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ORIGINAL RESEARCH

Stability analysis of a three-dimensional system of difference equations with quadratic terms

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Abstract

This study is involved with a class of three-dimensional system of difference equations incorporating quadratic term, which naturally extends and improve several results in the literature. Firstly, we demonstrate the existence of fixed points, the boundedness, persistence and invariance of positive solution of the mentioned system. Later, for this system, we give the global asymptotic stability at fixed point and the rate of convergence result which play an important role in the discrete dynamical systems. And lastly, some numerical examples are given to validate the effectiveness and feasibility of the theoretical findings.

Keywords System of difference equations · Global asymptotic stability · Boundedness · Rate of convergence

Mathematics Subject Classification 39A10 · 39A23 · 39A30

1 Introduction

Let's say that $\mathbb{N}_0$ is the set of all nonnegative integers, $\mathbb{N}$ is the set of all natural numbers, $\mathbb{Z}$ is the set of all integers, $\mathbb{R}$ is the set of all real numbers, and for $k \in \mathbb{Z}$ the notation $\mathbb{N}_k$ denotes the set of $\{n \in \mathbb{Z} : n \geq k\}$.

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A popular area of study in applied sciences is nonlinear difference equations or systems of difference equations with order greater than one. In nature, these equations or systems also exist as discrete analogues of numerical solutions of delay differential equations, which represent various phenomena in fields such as biology, geometry, probability theory, stochastic time series, physics, ecology, neural networks and engineering. The behavior of solutions to concrete difference equations or systems of difference equations with orders greater than one, as well as the asymptotic stability of their equilibrium points, have caught the attention of numerous authors recently. For example, in [1], Bešo et al. considered the following nonlinear second order difference equation

$$u_{n+1} = \gamma + \delta \frac{u_n}{u_{n-1}^2}, \quad n \in \mathbb{N}_0, \tag{1}$$

where the parameters γ, δ and the initial conditions u_{-i}, $i \in \{0, 1\}$, are positive real numbers. Boundedness, global attractivity and Neimark-Sacker bifurcation results were obtained. Subsequently, Tasdemir, in [2], generalized some results of Eq. (1) to the following higher-order difference equation

$$u_{n+1} = \mu + \eta \frac{u_n}{u_{n-m}^2}, \quad n \in \mathbb{N}_0, \tag{2}$$

where the parameters μ, η and the initial conditions u_{-i}, $i \in \{0, 1, \ldots, m\}$, are positive real numbers. Also, in [3], the equation in (1) was extended to the following discrete two-dimensional system of difference equations

$$u_{n+1} = \mu + \eta \frac{v_n}{v_{n-1}^2}, \quad v_{n+1} = \mu + \eta \frac{u_n}{u_{n-1}^2}, \quad n \in \mathbb{N}_0, \tag{3}$$

where the parameters μ, η and the initial conditions u_{-i}, v_{-i}, $i \in \{0, 1\}$, are positive real numbers. Further, Khan, in [4], studied the asymptotic properties of the following discrete difference equations system

$$u_{n+1} = \mathcal{B}_1 + \mathcal{B}_2 \frac{v_n}{v_{n-1}^2}, \quad v_{n+1} = \mathcal{B}_3 + \mathcal{B}_4 \frac{u_n}{u_{n-1}^2}, \quad n \in \mathbb{N}_0, \tag{4}$$

where $\mathcal{B}_i$, for $i \in \{1, 2, 3, 4\}$, are positive real numbers and the initial conditions u_{-j}, v_{-j}, for $j \in \{0, 1\}$ may be positive or negative real numbers, is a natural generalization of both the equation given in (1) and the system given in (2). Motivated by aforementioned studies, we consider the following nonlinear three-dimensional system of difference equations with quadratic terms

$$x_{n+1} = \mathcal{A}_1 + \mathcal{B}_1 \frac{y_n}{y_{n-1}^2}, \quad y_{n+1} = \mathcal{A}_2 + \mathcal{B}_2 \frac{z_n}{z_{n-1}^2}, \quad z_{n+1} = \mathcal{A}_3 + \mathcal{B}_3 \frac{x_n}{x_{n-1}^2}, \quad n \in \mathbb{N}_0, \tag{5}$$

where the parameters $\mathcal{A}_i$, $\mathcal{B}_i$, $i \in \{1, 2, 3\}$, and the initial conditions x_{-j}, y_{-j}, z_{-j}, $j \in \{0, 1\}$, are positive real numbers. In this paper we study the equilibrium point, the boundedness character, local asymptotic stability and global behavior of equilibrium point of system (5), the rate of convergences of the solutions and confirmation of theoretical results of mentioned system numerically. It is important to note that the results derived from this article are a generalization and extension of above mentioned articles. Other related difference equations and systems of difference equations can be found in references [4–24].

Let us consider a three-dimensional discrete dynamical system with second-order of the following form

$$\begin{cases} u_{n+1} &= f_1\left(v_n, v_{n-1}\right), \\ v_{n+1} &= f_2\left(w_n, w_{n-1}\right), \quad n \in \mathbb{N}_0, \\ w_{n+1} &= f_3\left(u_n, u_{n-1}\right), \end{cases} \tag{6}$$

where $f_1 : I_2 \times I_2 \to I_1$, $f_2 : I_3 \times I_3 \to I_2$ and $f_3 : I_1 \times I_1 \to I_3$, are continuously differentiable functions and I_j, $j \in \{1, 2, 3\}$, are some intervals of real numbers. Moreover, the solution $\{(u_n, v_n, w_n)\}_{n=-1}^{\infty}$ of corresponding system is uniquely determined by certain initial conditions.

Definition 1 [25] Let f_i, $i \in \{1, 2, 3\}$, be continuously differentiable functions at the equilibrium $(\bar{u}, \bar{v}, \bar{w})$ that is an equilibrium point of the map Φ. The linearized system of (6) about the equilibrium point $(\bar{u}, \bar{v}, \bar{w})$ is

$$U_{n+1} = \Phi\left(U_n\right) = \mathcal{F}_J U_n, \quad n \in \mathbb{N}_0, \tag{7}$$

where $U_n = (u_n, u_{n-1}, v_n, v_{n-1}, w_n, w_{n-1})^T$ and $\mathcal{F}_J$ is a Jacobian matrix of system (6) related to equilibrium point $\bar{U} = (\bar{u}, \bar{v}, \bar{w})$.

Theorem 1 [17] *Consider system (7), where $\bar{U}$ is a fixed point of Φ. If all eigenvalues of the Jacobian matrix $\mathcal{F}_J$ about $\bar{U}$ lie inside the open unit disk $\|\rho\| < 1$, that is, if all of them have absolute value less than one, then $\bar{U}$ is locally asymptotically stable. If at least one of the eigenvalues has a modulus greater than 1, than $\bar{U}$ is unstable.*

2 Linearized stability system

First of all, by employing the change of variables $\alpha_n = \frac{x_n}{\mathcal{A}_1}$, $\beta_n = \frac{y_n}{\mathcal{A}_2}$, $\gamma_n = \frac{z_n}{\mathcal{A}_3}$, for $n \geq -1$, system (5) becomes

$$\alpha_{n+1} = 1 + p\frac{\beta_n}{\beta_{n-1}^2}, \quad \beta_{n+1} = 1 + q\frac{\gamma_n}{\gamma_{n-1}^2}, \quad \gamma_{n+1} = 1 + r\frac{\alpha_n}{\alpha_{n-1}^2}, \quad n \in \mathbb{N}_0, \tag{8}$$

where

$$p = \frac{\mathcal{B}_1}{\mathcal{A}_1\mathcal{A}_2} > 0, \quad q = \frac{\mathcal{B}_2}{\mathcal{A}_2\mathcal{A}_3} > 0, \quad r = \frac{\mathcal{B}_3}{\mathcal{A}_1\mathcal{A}_3} > 0. \tag{9}$$

From here on, we will study on the equivalent system (8).

Lemma 1 *System* (8) *has two equilibrium points* $\Gamma_1 = (\xi_{11}, \xi_{12}, \xi_{13})$ *and* $\Gamma_2 = (\xi_{21}, \xi_{22}, \xi_{23})$*, where*

$$\begin{cases} \xi_{11} = \frac{1+p+q-r+\sqrt{(1+p+q-r)^2+4(1+p)(1+q)r}}{2(1+q)}, \\ \xi_{12} = \frac{1-p+q+r+\sqrt{(1+p+q-r)^2+4(1+p)(1+q)r}}{2(1+r)}, \\ \xi_{13} = \frac{1+p-q+r+\sqrt{(1+p+q-r)^2+4(1+p)(1+q)r}}{2(1+p)}, \end{cases} \tag{10}$$

and

$$\begin{cases} \xi_{21} = \frac{1+p+q-r-\sqrt{(1+p+q-r)^2+4(1+p)(1+q)r}}{2(1+q)}, \\ \xi_{22} = \frac{1-p+q+r-\sqrt{(1+p+q-r)^2+4(1+p)(1+q)r}}{2(1+r)}, \\ \xi_{23} = \frac{1+p-q+r-\sqrt{(1+p+q-r)^2+4(1+p)(1+q)r}}{2(1+p)}. \end{cases} \tag{11}$$

Proof Let $\Gamma = \left(\bar{\alpha}, \bar{\beta}, \bar{\gamma}\right)$ be equilibrium point of system (8). Then, from system (8), we have

$$\bar{\alpha} = 1 + p\frac{\bar{\beta}}{\bar{\beta}^2}, \quad \bar{\beta} = 1 + q\frac{\bar{\gamma}}{\bar{\gamma}^2}, \quad \bar{\gamma} = 1 + r\frac{\bar{\alpha}}{\bar{\alpha}^2}, \tag{12}$$

from which it follows that

$$\bar{\beta} = \frac{p}{\bar{\alpha} - 1}, \quad \bar{\gamma} = \frac{q}{\bar{\beta} - 1}, \quad \bar{\alpha} = \frac{r}{\bar{\gamma} - 1}. \tag{13}$$

By substituting the second equation in (13) into the third one in (13) and then the first equation in (13) into the third one in (13), it follows that

$$(q+1)\,\bar{\alpha}^2 - (p+q-r+1)\,\bar{\alpha} - r\,(p+1) = 0, \tag{14}$$

whose roots are

$$\bar{\alpha}_{1,2} = \frac{1+p+q-r \pm \sqrt{(1+p+q-r)^2 + 4\,(1+p)\,(1+q)\,r}}{2\,(1+q)}, \tag{15}$$

where $p = \frac{\mathcal{B}_1}{\mathcal{A}_1\mathcal{A}_2} > 0$, $q = \frac{\mathcal{B}_2}{\mathcal{A}_2\mathcal{A}_3} > 0$ and $r = \frac{\mathcal{B}_3}{\mathcal{A}_1\mathcal{A}_3} > 0$. Similarly, by substituting the third equation in (13) into the first one in (13) and then the second equation in (13) into the first one in (13), by keeping in mind the truth of (9) and after manipulation, we get

$$(r+1)\,\bar{\beta}^2 - (-p+q+r+1)\,\bar{\beta} - p\,(q+1) = 0, \tag{16}$$

whose roots are

$$\bar{\beta}_{1,2} = \frac{1 - p + q + r \pm \sqrt{(1 + p + q - r)^2 + 4\,(1 + p)\,(1 + q)\,r}}{2\,(1 + r)}, \tag{17}$$

where $p = \frac{\mathcal{B}_1}{\mathcal{A}_1\mathcal{A}_2} > 0$, $q = \frac{\mathcal{B}_2}{\mathcal{A}_2\mathcal{A}_3} > 0$ and $r = \frac{\mathcal{B}_3}{\mathcal{A}_1\mathcal{A}_3} > 0$. Analogously, by substituting the first equation in (13) into the second equation in (13) and the third equation in (13) into the second equation in (13), later by keeping in mind the truth of (9) and, finally, after manipulation, we have

$$(p + 1)\,\bar{\gamma}^2 - (p - q + r + 1)\,\bar{\gamma} - q\,(r + 1) = 0, \tag{18}$$

whose roots are

$$\bar{\gamma}_{1,2} = \frac{1 + p - q + r \pm \sqrt{(1 + p + q - r)^2 + 4\,(1 + p)\,(1 + q)\,r}}{2\,(1 + p)}, \tag{19}$$

where $p = \frac{\mathcal{B}_1}{\mathcal{A}_1\mathcal{A}_2} > 0$, $q = \frac{\mathcal{B}_2}{\mathcal{A}_2\mathcal{A}_3} > 0$ and $r = \frac{\mathcal{B}_3}{\mathcal{A}_1\mathcal{A}_3} > 0$. From (15), (17) and (19), one deduces that system (8) has two equilibrium points such as $\Gamma_1 = \left(\bar{\alpha}, \bar{\beta}, \bar{\gamma}\right) = (\xi_{11}, \xi_{12}, \xi_{13})$ and $\Gamma_2 = \left(\bar{\alpha}, \bar{\beta}, \bar{\gamma}\right) = (\xi_{21}, \xi_{22}, \xi_{23})$, where is described in (10) and (11). $\square$

Now, we will carry out the linearized form of system (8) related to the equilibrium point $\Gamma_1 = (\xi_{11}, \xi_{12}, \xi_{13})$. Firstly, we will write system (8) in vectorial form. To do this, we define the function $\mathcal{F} : (0, \infty)^6 \to (0, \infty)^6$ by

$$\mathcal{F}\,(X) = (g_1\,(X)\,, u_2, g_2\,(X)\,, v_2, g_3\,(X)\,, w_2)\,, \tag{20}$$

where $X = (u_1, u_2, v_1, v_2, w_1, w_2)\,,\ g_1\,(X) = 1 + p\frac{v_1}{v_2^2},\ g_2\,(X) = 1 + q\frac{w_1}{w_2^2}$ and $g_3\,(X) = 1 + r\frac{u_1}{u_2^2}$. From (20), one can write the vector form and linearized form of system (8) as follows

$$X_{n+1} = \mathcal{F}\,(X_n) = \mathcal{J}\,|_{\Gamma}\ X_n, \tag{21}$$

where $X_n = (u_n, u_{n-1}, v_n, v_{n-1}, w_n, w_{n-1})^T$ and $\mathcal{J}\,|_{\Gamma}$ is a Jacobian matrix of the system (8) about equilibirum point $\Gamma_1 = (\xi_{11}, \xi_{12}, \xi_{13})$, which is given by

$$\mathcal{J}\,|_{\Gamma} = \begin{pmatrix} 0 & 0 & \frac{p}{\xi_{12}^2} & \frac{-2p}{\xi_{12}^2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{q}{\xi_{13}^2} & \frac{-2q}{\xi_{13}^2} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{r}{\xi_{11}^2} & \frac{-2r}{\xi_{11}^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{22}$$

3 Boundedness and persistence of system (8)

In the following result, we prove that system (8) is bounded and persists.

Theorem 2 *If* $pqr < 1$, *then the solution* $\{(\alpha_n, \beta_n, \gamma_n)\}_{n=-1}^{\infty}$ *of system* (8) *is bounded and persists.*

Proof Let $\{(\alpha_n, \beta_n, \gamma_n)\}_{n=-1}^{\infty}$ be a positive solution of system (8). Then, from (8) we have

$$\alpha_n \geq 1, \quad \beta_n \geq 1, \quad \gamma_n \geq 1, \quad n \in \mathbb{N}. \tag{23}$$

Further, from (23) and system (8), the following inequalities can be easily obtained

$$\begin{cases} \alpha_{n+1} \leq 1 + p + pq + pqr\alpha_{n-2}, \\ \beta_{n+1} \leq 1 + q + qr + pqr\beta_{n-2}, \\ \gamma_{n+1} \leq 1 + r + pr + pqr\gamma_{n-2}. \end{cases} \tag{24}$$

From the first inequality in (24) one has

$$\hat{\alpha}_{n+1} = 1 + p + pq + pqr\hat{\alpha}_{n-2}, \tag{25}$$

such that $\hat{\alpha}_j = \alpha_j$, $j \in \{-1, 0, \ldots, 3\}$, whose solution is

$$\hat{\alpha}_n = \frac{1 + p + pq}{1 - pqr} + \left(\sqrt[3]{pqr}\right)^n \left(C_{11} + C_{12}\cos\left(\frac{2\pi}{3}n\right) + C_{13}\sin\left(\frac{2\pi}{3}n\right)\right), \tag{26}$$

where C_{1i}, for $i \in \{1, 2, 3\}$, are bounded up with $\hat{\alpha}_{-j}$, for $j \in \{-1, 0, 1\}$. From the second inequality in (24) one gets

$$\hat{\beta}_{n+1} = 1 + q + qr + pqr\hat{\beta}_{n-2}, \tag{27}$$

such that $\hat{\beta}_j = \beta_j$, $j \in \{-1, 0, \ldots, 3\}$, whose solution is

$$\hat{\beta}_n = \frac{1 + q + qr}{1 - pqr} + \left(\sqrt[3]{pqr}\right)^n \left(C_{21} + C_{22}\cos\left(\frac{2\pi}{3}n\right) + C_{23}\sin\left(\frac{2\pi}{3}n\right)\right), \tag{28}$$

where C_{2i}, for $i \in \{1, 2, 3\}$, are bounded up with $\hat{\beta}_{-j}$, for $j \in \{-1, 0, 1\}$. Similarly, from the third equality in (24) one obtains

$$\hat{\gamma}_{n+1} = 1 + r + pr + pqr\hat{\gamma}_{n-2}, \tag{29}$$

such that $\hat{\gamma}_j = \gamma_j$, $j \in \{-1, 0, \ldots, 3\}$, whose solution is

$$\hat{\gamma}_n = \frac{1+r+pr}{1-pqr} + \left(\sqrt[3]{pqr}\right)^n \left(C_{31} + C_{32}\cos\left(\frac{2\pi}{3}n\right) + C_{33}\sin\left(\frac{2\pi}{3}n\right)\right), \tag{30}$$

where C_{3i}, for $i \in \{1, 2, 3\}$, are depicted in $\hat{\gamma}_{-j}$, for $j \in \{-1, 0, 1\}$. By considering $\hat{\alpha}_j = \alpha_j$, $\hat{\beta}_j = \beta_j$, and $\hat{\gamma}_j = \gamma_j$, for $j \in \{-1, 0, \ldots, 3\}$, and from the assumption $pqr < 1$, then one has the following inequalities

$$\alpha_n \le \frac{1+p+pq}{1-pqr}, \quad \beta_n \le \frac{1+q+qr}{1-pqr}, \quad \gamma_n \le \frac{1+r+pr}{1-pqr}, \tag{31}$$

from which along with (23), it follows that

$$1 \le \alpha_n \le \frac{1+p+pq}{1-pqr}, \quad 1 \le \beta_n \le \frac{1+q+qr}{1-pqr}, \quad 1 \le \gamma_n \le \frac{1+r+pr}{1-pqr}. \tag{32}$$

□

Theorem 3 *System (8) has an invariant interval when $0 < pqr < 1$. Further the set $[1, \frac{1+p+pq}{1-pqr}] \times [1, \frac{1+q+qr}{1-pqr}] \times [1, \frac{1+r+pr}{1-pqr}]$ is an invariant.*

Proof Assume that $\{(\alpha_n, \beta_n, \gamma_n)\}_{n=-1}^{\infty}$ is the solution of system (8) such that $\alpha_{-i} \in [1, \frac{1+p+pq}{1-pqr}]$, $\beta_{-i} \in [1, \frac{1+q+qr}{1-pqr}]$ and $\gamma_{-i} \in [1, \frac{1+r+pr}{1-pqr}]$, for $i \in \{0, 1\}$. Then, from system (8) and equalities in (23) one gets

$$\begin{cases} 1 \le \alpha_1 = 1 + p\frac{\beta_0}{\beta_{-1}^2} \le \frac{1+p+pq}{1-pqr}, \\ 1 \le \beta_1 = 1 + q\frac{\gamma_0}{\gamma_{-1}^2} \le \frac{1+q+qr}{1-pqr}, \\ 1 \le \gamma_1 = 1 + r\frac{\alpha_0}{\alpha_{-1}^2} \le \frac{1+r+pr}{1-pqr}, \end{cases} \tag{33}$$

from which deduce that $\alpha_1 \in [1, \frac{1+p+pq}{1-pqr}]$, $\beta_1 \in [1, \frac{1+q+qr}{1-pqr}]$ and $\gamma_1 \in [1, \frac{1+r+pr}{1-pqr}]$. Finally, by using induction method, one easily shows that $\alpha_{k+1} \in [1, \frac{1+p+pq}{1-pqr}]$, $\beta_{k+1} \in [1, \frac{1+q+qr}{1-pqr}]$ and $\gamma_{k+1} \in [1, \frac{1+r+pr}{1-pqr}]$ if $\alpha_k \in [1, \frac{1+p+pq}{1-pqr}]$, $\beta_k \in [1, \frac{1+q+qr}{1-pqr}]$ and $\gamma_k \in [1, \frac{1+r+pr}{1-pqr}]$. □

4 Stability analysis of system (8)

The global attractivity and the local asymptotic stability of the equilibrium point given in (10) of system (8) will be addressed in this section. Also, the globally asymptotic result will be presented by using the gained results.

Theorem 4 *If the following condition*

$$\max\left\{\frac{12r(1+q)^2}{\left(1+p+q-r+\sqrt{\Delta}\right)^2}, \frac{12p(1+r)^2}{\left(1-p+q+r+\sqrt{\Delta}\right)^2}, \frac{12q(1+p)^2}{\left(1+p-q+r+\sqrt{\Delta}\right)^2}\right\} < 1, \tag{34}$$

where $\Delta = (1+p+q-r)^2 + 4(1+p)(1+q)r$*, holds, then the equilibrium point* $\Gamma_1 = (\xi_{11}, \xi_{12}, \xi_{13})$ *of system (8) is locally asymptotically stable.*

Proof From (21), the linearized equation of system (8) about equilibrium point $\Gamma_1 = (\xi_{11}, \xi_{12}, \xi_{13})$ is

$$X_{n+1} = \mathcal{J}\,|_{\Gamma_1}\, X_n, \tag{35}$$

where $X_n = (u_n, u_{n-1}, v_n, v_{n-1}, w_n, w_{n-1})^T$ and

$$\mathcal{J}\,|_{\Gamma_1} = \begin{pmatrix} 0 & 0 & \frac{4p(1+r)^2}{\left(1-p+q+r+\sqrt{\Delta}\right)^2} & \frac{-8p(1+r)^2}{\left(1-p+q+r+\sqrt{\Delta}\right)^2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{4q(1+p)^2}{\left(1+p-q+r+\sqrt{\Delta}\right)^2} & \frac{-8q(1+p)^2}{\left(1+p-q+r+\sqrt{\Delta}\right)^2} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{4r(1+q)^2}{\left(1+p+q-r+\sqrt{\Delta}\right)^2} & \frac{-8r(1+q)^2}{\left(1+p+q-r+\sqrt{\Delta}\right)^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{36}$$

where $\Delta = (1+p+q-r)^2 + 4(1+p)(1+q)r$. Let λ_i, for $i \in \{1, 2, \ldots, 6\}$, be the eigenvalues of $\mathcal{J}\,|_{\Gamma_1}$. From (34) we can choose an $\epsilon > 0$ such that

$$\max\left\{\sqrt[3]{\frac{12p(1+r)^2}{\left(1-p+q+r+\sqrt{\Delta}\right)^2}}, \sqrt[3]{\frac{12q(1+p)^2}{\left(1+p-q+r+\sqrt{\Delta}\right)^2}}, \sqrt[3]{\frac{12r(1+q)^2}{\left(1+p+q-r+\sqrt{\Delta}\right)^2}}\right\} < \epsilon < 1, \tag{37}$$

where $\Delta = (1+p+q-r)^2 + 4(1+p)(1+q)r$. If

$$\mathcal{D} = diag\left(1, \epsilon, \epsilon^{-1}, \epsilon^{-2}, \epsilon^{-3}, \epsilon^{-4}, \epsilon^{-5}\right), \tag{38}$$

then

$$
\begin{aligned}
&\mathcal{D}^{-1}\mathcal{J}\mid_{\Gamma_1}\mathcal{D} \\
&= \begin{pmatrix}
0 & 0 & \frac{4p(1+r)^2\epsilon^{-2}}{\left(1-p+q+r+\sqrt{\Delta}\right)^2} & \frac{-8p(1+r)^2\epsilon^{-3}}{\left(1-p+q+r+\sqrt{\Delta}\right)^2} & 0 & 0 \\
\epsilon & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{4q(1+p)^2\epsilon^{-2}}{\left(1+p-q+r+\sqrt{\Delta}\right)^2} & \frac{-8q(1+p)^2\epsilon^{-3}}{\left(1+p-q+r+\sqrt{\Delta}\right)^2} \\
0 & 0 & 0 & \epsilon & 0 & 0 \\
\frac{4r(1+q)^2\epsilon^{4}}{\left(1+p+q-r+\sqrt{\Delta}\right)^2} & \frac{-8r(1+q)^2\epsilon^{3}}{\left(1+p-q-r+\sqrt{\Delta}\right)^2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \epsilon & 0
\end{pmatrix}
\end{aligned}
\tag{39}
$$

where $\Delta = (1+p+q-r)^2 + 4\,(1+p)\,(1+q)\,r$. Then, we can obtain the next norm of the matrix $\mathcal{D}^{-1}\mathcal{J}\mid_{\Gamma_1}\mathcal{D}$ as

$$
\begin{aligned}
\|\mathcal{D}^{-1}\mathcal{J}\mid_{\Gamma_1}\mathcal{D}\| = \max\Bigg\{ & \frac{4p(1+r)^2\epsilon^{-2}}{\left(1-p+q+r+\sqrt{\Delta}\right)^2} + \frac{8p\,(1+r)^2\,\epsilon^{-3}}{\left(1-p+q+r+\sqrt{\Delta}\right)^2}, \\
& \frac{4q\,(1+p)^2\,\epsilon^{-2}}{\left(1+p-q+r+\sqrt{\Delta}\right)^2} + \frac{8q\,(1+p)^2\,\epsilon^{-3}}{\left(1+p-q+r+\sqrt{\Delta}\right)^2}, \\
& \frac{4r\,(1+q)^2\,\epsilon^{4}}{\left(1+p+q-r+\sqrt{\Delta}\right)^2} + \frac{8r\,(1+q)^2\,\epsilon^{3}}{\left(1+p-q-r+\sqrt{\Delta}\right)^2}, \epsilon \Bigg\}.
\end{aligned}
\tag{40}
$$

Since $\epsilon < 1$ and the inequality in (37) satisfies, we can write the following inequalities

$$
\begin{aligned}
\|\mathcal{D}^{-1}\mathcal{J}\mid_{\Gamma_1}\mathcal{D}\| \le \max\Bigg\{ & \frac{12p\,(1+r)^2\,\epsilon^{-3}}{\left(1-p+q+r+\sqrt{\Delta}\right)^2}, \frac{12q\,(1+p)^2\,\epsilon^{-3}}{\left(1+p-q+r+\sqrt{\Delta}\right)^2}, \\
& \frac{12r\,(1+q)^2\,\epsilon^{-3}}{\left(1+p-q-r+\sqrt{\Delta}\right)^2}, \epsilon \Bigg\} < 1.
\end{aligned}
\tag{41}
$$

Since $\mathcal{J}\mid_{\Gamma_1}$ possesses the same eigenvalues as $\mathcal{D}^{-1}\mathcal{J}\mid_{\Gamma_1}\mathcal{D}$, we have that $|\lambda_i| \le \|\mathcal{D}^{-1}\mathcal{J}\mid_{\Gamma_1}\mathcal{D}\| < 1$, where λ_i, for $i \in \{1, 2, \ldots, 6\}$, are the eigenvalues of $\mathcal{J}\mid_{\Gamma_1}$. From Theorem (1), the equilibrium point given in (10) of system (8) is locally asymptotically stable, which is desired. □

Theorem 5 *Assume that $p, q, r \in \left(0, \frac{1}{2}\right)$. Then, the equilibrium point Γ_1 of system (8) is global attractor.*

Proof Let $\{(\alpha_n, \beta_n, \gamma_n)\}_{n=-1}^{\infty}$ be a positive solution of system (8) and be $p, q, r \in \left(0, \frac{1}{2}\right)$. From Theorem (2), there exists

$$\begin{aligned} \lim_{n\to\infty}\sup\alpha_n = U_1,\ \lim_{n\to\infty}\sup\beta_n = U_2,\ \lim_{n\to\infty}\sup\gamma_n = U_3, \\ \lim_{n\to\infty}\inf\alpha_n = l_1,\quad \lim_{n\to\infty}\inf\beta_n = l_2,\quad \lim_{n\to\infty}\inf\gamma_n = l_3, \end{aligned} \tag{42}$$

where $U_1, U_2, U_3, l_1, l_2, l_3 \in (0, \infty)$. Then, from system (8) and the relations in (42), one gets the following inequalities

$$\begin{aligned} U_1 \le 1 + p\tfrac{U_2}{l_2^2},\ U_2 \le 1 + q\tfrac{U_3}{l_3^2},\ U_3 \le 1 + r\tfrac{U_1}{l_1^2}, \\ l_1 \ge 1 + p\tfrac{l_2}{U_2^2},\ \ l_2 \ge 1 + q\tfrac{l_3}{U_3^2},\ \ l_3 \ge 1 + r\tfrac{l_1}{U_1^2}, \end{aligned} \tag{43}$$

from which it follows that

$$U_2 + p\frac{l_2}{U_2} \le l_1 U_2, \tag{44}$$

$$U_3 + q\frac{l_3}{U_3} \le l_2 U_3, \tag{45}$$

$$U_1 + r\frac{l_1}{U_1} \le l_3 U_1, \tag{46}$$

$$U_1 l_2 \le l_2 + p\frac{U_2}{l_2}, \tag{47}$$

$$U_2 l_3 \le l_3 + q\frac{U_3}{l_3}, \tag{48}$$

$$U_3 l_1 \le l_1 + r\frac{U_1}{l_1}. \tag{49}$$

By multiplying both sides of inequality in (44) by U_3 and both sides of inequality in (49) by U_2, one has

$$U_3 l_1 U_2 \ge U_2 U_3 + p\frac{l_2 U_3}{U_2},\quad U_2 U_3 l_1 \le U_2 l_1 + r\frac{U_2 U_1}{l_1},$$

from which it follows that

$$U_2 U_3 + p\frac{l_2 U_3}{U_2} \le U_2 l_1 + r\frac{U_2 U_1}{l_1}. \tag{50}$$

Similarly, multiplying both sides of inequality in (45) by U_1 and both sides of inequality in (47) by U_3, one gets

$$U_1 U_3 l_2 \ge U_1 U_3 + q\frac{l_3 U_1}{U_3},\quad U_1 U_3 l_2 \le U_3 l_2 + p\frac{U_2 U_3}{l_2}$$

from which it follows that

$$U_1U_3 + q\frac{l_3U_1}{U_3} \le U_3l_2 + p\frac{U_2U_3}{l_2}. \tag{51}$$

Analogously, multiplying both sides of inequality in (46) by U_2 and both sides of inequality in (48) by U_1, one obtains

$$U_1U_2l_3 \ge U_1U_2 + r\frac{l_1U_2}{U_1}, \quad U_2l_3U_1 \le l_3U_1 + q\frac{U_1U_3}{l_3},$$

from which it follows that

$$U_1U_2 + r\frac{l_1U_2}{U_1} \le l_3U_1 + q\frac{U_1U_3}{l_3}. \tag{52}$$

From (50), (51) and (52), one can write

$$U_2U_3 + p\frac{l_2U_3}{U_2} + U_1U_3 + q\frac{l_3U_1}{U_3} + U_1U_2 + r\frac{l_1U_2}{U_1} \le U_2l_1 + r\frac{U_1U_2}{l_1} + U_3l_2 + p\frac{U_2U_3}{l_2} + l_3U_1 \\ +q\frac{U_1U_3}{l_3}, \tag{53}$$

which implies that

$$U_2U_3 + p\frac{l_2U_3}{U_2} + U_1U_3 + q\frac{l_3U_1}{U_3} + U_1U_2 + r\frac{l_1U_2}{U_1} - U_2l_1 - r\frac{U_1U_2}{l_1} - U_3l_2 - p\frac{U_2U_3}{l_2} - l_3U_1 \\ - q\frac{U_1U_3}{l_3} \le 0 \tag{54}$$

and consequently

$$U_3(U_2 - l_2) + U_2(U_1 - l_1) + U_1(U_3 - l_3) + pU_3(\frac{l_2}{U_2} - \frac{U_2}{l_2}) + qU_1(\frac{l_3}{U_3} - \frac{U_3}{l_3}) \\ + rU_2(\frac{l_1}{U_1} - \frac{U_1}{l_1}) \le 0. \tag{55}$$

From (55) and after some basic calculation, one has

$$U_3(U_2 - l_2)\left(1 - p(\frac{1}{l_2} + \frac{1}{U_2})\right) + U_2(U_1 - l_1)\left(1 - r(\frac{1}{l_1} + \frac{1}{U_1})\right) \\ + U_1(U_3 - l_3)\left(1 - q(\frac{1}{l_3} + \frac{1}{U_3})\right) \le 0. \tag{56}$$

From the fact that $1 \le l_1$, $1 \le l_2$, $1 \le l_3$ and from the assumption $p, q, r \in \left(0, \frac{1}{2}\right)$, then one gets

$$1 - p\left(\frac{1}{l_2} + \frac{1}{U_2}\right) > 0, \quad 1 - r\left(\frac{1}{l_1} + \frac{1}{U_1}\right) > 0, \quad 1 - q\left(\frac{1}{l_3} + \frac{1}{U_3}\right) > 0, \tag{57}$$

from which it follows that

$$U_1 - l_1 = 0, \quad U_2 - l_2 = 0, \quad U_3 - l_3 = 0, \tag{58}$$

and so, $U_1 = l_1$, $U_2 = l_2$ and $U_3 = l_3$, which completes the proof. □

Taking into account Theorem (4) and Theorem (5), the following theorem gives the main result of this article.

Theorem 6 *If the condition in* (34) *and* $p, q, r \in \left(0, \frac{1}{2}\right)$ *are hold, then the equilibrium point given in* (10) *of system* (8) *is globally asymptotically stable.*

5 Rate of convergence

In this section, we study the rate of convergence of a solutions which converges to the equilibrium point $\Gamma_1 = (\xi_{11}, \xi_{12}, \xi_{13})$ of the system (8) in the region of parameters described by $p, q, r \in (0, \infty)$. The following result gives the rate of convergence of solutions of difference equations system

$$\Psi_{n+1} = [M + N(n)]\Psi_n \tag{59}$$

where Ψ_n is a k-dimensional vector, $M \in C^{k\times k}$ is a constant matrix and $N : \mathbb{Z}^+ \to C^{k\times k}$ is a matrix function with

$$\|N(n)\| \to 0, \quad \text{as} \quad n \to \infty, \tag{60}$$

where $\|.\|$ denotes any matrix norm.

Theorem 7 (Perron's Theorem, see, [26]) *Assume that condition in* (60) *holds. If* Ψ_n *is a solution of* (59), *then either* $\Psi_n = 0$ *for all large n or*

$$\vartheta = \lim_{n\to\infty} \frac{\|\Psi_{n+1}\|}{\|\Psi_n\|}$$

or

$$\vartheta = \lim_{n\to\infty} (\|\Psi_n\|)^{\frac{1}{n}}$$

exists and ϑ *is equal to the modulus of one of the eigenvalues of matrix* M.

Theorem 8 *Assume that* $p, q, r \in \left(0, \frac{1}{2}\right)$ *and the solution* $\{(\alpha_n, \beta_n, \gamma_n)\}_{n=-1}^{\infty}$ *of system* (8) *tends to* $\Gamma_1 = (\xi_{11}, \xi_{12}, \xi_{13})$*. Then, the error vector*

$$\mathcal{E}_n = \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ e_n^2 \\ e_{n-1}^2 \\ e_n^3 \\ e_{n-1}^3 \end{pmatrix} = \begin{pmatrix} \alpha_n - \xi_{11} \\ \alpha_{n-1} - \xi_{11} \\ \beta_n - \xi_{12} \\ \beta_{n-1} - \xi_{12} \\ \gamma_n - \xi_{13} \\ \gamma_{n-1} - \xi_{13} \end{pmatrix}$$

of every solution of system (8) *satisfies both of the asymptotic relations*

$$\vartheta = \lim_{n\to\infty} (||\Psi_n||)^{\frac{1}{n}} = |\lambda_{1,2,3,4,5,6} J_F\, (\xi_{11}, \xi_{12}, \xi_{13})\,|,$$
$$\vartheta = \lim_{n\to\infty} \frac{||\Psi_{n+1}||}{||\Psi_n||} = |\lambda_{1,2,3,4,5,6} J_F\, (\xi_{11}, \xi_{12}, \xi_{13})\,|,$$

where ϑ *is equal to the modulus of one of the eigenvalues of* $\mathcal{J}_{\mathcal{F}}$ *about* $(\xi_{11}, \xi_{12}, \xi_{13})$*. and* $\lambda_{1,2,3,4,5,6} J_F\, (\xi_{11}, \xi_{12}, \xi_{13})$ *are the characteristic roots of the Jacobian matrix* $\mathcal{J}_{\mathcal{F}}\, (\xi_{11}, \xi_{12}, \xi_{13})$*.*

Proof Let $\{(\alpha_n, \beta_n, \gamma_n)\}_{n=-1}^{\infty}$ be a positive solution of system (8) such that the following conditions hold

$$\lim_{n\to\infty} \alpha_n = \xi_{11}, \quad \lim_{n\to\infty} \beta_n = \xi_{12}, \quad \lim_{n\to\infty} \gamma_n = \xi_{13}. \tag{61}$$

In order for the error terms of system, from (12) with depicting on $\Gamma_1 = \left(\bar{\alpha}, \bar{\beta}, \bar{\gamma}\right) = (\xi_{11}, \xi_{12}, \xi_{13})$, one has

$$\begin{cases} \alpha_{n+1} - \xi_{11} &= p\frac{\beta_n}{\beta_{n-1}^2} - \frac{p}{\xi_{12}} = \frac{p}{\beta_{n-1}^2}\Big(\beta_n - \xi_{12}\Big) - \frac{p(\beta_{n-1}+\xi_{12})}{\xi_{12}\beta_{n-1}^2}\Big(\beta_{n-1} - \xi_{12}\Big), \\ \beta_{n+1} - \xi_{12} &= q\frac{\gamma_n}{\gamma_{n-1}^2} - \frac{q}{\xi_{13}} = \frac{q}{\gamma_{n-1}^2}\Big(\gamma_n - \xi_{13}\Big) - \frac{q(\gamma_{n-1}+\xi_{13})}{\xi_{13}\gamma_{n-1}^2}\Big(\gamma_{n-1} - \xi_{13}\Big), \\ \gamma_{n+1} - \xi_{13} &= r\frac{\alpha_n}{\alpha_{n-1}^2} - \frac{r}{\xi_{11}} = \frac{r}{\alpha_{n-1}^2}\Big(\alpha_n - \xi_{11}\Big) - \frac{r(\alpha_{n-1}+\xi_{11})}{\xi_{11}\alpha_{n-1}^2}\Big(\alpha_{n-1} - \xi_{11}\Big). \end{cases} \tag{62}$$

Set

$$e_n^1 = \alpha_n - \xi_{11}, \quad e_n^2 = \beta_n - \xi_{12}, \quad e_n^3 = \gamma_n - \xi_{13}. \tag{63}$$

From (62) and (63), one gets

$$\begin{cases} e_{n+1}^1 = a_{11}e_n^2 + a_{12}e_{n-1}^2, \\ e_{n+1}^2 = a_{21}e_n^3 + a_{22}e_{n-1}^3, \\ e_{n+1}^3 = a_{31}e_n^1 + a_{32}e_{n-1}^1, \end{cases} \tag{64}$$

where $a_{11}=\frac{p}{\beta_{n-1}^2}$, $a_{12}=-\frac{p(\beta_{n-1}+\xi_{12})}{\xi_{12}\beta_{n-1}^2}$, $a_{21}=\frac{q}{\gamma_{n-1}^2}$, $a_{22}=-\frac{q(\gamma_{n-1}+\xi_{13})}{\xi_{13}\gamma_{n-1}^2}$, $a_{31}=\frac{r}{\alpha_{n-1}^2}$ and $a_{32}=-\frac{r(\alpha_{n-1}+\xi_{11})}{\xi_{11}\alpha_{n-1}^2}$, from which it follows that

$$\begin{cases}\lim_{n\to\infty} a_{11}=\frac{p}{\xi_{12}^2}, & \lim_{n\to\infty} a_{12}=\frac{-2p}{\xi_{12}^2},\\ \lim_{n\to\infty} a_{21}=\frac{q}{\xi_{13}^2}, & \lim_{n\to\infty} a_{22}=\frac{-2q}{\xi_{13}^2},\\ \lim_{n\to\infty} a_{31}=\frac{r}{\xi_{11}^2}, & \lim_{n\to\infty} a_{32}=\frac{-2r}{\xi_{11}^2}.\end{cases} \tag{65}$$

That is,

$$\begin{cases}a_{11}=\frac{p}{\xi_{12}^2}+\rho_{11}, & a_{12}=\frac{-2p}{\xi_{12}^2}+\rho_{12},\\ a_{21}=\frac{q}{\xi_{13}^2}+\rho_{21}, & a_{22}=\frac{-2q}{\xi_{13}^2}+\rho_{22},\\ a_{31}=\frac{r}{\xi_{11}^2}+\rho_{31}, & a_{32}=\frac{-2r}{\xi_{11}^2}+\rho_{32},\end{cases} \tag{66}$$

where $\rho_{ij}\to 0$ as $n\to\infty$. Then, one possesses the following system of the form in (59)

$$\mathcal{E}_{n+1}=(M+N(n))\,\mathcal{E}_n, \tag{67}$$

where $\mathcal{E}_n=\left(e_n^1, e_{n-1}^1, e_n^2, e_{n-1}^2, e_n^3, e_{n-1}^3\right)^T$ and

$$M=\begin{pmatrix}0&0&\frac{p}{\xi_{12}^2}&\frac{-2p}{\xi_{12}^2}&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&\frac{q}{\xi_{13}^2}&\frac{-2q}{\xi_{13}^2}\\ 0&0&1&0&0&0\\ \frac{r}{\xi_{11}^2}&\frac{-2r}{\xi_{11}^2}&0&0&0&0\\ 0&0&0&0&1&0\end{pmatrix},\quad N(n)=\begin{pmatrix}0&0&\rho_{11}&\rho_{12}&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&\rho_{21}&\rho_{22}\\ 0&0&1&0&0&0\\ \rho_{31}&\rho_{32}&0&0&0&0\\ 0&0&0&0&1&0\end{pmatrix}, \tag{68}$$

where $\|N(n)\|\to 0$ as $n\to\infty$. The matrix M is equal to $\mathcal{J}_{\mathcal{F}}(\xi_{11},\xi_{12},\xi_{13})$. So, by using Theorem (7) to system (8), the result easily follows. $\square$

6 Numerical simulations

In this section, we verify the above mathematical discussion and represent some interesting dynamical properties of system (8) through numerical simulations. For this, certain parametric values are taken into account for system (8).

Example 1 Consider system (8) with $p=0.49$, $q=0.48$, $r=0.41$. Then, system (8) can be written as

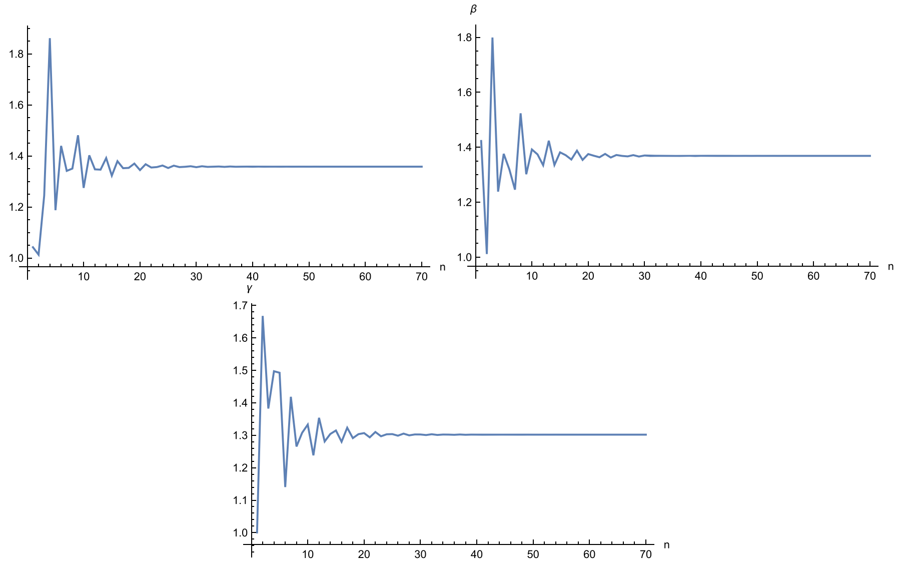

Fig. 1 Plot of α_n, β_n and γ_n in system (69) with $p, q, r \in \left(0, \frac{1}{2}\right)$.

$$\alpha_{n+1} = 1 + 0.49\frac{\beta_n}{\beta_{n-1}^2}, \quad \beta_{n+1} = 1 + 0.48\frac{\gamma_n}{\gamma_{n-1}^2},$$
$$\gamma_{n+1} = 1 + 0.41\frac{\alpha_n}{\alpha_{n-1}^2}, \quad n \in \mathbb{N}_0, \tag{69}$$

with the initial conditions $x_{-1} = 21.2$, $x_0 = 0.8$, $y_{-1} = 9.2$, $y_0 = 7.3$, $z_{-1} = 2.71$, $z_0 = 6.47$. From Theorem (5), one easily sees that every positive solution of system (8) is bounded and the equilibrium point $\Gamma_1 = (1.35801, 1.36869, 1.30191)$ of system (69) is global attractor (See, Figs. 1 and 2).

Example 2 Consider system (8) with $p = 1.7$, $q = 0.6$, $r = 3.4$. Then, system (8) can be written as

$$\alpha_{n+1} = 1 + 1.7\frac{\beta_n}{\beta_{n-1}^2}, \quad \beta_{n+1} = 1 + 0.6\frac{\gamma_n}{\gamma_{n-1}^2},$$
$$\gamma_{n+1} = 1 + 3.4\frac{\alpha_n}{\alpha_{n-1}^2}, \quad n \in \mathbb{N}_0, \tag{70}$$

with the initial conditions $x_{-1} = 4.2$, $x_0 = 5.1$, $y_{-1} = 0.6$, $y_0 = 3.1$, $z_{-1} = 0.6$, $z_0 = 14.6$. The the positive equilibrium point $\Gamma_1 = (2.36426, 1.2461, 2.43808)$ of system (70) is not global attractor. (See, Figs. 3 and 4).

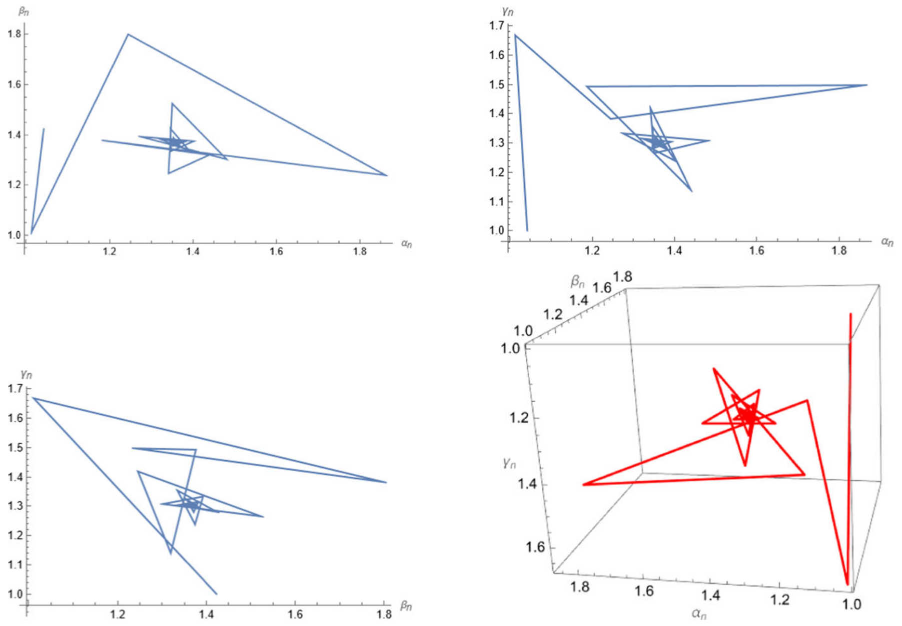

Fig. 2 Plot of attractor of system (69) with $p, q, r \in \left(0, \frac{1}{2}\right)$.

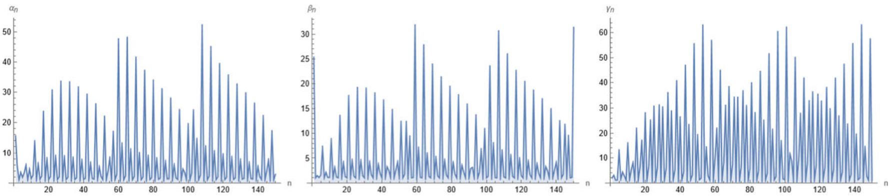

Fig. 3 Plot of α_n, β_n and γ_n in system (70)

7 Conclusion

This study represents a contribution to the analysis of three-dimensional concrete non-linear system of difference equations, with arbitrary constant and different parameters. This paper mainly discusses the dynamic properties of a class of second-order system of difference equations by utilizing stability theory and rate of convergence. The main results are as follows.

i. When $pqr < 1$, then the solution of system (8) is bounded and persists. Further for under this condition, system (8) has an invariant interval.

ii. When $\frac{12r(1+q)^2}{\left(1+p+q-r+\sqrt{\Delta}\right)^2} < 1$, $\frac{12p(1+r)^2}{\left(1-p+q+r+\sqrt{\Delta}\right)^2} < 1$ and $\frac{12q(1+p)^2}{\left(1+p-q+r+\sqrt{\Delta}\right)^2} < 1$, where $\Delta = (1+p+q-r)^2 + 4(1+p)(1+q)r$, then the equilibrium point $\Gamma_1 = (\xi_{11}, \xi_{12}, \xi_{13})$ of system (8) is locally asymptotic stable.

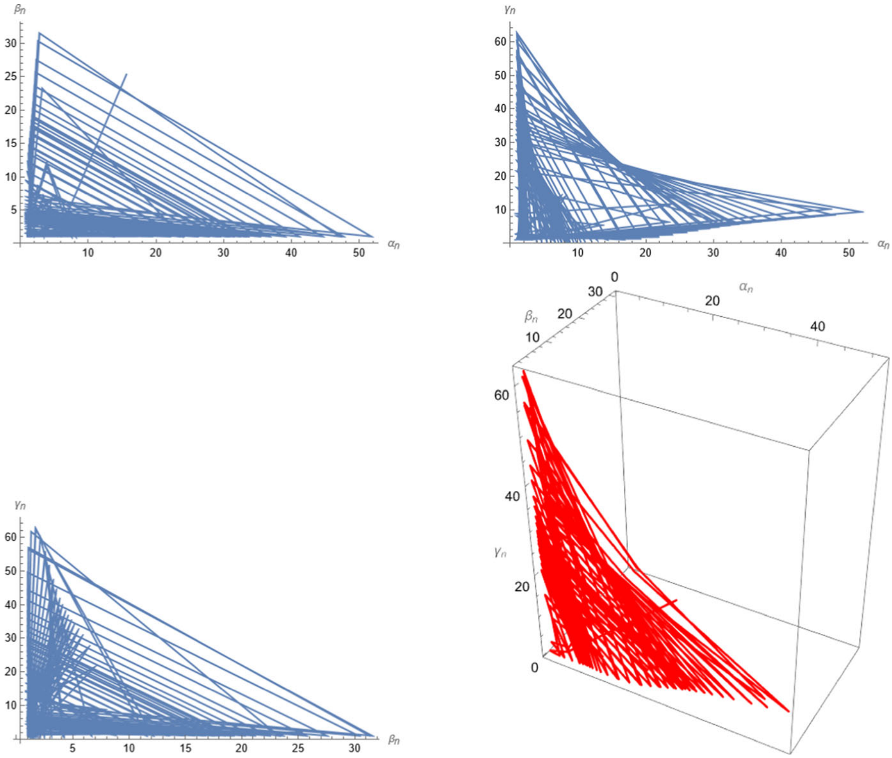

Fig. 4 The plot of system (70) with $p, r \notin \left(0, \frac{1}{2}\right)$ and $q \in \left(0, \frac{1}{2}\right)$

iii. When $p, q, r \in \left(0, \frac{1}{2}\right)$, then the equilibrium point Γ_1 of system (8) is global attractor.

The results imply that this approach might also be helpfully expanded to $k-$dimensional system of difference equations, or to system of difference equations with higher-order, or to system of difference equations with arbitrary powers. Thereby, we are going to offer a significant unresolved problem for scholars studying difference equations theory.

Open Problem. One can study the dynamical proporties of the following $k-$dimesional system of difference equations with quadratic terms

$$x_{n+1}^{(1)} = \mathcal{A}_1 + \mathcal{B}_1 \frac{x_n^{(2)}}{\left(x_{n-1}^{(2)}\right)^2},\ x_{n+1}^{(2)} = \mathcal{A}_2 + \mathcal{B}_2 \frac{x_n^{(3)}}{\left(x_{n-1}^{(3)}\right)^2}, \ldots, x_{n+1}^{(k)} = \mathcal{A}_k + \mathcal{B}_k \frac{x_n^{(1)}}{\left(x_{n-1}^{(1)}\right)^2},$$

where $n \in \mathbb{N}_0$, $k \in \mathbb{N}_4$, the parameters $\mathcal{A}_i$, $\mathcal{B}_i$, for $i \in \{1, 2, \ldots, k\}$, and the initial conditions $x_{-j}^{(i)}$, for $i \in \{1, 2, \ldots, k\}$ and $j \in \{0, 1\}$, are positive real numbers.

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Declarations

Conflict of interest The author declares that there is no Conflict of interest.

References

1. Bešo, E., Kalabušić, S., Mujić, N., Pilav, E.: Boundedness of solutions and stability of certain second-order difference equation with quadratic term. Adv. Differ. Equ. **2020**(1), 1–22 (2020)
2. Taşdemir, E.: Global dynamics of a higher order difference equation with a quadratic term. J. Appl. Math. Comput. 1–15 (2021)
3. Taşdemir, E.: On the global asymptotic stability of a system of difference equations with quadratic terms. J. Appl. Math. Comput. **66**, 423–437 (2021)
4. Khan, A.Q., et al.: Global dynamics of a nonsymmetric system of difference equations. Math. Prob. Eng. **2022** (2022)
5. Abualrub, S., Aloqeili, M.: Dynamics of positive solutions of a system of difference equations. J. Comput. Appl. Math. **392**, 113489 (2021)
6. Abu-Saris, R.M., DeVault, R.: Global stability of $y_{n+1} = A + \frac{y_n}{y_{n-1}}$. Appl. Math. Lett. **16**(2), 173–178 (2003)
7. Bao, H., et al.: Dynamical behavior of a system of second-order nonlinear difference equations. Int. J. Differ. Equ. **2015** (2015)
8. Din, Q.: Dynamics of a discrete Lotka–Volterra model. Adv. Differ. Equ. **2013**, 1–13 (2013)
9. Gümüş, M.: The global asymptotic stability of a system of difference equations. J. Differ. Equ. Appl. **24**(6), 976–991 (2018)
10. Gümüş, M.: Global asymptotic behavior of a discrete system of difference equations with delays. Filomat **37**(1), 251–264 (2023)
11. Khan, A.Q., Sharif, K.: Global dynamics, forbidden set, and transcritical bifurcation of a one-dimensional discrete-time laser model. Math. Methods Appl. Sci. **43**(7), 4409–4421 (2020)
12. Amira, K., Yacine, H.: Global behavior of p-dimensional difference equations system. Electron. Res. Arch. **29**(5), 3121–3139 (2021)
13. Kulenovic, M.R., Ladas, G.: Dynamics of Second Order Rational Difference Equations: With Open Problems and Conjectures. CRC Press, Boca Raton (2001)
14. Abd El-Moneam, M.: Global asymptotic stability of a system of difference equations with quadratic terms. Commun. Adv. Math. Sci. **6**(1), 31–43 (2023)
15. Papaschinopoulos, G., Schinas, C.: On a system of two nonlinear difference equations. J. Math. Anal. Appl. **219**(2), 415–426 (1998)
16. Papaschinopoulos, G., Schinas, C.: On the system of two nonlinear difference equations $x_{n+1} = A + \frac{x_{n-1}}{y_n}$, $y_{n+1} = A + \frac{y_{n-1}}{x_n}$. Int. J. Math. Math. Sci. **23**(12), 839–848 (2000)
17. Sedaghat, H.: Nonlinear Difference Equations: Theory with Applications to Social Science Models, vol. 15. Springer, Berlin (2013)
18. Taskara, N., Tollu, D., Yazlik, Y.: Solutions of rational difference system of order three in terms of Padovan numbers. J. Adv. Res. Appl. Math. **7**(3), 18–29 (2015)
19. Tollu, D., Yazlik, Y., Taskara, N.: Behavior of positive solutions of a difference equation. J. Appl. Math. Inf. **35**(3), 217–230 (2017)
20. Zhang, D., Ji, W., Wang, L., Li, X.: On the symmetrical system of rational difference equation $x_{n+1} = A + \frac{y_{n-k}}{y_n}$, $y_{n+1} = A + \frac{x_{n-k}}{x_n}$. Appl. Math. **4**, 834–837 (2013)

21. Huang, C., Liu, B., Yang, H., Cao, J.: Positive almost periodicity on SICNNs incorporating mixed delays and D operator. Nonlinear Anal. Modell. Control **27**(4), 719–739 (2022)
22. Zhao, X., Huang, C., Liu, B., Cao, J.: Stability analysis of delay patch-constructed Nicholson's blowflies system. Mathematics and Computers in Simulation (2023)
23. Huang, C., Liu, B., Qian, C., Cao, J.: Stability on positive pseudo almost periodic solutions of HPD-CNNs incorporating D operator. Math. Comput. Simul. **190**, 1150–1163 (2021)
24. Merve, K., Yazlik, Y., Tollu, D.T.: Solvability of a system of higher order nonlinear difference equations. Hacettepe J. Math. Stat. **49**(5), 1566–1593 (2020)
25. Kocic, V.L., Ladas, G.: Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, vol. 256. Springer, Berlin (1993)
26. Pituk, M.: More on Poincaré's and Perron's theorems for difference equations. J. Differ. Equ. Appl. **8**(3), 201–216 (2002)