h_{k+1} \in \mathcal{Z}_\ell(hR, \alpha)$ is of shorter length than f , so $f - h'_{f(k)} h_{k+1} \in (hA, \alpha)$. Consequently, $f \in (hA, \alpha)$, this contradicts the choice of f and so $\mathcal{Z}_\ell(hR, \alpha) = (hA, \alpha)$ follows. $\square$

As a consequence of Lemma 6 and Theorem 7, we obtain the following corollaries.

Corollary 8 For any automorphism α of R , $\mathcal{Z}_\ell(hR, \alpha) = (h(\mathcal{Z}_\ell(R)), \alpha)$.

Corollary 9 Let R be a ring and α be an automorphism of R . Then R is left nonsingular if and only if (hR, α) is left nonsingular.

In the following example we present a ring R such that (hR, α) is left nonsingular but $\mathcal{Z}_r(hR, \alpha) \neq 0$.

Example 10 Let F be a field and $R = F\langle x_0, x_1, \dots \mid x_k x_\ell = 0 \text{ for all } k \leq \ell \rangle$. Let α denote the F -endomorphism of R given by $\alpha(x_k) = x_{k+1}$ for all $k \geq 0$. Then:

- (1) $\mathcal{Z}_r(hR, \alpha) = \sum_{k=0}^{\infty} I'_k h_{k+1}$, where $I_k = Rx_0 + Rx_1 + \dots + Rx_k$ for all $k \geq 0$;
- (2) $\mathcal{Z}_r(R) = Rx_0$;
- (3) $\mathcal{Z}_\ell(hR, \alpha) = 0$ and $\mathcal{Z}_\ell(R) = 0$.

Proof (1) In view of Theorem 2, it is enough to show that $I_k = T_k$ for each $k \geq 0$. Let $J = r_R(x_0)$. Notice that $J = x_0 R + x_1 R + \dots$. Let $0 \neq r \in R$, if $r \in J$ then clearly $r \in J \cap r\alpha^m(R) \neq 0$, for each $m \geq 0$. If $r \notin J$, then $r = k + w$ for some $k \in F$ and $w \in J$ and for each $m \geq 0$, $0 \neq rx_m \in r\alpha^m(R) \cap J$. This shows that $x_0 \in T_0$. Now, Corollary 3 yields that $I_k = Rx_0 + \dots + Rx_k \subseteq T_k$.

Let $a \in R \setminus I_k$. Then $ax_k \neq 0$ as otherwise all monomials appearing in a would end with some x_i 's where $i \leq k$. Hence, as $x_0 R = Fx_0$, $r_R(a) \cap \alpha^k(x_0 R) = r_R(a) \cap Fx_k = 0$. Therefore $a \notin T_k$. This shows that $T_k \subseteq I_k$ and completes the proof of (1).

(2) As we remarked earlier, $Rx_0 = T_0 \subseteq \mathcal{Z}_r(R)$. Let $a \in R \setminus Rx_0$. Then, since $x_0 R = Fx_0$, it is clear that $r_R(a) \cap x_0 R = 0$ and $\mathcal{Z}_r(R) = Rx_0$ follows.

(3) In view of Theorem 7, it is enough to show that $A = 0$. Let $a \in R \setminus F$ and $n \geq 0$ be the biggest index such that x_n appears in some monomial from a . Then for each $v \geq 0$, $\ell_R(\alpha^v(a)) \subseteq \sum_{i \leq n+v} Rx_i$ and $R\alpha^v(x_{n+1}) = Rx_{n+v+1}$. Therefore $\ell_R(\alpha^v(a)) \cap R\alpha^v(x_{n+1}) \subseteq \sum_{i \leq n+v} Rx_i \cap Rx_{n+v+1} = 0$. This shows that $a \notin A$ and $A = 0$ follows.

Let $a \in R \setminus F$ and k denote the biggest index such that x_k appears in some monomial from a . Then $\ell_R(a) \cap Rx_n = 0$ for each $n > k$. Hence $\mathcal{Z}_\ell(R) = 0$. $\square$

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