Hopf bifurcations in plasma layers between rigid planes in thermal magnetohydrodynamics, via a simple formula

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Abstract

The phenomenon produced by the Hopf bifurcations is of notable importance. In fact, a Hopf bifurcation—guaranteeing the existence of an unsteady periodic solution of the linearized problem at stake—is also an optimum limit cycle candidate of the nonlinear associated problem and, if non linearly globally attractive, is an absorbing set and an effective limit cycle. The present paper deals with the onset of Hopf bifurcations in thermal magnetohydrodynamics (MHD). Precisely, it is devoted to characterization—via a simple formula—of the Hopf bifurcations threshold in horizontal plasma layers between rigid planes, heated from below and embedded in a constant transverse magnetic field. This problem, remarked clearly and notably by the Nobel Laureate Chandrasekhar (Nature 175:417–419, 1955), constitutes a difficulty met by him and—for plasma layers between rigid planes electricity perfectly conducting—is, as far as we know, still not removed. Let m_0 be the thermal conduction rest state and let P_r, P_m, R, Q , be the Prandtl, the Prandtl magnetic, the Rayleigh and the Chandrasekhar number, respectively. Recognized (according to Chandrasekhar) that the instability of m_0 via Hopf bifurcation can occur only in a plasma with $P_m > P_r$, in this paper it is shown that the Hopf bifurcations occur if and only if

$$Q > Q_c = \frac{4\pi^2[1 + P_r(\mu/2\pi)^4]}{P_m - P_r},$$

with $\mu = 7.8532$. Moreover, the critical value of R at which the Hopf bifurcation occurs is characterized via the smallest zero of the second invariant of the spectrum equation governing the most destabilizing perturbation. The critical value of Q , in the free-rigid and rigid-free cases is shown to be $\frac{1}{4}$ of the previous value.

Keywords Thermal MHD · Instability · Hopf bifurcations · Plasma layers between rigid planes

1 Introduction

Let L be a horizontal layer heated from below, filled by a plasma and embedded in a transverse uniform magnetic field and let m_0 denote the thermal conduction in L : the temperature field arising when the fluid is in rest and L is heated from below by a constant transverse gradient of temperature. The linear stability of m_0 —in the scheme of the nonrelativistic thermal MHD—has been analyzed by Chandrasekhar in the early of 1950 (Chandrasekhar 1952, 1954) and it appeared in 1961 in the celebrated monograph (Chandrasekhar 1981). The Chandrasekhar results—for any type of boundaries (rigid–rigid, rigid–free, free–rigid, free–free)—have been since then a basic paradigm for all the subsequent researchers. In particular, he obtained a relevant inhibition of convection (stabilizing effect) by a magnetic

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field, verified successively experimentally (Nakagawa 1955). As concerns the occurring of the Hopf bifurcations at the onset of m_0 instability, he restricted himself mostly to the case when the fluid is confined between two free planes, showing that bifurcations can occur only if $P_m > P_r$ and envisaging the existence of a bound $Q_c(P_r, P_m) > 0$ such that if and only if

$$P_m > P_r, \quad Q > Q_c$$

Hopf bifurcations occur. He wrote: “there is no simple formula which gives Q_c as function of P_r and P_m ” (see Chandrasekhar 1981, p. 184). Recently, Rionero (2019a) has shown that in the free–free case, this difficulty can be removed and that the Hopf bifurcations occur if and only if

$$P_m > P_r, \quad Q > Q_c = \frac{1 + P_r}{P_m - P_r} \pi^2. \quad (1)$$

Let

$$\mathcal{P}(\lambda) = \lambda^n + A_1 \lambda^{n-1} + \dots + A_{n-1} \lambda + A_n = 0$$

be the spectrum equation governing the linear stability of a steady state of a dynamical system and let the coefficients depend on a positive bifurcation parameter R and on the wave number a^2 of the perturbations in such a way that

$$A_k = 0 \Leftrightarrow R = \mathcal{F}_k(a^2), \quad k \in \{1, 2, 3, \dots, n\} \quad (2)$$

with $\mathcal{F}_k(a^2)$ differentiable function of $a^2 \in \mathbb{R}^+$. Taking into account that the instability can occur only via a zero eigenvalue ($\lambda = 0 \Leftrightarrow A_n = 0$), or via a couple of pure imaginary eigenvalues ($\lambda_{1,2} = \pm i\omega$)—with i imaginary unit and $\omega = \text{const.} \in \mathbb{R}^+$ such that $\mathcal{P}(i\omega) = 0$ —the ideas that lead Rionero to the solution of the problem can be summarized as follows:

1. the application of the critical numbers

$$R_{c_k} = \mathcal{F}_k(a_{c_k}^2) = \min_{a^2 \in \mathbb{R}^+} \mathcal{F}_k(a^2) \quad (3)$$

introduced by him, for $k < n$, about 10 years ago (Rionero 2012, 2013, 2019a, b, 2020; Flavin and Rionero 1996);

2. at the growing of R , from a stable state, the Hopf bifurcations occur if and only if exists at least a $k < n$ such that $R_{c_k} < R_{c_n}$, i.e. A_k becomes zero before A_n : this is because—as it is well known— $A_k > 0, \forall k \in \{1, 2, \dots, n\}$, is necessary for the linear stability, i.e. for letting all the eigenvalues have negative real part;

3. if the coefficients depend also on other parameters Q, P_m, P_r , then one has $R_{c_k} = R_{c_k}(Q, P_m, P_r)$ and the Hopf bifurcations can occur if and only if the inequality

$$R_{c_k}(Q, P_m, P_r) < R_{c_n}(Q, P_m, P_r),$$

for at least a $k < n$, holds.

The free–free layers—of relevant interest in astrophysics applications—do not have the same interest in terrestrial—geophysics and industrial—applications when rigid boundary planes occur. In the present paper, aimed to remove the difficulty remarked by Chandrasekhar, in the presence of rigid boundary, we return to look for the Hopf bifurcations threshold in plasma layers between rigid planes, electricity perfectly conducting, via the guidelines (1)–(3).

In the case, at stake, the spectrum equation is

$$\mathcal{P}(\lambda) = \lambda^3 - \mathcal{I}_1 \lambda^2 + \mathcal{I}_2 \lambda - \mathcal{I}_3 = 0, \quad (4)$$

with the coefficients such that

$$\mathcal{I}_k = \mathcal{I}_k(a^2, P_m, P_r, R, Q) \Leftrightarrow R = \mathcal{F}_k(a^2, P_m, P_r, Q), \quad k = 1, 2, 3 \quad (5)$$

with $\mathcal{F}_k$ differentiable function of a^2, P_m, P_r, Q . Choosing—as it is standard in the convection problems— R as bifurcation parameter, via (1)–(3) the properties (1)–(3) will be proven in the rest of the paper, with A_2 and A_3 given in (51) and (52).

Property 1 *Let m_0 be linearly stable, for any wave number, at $R = 0$. Then, if and only if*

$$R_{c_2}(Q, P_m, P_r) < R_{c_3}(Q, P_m, P_r), \quad (6)$$

an Hopf bifurcation occurs and occurs at $R_ \in]0, R_{c_2}[$ lowest root of*

$$\mathcal{I}_1 \mathcal{I}_2 - \mathcal{I}_3 = 0, \quad (7)$$

with $a^2 = a_{c_2}^2$ and has the frequency

$$\omega(Q, P_m, P_r) = (\sqrt{\mathcal{I}_2})_{(R=R_*)}^{(a^2=a_{c_2}^2)}. \quad (8)$$

It remains to satisfy (6). In other words, since P_m and P_r are structural parameters characterizing the fluid, the question is: in which fluids the Hopf bifurcations occur and for which values of Q ? The answer is given by the following property.

Property 2 (Main result) *If and only if*

$$P_m > P_r, \quad Q \geq Q_c = \frac{4\pi^2 [1 + P_r(\mu/2\pi)^4]}{P_m - P_r}, \quad \mu = 7.8532 \quad (9)$$

$R_{c_2} < R_{c_3}$ holds.

Property 2 reduces the linear stability of m_0 to a standard type of stability in convection problem. In fact, setting

$$\xi = a^2 + 4\pi^2, \quad \gamma = a^2 + 8a^2\pi^2 + \mu^4, \quad (10)$$

one has

Property 3 *The steady state m_0 is linearly stable if and only if $P_m < P_r$ and*

$$R < R_{c_2} = \min_{a^2 \in \mathbb{R}^+} \frac{\xi^4 + [4\pi^2 Q P_r + \gamma(P_m + P_r)]}{a^2}, \quad Q \geq Q_c, \quad (11)$$

or

$$R < R_{c_3} = \min_{a^2 \in \mathbb{R}^+} \frac{(\gamma + 4\pi^2 Q)\xi}{a^2}, \quad P_m < P_r, \quad \forall Q \quad (12)$$

or

$$R < R_{c_3}, \quad P_m > P_r, \quad Q < Q_c. \quad (13)$$

The plan of the paper is the following. Section 2 is devoted to preliminaries. After having recalled the m_0 thermal conduction rest state, the nonlinear MHD equations governing the perturbations to m_0 are introduced. In Sect. 3, the linear stability of m_0 is introduced and it is shown that it depends on the behaviour of the essential variables w, h, θ with w, h vertical perturbations to velocity and magnetic fields, respectively, and θ perturbation to the thermal field. The boundary conditions for w, h, θ , in the presence of rigid perfect conductor boundary planes, are considered in Sect. 4. In Sect. 5, the functional spaces of w, h, θ —admissible by the boundary conditions—are introduced and the Sturm–Liouville type problem associated with w is taken into account: w can be expanded either in a Fourier series of even or odd functions. In Sect. 6, it is shown that m_0 is linearly stable with respect to the perturbations (w, h, θ) with w even function. Successively, in Sect. 7, w odd is taken into account and the spectrum equation for the n th ($n \in \mathbb{N}$), Fourier components of the perturbations, is obtained. Property 1 is obtained in Sect. 8, while the other two are obtained in the subsequent section. The onset of Hopf bifurcations in the free-rigid and rigid-free cases is studied in Sect. 10. Final remarks are given in Sect. 11. The paper ends with an Appendix (Sect. 12) with three subsections concerning the invariants

of 3×3 matrices, the Hurwitz criterion and the linear and nonlinear energy decay.

2 Preliminaries

Misregarding the displacement currents, the equations governing the thermal MHD under the validity of the state equation for the density ρ ,

$$\rho = \rho_0 [1 - \alpha(T - T_0)],$$

are (see Chandrasekhar 1981, chapter 5)

$$\begin{cases} \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla \frac{p}{\rho_0} + [1 - \alpha(T - T_0)]\mathbf{g} + \frac{\mu}{\rho_0} \nabla \mathbf{H} \times \mathbf{H} + \nu \Delta \mathbf{v}, \\ \frac{\partial \mathbf{H}}{\partial t} + \nabla \times (\mathbf{H} \times \mathbf{v}) = \eta \Delta \mathbf{H}, \\ \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = k \Delta T, \\ \nabla \cdot \mathbf{H} = \nabla \cdot \mathbf{v} = 0, \end{cases}$$

with

- $\mathbf{v}$ velocity field
- $\mathbf{H}$ magnetic field
- T temperature field
- T_0 assigned reference temperature
- ν kinematic viscosity
- k thermal diffusivity coefficient
- η resistivity
- μ magnetic permeability
- g gravitational acceleration
- α coefficient of volume expansion
- p pressure field.

Let L be an infinite horizontal plasma layer, permeated by an imposed uniform magnetic field $\mathbf{H}_0$ normal to the layer, under the action of a vertical gravity field $\mathbf{g} = -g\mathbf{k}$, and in which a constant adverse temperature gradient β is maintained. Let $d > 0$, $\Omega_d = \mathbb{R}^2 \times (0, d)$ and $Oxyz$ be a Cartesian frame of reference with unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$, respectively. We assume that the plasma is confined between the planes $z = 0$ and $z = d$, with assigned temperatures $\tilde{T}(x, y, 0) = \tilde{T}_0$, $\tilde{T}(x, y, d) = -\beta d + \tilde{T}_0$. Here we consider the rest state $m_0 = (\tilde{\mathbf{v}}, \tilde{\mathbf{H}}, \tilde{T}, \tilde{p}) = (0, H_0\mathbf{k}, -\beta z + \tilde{T}_0, \tilde{p})$ (thermal conduction), which is a solution to the stationary previous equations. In the sequel, $\mathbf{x} = (x, y, z)$ will denote the coordinates in the unity of length d and, in view of the symmetry with respect to the two bounding planes, the origin of z is fixed at

the midway between the two planes and one has then $z \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. The following symbols are also used:

$\mathbf{u} = (u, v, w)$	perturbation in the velocity field
$\mathbf{h} = (h_1, h_2, h)$	perturbation in the magnetic field
θ	perturbation in the temperature field
$f_t = \frac{\partial f}{\partial t}, f_z = \frac{\partial f}{\partial z}$	$\Delta_1 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \Delta = \Delta_1 + \frac{\partial^2}{\partial z^2}$
$P_r = \frac{\nu}{k}, \text{ Prandtl number}$	$P_m = \frac{\nu}{\eta}, \text{ magnetic Prandtl number}$
$R = \frac{g\alpha\beta d^4}{k\nu}, \text{ Rayleigh number}, Q = \frac{\mu H_0^2 d^2}{4\pi\rho\nu\eta} \text{ Chandrasekhar number}$	
$\langle \cdot, \cdot \rangle_{L^2}$ -scalar product,	$\ \cdot\ _{L^2}$ -norm.

The (non-dimensional) equations for a perturbation $(\mathbf{u}, \mathbf{h}, \theta, p_1)$ to m_0 are

$$\begin{cases} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - P_m \mathbf{h} \cdot \nabla \mathbf{h} = -\nabla p_1 + \sqrt{R} \theta \mathbf{k} + \Delta \mathbf{u} + \sqrt{Q} \mathbf{h}_z, \\ \nabla \cdot \mathbf{u} = 0, \\ P_m (\mathbf{h}_t + \mathbf{u} \cdot \nabla \mathbf{h} - \mathbf{h} \cdot \nabla \mathbf{u}) = \sqrt{Q} \mathbf{u}_z + \Delta \mathbf{h}, \\ \nabla \cdot \mathbf{h} = 0, \\ P_r (\theta_t + \mathbf{u} \cdot \nabla \theta) = \sqrt{R} w + \Delta \theta. \end{cases} \quad (14)$$

To system (14), we add the *kinematically admissible initial conditions*

$$\begin{cases} \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \mathbf{h}(\mathbf{x}, 0) = \mathbf{h}_0(\mathbf{x}), \theta(\mathbf{x}, 0) = \theta_0(\mathbf{x}), \\ \nabla \cdot \mathbf{u}_0 = \nabla \cdot \mathbf{h}_0 = 0, \end{cases} \quad (15)$$

where $\theta_0, \mathbf{u}_0, \mathbf{h}_0$ are assigned initial fields. As concerns the boundary conditions, we assume that the two planes bounding L are rigid and electricity perfectly conducting {case B of Chandrasekhar (1981)} and at assigned temperatures. Then in view of the fluid viscosity and its complete adherence to the rigid boundaries, there will be no motions on the boundaries:

$$u = v = w = 0, \quad \forall t \geq 0, z \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}. \quad (16)$$

On the other hand, in view of the perfect electric conductivity, no magnetic field can cross the boundary

$$h_1 = h_2 = h = 0, \quad \forall t \geq 0, z \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}. \quad (17)$$

Finally, since the boundaries are at assigned temperatures, one has

$$\theta = 0, \quad \forall t \geq 0, z \in \left[-\frac{1}{2}, \frac{1}{2}\right]. \quad (18)$$

We assume that

- (i) the perturbations $(\nabla \pi, u, v, w, \theta, h_1, h_2, h)$ are periodic in the x and y directions of periods $2\pi/a_x, 2\pi/a_y$, respectively;
- (ii) $\Omega = [0, 2\pi/a_x] \times [0, 2\pi/a_y] \times \left[-\frac{1}{2}, \frac{1}{2}\right]$ is the periodicity cell;
- (iii) $u, v, w, \theta, h_1, h_2, h$ are such that together with all their first derivatives and second spatial derivatives are square integrable in $\Omega, \forall t \in \mathbb{R}^+$ and can be expanded in a Fourier series uniformly convergent in Ω .

We end by recalling that, along the time, many theorems of existence of weak and strong solution for MHD equations, under various circumstances, have been given. We confine ourselves to mentioning the recent papers concerning the existence of smooth solutions for large data {see Lin et al. (2016), Ren et al. (2014), Zhou and Zhu (2018), Zhang (2019) and reference therein}.

3 Essential fields

Neglecting the non-linear terms of (14), one obtains

$$\begin{cases} \hat{\mathbf{u}}_t = -\nabla \hat{\pi} + \sqrt{R} \hat{\theta} \mathbf{k} + \Delta \hat{\mathbf{u}} + \sqrt{Q} \hat{\mathbf{h}}_z, \\ \nabla \cdot \hat{\mathbf{u}} = 0, \quad \nabla \cdot \hat{\mathbf{h}} = 0, \\ P_m \hat{\mathbf{h}}_t = \sqrt{Q} \hat{\mathbf{u}}_z + \Delta \hat{\mathbf{h}}, \\ P_r \hat{\theta}_t = \sqrt{R} \hat{w} + \Delta \hat{\theta} \end{cases} \quad (19)$$

under the initial-boundary conditions analogous to (15) and (16). Since the vertical component of the double curl of (19)₁ is

$$\frac{\partial}{\partial t} \Delta \hat{w} = \sqrt{R} \Delta_1 \hat{\theta} + \Delta \Delta \hat{w} + \sqrt{Q} \Delta \frac{\partial \hat{h}}{\partial z}, \quad (20)$$

one has the linear system in the essential fields $(\hat{w}, \hat{h}, \hat{\theta})$ in $\Omega \times [0, \infty]$

$$\begin{cases} \frac{\partial}{\partial t} \Delta \hat{w} = \sqrt{R} \Delta_1 \hat{\theta} + \Delta \Delta \hat{w} + \sqrt{Q} \Delta \frac{\partial \hat{h}}{\partial z}, \\ P_m \frac{\partial \hat{h}}{\partial t} = \sqrt{Q} \frac{\partial \hat{w}}{\partial z} + \Delta \hat{h}, \\ P_r \frac{\partial \hat{\theta}}{\partial t} = \sqrt{R} \hat{w} + \Delta \hat{\theta}, \end{cases} \quad z \in \left[-\frac{1}{2}, \frac{1}{2}\right] \quad (21)$$

with

$$\begin{cases} \hat{w} = \sum_{n=1}^{\infty} \hat{w}_n, \quad \hat{w}_n = \tilde{w}_n(t) F(x, y) F_{1n}(z), \\ \hat{h} = \sum_{n=1}^{\infty} \hat{h}_n, \quad \hat{h}_n = \tilde{h}_n(t) F(x, y) F_{2n}(z), \\ \hat{\theta} = \sum_{n=1}^{\infty} \hat{\theta}_n, \quad \hat{\theta}_n = \tilde{\theta}_n(t) F(x, y) F_{3n}(z), \\ F(x, y) = \exp[i(a_x x + a_y y)], \end{cases} \quad (22)$$

which is the same class of perturbations considered in Chandrasekhar (1981) (normal modes). Therefore, setting

$$\hat{Z} = \frac{\partial \hat{h}}{\partial z}, \quad (23)$$

in view of the linearity of (21) one has

$$\begin{cases} \frac{\partial}{\partial t} \Delta \hat{w}_n = \sqrt{R} \Delta_1 \hat{\theta}_n + \Delta \Delta \hat{w}_n + \sqrt{Q} \Delta \hat{Z}_n, \\ P_m \frac{\partial \hat{Z}_n}{\partial t} = \sqrt{Q} \frac{\partial^2 \hat{w}_n}{\partial z^2} + \Delta \hat{Z}_n, \\ P_r \frac{\partial \hat{\theta}_n}{\partial t} = \sqrt{R} \hat{w}_n + \Delta \hat{\theta}_n, \end{cases} \quad z \in \left[-\frac{1}{2}, \frac{1}{2}\right] \quad (24)$$

with

$$\hat{Z}_n = \tilde{Z}_n(t) F \frac{\partial}{\partial z} F_{2n}, \quad \tilde{Z}_n(t) = \tilde{h}_n(t). \quad (25)$$

4 Additional boundary conditions

Requiring that (19)₂ continues to hold also for $z = \pm \frac{1}{2}$, the following supplementary boundary condition holds: $\frac{\partial \hat{w}}{\partial z} = 0$, at $z = \pm \frac{1}{2}$.

In fact, (19)₂ implies

$$\frac{\partial \hat{u}}{\partial x} = \frac{\partial \hat{v}}{\partial y} = 0, \quad \forall t \geq 0, z = \pm \frac{1}{2}.$$

Then

$$\nabla \cdot \hat{\mathbf{u}} = 0, \quad \left(\forall t \geq 0, z = \pm \frac{1}{2}\right) \Rightarrow \frac{\partial \hat{w}}{\partial z} = 0, \quad \left(\forall t \geq 0, z = \pm \frac{1}{2}\right). \quad (26)$$

Moreover, in view of (16) and (18), (21)₂ and (21)₃ imply, respectively,

$$\Delta \hat{h} = 0, \quad \forall t \geq 0, \quad z = \pm \frac{1}{2}, \quad (27)$$

$$\Delta \hat{\theta} = 0, \quad \forall t \geq 0, \quad z = \pm \frac{1}{2}. \quad (28)$$

Collecting all the boundary conditions, one has

$$\hat{h}_1 = \hat{h}_2 = \hat{w} = \frac{\partial \hat{w}}{\partial z} = \hat{h} = \Delta \hat{h} = \hat{\theta} = \Delta \hat{\theta} = 0, \quad \forall t \geq 0, z = \pm \frac{1}{2}. \quad (29)$$

5 Admissible perturbations

One easily verifies that (29) is verified by (22), on choosing $\forall z \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

$$\begin{cases} F_{2n} = \frac{1}{2n\pi} (\cos n\pi - \cos 2n\pi z), \\ F_{3n} = \sin 2n\pi z = Z_{2n}, \end{cases} \quad (30)$$

and F_{1n} solution of the Sturm–Liouville type problem

$$\begin{cases} f^{(iv)} = \mu^4 f, \quad z \in \left[-\frac{1}{2}, \frac{1}{2}\right], \\ f = f' = 0, \quad z = \pm \frac{1}{2} \end{cases} \quad (31)$$

whose solutions are given by the set of the even orthogonal functions {Chandrasekhar (1981), p. 635}

$$\begin{cases} f_n = \frac{\cosh \mu_n z}{\cosh \mu_n} - \frac{\cos \mu_n z}{\cos \mu_n}, & z \in \left[-\frac{1}{2}, \frac{1}{2}\right], \\ \mu_n : \tanh \frac{\mu_n}{2} + \tan \frac{\mu_n}{2} = 0 \end{cases} \quad (32)$$

and the set of the odd orthogonal functions

$$\begin{cases} f_n^* = \frac{\sinh \mu_n z}{\sinh \mu_n} - \frac{\sin \mu_n z}{\sin \mu_n}, \\ \mu_n : \coth \frac{\mu_n}{2} - \cot \frac{\mu_n}{2} = 0. \end{cases} \quad (33)$$

$$\begin{cases} \Delta_1 = -a^2, \quad a^2 = a_x^2 + a_y^2, \quad \Delta = a^2 + \frac{\partial^2}{\partial z^2}, \quad \frac{\partial^2}{\partial z^2} \Delta = -a^2 \frac{\partial^2}{\partial z^2} + \frac{\partial^4}{\partial z^4}, \\ \Delta \Delta = \left(-a^2 + \frac{\partial^2}{\partial z^2}\right) = \left(a^2 - 2a^2 \frac{\partial^2}{\partial z^2} + \frac{\partial^4}{\partial z^4}\right), \end{cases} \quad (36)$$

one has

$$\begin{cases} \Delta \Delta \hat{w}_n = \tilde{w}_n \Delta \Delta (F F_{1n}) = \tilde{w}_n F \left(a^4 - 2a^2 \frac{d^2}{dz^2} + \frac{d^4}{dz^4} \right) F_{1n}, \\ \Delta \hat{Z}_n = -\xi_n \hat{Z}_n, \quad \Delta \hat{\theta}_n = -\xi_n \hat{\theta}_n, \quad \xi_n = a^2 + 4n^2 \pi^2 \end{cases} \quad (37)$$

and (24) implies

$$\begin{cases} \left[\left(-a^2 + \frac{d^2}{dz^2} \right) \frac{d}{dt} - \left(a^4 - 2a^2 \frac{d^2}{dz^2} + \frac{d^4}{dz^4} \right) \right] F_{1n} \tilde{w}_n = -\sin 2n\pi z \left(a^2 \sqrt{R} \tilde{\theta}_n + \xi_n \sqrt{Q} \tilde{Z}_n \right), \\ \sin 2n\pi z \left[\frac{d\tilde{Z}_n}{dt} + \frac{\xi_n}{P_m} \tilde{Z}_n \right] = \frac{\sqrt{Q}}{P_m} \tilde{w}_n F''_{1n}, \\ \sin 2n\pi z \left(\frac{d\tilde{\theta}_n}{dt} + \frac{\xi_n \tilde{\theta}_n}{P_r} \right) = \frac{\sqrt{R}}{P_r} F_{1n} \tilde{w}_n. \end{cases} \quad (38)$$

It is found that Chandrasekhar (1981)

$$\begin{cases} \mu_1 = 4.73004074, \quad \mu_2 = 10.99560784, \quad \mu_3 = 17.27875066, \\ \mu_4 = 23.5619449, \quad n > 4 \Rightarrow \mu_n \Rightarrow \left(2n - \frac{1}{2} \right) \pi, \quad n \rightarrow \infty \end{cases} \quad (34)$$

in the case (32) and by

$$\begin{cases} \mu_1 = 7.85320462, \quad \mu_2 = 14.13716549, \quad \mu_3 = 26.70353726, \\ \mu_4 = 20.42035225, \quad n > 4 \Rightarrow \mu_n \Rightarrow \left(2n + \frac{1}{2} \right) \pi, \quad n \rightarrow \infty \end{cases} \quad (35)$$

in the case (33).

6 Stability with $\hat{w}$ even

In view of

In the case $F_{1n} = f_n$, the functions

$$F_{1n}, \quad \left(-a^2 + \frac{d^2}{dz^2} \right) F_{1n}, \quad \left(a^4 - 2a^2 \frac{d^2}{dz^2} + \frac{d^4}{dz^4} \right) F_{1n} = \left(a^4 + \mu_n^4 + \frac{d^2}{dz^2} \right) F_{1n} \quad (39)$$

are even in $\left[-\frac{1}{2}, \frac{1}{2}\right]$ and one has

$$\langle F_{1n}, \sin 2m\pi z \rangle = \langle F''_{1n}, \sin 2m\pi z \rangle = 0, \quad \forall m \in \mathbb{N}. \quad (40)$$

Then (38) implies

$$\frac{d\tilde{Z}_n}{dt} = -\frac{\xi_n}{P_m} \tilde{Z}_n, \quad \frac{d\tilde{\theta}_n}{dt} = -\frac{\xi_n}{P_r} \tilde{\theta}_n,$$

i.e.

$$\tilde{Z}_n = Z_n^{(0)} e^{-\xi_n/P_m t}, \quad \tilde{\theta}_n = \tilde{\theta}_n^{(0)} e^{-\xi_n/P_r t},$$

for any initial data and $P_m, P_r, \sqrt{R}$ and $\sqrt{Q}$. In view of (38)₂ the exponential decay of $\tilde{w}$ —for any initial data—immediately follows. In Appendix, the linear and nonlinear $L^2(\Omega)$ -energy decay is investigated.

7 Stability with $\hat{w}$ odd

Setting

$$A_n = \langle \sin 2n\pi z, f_n^* \rangle, \quad B_n = \|\sin 2n\pi z\|^2, \quad (41)$$

one has

$$\begin{aligned} \left\langle \sin 2n\pi z, \frac{d^2 f_n^*}{dz^2} \right\rangle &= -2n\pi \left\langle \cos 2n\pi z, \frac{d f_n^*}{dz} \right\rangle \\ &= -4n^2 \pi^2 A_n, \\ \left\langle \left(a^4 - 2a^2 \frac{d^2}{dz^2} + \mu_n^4 \right) f_m^*, \sin 2n\pi z \right\rangle &= (a^4 + 8a^2 n^2 \pi^2 + \mu_n^4) A_n, \\ \left\langle \left(-a^2 + \frac{d^2}{dz^2} \right) f_n^*, \sin 2n\pi z \right\rangle &= -\xi_n A_n. \end{aligned}$$

Then (38) gives

$$\frac{d}{dt} \begin{pmatrix} \tilde{w}_n \\ \tilde{z}_n \\ \tilde{\theta}_n \end{pmatrix} = L_n \begin{pmatrix} \tilde{w}_n \\ \tilde{z}_n \\ \tilde{\theta}_n \end{pmatrix}, \quad (42)$$

with L_n given by

$$L_n = \begin{pmatrix} -\frac{\alpha_n}{\xi_n} & \frac{\sqrt{Q}B_n}{A_n} & \frac{a^2\sqrt{R}B_n}{A_n\xi_n} \\ -\frac{4n^2\pi^2\sqrt{Q}A_n}{P_m B_n} & -\frac{\xi_n}{P_m} & 0 \\ \frac{\sqrt{R}A_n}{P_r B_n} & 0 & -\frac{\xi_n}{P_r} \end{pmatrix} \quad (43)$$

and

$$\alpha_n = a^4 + 8a^2 n^2 \pi^2 + \mu_n^4. \quad (44)$$

The spectrum equation of (42) is

$$\lambda^3 - I_{1n}\lambda^2 + I_{2n}\lambda - I_{3n} = 0, \quad (45)$$

with the characteristic values I_{rn} , ($r = 1, 2, 3$), given by {see Appendix }

$$\begin{cases} I_{1n} = -\left(\frac{\alpha_n}{\xi_n} + \frac{P_m + P_r}{P_m P_r} \xi_n \right), \\ I_{2n} = \frac{1}{P_r \xi_n} \left[(4n^2 \pi^2 Q \xi_n) \frac{P_r}{P_m} + \frac{1}{P_m} \xi_n^3 + \alpha_n \xi_n \left(1 + \frac{P_r}{P_m} \right) - a^2 R \right], \\ I_{3n} = \frac{1}{P_r P_m} [a^2 R - (\alpha_n + 4n^2 \pi^2 Q) \xi_n]. \end{cases} \quad (46)$$

8 Proof of property 1

The following properties hold.

(i) Only if

$$I_{1n} < 0, \quad I_{2n} > 0, \quad I_{3n} < 0, \quad \forall (a^2, n) \in \mathbb{R}^+ \times \mathbb{N}, \quad (47)$$

the roots of (44) (eigenvalues of L) have all negative real parts (Appendix);

(ii) if and only if the Liénard–Chipart conditions

$$I_{1n} < 0, \quad I_{2n} > 0, \quad I_{3n} < 0, \quad I_{1n}I_{2n} - I_{3n} < 0, \quad \forall n \in \mathbb{N} \quad (48)$$

hold, all the eigenvalues have negative real part (Appendix 79).

Since $\xi_n = a^2 + 4n^2 \pi^2$, I_{1n} , I_{3n} are decreasing functions of n while I_{2n} is an increasing function of n . Therefore, to satisfy i), it is necessary and sufficient to consider $n = 1$ (most destabilizing perturbation). Setting

$$I_{k1} = I_k, \quad \xi = \xi_1 = a^2 + 4\pi^2, \quad \alpha_1 = \gamma \quad (49)$$

(45) reduces to (4) with

1. I_1 independent of Q and R given by

$$\begin{cases} I_1 = -\left(\frac{\gamma}{\xi} + \frac{P_m + P_r}{P_m P_r} \right) < 0, \\ \gamma = a^4 + 8a^2 \pi^2 + \mu^4, \end{cases} \quad (50)$$

2. I_2 is a decreasing function of R , $\forall (a^2, P_m, P_r, Q) \in (\mathbb{R}^+)^4$ and

$$I_2 = 0 \Leftrightarrow \begin{cases} R = R_2 = \frac{F_2}{a^2}, \\ F_2 = \xi^3 + [4\pi^2 Q P_r + \gamma(P_m + P_r)]\xi; \end{cases} \quad (51)$$

3. I_3 is an increasing function of R , $\forall (a^2, P_m, P_r, Q) \in (\mathbb{R}^+)^4$ and

$$I_3 = 0 \Leftrightarrow \begin{cases} R = R_3 = \frac{F_3}{a^2}, \\ F_3 = (\gamma + 4\pi^2 Q)\xi. \end{cases} \quad (52)$$

Property 4 Let

$$\exists (a^2, P_m, P_r, Q, R) \in (\mathbb{R}^+)^5 \Rightarrow I_2 > 0, I_3 < 0. \quad (53)$$

Then at (a^2, P_m, P_r, Q, R) an Hopf bifurcation occurs if and only if one has that

$$I_1 I_2 - I_3 = 0 \quad (54)$$

and has the frequency $\omega^2 = I_2 = \frac{I_3}{I_1}$.

Proof Let (4) admit the couple of pure imaginary roots $\lambda = \pm i\omega$. Then one has

$$\begin{cases} \omega(\omega^2 - I_2) = 0, \\ I_1 \omega^2 - I_3 = 0. \end{cases} \quad (55)$$

Vice versa, let (54) holds. Then (4) becomes

$$\lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_1 I_3 = 0,$$

i.e.

$$(\lambda - I_1)(\lambda^2 + I_2) = 0 \quad (56)$$

having the roots $\lambda_1 = I_1$, $\lambda_2 = i\sqrt{I_2}$, $\lambda_3 = -i\sqrt{I_2}$. $\square$

Property 5 At $R = 0$, m_0 is stable for any a^2 .

Proof In fact, (42) reduces to

$$\frac{d}{dt} \begin{pmatrix} \tilde{w}_1 \\ \tilde{z}_1 \end{pmatrix} = \begin{pmatrix} -\gamma & \sqrt{Q} \frac{B_1}{A_1} \\ -\frac{4\pi^2 \sqrt{Q} A_1}{P_m B_1} & -\frac{\xi_n}{P_m} \end{pmatrix} \begin{pmatrix} \tilde{w}_1 \\ \tilde{z}_1 \end{pmatrix}, \quad \frac{d\tilde{\theta}_1}{dt} = -\frac{\xi}{P_r} \tilde{\theta}_1. \quad (57)$$

Being $\gamma + 4\pi^2 Q > 0$, $\forall a^2$, the linear stability of $m_0 \forall a^2$, immediately follows. $\square$

Property 6 The critical numbers R_{c_2}, R_{c_3} are given by

$$\begin{aligned} R_{c_2} &= \min_{a^2 \in \mathbb{R}^+} \frac{\xi^3 + [4\pi^2 Q P_r + \gamma(P_r + P_m)]\xi}{a^2}, \\ R_{c_3} &= \min_{a^2 \in \mathbb{R}^+} \frac{(\gamma + 4\pi^2 Q)\xi}{a^2}. \end{aligned} \quad (58)$$

Proof In view of

$$F_2 > 0, \quad \forall a^2 \in \mathbb{R}^+, \quad \lim_{a^2 \rightarrow 0} \frac{F_2}{a^2} = \lim_{a^2 \rightarrow \infty} \frac{F_2}{a^2} = \infty, \quad (59)$$

it follows that exists a finite wave number $a_{c_2}^2$, root of $a^2 \frac{dF_2}{da^2} = F_2$, such that

$$\begin{cases} R_{c_2} = \left\{ (a_{c_2}^2 + 4\pi^2)^3 + [4\pi^2 Q P_r + \gamma_{c_2}(P_r + P_m)](a_{c_2}^2 + 4\pi^2) \right\} (a_{c_2}^2)^{-1}, \\ \gamma_{c_2} = (\gamma)_{(a^2=a_{c_2}^2)}. \end{cases} \quad (60)$$

Analogously one has that exists a finite wave number $a_{c_3}^2$ —root of the equation $a^2 \frac{dF_3}{da^2} = F_3$ —such that

$$R_{c_3} = \frac{(\gamma_{c_3} + 4\pi^2 \sqrt{Q})\xi}{a_{c_3}^2}, \quad \gamma_{c_3} = (\gamma)_{(a^2=a_{c_3}^2)}. \quad (61)$$

$\square$

Property 7 Property 1 holds.

Proof In view of properties 4 and 5, it remains only to show its sufficiency. Let then (6) holds. On the other hand, the stability of m_0 at $R = 0$ for any wave number, implies—in view of the Liénard–Chipart conditions

$$I_1 < 0, \quad I_2 > 0, \quad I_3 < 0, \quad I_1 I_2 - I_3 < 0, \quad \text{at } R = 0. \quad (62)$$

But, at $\{R = R_{c_2}, a^2 = a_{c_2}^2\}$, one has

$$I_1 < 0, \quad I_2 = 0, \quad I_3 < 0, \quad I_1 I_2 - I_3 = -I_3 > 0; \quad (63)$$

therefore,

$$(I_1 I_2 - I_3)_{(R=0)}^{(a^2=a_{c_2}^2)} < 0, \quad (I_1 I_2 - I_3)_{(R=R_{c_2})}^{(a^2=a_{c_2}^2)} > 0 \quad (64)$$

implies the existence in $]0, R_{c_2}[$ of an R_* such that

$$(I_1 I_2 - I_3)_{(R=R_*)}^{(a^2=a_{c_2}^2)} = 0. \quad (65)$$

$\square$

9 Proof of Properties 2 and 3

Property 8 In plasma layers between two electricity perfectly conducting rigid planes, the MHD–Hopf bifurcations occur only in the plasma with $P_m > P_r$ and only for Q sufficiently high. Precisely if and only if

$$P_r < P_m, \quad Q > Q_c = \frac{4\pi^2}{P_m - P_r} \left[1 + P_r \left(\frac{\mu}{2\pi} \right)^4 \right], \quad \mu = 7.8532. \quad (66)$$

Proof (66) are necessary. In fact,

$$P_r \geq P_m \Rightarrow R_{c_2} > R_{c_3}, \quad \forall Q > 0. \quad (67)$$

Let then $P_r < P_m$ and set

$$\Psi = \frac{P_m}{\xi_1}(F_2 - F_3) = (P_r - P_m)4\pi^2 Q + \xi_1^2 + P_r \gamma. \quad (68)$$

In view of

$$\frac{\partial \Psi}{\partial a^2} = 2\xi_1 > 0, \quad (69)$$

one has

$$(\Psi)_{(a^2=0)} \geq 0 \Rightarrow \frac{F_2}{a^2} > \frac{F_3}{a^2}, \quad \forall a^2 > 0, \quad (70)$$

which implies $R_{c_2} > R_{c_3}$. Therefore, $(\Psi)_{(a^2=0)} < 0$, i.e. (66)₂ is necessary too. Vice versa, let (66) hold. Then

$$\begin{cases} (\Psi)_{(a^2=0)} < 0, & \frac{\partial \Psi}{\partial a^2} > 0, \quad \forall a^2, \\ \lim_{a^2 \rightarrow \infty} \Psi = \infty \end{cases} \quad (71)$$

implies the existence of a positive number a_*^2 such that

$$\frac{F_2}{a^2} \leq (\geq) \frac{F_3}{a^2}, \quad (72)$$

according to $a^2 \leq (\geq) a_*^2$ respectively, and one has

$$R_{c_2} = \min_{a^2 \in \mathbb{R}^+} \frac{F_2}{a^2} < \min_{a^2 < a_*^2} \frac{F_3}{a^2} = R_{c_3}. \quad (73)$$

□

Property 9 Property 3 holds.

Proof Property defines precisely the frontier between the onset of Hopf bifurcation (and hence the instability for $R \geq R_{c_2}$) and the onset of steady bifurcation (and hence the

Fig. 1 Asymptotic behaviour of Q_c for $P_r = 1$ (continuous line); $P_r = 2$ (dashed line); $P_r = 4$ (dotted line)

Fig. 2 Asymptotic behaviour of $\mathcal{R}$ versus P_r

instability for $R \geq R_{c_3}$). Property 3 immediately follows. □

Remark 1 We remark that a_*^2 is obtained requiring $(\Psi)_{(a^2=a_*^2)} = 0 \Leftrightarrow (F_2 - F_3)_{(a^2=a_*^2)} = 0$. One has

$$(a_*^2 + 4\pi^2)^2 + (a_*^4 + 8a_*^2\pi^2 + \mu_1^4)P_r = (P_m - P_r)4\pi^2 Q,$$

i.e.

$$(1 + P_r)(a_*^2 + 4\pi^2)^2 = (P_m - P_r)4\pi^2 Q - P_r[\mu_1^4 - (4\pi^2)^2].$$

Therefore,

$$\begin{aligned} (a_*^2 + 4\pi^2)^2 &= \frac{(P_m - P_r)4\pi^2 Q - P_r[\mu_1^4 - (4\pi^2)^2]}{1 + P_r} \\ &> (P_m - P_r)4\pi^2 Q_c - P_r[\mu_1^4 - (4\pi^2)^2] \end{aligned}$$

and—in view of (66)₂—is guaranteed

$$(a_*^2 + 4\pi^2)^2 > (4\pi^2)^2 + P_r\mu_1^4 - P_r\mu_1^4 + (4\pi^2)^2 = 2(4\pi^2)^2$$

and hence

$$a_*^2 > 4\sqrt{2}\pi^2 - 4\pi^2 = (\sqrt{2} - 1)4\pi^2.$$

Remark 2 In Fig. 1, the behaviour of Q_c at the increasing of $P_m (> P_r)$ for $P_r \in \{1, 2, 4\}$ is evaluated.

In Fig. 2, the behaviour of the ratio $\mathcal{R}$,

$$\mathcal{R} = \frac{4[1 + Pr(\mu/2\pi)^4]}{1 + Pr}, \quad \mu = 7.8532,$$

between the critical values of Q in the rigid–rigid case (given by (66)) and free–free case (given by (1)) at the growing of P_r is considered.

10 Hopf bifurcation threshold in the rigid-free and free-rigid cases in MHD

In view of

$$f_n^*(0) = (\sin 2n\pi z)_{(z=0)} = 0,$$

it follows that w_n, θ_n, Z_n verify the conditions

$$\begin{cases} w_n = w'_n = \theta_n = Z_n = 0, & z = \frac{1}{2}, \\ w_n = \theta_n = Z_n = 0, & z = 0, \end{cases} \quad (74)$$

$$\begin{cases} w_n = \theta_n = Z_n = 0, & z = 0, \\ w_n = w'_n = \theta_n = Z_n = 0, & z = -\frac{1}{2}. \end{cases} \quad (75)$$

One easily realizes that (74) and (75) are, respectively, the boundary MHD conditions for a layer of depth $\frac{1}{2}$ in the free-rigid and rigid-free cases respectively, when the rigid plane bounding the layer is electricity perfectly conducting. Therefore, the results of the previous sections can be used but taking into account of the different depth. The Rayleigh and Chandrasekhar numbers of the free-rigid and rigid-free cases, $\bar{R}, \bar{Q}$ are defined by

$$\bar{R} = \frac{g\alpha\beta\left(\frac{d}{2}\right)^4}{k\nu} = \frac{1}{16}R, \quad \bar{Q} = \frac{\mu H^2\left(\frac{d}{2}\right)^2}{4\pi\rho\nu} = \frac{1}{4}Q_*, \quad (76)$$

which implies

$$\begin{cases} \sqrt{\bar{R}_i} = \frac{1}{16}\sqrt{R_i}, & (i = 1, 2, 3, 4), \\ \bar{Q} = \frac{1}{4}Q, \quad \bar{Q}_c = \frac{1}{4}Q_c. \end{cases} \quad (77)$$

11 Discussion, final remarks and perspectives

1. The onset of Hopf bifurcations in plasma horizontal layers between two perfect conductor rigid planes and embedded in a constant transverse magnetic field, in non-relativistic thermal MHD, is investigated.
2. The threshold of Hopf bifurcation characterization—via a simple formula—problem introduced by Chandrasekhar (1981)—is solved by intro-

ducing a critical value R_{c_2} for the Rayleigh number and requiring $R_{c_2} < R_{c_3}$ with R_{c_3} steady convection critical value.

3. The property of each coefficient of the spectrum equation to drive the instability and the type of bifurcation, is shown and applied.
4. The threshold Q_c that, in the structural assumption $P_r < P_m$, the Chandrasekhar number Q has to cross for the onset of Hopf bifurcations, is given by (66) in the rigid-rigid case and by (10.4) in the rigid-free and free-rigid cases.
5. The onset of Hopf bifurcations in rotating layers between perfect conductor rigid planes, in MHD and in presence of Hall currents, is under investigation.

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Compliance with ethical standards

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Appendix

Characteristic values of a 3×3 matrix

Let $\mathcal{L} = \|a_{ij}\|$ be a quadratic matrix of order 3, with real entries. The algebraic cubic equation

$$P(\lambda) = 0, \quad (78)$$

with

$$P(\lambda) = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix}, \quad (79)$$

is the spectral equation of $\mathcal{L}$ and its roots are the eigenvalues of $\mathcal{L}$. $P(\lambda)$, being a polynomial of 3 degree, can be written (McLaurin)

$$P(\lambda) = \sum_{n=1}^3 \frac{P(0)^{(n)}}{n!} \lambda^n. \quad (80)$$

In view of

$$\begin{cases} \frac{dP}{d\lambda} = - \left[\begin{vmatrix} a_{22} - \lambda & a_{23} \\ a_{32} & a_{33} - \lambda \end{vmatrix} + \begin{vmatrix} a_{11} - \lambda & a_{13} \\ a_{31} & a_{33} - \lambda \end{vmatrix} + \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} \right], \\ \frac{d^2 P(\lambda)}{d\lambda^2} = 2[a_{11} + a_{22} + a_{33} - 3\lambda], \quad \frac{d^3 P}{d\lambda^3} = -6 \end{cases} \quad (81)$$

setting

$$P(0) = I_3, \quad P'(0) = -I_2, \quad P''(0) = 2I_1, \quad P'''(0) = -6 \quad (82)$$

one has

$$\begin{cases} I_1 = a_{11} + a_{22} + a_{33}, \quad I_3 = \det L, \\ I_2 = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{cases} \quad (83)$$

and the spectral equation becomes

$$\lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3 = 0. \quad (84)$$

The quantities I_1, I_2, I_3 are called *characteristic* values or invariants of $\mathcal{L}$ since are invariant with respect to nonsingular linear transformations.

Liénard–Chipart criterion

Let a_{ij} be real numbers and let

$$\frac{d}{dt} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \mathcal{L} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad (85)$$

with

$$\mathcal{L} = \|a_{ij}\|. \quad (86)$$

Then (84) is the spectral equation of (85) i.e. the roots of (84) are the eigenvalues of (85). Therefore, the zero solution is stable if and only if all the eigenvalues have all negative real parts. The algebraic Liénard–Chipart’s criterion Merkin (1997); Gantmacher (2000), applied to (84), allows to determine the stability of the zero solution of (85) without solving the spectral equation. In fact, introducing the Hurwitz matrix of (84)

$$\text{Hurwitz matrix} = \begin{vmatrix} -I_1 & -I_3 & 0 \\ 1 & I_2 & 0 \\ 0 & -I_1 & -I_3 \end{vmatrix}, \quad (87)$$

the criterion is the roots of (84) have negative real part if

and only if

$$I_1 < 0, \quad I_2 > 0, \quad I_3 < 0, \quad - \begin{vmatrix} -I_1 & I_3 \\ 1 & I_2 \end{vmatrix} = I_1 I_2 - I_3 < 0. \quad (88)$$

In particular, it follows that each one of the conditions

$$I_1 < 0, \quad I_2 > 0, \quad I_3 < 0 \quad (89)$$

is necessary for all the roots have negative real parts (otherwise some roots will have positive real parts) (Merkin 1997; Gantmacher 2000).

$L^2(\Omega)$ -linear and nonlinear energy decay

Let R_L be the R critical value of linear stability of m_0 and let λ_i , ($i = 1, 2, 3$), be the eigenvalues associated to $R < R_L$. Denoting by $R_e(\lambda)$ the real part of an eigenvalue λ and setting

$$-\bar{\alpha} = \sup_{i \in \{1, 2, 3\}} R_e(\lambda_i), \quad (90)$$

one has not only $\bar{\alpha} > 0$ but also

$$\left(\frac{d\hat{\mathcal{E}}}{dt} \right)_{(t=0)} < 0, \quad \forall (\hat{w}_0, \hat{h}_0, \hat{\theta}_0), \quad (91)$$

with $\hat{\mathcal{E}}$ partial $L^2(\Omega)$ linear energy given by

$$\hat{\mathcal{E}} = \frac{1}{2} \left(\|\hat{w}\|^2 + \frac{1}{P_m} \|\hat{h}\|^2 + \frac{1}{P_r} \|\hat{\theta}\|^2 \right). \quad (92)$$

In fact, setting

$$\begin{cases} \mathbf{X} = (X_1, X_2, X_3)^T = (\tilde{w}_1, \tilde{h}_1, \tilde{\theta}_1)^T, \\ |\mathbf{X}| = \left(\sum_{i=1}^3 X_i^2 \right)^{\frac{1}{2}}, \end{cases} \quad (93)$$

(42) and (43), for $n = 1$, give

$$\frac{d\mathbf{X}}{dt} = L\mathbf{X}, \quad L = L_1. \quad (94)$$

Since there exists a nonsingular transformation

$$\mathbf{X} = U\mathbf{Y}, \quad \det U \neq 0, \quad (95)$$

such that

$$\frac{d}{dt}|\mathbf{Y}| \leq -\frac{\bar{\alpha}}{2}|\mathbf{Y}|, \quad (96)$$

{see Gantmacher (1975), p. 196, formula (10) with $\delta = \frac{\alpha}{2}, \eta = 0$ }, then (95) and (96) imply

$$\begin{cases} |\mathbf{Y}| \leq |\mathbf{Y}_0|e^{-\frac{\bar{\alpha}}{2}t}, & \forall |\mathbf{Y}_0| \\ |\mathbf{X}| \leq |\mathbf{X}_0|e^{-\frac{\bar{\alpha}}{2}t}, & \forall |\mathbf{X}_0| \end{cases} \quad (97)$$

with $|\mathbf{X}_0| \geq |U||\mathbf{Y}_0|$ and hence $\forall |\mathbf{X}_0|$, since $|\mathbf{Y}_0|$ is arbitrary. Setting

$$|\Psi| = |\mathbf{X}_0|e^{-\frac{\bar{\alpha}}{2}t}, \quad (98)$$

one has

$$\left(\frac{d|\mathbf{X}|}{dt} \right)_{(t=0)} \leq \lim_{t \rightarrow 0^+} \frac{|\Psi| - |\Psi_0|}{t} = -\frac{\alpha}{2}|\mathbf{X}_0| < 0, \quad \forall |\mathbf{X}_0| \neq 0, \quad (99)$$

and (91) immediately follows together with

$$\hat{\mathcal{E}} \leq \hat{\mathcal{E}}_0 e^{-\bar{\alpha}t}. \quad (100)$$

Finally, introducing the total $L^2(\Omega)$ -linear energy

$$\hat{E} = \frac{1}{2} \left(\|\hat{\mathbf{u}}\|^2 + \frac{1}{P_m} \|\hat{\mathbf{h}}\|^2 + \frac{1}{P_r} \|\hat{\theta}\|^2 \right), \quad (101)$$

one easily obtains

$$\frac{d\hat{E}}{dt} = 2\sqrt{R}\langle \hat{w}, \hat{\theta} \rangle + \langle \hat{\mathbf{u}}, \Delta \hat{\mathbf{u}} \rangle + \frac{1}{P_m} \langle \hat{\mathbf{h}}, \Delta \hat{\mathbf{h}} \rangle + \frac{1}{P_r} \langle \hat{\theta}, \Delta \hat{\theta} \rangle, \quad (102)$$

and taking into account that the boundary conditions and the Poincaré inequality Flavin and Rionero (1996) imply

$$\langle \varphi, \Delta \varphi \rangle = -\|\nabla \varphi\|^2 \leq -\pi^2 \|\varphi\|^2, \quad \forall \varphi \in \{\hat{\mathbf{u}}, \hat{\mathbf{h}}, \hat{\theta}\}$$

one easily obtains

$$\frac{d\hat{E}}{dt} < 2\sqrt{R}\hat{\mathcal{E}} - \gamma\hat{E}, \quad (103)$$

and hence

$$\hat{E} \leq \hat{E}_0 e^{-\gamma t} + 2\sqrt{R}e^{-\gamma t} \int_0^t e^{\gamma \tau} \mathcal{E}(\tau) d\tau. \quad (104)$$

In view of (100), one has

$$\hat{E} \leq \hat{E}_0 \left(1 + \frac{2\sqrt{R_L}}{\gamma - \alpha} \right) e^{-\bar{\alpha}t}, \quad \bar{\alpha} < \min(\alpha, \gamma). \quad (105)$$

Passing to the nonlinear $L^2(\Omega)$ -energy decay and setting

$$E = \frac{1}{2} \left(\|\mathbf{u}\|^2 + \frac{1}{P_m} \|\mathbf{h}\|^2 + \frac{1}{P_r} \|\theta\|^2 \right), \quad (106)$$

in view of

$$\begin{cases} \langle \mathbf{u}, \mathbf{u} \cdot \nabla \mathbf{u} \rangle = \langle \mathbf{h}, \mathbf{h} \cdot \nabla \mathbf{h} \rangle = \langle \theta, \mathbf{u} \cdot \nabla \theta \rangle = 0, \\ \langle \mathbf{h}, \mathbf{u} \cdot \nabla \mathbf{h} \rangle = \langle \mathbf{h}, \mathbf{h} \cdot \nabla \mathbf{u} \rangle = 0, \\ \langle \mathbf{h}_z, \mathbf{u} \rangle + \langle \mathbf{h}, \mathbf{u}_z \rangle = 0, \end{cases} \quad (107)$$

one has

$$\frac{dE}{dt} = 2\sqrt{R}\langle w, \theta \rangle + \langle \mathbf{u}, \Delta \mathbf{u} \rangle + \frac{1}{P_m} \langle \mathbf{h}, \Delta \mathbf{h} \rangle + \frac{1}{P_r} \langle \theta, \Delta \theta \rangle. \quad (108)$$

On the other hand, (103) and (106) imply

$$\frac{d\hat{E}}{dt} < 2\sqrt{R}\hat{\mathcal{E}} \leq 2\sqrt{R_L}\hat{\mathcal{E}}_0 e^{-\frac{\bar{\alpha}}{2}t}, \quad \forall \hat{\mathcal{E}}_0 \quad (109)$$

and hence

$$\frac{d\hat{E}}{dt} \leq \hat{E}_0 e^{-\frac{\bar{\alpha}}{2}t}, \quad (110)$$

with $\hat{E}_0 > 2\sqrt{R_L}\hat{\mathcal{E}}_0$ and hence $\forall \hat{E}_0$ since $\hat{\mathcal{E}}_0$ is arbitrary. Setting

$$\hat{\Psi} = E_0 e^{-\frac{\bar{\alpha}}{2}t}, \quad (111)$$

one has

$$\left(\frac{d\hat{E}}{dt} \right)_{(t=0)} \leq \lim_{t \rightarrow 0^+} \frac{\hat{\Psi}(t) - \hat{\Psi}(0)}{t} = -\frac{\bar{\alpha}}{2}\hat{E}_0 < 0, \quad \forall \hat{E}_0. \quad (112)$$

On choosing

$$\hat{\mathbf{u}}_0 = \mathbf{u}(\tau), \quad \hat{\mathbf{h}}_0 = \mathbf{h}(\tau), \quad \hat{\theta}_0 = \theta(\tau), \quad (113)$$

one has

$$\left(\frac{dE}{dt}\right)_{(t=\tau)} = \left(\frac{d\hat{E}}{dt}\right)_{(t=0)} < 0, \quad \forall \tau \geq 0, \quad (114)$$

i.e. $R < R_L$ implies the global nonlinear asymptotic $L^2(\Omega)$ -stability.

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