ORIGINAL RESEARCH

New analytical wave structures of the (3 + 1)-dimensional extended modified Ito equation of seventh-order

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Abstract

Partial differential equations are frequently employed to depict issues arising across various scientific and engineering domains. Efforts have been made to analytically solve these equations, revealing shortcomings in some widely utilized methods, including modeling deficiencies and intricate solution processes. To address these limitations, diverse analytical methods have been explored. The Ito equation, introduced in 1980, underwent development, leading to the formulation of a fifth-order Ito equation. A seventh-order integrable (3 + 1)-dimensional extended modified Ito equation emerged by augmenting this equation with three additional terms. In this study, novel exact solutions for the equation, absent in existing literature, were derived using the extended hyperbolic function and modified Kudryashov methods. To scrutinize the dynamic behavior of these findings, we presented 3D, contour, and 2D visualizations of select solutions. The results showcase numerous new solutions, underscoring the reliability and efficacy of the employed methods.

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1 Introduction

Today, many events that affect and direct our lives can be explained by ordinary and partial differential equations (PDEs). The mathematical modeling of many phenomena in our lives has been modeled and solved by differential equations. The solutions of these differential equations make a significant contribution to scientists about the nature of the events being modeled. Researching, examining, interpreting, and presenting different physical properties of these problems in science and engineering has become one of many researchers' primary areas of interest in recent years. PDEs with more successful results in describing many different physical problems brought by modern times have been presented in the literature. Besides, obtaining the solutions to such differential equations and creating different results have become more critical. Most physics and engineering problems naturally fall into one of three physical categories: equilibrium, eigenvalue, and diffusion. The study of differential equations arising from these problems falls within the field of PDEs. Surface studies in geometry and a diverse array of mechanical problems marked the initial appearance of PDEs. Subsequent research demonstrated that a multitude of chemical, physical, and biological phenomena could be effectively represented using PDEs. Consequently, a majority of scientists directed their interest towards the challenges posed by these equations. With the development of computer technology, studies on PDEs have increased. These studies have an important place in many branches of science. They have become very attractive to scientists in recent years due to their applications in various fields such as fluid dynamics [1], image processing [2], physics [3], wave theory [4], mathematical biology [5], viscoelasticity theory [6] and so on. Therefore, new PDEs find a place in nonlinear phenomena, and new analytical approaches are regularly proposed in parallel with this. These include, trial equation method [7, 8], (G'/G)-expansion method and its modification [9–11], extended hyperbolic function method [12], sub-equation method [13], the generalized Riccati equation mapping method [14], modified Kudryashov method [15, 16], generalized Kudryashov Method [17, 18], sine-cosine method [19, 20], extended tanh-coth expansion method [21], Hirota bilinear method [22], the extended sinh-Gordon equation expansion method [23, 24], The unified method [25, 26], ansatz method [27] Laplace transform method [28], modified extended auxiliary equation mapping method [29], F-expansion method [30, 31], and the new extended direct algebraic method [32, 33], Lie symmetry method [34] improved Bernoulli sub-equation function method [35], etc.

This paper deals with the integrable Ito equation, an essential equation as a general form of the bilinear KdV equation that has many applications in quantum mechanics and nonlinear optics. The equation first appeared in 1980 [36] and has been solved by many numerical and analytical approaches. The earlier version of the equation is

$$v_{tt} + 6(v_x v_t)_x + v_{xxxt} = 0. \tag{1}$$

Then, this equation was developed, and the fifth-order Ito equation was created as [37, 38]:

$$v_t + (6v^5 + 10(v^2 v_{xx} + v v_x^2) + v_{xxxx})_x = 0. \tag{2}$$

Hence, the modified Ito equation Eq. (2) is combined with the Painlevé integrability, resulting in the seventh-order modified Ito equation. Very recently, this equation is extended to the following form by adding αv_x, βv_y, and γv_z terms into it [39].

$$v_t + \left(20v^7 + 70(v^4 v_{2x} + 2v^3 v_x^2) + 14(v^2 v_{4x} + 3v v_{2x}^2 + 4v v_x v_{3x} + 5v_x^2 v_{2x}) + v_{6x}\right)_x$$
$$+ \alpha v_x + \beta v_y + \gamma v_z = 0. \tag{3}$$

This study [39] is devoted to proposing a new seventh-order extended modified Ito equation in $(3 + 1)$-dimensions. The classic seventh-order integrable $(3 + 1)$-dimensional Ito equation is also established. The extended model's whole integrability is tested using Painlevé analysis. For the model, three branches of resonance spots are obtained. Simplified Hirota's method and unique ansatz techniques are applied to obtain multi-soliton and multi-singular soliton solutions and various other solutions. Extensive research has been conducted on the previously mentioned Ito equations, and the scientific community has contributed valuable insights. However, to the best of the authors' knowledge, the extended modified Ito equation has not been thoroughly investigated subsequent to the primary publication. This study aims to fill this gap by obtaining a wealth of new exact solutions for the equation through the application of a new extended hyperbolic function and modified Kudryashov methods. These methods have computational efficiency, accuracy in obtaining solutions, and versatility across different types of problems, and they produce abundant, exact solutions when compared to the other methods that exist in the literature. Besides, they have the ability to handle complex scenarios that arise in many problems in mathematics, physics, and engineering. Both methods have been recently developed and proposed. Therefore, they produce more accurate solutions than in their previous states. Also, they are flexible and applicable to a wide range of PDEs, especially nonlinear ones.

The paper is structured as follows: in Sect. 2, the new extended hyperbolic function and the modified Kudryashov methods are described. Section 3 provides the solutions to the given equation. Lastly, in Sect. 4 presents our concluding remarks.

2 Methodology of the proposed methods

Analytical methods for solving PDEs have both strengths and limitations. While they can provide exact solutions for certain types of equations, there are several challenges and limitations associated with these methods: Analytical methods are often limited to specific types of PDEs that have well-defined solutions. Many real-world prob-

lems involve complex geometries and boundary conditions, making it difficult to find exact solutions. PDEs that describe real-world phenomena can be highly complex and nonlinear. Finding analytical solutions for such equations is often mathematically challenging or even impossible. For certain PDEs, analytical solutions may involve complex mathematical operations, making them computationally intensive. This situation can limit the feasibility of using these methods for large-scale problems. In addition, analytical methods become increasingly challenging as the problem's dimensionality and derivative order of the problem increases. While some methods work well for one or two dimensions, extending them to three or more dimensions can be impractical.

The presented methods work well in our case, although the equation is $(3 + 1)$-dimensional and even seventh order. Besides, both methods start with the balancing principle and find an M value. However, when confronting equations with non-integer balance numbers, the methods will not function. Moreover, when addressing certain higher-order partial differential equations (PDEs), the applicability of these methods might be limited. Consequently, these case-specific techniques demonstrate efficacy for certain PDEs while falling short for others.

2.1 New extended hyperbolic function method

In this particular section, we are set to introduce a sophisticated approach known as the extended hyperbolic function method. By enabling us to create wave solutions for the associated differential equation and expand the solution functions of a trial equation into a finite series, this approach has shown to be incredibly effective. It may be applied with ease to both nonlinear PDEs and nonlinear PDEs with complex coefficients. Thus, in this investigation, we will utilize the recently developed extended hyperbolic function method to explore the wave solutions of a nonlinear PDE. As a result, we consider a nonlinear PDE given in the general form as:

$$\mathcal{Q}(v, v_t, v_x, v_y, v_x^2, v_y^2, \ldots) = 0, \tag{4}$$

where $v = v(x, \ldots, t)$ and perform the following steps:

step 1: assume the wave transform is in this form

$$v(x, \ldots, t) = v(\xi), \quad \xi = kx + \cdots + ct, \tag{5}$$

where the arbitrary constant values k and c will be determined later.
step 2: substituting the transformation (5) into the Eq. (4), it is converted to the following ordinary differential equation (ODE):

$$\mathcal{V}(v(\xi), v'(\xi), v''(\xi), \ldots) = 0, \tag{6}$$

where $\mathcal{V}$ is a polynomial in $v(\xi)$ and the superscripts represent the regular derivatives of $v(\xi)$ with regard to ξ.

step 3: we propose a trial solution $v(\xi)$ to solve the Eq. (6) as a finite series expansion:

$$v(\xi) = \sum_{m=0}^{M} a_m \varphi^m(\xi), \quad a_M \neq 0, \tag{7}$$

where a_m, $(0 \leq m \leq M)$ represents the arbitrary constants to be found later.

Family 1. $\varphi(\xi)$ will satisfy the following ODE in the first form, as

$$\varphi'(\xi) = \varphi(\xi)\sqrt{\lambda + \mu\varphi(\xi)^2}, \quad \lambda, \mu \in \mathbb{R}. \tag{8}$$

Thus, the solutions of Eq. (8) are given as follows:
Set 1. For $\lambda > 0$ and $\mu > 0$,

$$\varphi(\xi) = -\sqrt{\frac{\lambda}{\mu}}\, \mathrm{csch}\left(\sqrt{\lambda}(\xi + \xi_0)\right), \tag{9}$$

Set 2. For $\lambda < 0$ and $\mu > 0$,

$$\varphi(\xi) = \sqrt{\frac{-\lambda}{\mu}}\, \sec\left(\sqrt{-\lambda}(\xi + \xi_0)\right), \tag{10}$$

Set 3. For $\lambda > 0$ and $\mu < 0$,

$$\varphi(\xi) = \sqrt{\frac{\lambda}{-\mu}}\, \mathrm{sech}\left(\sqrt{\lambda}(\xi + \xi_0)\right), \tag{11}$$

Set 4. For $\lambda < 0$ and $\mu > 0$,

$$\varphi(\xi) = \sqrt{\frac{-\lambda}{\mu}}\, \csc\left(\sqrt{-\lambda}(\xi + \xi_0)\right), \tag{12}$$

Set 5. For $\lambda > 0$ and $\mu = 0$,

$$\varphi(\xi) = \exp\left(\sqrt{\lambda}(\xi + \xi_0)\right), \tag{13}$$

Set 6. For $\lambda < 0$ and $\mu = 0$,

$$\varphi(\xi) = \cos\left(\sqrt{-\lambda}(\xi + \xi_0)\right) + i\sin\left(\sqrt{-\lambda}(\xi + \xi_0)\right), \tag{14}$$

Set 7. For $\lambda = 0$ and $\mu > 0$,

$$\varphi(\xi) = \pm \frac{1}{\left(\sqrt{\mu}(\xi + \xi_0)\right)}, \tag{15}$$

Set 8. For $\lambda = 0$ and $\mu < 0$,

$$\varphi(\xi) = \pm \frac{i}{\left(\sqrt{-\mu}(\xi + \xi_0)\right)}. \tag{16}$$

Family 2. By the established pattern, we suppose that $\varphi(\xi)$ conforms to the ODE in the following manner:

$$\varphi'(\xi) = \lambda + \mu \varphi(\xi)^2, \quad \lambda, \mu \in \mathbb{R}. \tag{17}$$

The solutions for the Eq. (17) are considered as follows:
Set 1. For $\lambda \mu > 0$,

$$\varphi(\xi) = sgn(\lambda) \sqrt{\frac{\lambda}{\mu}} \tan \left(\sqrt{\lambda \mu}(\xi + \xi_0) \right), \tag{18}$$

Set 2. For $\lambda \mu > 0$,

$$\varphi(\xi) = -sgn(\lambda) \sqrt{\frac{\lambda}{\mu}} \cot \left(\sqrt{\lambda \mu}(\xi + \xi_0) \right), \tag{19}$$

Set 3. For $\lambda \mu < 0$,

$$\varphi(\xi) = sgn(\lambda) \sqrt{\frac{\lambda}{-\mu}} \tanh \left(\sqrt{-\lambda \mu}(\xi + \xi_0) \right), \tag{20}$$

Set 4. For $\lambda \mu < 0$,

$$\varphi(\xi) = sgn(\lambda) \sqrt{\frac{\lambda}{-\mu}} \coth \left(\sqrt{-\lambda \mu}(\xi + \xi_0) \right), \tag{21}$$

Set 5. For $\lambda = 0$ and $\mu >, 0$

$$\varphi(\xi) = -\frac{1}{\mu(\xi + \xi_0)}, \tag{22}$$

Set 6. For $\lambda \in \mathbb{R}$ and $\mu = 0$,

$$\varphi(\xi) = \lambda(\xi + \xi_0), \tag{23}$$

where sgn is the sign function.

step 4: Applying the balancing rule to Eq. (6) will provide the balancing constant $M \in \mathbb{Z}^+$. We obtain an algebraic equation in the form of $\varphi^m(\xi)$ by substituting Eqs. (7)–(6). We then balance this equation by setting the powers of $\varphi^m(\xi)$, $m = (0, 1, 2, \ldots)$ equal to zero, resulting in a set of algebraic equations. These equations provide the necessary inputs and the exact solutions to the provided equation.

2.2 Modified Kudryashov method

Consider the nonlinear PDE as

$$\mathscr{Q}\left(v, \frac{\partial v}{\partial \xi}, \frac{\partial^2 v}{\partial \xi^2}, \frac{\partial^3 v}{\partial \xi^3} \cdots \right) = 0, \tag{24}$$

where $v = v(x, \ldots, t)$. The transformation

$$\xi = kx + \cdots + ct, \quad v(x, \ldots, t) = v(\xi), \tag{25}$$

will transform Eq. (25) in the forms of an ODE:

$$\mathscr{F}(v(\xi), v'(\xi), v''(\xi), v'''(\xi), \ldots) = 0. \tag{26}$$

Assume the structure of the solution to Eq. (26) is

$$v(\xi) = \sum_{j=0}^{M} B_j \varphi^j(\xi), \quad B_M \neq 0, \tag{27}$$

where the function $\varphi(\xi)$ satisfies the ODE:

$$\varphi'(\xi) = \log(a)\varphi(\xi)(\varphi(\xi) - 1). \tag{28}$$

The solution to the Eq. (28) is provided by

$$\varphi(\xi) = \frac{1}{da^\xi + 1}, \quad a > 0, \quad d = \text{const.} \tag{29}$$

When (27) and (28) are substituted into Eq. (26), a polynomial in $\varphi^j(\xi)$ is obtained for $(j = 0, 1, 2, \ldots, M)$. Setting all of the coefficients of $\varphi^j(\xi)$ to zero [40], results in a set of algebraic equations in k, c and B_j. One can acquire the variables of the equation after solving this system. Eventually, by plugging these values into Eqs. (27) and (29), the wave solutions to Eq. (24) are produced.

3 Solutions to $(3 + 1)$dimensional extended modified Ito equation of seventh-order

In this section, we provide analytical solutions to the governing equation

$$
v_t + \left(20v^7 + 70(v^4 v_{2x} + 2v^3 v_x^2) + 14(v^2 v_{4x} + 3vv_{2x}^2 + 4vv_x v_{3x} + 5v_x^2 v_{2x}) + v_{6x}\right)_x
$$
$$
+\alpha v_x + \beta v_y + \gamma v_z = 0. \tag{30}
$$

For $\xi = kx + wy + sz + ct$, the transformation $v(x, y, z, t) = v(\xi)$ and integrating the resulting equation, reduces the above PDE to following ODE:

$$
v(c + \alpha k + \gamma s + \beta w) + k^7 v^{(6)} + 42k^5 v(v'')^2 + 14k^5 v^2 v^{(4)}
$$
$$
+ 56k^5 vv^{(3)} v' + 140k^3 v^3 (v')^2 + 70k^3 v''(k^2 (v')^2 + v^4) + 20kv^7 = 0. \tag{31}
$$

Balancing $v^{(6)} = M + 6$ with $v^7 = 7M$ gives $M = 1$. Using it in Eqs. (7) and (27), the analytical solutions to the equation is comes as follows.

3.1 New extended hyperbolic function method solutions

Family 1. According to Eq. (7) for $M = 1$, we should seek the solutions in the form of,

$$
v = a_0 + a_1 \varphi(\xi), \quad a_1 \neq 0, \tag{32}
$$

where a_0 and a_1 are constants. Substituting Eq. (32) into Eq. (31) and equating the coefficients polynomials of $\varphi(\xi)$ to zero, produces the following equations together with Eq. (8).

$$
\begin{aligned}
\varphi^0(\xi): \quad & a_0 c + \alpha a_0 k + 20a_0^7 k + a_0 \gamma s + a_0 \beta w = 0, \\
\varphi^1(\xi): \quad & a_1 c + a_1 \lambda^3 k^7 + 14a_0^2 a_1 \lambda^2 k^5 + 70a_0^4 a_1 \lambda k^3 + \alpha a_1 k \\
& + 140a_0^6 a_1 k + a_1 \gamma s + a_1 \beta w = 0, \\
\varphi^2(\xi): \quad & 126a_0 a_1^2 \lambda^2 k^5 + 420a_0^3 a_1^2 \lambda k^3 + 420a_0^5 a_1^2 k = 0, \\
\varphi^3(\xi): \quad & 182a_1 \lambda^2 k^7 \mu + 182a_1^3 \lambda^2 k^5 + 280a_0^2 a_1 \lambda k^5 \mu \\
& + 840a_0^2 a_1^3 \lambda k^3 + 140a_0^4 a_1 k^3 + 700a_0^4 a_1^3 k = 0, \\
\varphi^4(\xi): \quad & 1120a_0 a_1^2 \lambda k^5 \mu + 700a_0 a_1^4 \lambda k^3 + 700a_0^3 a_1^2 k^3 \mu \\
& + 700a_0^3 a_1^4 k = 0, \\
\varphi^5(\xi): \quad & 840a_1 \lambda k^7 \mu^2 + 1050a_1^3 \lambda k^5 \mu + 336a_0^2 a_1 k^5 \mu^2 \\
& + 210a_1^5 \lambda k^3 + 1260a_0^2 a_1^3 k^3 \mu + 420a_0^2 a_1^5 k = 0, \\
\varphi^6(\xi): \quad & 1176a_0 a_1^2 k^5 \mu^2
\end{aligned}
$$

$$+ 980 a_0 a_1^4 k^3 \mu + 140 a_0 a_1^6 k = 0,$$

$$\varphi^7(\xi): \quad 720 a_1 k^7 \mu^3$$

$$+ 980 a_1^3 k^5 \mu^2 + 280 a_1^5 k^3 \mu + 20 a_1^7 k = 0. \tag{33}$$

Here, we obtain one set of solutions for a_0, a_1, and c. **case 1.**

$$a_0 = 0, \quad a_1 = \mp i k \sqrt{\mu}, \quad c = -\lambda^3 k^7 - \alpha k - \gamma s - \beta w. \tag{34}$$

Substitute Eq. (34) into Eqs. (9)–(16), respectively, we get the following wave the solutions of Eq. (3):

Set 1. When $\lambda > 0$ and $\mu > 0$, we get

$$v_1^{\mp} = \mp i k \sqrt{\lambda \mu} s \sqrt{\frac{\lambda}{\mu}} \coth \left(\sqrt{\lambda} \left(d + kx + wy + sz + \left(-\lambda^3 k^7 - \alpha k - \gamma s - \beta w \right) t \right) \right)$$

$$\times \operatorname{csch} \left(\sqrt{\lambda} \left(d + kx + wy + sz + \left(-\lambda^3 k^7 - \alpha k - \gamma s - \beta w \right) t \right) \right). \tag{35}$$

Set 2. When $\lambda < 0$ and $\mu > 0$, we get

$$v_2^{\mp} = \mp i k \sqrt{-\lambda} \sqrt{\mu} s \sqrt{-\frac{\lambda}{\mu}} \tan \left(\sqrt{-\lambda} \left(d + kx + wy + sz + \left(-\lambda^3 k^7 - \alpha k - \gamma s - \beta w \right) t \right) \right)$$

$$\times \sec \left(\sqrt{-\lambda} \left(d + kx + wy + sz + \left(-\lambda^3 k^7 - \alpha k - \gamma s - \beta w \right) t \right) \right). \tag{36}$$

Set 3. When $\lambda > 0$ and $\mu < 0$, we get

$$v_3^{\mp} = \pm i k \sqrt{\lambda} \sqrt{\mu} s \sqrt{-\frac{\lambda}{\mu}} \tanh \left(\sqrt{\lambda} \left(d + kx + wy + sz + \left(-\lambda^3 k^7 - \alpha k - \gamma s - \beta w \right) t \right) \right)$$

$$\times \operatorname{sech} \left(\sqrt{\lambda} \left(d + kx + wy + sz + \left(-\lambda^3 k^7 - \alpha k - \gamma s - \beta w \right) t \right) \right). \tag{37}$$

Set 4. When $\lambda < 0$ and $\mu > 0$, we obtain

$$v_4^{\mp} = \mp i k \sqrt{\mu} \sqrt{-\frac{\lambda}{\mu}} \csc \left(\sqrt{-\lambda} \left(d + kx + wy + sz + \left(-\lambda^3 k^7 - \alpha k - \gamma s - \beta w \right) t \right) \right). \tag{38}$$

Set 5. When $\lambda > 0$ and $\mu = 0$, trivial solutions are encountered.
Set 6. When $\lambda < 0$ and $\mu = 0$, trivial solutions are encountered.
Set 7. When $\lambda = 0$ and $\mu > 0$, we obtain

$$
\begin{cases}
v_{5.1}^{\mp} = \mp \dfrac{ik}{d+kx+wy+sz+(-\lambda^3 k^7 - \alpha k - \gamma s - \beta w)t}, & \varphi(\xi) = \dfrac{1}{\sqrt{\mu}(\xi+\xi_0)}. \\
v_{5.2}^{\mp} = \pm \dfrac{ik}{d+kx+wy+sz+(-\lambda^3 k^7 - \alpha k - \gamma s - \beta w)t}, & \varphi(\xi) = -\dfrac{1}{\sqrt{\mu}(\xi+\xi_0)}.
\end{cases}
\tag{39}
$$

Set 8. When $\lambda = 0$ and $\mu < 0$, we obtain

$$
\begin{cases}
v_{6.1}^{\mp} = \pm \dfrac{k\sqrt{\mu}}{d+\sqrt{-\mu}(kx+wy+sz+(-\lambda^3 k^7 - \alpha k - \gamma s - \beta w)t)}, & \varphi(\xi) = \dfrac{i}{\sqrt{-\mu}(\xi+\xi_0)}. \\
v_{6.2}^{\mp} = \mp \dfrac{ik}{d+kx+wy+sz+(-\lambda^3 k^7 - \alpha k - \gamma s - \beta w)t}, & \varphi(\xi) = -\dfrac{i}{\sqrt{-\mu}(\xi+\xi_0)}.
\end{cases}
\tag{40}
$$

Family 2. According to Eq. (7) for $N = 1$, we should seek the solutions in the form of

$$
v = a_0 + a_1 \varphi(\xi),
\tag{41}
$$

where a_0 and a_1 are constants. Substituting Eq. (41) into the Eq. (31) and equating the coefficients polynomials of $\varphi(\xi)$ to zero, we get an algebraic system of equations. By solving the system via software, one may obtain the value of a_0, a_1, and c.
case 2.

$$
a_0 = 0, \quad a_1 = \mp ik\mu, \quad c = -20\lambda^3 k^7 \mu^3 - \alpha k - \gamma s - \beta w.
\tag{42}
$$

Substitute Eq. (42) into Eqs. (18)–(23), respectively, the solutions of Eq. (3) are obtained as:
Set 1. When $\lambda\mu > 0$, we get

$$
\begin{aligned}
v_7^{\mp} = \mp ik\mu \sqrt{\frac{\lambda}{\mu}} \mathrm{sgn}(\lambda) \tan \Big(\sqrt{\lambda\mu}\, (d + kx + wy + sz \\
+ \Big(-20\lambda^3 k^7 \mu^3 - \alpha k - \gamma s - \beta w\Big) t\Big)\Big).
\end{aligned}
\tag{43}
$$

Set 2. When $\lambda\mu > 0$, we get

$$
\begin{aligned}
v_8^{\mp} = \pm ik\mu \sqrt{\frac{\lambda}{\mu}} \mathrm{sgn}(\lambda) \cot \Big(\sqrt{\lambda\mu}\, (d + kx + wy + sz \\
+ \Big(-20\lambda^3 k^7 \mu^3 - \alpha k - \gamma s - \beta w\Big) t\Big)\Big).
\end{aligned}
\tag{44}
$$

Set 3. When $\lambda\mu < 0$, we obtain

$$
\begin{aligned}
v_9^{\mp} = \mp - ik\mu \sqrt{-\frac{\lambda}{\mu}} \mathrm{sgn}(\lambda) \tanh \\
\Big(\sqrt{-\lambda\mu}\, \big(d + + kx + wy + sz + \big(-20\lambda^3 k^7 \mu^3 - \alpha k - \gamma s - \beta w\big) t\big)\Big).
\end{aligned}
\tag{45}
$$

Set 4. When $\lambda\mu < 0$, we obtain

$$v_{10}^{\mp} = \mp - ik\mu\sqrt{-\frac{\lambda}{\mu}\text{sgn}(\lambda)}\coth$$
$$\left(\sqrt{-\lambda\mu}\left(d + kx + wy + sz + \left(-20\lambda^3 k^7\mu^3 - \alpha k - \gamma s - \beta w\right)t\right)\right). \tag{46}$$

Set 5. When $\lambda = 0$ and $\mu > 0$, we get

$$v_{11}^{\mp} = \pm\frac{ik}{d + kx + wy + sz + (\alpha(-k) - \gamma s - \beta w)t}. \tag{47}$$

Set 6. When $\lambda \in \mathbb{R}$ and $\mu = 0$, trivial solutions are encountered.

3.2 Modified Kudryashov method solutions

According to Eq. (27) for $M = 1$, we should seek the solutions in the form of,

$$v = B_0 + B_1\varphi(\xi), \quad B_1 \neq 0. \tag{48}$$

When associated with Eq. (28), the following set of equations emerges.

$$\begin{aligned}
&B_1 k^7 \log^6(a) + 14B_0^2 B_1 k^5 \log^4(a) + 70B_0^4 B_1 k^3 \log^2(a) + B_1 c + \alpha B_1 k \\
&+ 140B_0^6 B_1 k + B_1\gamma s + \beta B_1 w = 0, -63B_1 k^7 \log^6(a) + 126B_0 B_1^2 k^5 \log^4(a) \\
&-210B_0^2 B_1 k^5 \log^4(a) + 420B_0^3 B_1^2 k^3 \log^2(a) \\
&-210B_0^4 B_1 k^3 \log^2(a) + 420B_0^5 B_1^2 k = 0, \\
&602B_1 k^7 \log^6(a) + 182B_1^3 k^5 \log^4(a) - 1120B_0 B_1^2 k^5 \log^4(a) + 700B_0^2 B_1 k^5 \log^4(a) \\
&+ 840B_0^2 B_1^3 k^3 \log^2(a) - 1120B_0^3 B_1^2 k^3 \log^2(a) + 140B_0^4 B_1 k^3 \log^2(a) + 700B_0^4 B_1^3 k = 0, \\
&-2100B_1 k^7 \log^6(a) - 1260B_1^3 k^5 \log^4(a) + 3010B_0 B_1^2 k^5 \log^4(a) - 840B_0^2 B_1 k^5 \log^4(a) \\
&+ 700B_0 B_1^4 k^3 \log^2(a) - 2100B_0^2 B_1^3 k^3 \log^2(a) + 700B_0^3 B_1^2 k^3 \log^2(a) + 700B_0^3 B_1^4 k = 0, \\
&3360B_1 k^7 \log^6(a) + 2940B_1^3 k^5 \log^4(a) - 3192B_0 B_1^2 k^5 \log^4(a) + 336B_0^2 B_1 k^5 \log^4(a) \\
&+ 210B_1^5 k^3 \log^2(a) - 1680B_0 B_1^4 k^3 \log^2(a) + 1260B_0^2 B_1^3 k^3 \log^2(a) + 420B_0^2 B_1^5 k = 0, \\
&-2520B_1 k^7 \log^6(a) - 2842B_1^3 k^5 \log^4(a) + 1176B_0 B_1^2 k^5 \log^4(a) - 490B_1^5 k^3 \log^2(a) \\
&+ 980B_0 B_1^4 k^3 \log^2(a) + 140B_0 B_1^6 k = 0, \\
&720B_1 k^7 \log^6(a) + 980B_1^3 k^5 \log^4(a) + 280B_1^5 k^3 \log^2(a) + 20B_1^7 k = 0. \tag{49}
\end{aligned}$$

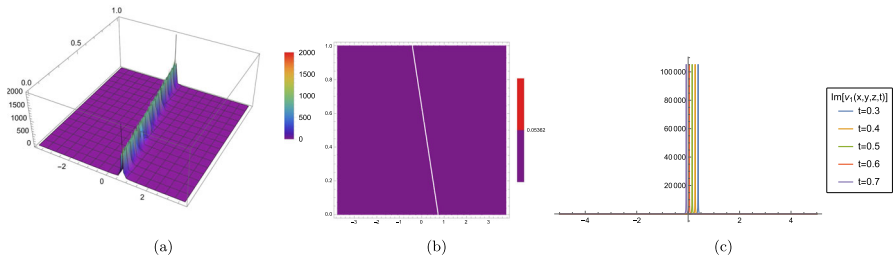For B_0, B_1, and c, we get two cases and sets of solutions in this instance.

Fig. 1 Graphical illustration of $Im[v_1]$ of Eq. (35)

Case 3.

$$B_0 = -\frac{1}{2}ik \log(a), \quad B_1 = ik \log(a),$$

$$c = \frac{5}{16}k^7 \log^6(a) - \alpha k - \gamma s - \beta w. \tag{50}$$

Inserting these values into Eq. (48), using Eq. (29), we have the solution as:
Set1.

$$v_{12} = \frac{ik \log(a)}{da^{t\left(\frac{5}{16}k^7 \log^6(a) - \alpha k - \gamma s - \beta w\right) + kx + sz + wy} + 1} - \frac{1}{2}ik \log(a). \tag{51}$$

Case 4.

$$B_0 = \frac{1}{2}ik \log(a), \quad B_1 = -ik \log(a),$$

$$c = \frac{5}{16}k^7 \log^6(a) - \alpha k - \gamma s - \beta w. \tag{52}$$

Inserting these values into Eq. (48), using Eq. (29), we have the solution as:
Set2.

$$v_{13} = \frac{1}{2}ik \log(a) - \frac{ik \log(a)}{da^{t\left(\frac{5}{16}k^7 \log^6(a) - \alpha k - \gamma s - \beta w\right) + kx + sz + wy} + 1}. \tag{53}$$

3.3 Graphical overview and interpretation

In this section, we aim to bridge the gap between theoretical solutions and practical applications by providing graphical representations. To enhance the reader's comprehension, we present 3D graphics, contour plot and 2D graphics denoted by figures (a)–(c), respectively.

In this study, a variety of graphs made with Mathematica are displayed in order to investigate the behavior of solitons and assess the physical significance of the solutions obtained by selecting suitable values for unknown parameters.
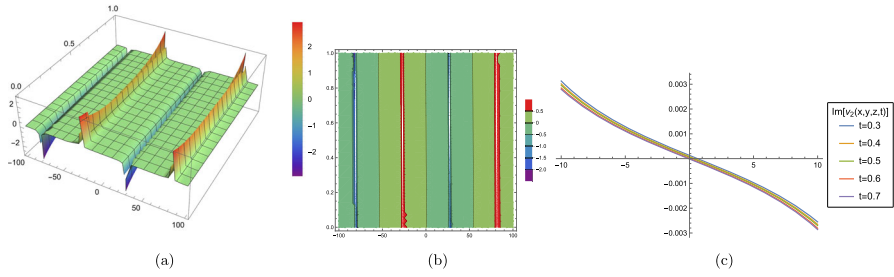
(a)   (b)   (c)

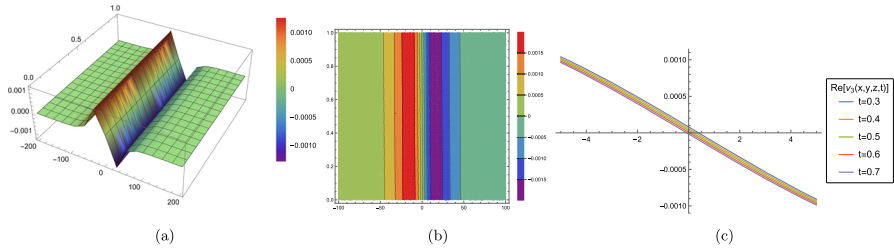**Fig. 2** Graphical illustration of $Im[v_2]$ of Eq. (36)

(a)   (b)   (c)

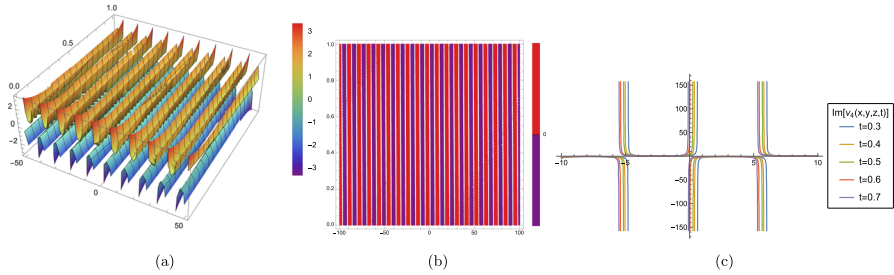**Fig. 3** Graphical illustration of $Re[v_3]$ of Eq. (37)

(a)   (b)   (c)
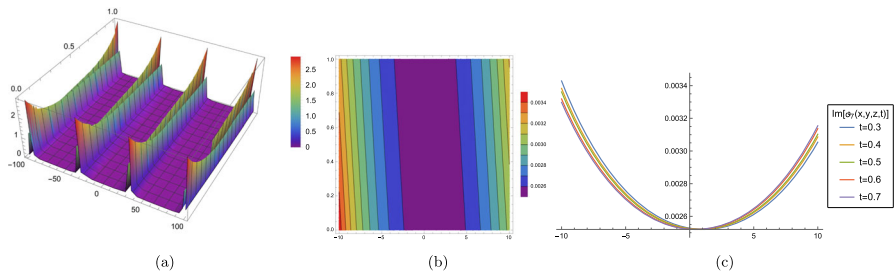
Fig. 4 Graphical illustration of $Im[v_4]$ of Eq. (38)

(a)   (b)   (c)

Fig. 5 Graphical illustration of $Im[v_7]$ of Eq. (43)
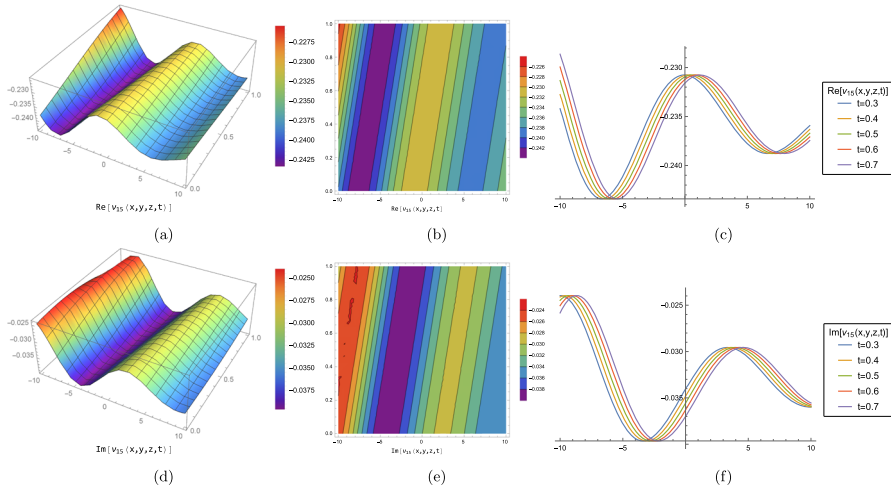
Fig. 6 Graphical illustration of $Re[v_{15}]$ and $Im[v_{15}]$ of Eq. (51)
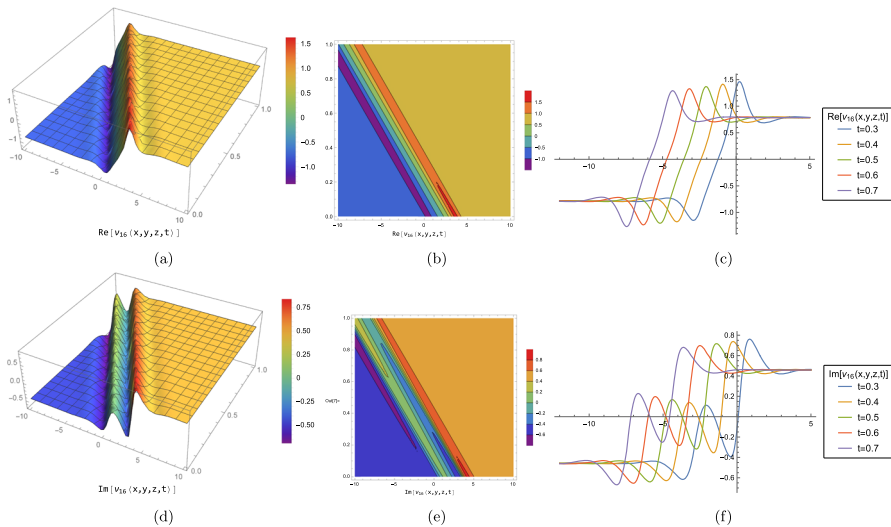
Fig. 7 Graphical illustration of $Re[v_{16}]$ and $Im[v_{16}]$ of Eq. (53)

- In Fig. 1 with (a), (b) for $\lambda = 0.15$, $\mu = 0.7$, $k = -0.15$, $y = 0.01$, $s = 0.16$, $z = 0.02$, $w = 0.7$, $\xi_0 = 0.1$, $\alpha = 0.1$, $\beta = 0.2$, $\gamma = 0.3$, singular soliton is observed.
- In Fig. 2 with (a), (b) for $\lambda = -0.15$, $\mu = 0.7$, $k = -0.15$, $y = 0.01$, $s = 0.16$, $z = 0.02$, $w = 0.7$, $\xi_0 = 0.1$, $\alpha = 0.1$, $\beta = 0.2$, $\gamma = 0.3$, periodic solution is observed.
- In Fig. 3 with (a), (b) for $\lambda = 0.15$, $\mu = -0.7$, $k = -0.15$, $y = 0.01$, $s = 0.16$, $z = 0.02$, $w = 0.7$, $\xi_0 = 0.1$, $\alpha = 0.1$, $\beta = 0.2$, $\gamma = 0.3$, dark-bright soliton is observed.

- In Fig. 4 with (a), (b) for $\lambda = -0.15, \mu = 0.7, k = -0.15, y = 0.01, s = 0.16, z = 0.02, w = 0.7, \xi_0 = 0.1, \alpha = 0.1, \beta = 0.2, \gamma = 0.3$, periodic solution is observed.
- In Fig. 5 with (a), (b) for $\lambda = 0.15, \mu = 0.7, k = -0.15, y = 0.1, s = 0.16, z = 0.1, w = 0.7, \xi_0 = 0.1, \alpha = 0.1, \beta = 0.2, \gamma = 0.3$, periodic solution is observed.
- In Fig. 6 with (a), (b), (d), (e) for $a = -0.64, k = 0.15, y = 0.05, s = 0.9, z = 0.03, w = 0.6, d = 0.01, \alpha = 0.01, \beta = 0.25, \gamma = 0.36$, lump soliton is observed.
- In Fig. 7 with (a), (b), (d), (e) for $a = -0.16, k = -0.5, y = -0.01, s = -0.16, z = -0.02, w = 0.8, d = -0.1, \alpha = 0.1, \beta = 0.2, \gamma = 0.3$, lump soliton is observed.

The graphical results reveal that the proposed approaches will assist the other related strong nonlinear models, leading to some novel soliton solutions. As a result, the findings in this study provide new knowledge to the existing literature because of their significance in the areas mentioned.

4 Conclusion

In this study, we examined the soliton properties of the integrable seventh-order extended modified Ito equation in $(3 + 1)$-dimensions, a generalized form of the bilinear KdV equation. The investigation employed the new extended hyperbolic function and modified Kudryashov methods. Subsequently, we presented 3D, contour, and 2D plots to visually convey certain solutions with their corresponding values. The accuracy of these approaches was affirmed through analytical results and graphical representations. Notably, the obtained solutions exhibited distinct physical characteristics highlighted in previous research. All solutions presented are novel and not documented in existing literature. Consequently, these methodologies offer potential applications for addressing and resolving various highly nonlinear PDEs. Some real-world scenarios or specific fields, such as quantum mechanics and nonlinear optics, may profit from the obtained solutions. In conclusion, this work has the potential to advance knowledge in the aforementioned areas and provide informative data for both theoretical and practical applications. As a future study, the authors might consider studying the fractional form of the equation.

Data availability statement Not applicable.

Declarations

Conflict of interest The authors declare that they have no Conflict of interest.

Ethical approval Not applicable.

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