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ORIGINAL RESEARCH

Dynamics and calculation of the basic reproduction number for a nonlocal dispersal epidemic model with air pollution

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Abstract

In order to reflect the dispersal of pollutants in non-adjacent areas and the large-scale movement of individuals, this paper proposes an epidemic model of nonlocal dispersal with air pollution, where the transmission rate is related to the concentration of pollutants. This paper checks the uniqueness and existence of the global positive solution and defines the basic reproduction number, R_0. We simultaneously explore the global dynamics: when $R_0 < 1$, the disease-free stable point is global asymptotic stability; when $R_0 > 1$, the disease is uniformly persistent. Additionally, in order to approximate R_0, a numerical method has been introduced. Illustrative examples are used to verify the theoretical outcomes and show the effect of the dispersal rate on the basic reproduction number R_0.

Keywords Dynamics · Numerical approximation · Nonlocal dispersal epidemic model · Basic reproduction number · Air pollution

Mathematics Subject Classification 35Q92 · 37N25 · 92D30

1 Introduction

In the process of human development of urbanization and industrialization, coal fire-power industries, automobile exhaust emissions, and the destruction of vegetation

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have caused air pollution [1, 2]. Nowadays, hundreds of millions of residents in China spend their days shrouded in serious air pollution, especially in winter [3], a situation that has attracted extensive attention from scholars. Studies using statistical and computational techniques have shown that air pollution threatens human health and affects the spread of communicable diseases [4, 5]. A large number of research results have also offered accumulated proof of the influence of pollutants in the air on the transmission of communicable diseases such as influenza, measles, COVID-19, and so on [6–9]. Therefore, it has been one of the most fun and significant topics to investigate the impact of pollutants in the air on the burst of communicable diseases [10].

Mathematical modeling plays an important role in better understanding the epidemic transmission mechanisms and more accurately predicting the trend of infectious diseases [11, 12]. Over the last decade, many ordinary differential equation models have been proposed to investigate infectious diseases influenced by air pollution [1, 13, 14]. In [15], Tang et al. characterized the dynamics of the air quality index (AQI) by a differential equation. Based on literature [15], He, Tang and Wang established a susceptible-infective-susceptible (SIS) model with air pollution [1]. When studying the dynamic behaviors of a system, the heterogeneity of each individual (position, age, sex, etc.) is a very important factor [16–19]. However, note that these studies did not consider spatial heterogeneity. In fact, there are some reports on the heterogeneity of air pollution (see, for instance, [20, 21] and the references therein). Additionally, individuals and air pollutants flow between areas [22, 23]. In epidemiology, more and more evidence shows that individual mobility and environmental heterogeneity have a outstanding effect on the transmission of communicable diseases [24, 25].

During the past few decades, some nonlocal dispersal infectious disease models have been established to discuss the impacts of environmental heterogeneity and individual mobility on the spread of communicable diseases (see, for instance, [22, 26–33] and the references therein). Among these works, Yang et al. found that the durability of communicable diseases can be enhanced through the nonlocal motion of infected or susceptible individuals [30]. In the case of heterogeneous circumstances, Wang et al. explored the dynamic behaviors of a foot-and-mouth model with nonlocal dispersal [31]. Under Dirichlet boundary condition, Yang and Li established a SIS epidemic model with nonlocal dispersal and discussed the dynamic behaviors in terms of R_0 [32]. Reference [23] showed that the pollutants in the air will move to non-adjacent areas. On the other hand, we know from the works of [22, 30, 32, 34] that individual movement is frequently unrestrained and should not be confined to a subregion. Therefore, considering nonlocal dispersal into epidemic models with air pollution can help us better understand the transmission mechanism of infectious diseases affected by air pollution. Nevertheless, it is worth pointing out that the nonlocal diffusion SIS infectious disease model with pollution in the air hasn't been established up to now.

By considering nonlocal dispersal into the model developed by He et al. [1], this paper establishes a nonlocal diffusion SIS infectious disease model with air pollution. Naturally, we focus on the definition of R_0 and how R_0 determines the dynamic behaviors of the proposed model. Additionally, R_0 for the model cannot be explicitly calculated in general since it has a complex form. Thus, approximating R_0 via a numerical method is also a significant and challenging work. The novelties and contributions of this paper are shown as follows:

- A nonlocal diffusion epidemic model with air pollution is formulated via taking into account the nonlocal dispersal phenomenon.
- The dynamic behaviors for the nonlocal diffusion infectious disease model with air pollution is explored in terms of R_0.
- A numerical method of approximating the basic reproduction number R_0 is introduced.

The rest of this article is planned as follows: Sect. 2 proposes the nonlocal dispersal epidemic model with air pollution and some preliminary knowledge. In Sect. 3, we demonstrate that the model has a unique global positive solution. Section 4 defines R_0 and analyzes the dynamic behaviors. Section 5 gives a numerical method to approximate R_0. Section 6 verifies the theoretical results by numerical simulations and finds that there is a negative correlation between the dispersal rate and R_0. Finally, Sect. 7 concludes the paper.

2 Model derivation and preliminary knowledge

It is well known that the occurrence of respiratory disease is closely related to air pollution, and the severity of air pollution has an impact on the risk of infection of respiratory disease. Based on this, He, Tang and Wang [1] established the SIS model with air pollution as follows:

$$\begin{cases} \dot{S}(t) = -\beta(F(t))S(t)I(t) + \gamma I(t), \\ \dot{F}(t) = -\theta F(t) + c, \\ \dot{I}(t) = -\gamma I(t) + \beta(F(t))S(t)I(t), \\ S(0) = S_0, \ F(0) = F_0, \ I(0) = I_0, \end{cases} \tag{2.1}$$

where $F(t)$, $S(t)$, and $I(t)$ denote AQI, the number of susceptible individuals, and the number of infected individuals at time t, respectively. c and θ represent the inflow and removal rates of pollutants, respectively. γ is the recovery rate for infected individuals. $\beta(F(t)) = \beta_0 F(t)$ is the infection rate. Since $\dot{S}(t) + \dot{I}(t) = 0$, one can obtain that $I(t) + S(t) = I_0 + S_0 = M$, where M is the population's size. β_0, θ, c, and γ are all positive numbers.

As we all know, the above model ignores the spatial heterogeneity and movement of individuals and pollutants, and of course does not show the spatial pattern of the diseases. In view of this, we will establish the corresponding partial differential equation model to address this shortcoming. Since individuals and pollutants are often dispersed to more distant locations with a certain probability, it is more appropriate to use an integral operator including probability density function to represent the movement of pollutants and individuals. There have been some works on SIS epidemic models with nonlocal dispersal, where the nonlocal dispersal is represented by an integral operator [30, 32]. Here, we introduce the nonlocal dispersal of pollutants in the air by using a method similar to that in the references [30, 32]. With the help of integral operator,

Table 1 List of notations and their meanings in model (2.3)

Notation	Biological meaning
$S(y,t)$	Population of susceptible individuals at position y and time t
$I(y,t)$	Population of infected individuals at position y and time t
$F(y,t)$	AQI at position y and time t
k_S	Dispersal rate for susceptible individuals
k_I	Dispersal rate for infected individuals
k_F	Dispersal rate for air pollutants
$\Lambda(y)$	Birth rate at position y
$\gamma(y)$	Recovery rate at position y
$\mu(y)$	Mortally rate at position y
$c(y)$	Emission rate of air pollutants at position y
$\theta(y)$	Removal rate of air pollutants at position y
β_0	Baseline transmission coefficient (see the definition in [13])

the movement of air pollutants in non-adjacent areas can be modeled as

$$\mathcal{L}_0\psi = \int_\Omega P(y-z)(\psi(z)-\psi(y))dz, \tag{2.2}$$

where $P(\cdot)$ is the function of probability density, which is a non negative even function integrating 1 in the whole space. $P(y-z)$ is the movement probability of air pollutants from place z to place y, the convolution $\int_\Omega P(y-z)\psi(z)dz$ is the rate at which the pollutants are reaching place y from Ω, and $-\int_\Omega P(z-y)\psi(y)dz = -\psi(y)$ is the rate at which pollutants in the air are leaving location y to travel to Ω. Based on model (2.1) and the nonlocal dispersal operator (2.2), a new nonlocal dispersal epidemic model including birth and death is established as follows:

$$\begin{cases} \dfrac{\partial S(y,t)}{\partial t} &= k_S \displaystyle\int_\Omega P(y-z)S(z,t)dz - k_S S(y,t) + \Lambda(y) + \gamma(y)I(y,t) \\ & \quad -\beta_0 F(y,t)S(y,t)I(y,t) - \mu(y)S(y,t), \qquad t>0,\ y\in\bar{\Omega}, \\ \dfrac{\partial F(y,t)}{\partial t} &= k_F \displaystyle\int_\Omega P(y-z)F(z,t)dz - k_F F(y,t) + c(y) - \theta(y)F(y,t), \\ & \qquad\qquad t>0,\ y\in\bar{\Omega}, \\ \dfrac{\partial I(y,t)}{\partial t} &= k_I \displaystyle\int_\Omega P(y-z)I(z,t)dz - k_I I(y,t) + \beta_0 F(y,t)S(y,t)I(y,t) \\ & \quad -\gamma(y)I(y,t) - \mu(y)I(y,t), \qquad t>0,\ y\in\bar{\Omega}, \\ S(y,0) &= S_0(y),\ F(y,0) = F_0(y),\ I(y,0) = I_0(y), \qquad x\in\bar{\Omega}, \end{cases} \tag{2.3}$$

where $\bar{\Omega}$ is the closure of Ω. We limit our focus to $\Omega \subset \mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. The meaning of all the notations in model (2.3) is listed in Table 1.

Remark 2.1 When the Lebesgue measure of the compact support set of the probability density function P tends to zero, the nonlocal convolution operator $k_u \int_\Omega P(y -$

$z)u(z,t)dz - k_u u(y,t)$ in model (2.3) degenerates to the local diffusion operator $\Delta u(x,t)$, where $u = S, F, I$. When the infection rate $\beta_0 F(y,t)$ is constantly equal to a constant, this model reduces to the classical nonlocal dispersal SIS epidemic model. When $k_S = k_F = k_I = \Lambda(y) = \mu(y) = 0$, and $S_0(y)$, $F_0(y)$, $I_0(y)$, $\gamma(y)$, $c(y)$, and $\theta(y)$ do not depend on the spatial location y, model (2.3) degenerates to model (2.1).

The assumptions on model (2.3) are given:

(a) $u(y)$ is Lipschitz continuous and strictly positive on $\bar{\Omega}$, and the following values exist as positive constants:

$$u^+ := \max_{y\in\bar{\Omega}} u(y),\ u^- := \min_{y\in\bar{\Omega}} u(y),$$

where $u = \Lambda, \gamma, \mu, c, \theta$.

(b) $K_i \geq 0$, where $i = S, F, I$.

(c) $P(y)$ is the function of Lipschitz continuity on $\bar{\Omega}$ and meets the properties:

$$P(y) = P(-y) \geq 0 \text{ on } \mathbb{R},\ P(y) > 0 \text{ on } \bar{\Omega},\ \int_{\mathbb{R}} P(y)dy = 1, \text{ and } P(0) > 0.$$

(d) $I(y,t) + S(y,t) = N(y)$, where $N(y)$ denotes the total number of people at position y.

Consider the following positive cones and function spaces:

$$\begin{aligned} X &:= C(\bar{\Omega}) = \{u : \bar{\Omega} \to \mathbb{R} : u \text{ is continuous}\},\ Z := [C(\bar{\Omega})]^3, \\ X_+ &:= C_+(\bar{\Omega}) = \{u \in C(\bar{\Omega}) : u \geq 0 \text{ on } \bar{\Omega}\},\quad Z_+ := [C_+(\bar{\Omega})]^3, \\ X_{++} &:= \{u \in C(\bar{\Omega}) : u > 0 \text{ on } \bar{\Omega}\}. \end{aligned}$$

X and Y are separately equipped with norms

$$\|\psi\|_X := \max_{y\in\bar{\Omega}} |\psi(y)|,\ \psi \in X \text{ and } \|\psi\|_Z := \max_{y\in\bar{\Omega}} \left(\sum_{i=1}^3 |\psi_i(y)|^2\right)^{\frac{1}{2}},\ \psi \in Z,$$

where $|\cdot|$ is the absolute value norm on $\mathbb{R}$.

On the space X, define the operators

$$A_S\psi(y) := k_S \int_\Omega P(y-z)\psi(z)dz - (\mu(y) + k_S)\psi(y), \tag{2.4}$$

$$\bar{A}_S\psi(y) := k_S \int_\Omega P(y-z)\psi(z)dz - (\mu(y) + \gamma(y) + k_S)\psi(y), \tag{2.5}$$

$$A_I\psi(y) := k_I \int_\Omega P(y-z)\psi(z)dz - (\gamma(y) + \mu(y) + k_I)\psi(y), \tag{2.6}$$

$$A_F\psi(y) := k_F \int_\Omega P(y-z)\psi(z)dz - (\theta(y) + k_F)\psi(y). \tag{2.7}$$

Under assumptions (a) and (c), A_S, $\bar{A}_S$, A_F, and A_I are all linear bounded operators. Therefore, it follows from [35] that they are infinitesimal generators of uniformly continuous semigroups $\{T_S(t)\}_{t\geq 0}$, $\{T_{\bar{S}}(t)\}_{t\geq 0}$, $\{T_F(t)\}_{t\geq 0}$, and $\{T_I(t)\}_{t\geq 0}$ on X, respectively. The variation of constants formula yields that the semigroups $\{T_S(t)\}_{t\geq 0}$, $\{T_{\bar{S}}(t)\}_{t\geq 0}$, $\{T_F(t)\}_{t\geq 0}$, and $\{T_I(t)\}_{t\geq 0}$ are all positive.

3 Uniqueness and existence of the global positive solution

This section proves that model (2.3) has a unique local positive solution and obtains that the solution is global with the help of the positive invariant set of model.

3.1 Uniqueness and existence of local positive solution

Theorem 3.1 *If $(S_0(y), F_0(y), I_0(y)) \in Z$, there exists $t_0 > 0$ such that model (2.3) admits the unique solution $(S(y,t), F(y,t), I(y,t))$ for all $t \in [0, t_0]$, and either $t_0 = +\infty$ or*

$$\limsup_{t\to t_0^-} \|(S(y,t), F(y,t), I(y,t))\|_Z = +\infty.$$

Proof Because A_S, A_F, and A_I are infinitesimal generators of uniformly continuous semigroups $\{T_S(t)\}_{t\geq 0}$, $\{T_F(t)\}_{t\geq 0}$, and $\{T_I(t)\}_{t\geq 0}$, respectively, the solution of model (2.3) can be expressed as

$$u(y,t) = T(t)u(\cdot,0)(y) + \int_0^t T(t-\tau)K(u(\cdot,\tau))(y)d\tau, \;\; t \geq 0, y \in \bar{\Omega},$$

where

$$u(y,t) := \begin{pmatrix} S(y,t) \\ F(y,t) \\ I(y,t) \end{pmatrix}, \qquad T(t) := \begin{pmatrix} T_S(t) \\ T_F(t) \\ T_I(t) \end{pmatrix},$$
$$K(u(\cdot,t))(y) := \begin{pmatrix} \Lambda(y) + (\gamma(y) - \beta_0 F(y,t)S(y,t))I(y,t) \\ c(y) \\ \beta_0 F(y,t)S(y,t)I(y,t) \end{pmatrix}.$$

For all $\psi = (\psi_1, \psi_2, \psi_3) \in Z$, let $K^{'}[\psi]$ be a linear operator on Z defined by

$$K^{'}[\psi](\varphi) := \begin{pmatrix} -\beta_0[\psi_2\psi_3\varphi_1 + \psi_1\psi_3\varphi_2 + (\psi_1\psi_2 - \gamma)\varphi_3] \\ 0 \\ \beta_0[\psi_2\psi_3\varphi_1 + \psi_1\psi_3\varphi_2 + \psi_1\psi_2\varphi_3] \end{pmatrix}, \tag{3.1}$$

where $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in Z$. Then, it is not hard to verify that

$$K(\varphi - \psi)(y) = K^{'}[\psi](\varphi - \psi)(y) + \begin{pmatrix} -\beta_0[\Pi_{i=1}^{3}(\varphi_i(y) - \psi_i(y)) + m(y)] \\ 0 \\ \beta_0[\Pi_{i=1}^{3}(\varphi_i(y) - \psi_i(y)) + m(y)] \end{pmatrix}, \tag{3.2}$$

where

$$\begin{aligned} m(y) =& (\varphi_2(y) - \psi_2(y))(\varphi_3(y) - \psi_3(y))\psi_1(y) + (\varphi_3(y) - \psi_3(y))\psi_2(y) \\ & \times (\varphi_1(y) - \psi_1(y)) + (\varphi_1(y) - \psi_1(y))(\varphi_2(y) - \psi_2(y))\psi_3(y). \end{aligned}$$

The definition of norm gives that the last term in the right-hand side of Eq. (3.2) is an infinitesimal of higher order than $\varphi - \psi$. This implies that K is Fréchet differentiable for each $\psi \in Z$, and $K'[\psi]$ defined by Eq. (3.1) is the Fréchet derivative of K at ψ. Furthermore, we conclude from observing Eq. (3.1) that $K'[\psi]$ is continuous with respect to ψ. Hence, K is continuously Fréchet differentiable on Z. With the help of [36, Proposition 4.16], the assertion can be obtained, and this ends the proof. □

Theorem 3.1 shows model (2.3) admits a unique local solution. Next, the forthcoming Lemma 3.1 will give its positivity.

Lemma 3.1 *Assume that* $(S(y,t), F(y,t), I(y,t)) \in Z$ *be the solution of model* (2.3) *with* $(S_0(y), F_0(y), I_0(y)) \in Z_+$*. Then,* $(S(y,t), F(y,t), I(y,t)) \in Z_+$ *for all* $t \in [0, t_0)$*, and either* $t_0 = +\infty$ *or*

$$\limsup_{t \to t_0^-} \|u(y,t)\|_Z = +\infty.$$

Proof By the same method as [22, Proposition 2.2], this proof can be completed. □

According to Theorem 3.1 and Lemma 3.1, for model (2.3), a positive continuous semiflow $\{\Phi_t\}_{t\in[0,t_0)} : Z_+ \to Z_+$ can be define as follows:

$$\Phi_t((S_0(y), F_0(y), I_0(y))) := u(\cdot, t),\ t \in [0, t_0),\ (S_0(y), F_0(y), I_0(y)) \in Z_+. \tag{3.3}$$

3.2 Positive invariant set

Let $E^0 := (S^0(y), F^0(y), 0)$ denote the disease-free stable point of model (2.3), where $S^0(y) \in X_+$ and $F^0(y) \in X_+$ meet the equations

$$0 = k_F \int_\Omega P(y-z)F^0(z)dz - (k_F + \theta(y))F^0(y) + c(y),\ y \in \bar{\Omega} \tag{3.4}$$

and

$$0 = k_S \int_\Omega P(y-z)S^0(z)dz - (k_S + \mu(y))S^0(y) + \Lambda(y), \ y \in \bar{\Omega}. \tag{3.5}$$

The following Lemma 3.2 will give the uniqueness and existence of the disease-free stable point E_0, the Lipschitz continuity of $S^0(y)$ and $F^0(y)$, and the boundedness of $F(y,t)$ and $S(y,t)$.

Lemma 3.2 *The following assertions hold.*

1. *model* (2.3) *exists the unique disease-free stable point* E^0.
2. $S^0(y)$ *and* $F^0(y)$ *are both Lipschitz continuous on* Ω.
3. *If* $F_0(y) \leq F^0(y)$ *and* $S_0(y) \leq S^0(y)$ *for all* $y \in \bar{\Omega}$*, then* $F(y,t) \leq F^0(y)$ *and* $S(y,t) \leq S^0(y)$ *for all* $(y,t) \in \bar{\Omega} \times [0, t_0)$.

Proof (1) With the help of the operator A_F defined by Eq. (2.7), Eq. (3.4) is equivalent to the following equation

$$A_F F^0(y) + c(y) = 0, \quad y \in \bar{\Omega}. \tag{3.6}$$

Through the previous discussion, A_F is the infinitesimal generator of positive uniformly continuous semigroup $\{T_F(t)\}_{t\geq 0}$ on X. According to [37, Theorem 3.12], it is easy to get that A_F is a resolvent-positive operator. Additionally, it follows from [32] that $s(A_F) < 0$, where $s(\cdot)$ is the spectral bound of an operator. Therefore, taking $\lambda = 0$ into $(\lambda I - A_F)^{-1}$, it follows from [37, Theorem 3.12] that

$$(-A_F)^{-1}\psi(y) = \lim_{b\to+\infty} \int_0^b e^{tA_F}\psi(y)dt, \quad \psi(y) \in X.$$

From Eq. (3.6), it is not hard to get that

$$F^0(y) = (-A_F)^{-1}c(y) = \lim_{b\to+\infty} \int_0^b e^{tA_F}c(y)dt, \quad y \in \bar{\Omega}. \tag{3.7}$$

By the same method, it follows that

$$S^0(y) = (-A_S)^{-1}\Lambda(y) = \lim_{b\to+\infty} \int_0^b e^{tA_S}\Lambda(y)dt, \quad y \in \bar{\Omega}.$$

Hence, $S^0(y)$ and $F^0(y)$ are all positive and uniquely exist.

(2) By virtue of Eq. (3.4), it follows that

$$F^0(y) = \frac{k_F \int_\Omega P(y-z)F^0(z)dz + c(y)}{k_F + \theta(y)}, \quad y \in \bar{\Omega}. \tag{3.8}$$

with the help of Assumption (c), it is easy to get that

$$
\begin{aligned}
&\|k_F \int_\Omega P(y-z)F^0(z)dz - k_F \int_\Omega P(\tilde{y}-z)F^0(z)dz\|_X \\
&\le k_F \int_\Omega \|P(y-z) - P(\tilde{y}-z)\|_X F^0(z)dz, \\
&\le k_F L_P \int_\Omega F^0(z)dz|y-\tilde{y}|, \qquad y, \tilde{y} \in \bar{\Omega},
\end{aligned}
$$

where $L_P > 0$ is the Lipschitz constant of $P(y)$. Hence, combining the Lipschitz continuity of $c(y)$ and the boundedness of $\theta(y)$, it follows that $F^0(y)$ given as Eq. (3.8) is Lipschitz continuous on $\bar{\Omega}$. Similarly,

$$
S^0(y) = \frac{k_S \int_\Omega P(y-z)S^0(z)dz + \Lambda(y)}{k_S + \mu(y)}, \quad y \in \bar{\Omega}
$$

is also Lipschitz continuous on $\bar{\Omega}$.

(3) Eq. (3.5) provides that

$$
\Lambda(y) = -k_S \int_\Omega P(y-z)S^0(z)dz + (k_S + \mu(y))S^0(y), \quad y \in \bar{\Omega}. \tag{3.9}
$$

Let $\tilde{S}(y,t) := S^0(y) - S(y,t)$, $(y,t) \in \bar{\Omega} \times [0, t_0)$. According to the second equation of model (2.3), Eq. (3.9), and assumption (d), it is not hard to get that

$$
\begin{aligned}
\frac{\partial}{\partial t}\tilde{S}(y,t) &= -\frac{\partial}{\partial t}S(y,t), \\
&= (k_S + \mu(y) + \beta_0 F(y,t)I(y,t))S(y,t) \\
&\quad -k_S \int_\Omega P(y-z)S(z,t)dz - \Lambda(y) - \gamma(y)I(y,t), \\
&= k_S \int_\Omega P(y-z)\tilde{S}(z,t)dz - (k_S + \mu(y))\tilde{S}(y,t) \\
&\quad -[\gamma(y) - \beta_0 F(y,t)S(y,t)]I(y,t), \\
&= \bar{A}_S \tilde{S}(y,t) + \beta_0 F(y,t)S(y,t)I(y,t), \quad (y,t) \in \bar{\Omega} \times [0, t_0).
\end{aligned} \tag{3.10}
$$

Since $\bar{A}_S$ is the infinitesimal generator of the positive semigroup $\{T_{\bar{S}}(t)\}_{t\ge 0}$. Therefore, the solution $\tilde{S}(y,t)$ of Eq. (3.10) is equivalent to

$$
\tilde{S}(y,t) = e^{t\bar{A}_S}\tilde{S}(\cdot,0)(y) + \int_0^t e^{(t-\tau)\bar{A}_S}[\beta_0 F(\cdot,\tau)S(\cdot,\tau)I(\cdot,\tau)](y)d\tau.
$$

In virtue of the positivity of the semigroup $\{T_{\bar{S}}(t)\}_{t\ge 0}$, the initial value $\tilde{S}(y,0) = S^0(y) - S_0(y)$ for all $y \in \bar{\Omega}$, and the function $\beta_0 F(y,t)S(y,t)I(y,t)$ for all

$(y,t) \in \bar{\Omega} \times [0, t_0)$, it follows $\tilde{S}(y,t) = S^0(y) - S(y,t) \geq 0$ for all $(y,t) \in \bar{\Omega} \times [0, t_0)$, i.e., $S(y,t) \leq S^0(y)$ for all $(y,t) \in \bar{\Omega} \times [0, t_0)$. Similarly, we can prove that if $F_0(y) \leq F^0(y)$, then $F(y,t) \leq F^0(y)$ for all $(y,t) \in \bar{\Omega} \times [0, t_0)$. This ends the proof.

□

The state space of model (2.3) can be defined as

$$D := \left\{(S, F, I) \in Z_+ : S \leq S^0, F \leq F^0, \int_\Omega (S + I)dy \leq \frac{\Lambda^+ \mathfrak{M}(\Omega)}{\mu^-}, \forall t \geq 0\right\},$$

where $\mathfrak{M}(\Omega)$ denotes the Lebesgue measure of Ω.

Theorem 3.2 *For model* (2.3), *D is positively invariant.*

Proof According to the first and third equations of model (2.3) and assumptions (a) and (c), it is not hard to get that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega S(y,t)dy \\
&\quad = \int_\Omega [k_S \int_\Omega P(y-z)S(z,t)dz - k_S S(y,t) + \Lambda(y) + \gamma(y)I(y,t)]dy \\
&\qquad - \int_\Omega [\beta_0 F(y,t)S(y,t)I(y,t) + \mu(y)S(y,t)]dy, \\
&\quad \leq k_S \int_\Omega \int_{\mathbb{R}} P(y-z)dyS(z,t)dz - k_S \int_\Omega S(y,t)dy + \int_\Omega \Lambda(y)dy \\
&\qquad + \int_\Omega [\gamma(y)I(y,t) - \beta_0 F(y,t)S(y,t)I(y,t) - \mu(y)S(y,t)]dy, \\
&\quad \leq \Lambda^+ \mathfrak{M}(\Omega) + \int_\Omega [I(y,t)(\gamma(y) - \beta_0 F(y,t)S(y,t)) - \mu^- S(y,t)]dy, \qquad (3.11)
\end{aligned}$$

and

$$\frac{d}{dt}\int_\Omega I(y,t)dy \leq \int_\Omega [I(y,t)(\beta_0 F(y,t)S(y,t) - \gamma(y)) - \mu^- I(y,t)]dy. \qquad (3.12)$$

Adding Eqs. (3.11) and (3.12), it is immediate to get that

$$\frac{d}{dt}\int_\Omega (S(y,t) + I(y,t))dy \leq \Lambda^+ \mathfrak{M}(\Omega) - \mu^- \int_\Omega (S(y,t) + I(y,t))dy.$$

If $\int_\Omega (S_0(y) + I_0(y))dy \leq \frac{\Lambda^+ \mathfrak{M}(\Omega)}{\mu^-}$, the comparison principle and the variation of constants formula give that

$$\begin{aligned}
\int_\Omega (S(y,t) + I(y,t))dy &\leq \int_\Omega (S_0(y) + I_0(y))dy e^{-\mu^- t} + \frac{\Lambda^+ \mathfrak{M}(\Omega)}{\mu^-}[1 - e^{-\mu^- t}], \\
&\leq \frac{\Lambda^+ \mathfrak{M}(\Omega)}{\mu^-}, \qquad t \in [0, t_0).
\end{aligned} \qquad (3.13)$$

Combining Eq. (3.13) with Lemma 3.2, it follows that for model (2.3), D is positively invariant. This completes the proof. □

Remark 3.3 It follows from Theorem 3.1, Lemma 3.1, and Theorem 3.2 that for all $(S_0(y), F_0(y), I_0(y)) \in D$ model (2.3) admits the unique global positive solution $(S(y,t), F(y,t), I(y,t))$ for all $(y,t) \in \bar{\Omega} \times [0, \infty)$.

4 Threshold dynamics

As we all know, when discussing the dynamics of infectious disease models, R_0 serves as a crucial threshold. To define R_0, we firstly linearize the third equation of model (2.3) around the disease-free stable point E^0 as follows:

$$\begin{aligned}\frac{\partial I(y,t)}{\partial t} =& k_I \int_\Omega P(y-z)I(z,t)dz - k_I I(y,t) + \beta_0 F^0(y)S^0(y)I(y,t) \\ &- \gamma(y)I(y,t) - \mu(y)I(y,t), \qquad (y,t) \in \bar{\Omega} \times [0, +\infty).\end{aligned} \tag{4.1}$$

On X, we define the linear operator $\mathcal{F}$ as follows:

$$\mathcal{F}\phi(y) := \beta_0 F^0(y)S^0(y)\phi(y), \quad \phi \in X.$$

Subsequently, with the help of the operator A_I defined by Eq. (2.6), Eq. (4.1) can be rewritten as the abstract form in X,

$$\frac{dI(t)}{dt} = A_I I(t) + \mathcal{F}I(t), \quad t \in (0, \infty).$$

With the help of the proof of Lemma 3.2, it is easy to get that A_I is resolvent-positive, $s(A_I) < 0$, and

$$(-A_I)^{-1}\phi(y) = \lim_{b \to +\infty} \int_0^b e^{tA_I}\phi(y)dt, \phi \in X.$$

Therefore, inspired by the work of [38], the operator of next generation $\hbar := \mathcal{F}(-A_I)^{-1}$ can be obtained via

$$\hbar\phi(y) = \lim_{a \to \infty} \beta_0 F^0(y)S^0(y) \int_0^a e^{tA_I}\phi(y)dt, \phi \in X,$$

and R_0 is obtained by

$$R_0 := r(\hbar), \tag{4.2}$$

where $r(\hbar)$ is the spectral radius of the next generation operator $\hbar$.

4.1 Disease extinction

Associated with Eq. (4.1), we discuss the eigenvalue problem

$$\begin{aligned} & k_I \int_\Omega P(y-z)u(z)dz - k_I u(y) + \beta_0 F^0(y) S^0(y) u(y) \\ & = (\lambda + \gamma(y) + \mu(y))u(y), \qquad y \in \bar{\Omega}. \end{aligned} \tag{4.3}$$

According to assumptions (a) and (b) and Lemma 3.2, it is immediate to get $\beta_0 F^0(y)S^0(y) - (k_I + \mu(y) + \gamma(y))$ is Lipschitz continuous on $\bar{\Omega}$. Therefore, from [39], it is not hard to get the forthcoming Lemma 4.1.

Lemma 4.1 *Eigenvalue problem (4.3) admits a principal eigenvalue λ_*, and it corresponds to a eigenfunction $u_0(y) \in X_{++}$. More specifically, λ_* is obtained via*

$$\begin{aligned} \lambda_* = \max_{\|u\|_{L^2(\Omega)}=1} \Big\{ & k_I \int_\Omega \int_\Omega P(y-z)u(y)u(z)dzdy \\ & + \int_\Omega [\beta_0 F^0(y)S^0(y) - (k_I + \gamma(y) + \mu(y))]u^2(y)dy \Big\}. \end{aligned}$$

From [40], we can get $\lambda_* = s(A_I + \mathcal{F})$. From [37], we have the following Lemma 4.2.

Lemma 4.2 *The sign of $R_0 - 1$ and λ_* are not different.*

Lemma 4.2 establishes the relationship between R_0 and λ_*. Next, when $R_0 < 1$, the global dynamic behavior of the disease-free stable point of model (2.3) will be analyzed with the help of Lemmas 4.1 and 4.2.

Theorem 4.1 *If $R_0 < 1$, then the disease-free steady state E^0 is globally asymptotically stable.*

Proof We choose the Lyapunov function

$$L(t) := \int_\Omega u_0(y) I(y,t)dy, \quad (y,t) \in \bar{\Omega} \times [0, +\infty),$$

where u_0 is the eigenfunction of eigenvalue problem (4.3), and its eigenvalue is λ_*. It is not hard to discover that $L \geq 0$ and $L = 0$ iff $I \equiv 0$. By direct calculation, it follows that

$$\begin{aligned} \frac{dL}{dt} &= \int_\Omega u_0(y) \frac{\partial}{\partial t} I(y,t)dy, \\ &= \int_\Omega u_0(y)[k_I \int_\Omega P(y-z)I(z,t)dz - k_I I(y,t) + \beta_0 F(y,t)S(y,t)I(y,t) \\ &\quad -(\gamma(y) + \mu(y))I(y,t)]dy. \end{aligned} \tag{4.4}$$

With the help of assumption (c) and Eq. (4.3), it follows that

$$\begin{aligned}\int_\Omega u_0(y)k_I \int_\Omega P(y-z)I(z,t)dzdy &= k_I \int_\Omega \int_\Omega P(y-z)u_0(y)I(z,t)dydz\\ &= k_I \int_\Omega I(z,t)\int_\Omega P(y-z)u_0(y)dydz = k_I \int_\Omega I(y,t)\int_\Omega P(y-z)u_0(z)dzdy,\\ &= \int_\Omega I(y,t)[k_I u_0(y) - \beta_0 F^0(y)S^0(y)u_0(y) + (\gamma(y)+\mu(y))u_0(y) + \lambda_* u_0(y)]dy.\end{aligned} \tag{4.5}$$

Then, substituting Eq. (4.5) into Eq. (4.4), we have

$$\frac{dL}{dt} = \int_\Omega u_0(y)[\lambda_* I(y,t) + [\beta_0 F(y,t)S(y,t) - \beta_0 F^0(y)S^0(y)]I(y,t)]dy.$$

By virtue of Lemma 4.2, it is easy to get that if $R_0 < 1$, then $\lambda_* < 0$. In addition, since $u_0 \in X_{++}$, $\frac{dL}{dt} = 0$ iff $I \equiv 0$. Hence, we can get that the disease-free steady state E^0 is globally asymptotically stable. This completes the proof. □

4.2 Uniform persistence

For the convenience of future discussion, define the following space:

$$\begin{aligned}D_0 &:= \left\{(S,F,I)\in D : I \not\equiv 0\right\},\ \partial D := \left\{(S,F,I)\in D : I \equiv 0\right\},\\ N_\partial &:= \left\{(S_0,F_0,I_0)\in \partial D : \Phi_t((S_0,F_0,I_0)) \in \partial D,\ t\in[0,\infty)\right\}.\end{aligned}$$

Before discussing uniform persistence of model (2.3), it is necessary to give the definition of uniform persistence.

Definition 4.2 If there admits a number $\zeta > 0$ such that

$$\min\left\{\liminf_{t\to+\infty} S(y,t), \liminf_{t\to+\infty} F(y,t), \liminf_{t\to+\infty} I(y,t)\right\} \geq \zeta$$

for any initial value $(S_0, F_0, I_0) \in D_0$, then model (2.3) is said to be uniformly persistent in D_0.

Remark 4.3 The uniform persistence of model (2.3) can be used to describe the long-term existence of airborne pollutants, susceptible individuals, and infected individuals. On the other hand, the work of [41] shows that a dissipative, uniformly persistent, and asymptotically compact model must admit a endemic equilibrium.

In order to discuss the uniform persistence of model (2.3) when $R_0 > 1$, we give Lemma 4.3 and Lemma 4.4.

Lemma 4.3 $\omega((S_0, F_0, I_0)) = \{(S^0, F^0, 0)\} = \{E^0\}$ *for any* $(S_0, F_0, I_0) \in N_\partial$, *where* $\omega((S_0, F_0, I_0))$ *denotes the* ω *limit set of the positive orbit* $\{\Phi_t((S_0, F_0, I_0))\}_{t\geq 0}$.

Proof According to model (2.3), it is not hard to get that for any $(S_0, F_0, I_0) \in N_\partial$,

$$\begin{cases} \dfrac{\partial S(y,t)}{\partial t} = k_S \displaystyle\int_\Omega P(y-z)S(z,t)dz - k_S S(y,t) + \Lambda(y) - \mu(y)S(y,t), \\ \dfrac{\partial F(y,t)}{\partial t} = k_F \displaystyle\int_\Omega P(y-z)F(z,t)dz - k_F F(y,t) + c(y) - \theta(y)F(y,t), \end{cases}$$

for all $(y,t) \in \bar{\Omega} \times [0, +\infty)$. In virtue of Eqs. (3.4) and (3.5), we have that

$$\begin{cases} \dfrac{\partial S(y,t)}{\partial t} = k_S \displaystyle\int_\Omega P(y-z)(S(z,t) - S^0(z))dz - (k_S + \mu(y))(S(y,t) - S^0(y)), \\ \dfrac{\partial F(y,t)}{\partial t} = k_F \displaystyle\int_\Omega P(y-z)(F(z,t) - F^0(z))dz - (k_F + \theta(y))(F(y,t) - F^0(y)), \end{cases} \tag{4.6}$$

for all $(y,t) \in \bar{\Omega} \times [0, +\infty)$. Set $u_1(y,t) := S^0(y) - S(y,t)$ and $u_2(y,t) := F^0(y) - F(y,t)$. Then, Eq. (4.6) can be rewritten as follow:

$$\begin{cases} \dfrac{\partial u_1(y,t)}{\partial t} = k_S \displaystyle\int_\Omega P(y-z)u_1(z,t)dz - (k_S + \mu(y))u_1(y.t), \\ \dfrac{\partial u_2(y,t)}{\partial t} = k_F \displaystyle\int_\Omega P(y-z)u_2(z,t)dz - (k_F + \theta(y))u_2(y.t), \end{cases} \tag{4.7}$$

for all $(y,t) \in \bar{\Omega} \times [0, +\infty)$. From Eqs. (2.4) and (2.7), Eq. (4.7) is equivalent to the following form in X_+:

$$\frac{d}{dt}u_1(t) = A_S u_1(t), \;\; \frac{d}{dt}u_2(t) = A_F u_2(t), \;\; t > 0.$$

Since $k_F + \theta(y)$ and $k_S + \mu(y)$ are both Lipschitz continuous, it follows from the same method as obtaining Lemma 4.1 that there exist the principal eigenvalues $\lambda_1^* = s(A_S) < 0$ and $\lambda_2^* = s(A_F) < 0$, and they correspond the eigenfunctions $v_1^*(y) \in X_{++}$ and $v_2^*(y) \in X_{++}$, respectively. For convenience, introduce the following marks:

$$v^*(y) = (v_1^*(y), v_2^*(y)), \;\; U(y,t) := (u_1(y,t), u_2(y,t))^\top.$$

Now, we choose Lyapunov function

$$L(t) = \int_\Omega v^*(y)U(y,t)dy.$$

Obviously, $L \geq 0$, and $L = 0$ iff $U(y,t) \equiv \mathbf{0}$, where $\mathbf{0}$ is $(0,0)^\top$. The direct calculation yields that

$$\begin{aligned}\frac{dL}{dt} &= \int_\Omega v^*(y)\frac{\partial}{\partial t}U(y,t)dy,\\ &= \int_\Omega v_1^*(y)[k_S\int_\Omega P(y-z)u_1(z,t)dz - (k_S+\mu(y))u_1(y,t)]dy \\ &\quad + \int_\Omega v_2^*(y)[k_F\int_\Omega P(y-z)u_2(z,t)dz - (k_F+\theta(y))u_2(y,t)]dy.\end{aligned} \tag{4.8}$$

With the help of assumption (c), it follows that

$$\begin{aligned}&\int_\Omega v_1^*(y)k_S\int_\Omega P(y-z)u_1(z,t)dzdy = k_S\int_\Omega\int_\Omega P(y-z)v_1^*(y)u_1(z,t)dydz,\\ &= k_S\int_\Omega u_1(z,t)\int_\Omega P(y-z)v_1^*(y)dydz = k_S\int_\Omega u_1(y,t)\int_\Omega P(y-z)v_1^*(z)dzdy,\\ &= \int_\Omega u_1(y,t)[(k_S+\mu(y)+\lambda_1^*)v_1^*(y)]dy.\end{aligned} \tag{4.9}$$

Similarly, it follows that

$$\int_\Omega v_2^*(y)k_F\int_\Omega P(y-z)u_2(z,t)dzdy = \int_\Omega u_2(y,t)[(k_F+\theta(y)+\lambda_2^*)v_2^*(y)]dy. \tag{4.10}$$

Then, substituting Eqs. (4.9) and (4.10) into Eq. (4.8), it follows that

$$\frac{dL}{dt} = \lambda_1^*\int_\Omega v_1^*(y)u_1(y,t)dy + \lambda_2^*\int_\Omega v_2^*(y)u_2(y,t)dy.$$

Since $-\lambda_1^*$, $-\lambda_2^*$, $v_1^*(y)$, and $v_2^*(y)$ are all strictly positive, it follows that $\frac{dL}{dt} \leq 0$, and $\frac{dL}{dt} = 0$ iff $U(y,t) \equiv \mathbf{0}$, that is, $S(y,t) = S^0(y)$ and $F(y,t) = F^0(y)$. This implies that $\omega((S_0,F_0,I_0)) = \{(S^0,F^0,0)\} = \{E^0\}$. This ends the proof. □

Definition 4.4 If there admits a constant $\epsilon > 0$ such that

$$\limsup_{t\to+\infty}\|\Phi(t)((S_0,F_0,I_0)) - (S^0,F^0,0)\|_Z \geq \epsilon$$

for any $(S_0,F_0,I_0) \in D_0$, then E^0 is a uniform weak repeller for D_0.

The following task is to establish that E^0 is a uniform weak repeller associated with D_0.

Lemma 4.4 *If $R_0 > 1$, then E^0 is a uniform weak repeller for D_0.*

Proof Lemma 4.2 gives that $R_0 > 1$ means $\lambda_* > 0$. This implies that there exists a constant $0 < \epsilon < \min_{y\in\bar{\Omega}}[F^0(y) + S^0(y)]$ such that the eigenvalue problem,

$$\begin{aligned}\vartheta\varphi(y) = & k_I \int_\Omega P(y-z)\varphi(z)dz - k_I\varphi(y) + \beta_0(F^0(y)-\epsilon)(S^0(y)-\epsilon)\varphi(y) \\ & - (\gamma(y)+\mu(y))\varphi(y),\ y \in \bar{\Omega},\end{aligned} \tag{4.11}$$

exists a principal eigenvalue $\vartheta_0 > 0$ corresponding to a eigenfunction $\varphi_0(y) \in X_{++}$. On the contrary, assume that

$$\limsup_{t\to+\infty} \|\Phi(t)((S_0, F_0, I_0)) - (S^0, F^0, 0)\|_Z < \epsilon \tag{4.12}$$

and show a contradiction. By the definition of upper limit, Eq. (4.12) means that there exists a constant $T_0 > 0$ such that

$$S(y,t) > S^0(y) - \epsilon \text{ and } F(y,t) > F^0(y) - \epsilon \tag{4.13}$$

for all $(y,t) \in \bar{\Omega} \times [T_0, +\infty)$. With the help of assumptions (a)-(c) and the variation of constants formula, it is not hard to get that if $(S_0, F_0, I_0) \in D_0$, then

$$S(y,t) > 0,\ F(y,t) > 0, \text{ and } I(y,t) > 0$$

for all $(y,t) \in \bar{\Omega} \times [T_0, +\infty)$. In virtue of the positivity of $\varphi_0(y)$ and $I(y, T_0)$ for all $y \in \bar{\Omega}$, it is immediate to get that there admits a positive constant $\bar{\epsilon}$ such that

$$I(y, T_0) \geq \bar{\epsilon}\varphi_0(y),\ \forall y \in \bar{\Omega}.$$

Denote the solution of the equation

$$\begin{cases} \dfrac{\partial I^\varsigma(y,t)}{\partial t} = k_I \displaystyle\int_\Omega P(y-z)I^\varsigma(z,t)dz - k_I I^\varsigma(y,t) - (\gamma(y)+\mu(y))I^\varsigma(y,t) \\ \qquad + \beta_0(F^0(y)-\epsilon)(S^0(y)-\epsilon)I^\varsigma(y,t), \qquad t \geq t_0,\ y \in \bar{\Omega}, \\ I^\varsigma(y, T_0) = \bar{\epsilon}\varphi_0(y), \qquad y \in \bar{\Omega}, \end{cases} \tag{4.14}$$

by $I^\varsigma(y,t)$. Combined with Eq. (4.11), it follows that

$$I^\varsigma(y,t) = e^{\vartheta_0(t-T_0)}\bar{\epsilon}\varphi_0(y),\ \forall\,(y,t) \in \bar{\Omega} \times [T_0, +\infty).$$

Set

$$I^\iota(y,t) := I(y,t) - I^\varsigma(y,t),\ \forall\,(y,t) \in \bar{\Omega} \times [T_0, +\infty).$$

Then, according to the third equation of model (2.3) and Eq. (4.14), it follows that

$$\frac{\partial I^\iota(y,t)}{\partial t} = \frac{\partial I(y,t)}{\partial t} - \frac{\partial I^\varsigma(y,t)}{\partial t}$$

$$
\begin{aligned}
&= k_I \int_\Omega P(y-z) I^\iota(z,t)dz - k_I I^\iota(y,t) - (\gamma(y)+\mu(y))I^\iota(y,t) + \beta_0[F(y,t)\\
&\quad \times S(y,t)I(y,t) - (F^0(y)-\epsilon)(S^0(y)-\epsilon)I^\varsigma(y,t)],\\
&= A_I I^\iota(\cdot,t)(y) + \beta_0[F(y,t)S(y,t)I(y,t) - (F^0(y)-\epsilon)(S^0(y)-\epsilon)I^\varsigma(y,t)],
\end{aligned} \tag{4.15}
$$

for all $(y,t) \in \bar{\Omega} \times [T_0, +\infty)$. Because positive semigroup $\{T_I(t)\}_{t\geq 0}$ is generated by A_I, the solution $I^\iota(y,t)$ of Eq. (4.15) can be written as

$$
\begin{aligned}
I^\iota(y,t) = &\int_{T_0}^t T_I(t-\kappa)\beta_0[F(\cdot,\kappa)S(\cdot,\kappa)I(\cdot,\kappa) - (F^0(\cdot)-\epsilon)(S^0(\cdot)-\epsilon)\\
&\times I^\varsigma(\cdot,\kappa)](y)d\kappa + T_I(t-T_0)I^\iota(\cdot,T_0)(y), \quad t \geq T_0,\ y \in \bar{\Omega}.
\end{aligned}
$$

Then, with the help of the [42, Theorem 2.1] and Eq. (4.13), it is easy to get that

$$
\begin{aligned}
&I^\iota(y,t) \geq T_I(t-T_0)I^\iota(\cdot,T_0)(y)\\
&+ \int_{T_0}^t T_I(t-\kappa)\beta_0[(F^0(\cdot)-\epsilon)(S^0(\cdot)-\epsilon)I^\iota(\cdot,\kappa)](y)d\kappa,
\end{aligned}
$$

for all $t \geq T_0$, $y \in \bar{\Omega}$. Hence, inspired by the proof of Lemma 3.2, it is not hard to get that

$$
I^\iota(y,t) = I(y,t) - I^\varsigma(y,t) = I(y,t) - e^{\vartheta_0(t-T_0)}\bar{\epsilon}\varphi_0(y) \geq 0,\ t \geq T_0,\ y \in \bar{\Omega}.
$$

Subsequently, we have

$$
I(y,t) \geq e^{\vartheta_0(t-T_0)}\bar{\epsilon}\varphi_0(y),\ t \geq T_0,\ y \in \bar{\Omega}.
$$

By taking limits on both sides of the above inequality, it is immediate to get that

$$
\liminf_{t\to+\infty} I(y,t) \geq \liminf_{t\to+\infty} e^{\vartheta_0(t-T_0)}\bar{\epsilon}\varphi_0(y) = +\infty.
$$

This is opposite to the boundedness shown by Theorem 3.2. This ends the proof. □

Remark 4.5 Lemma 4.4 shows that when $R_0 > 1$, the solution of model (2.3) with $(S_0, F_0, I_0) \in D_0$ does not tend to the disease-free steady state E^0. It means that for $\Phi(t)$ in D_0, $\{E^0\}$ is an isolated invariant set.

Theorem 4.6 *If $R_0 > 1$, then model (2.3) with initial condition $(S_0, F_0, I_0) \in D_0$ is uniformly persistent.*

Proof In virtue of Lemmas 4.3 and 4.4, it is not hard to get $E^0 = (S^0, F^0, 0)$ is isolated in D, and $D_0 \cap W^s((S^0, F^0, 0)) = \emptyset$, where $W^s((S^0, F^0, 0))$ and $\emptyset$ denote the stable manifold of $(S^0, F^0, 0)$ and empty set, respectively. Hence, with the help of the [42], there exists a constant $\eta > 0$ such that

$$\min_{(\varphi_1,\varphi_2,\varphi_3)\in\omega((S_0,F_0,I_0))}\left[\min(\inf_{y\in\bar{\Omega}}\varphi_1(y), \inf_{y\in\bar{\Omega}}\varphi_2(y), \inf_{y\in\bar{\Omega}}\varphi_3(y))\right] > \eta$$

for all $(S_0, F_0, I_0) \in D_0$. This implies that

$$\liminf_{t\to+\infty} S(y,t) \geq \eta, \quad \liminf_{t\to+\infty} F(y,t) \geq \eta, \quad \text{and} \quad \liminf_{t\to+\infty} I(y,t) \geq \eta$$

for all $y \in \bar{\Omega}$. Therefore, model (2.3) is uniformly persistent in D_0. This ends the proof. □

5 Calculation of the R_0

Although the basic reproduction number R_0 of model (2.3) is defined in Eq. (4.2), it has an abstract form and cannot be computed explicitly. In order to solve this problem, we discretize model (2.3) into a model of ordinary differential equational.

Take $\Omega = [-a/2, a/2] \in \mathbb{R}$. Let $\Delta y := a/n$, $y_i := -a/2 + i\Delta y$, $P_{ji} := P(y_j - y_i)$, $S_i^0 := S^0(y_i)$, $F_i^0 := F^0(y_i)$, $\Lambda_i := \Lambda(y_i)$, $\mu_i := \mu(y_i)$, $\gamma_i := \gamma(y_i)$, $c_i := c(y_i), \theta_i := \theta(y_i), i, j = 0, 1, 2, \cdots, n$, where $n \in \mathbb{N}$, and $\mathbb{N}$ is the set of natural numbers. Based on the definition of definite integral, we have the approximation as follows:

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} P(y_i - z)B(z,t)dz \approx \sum_{k=1}^{n} P_{ik}B(z_k,t)\Delta z, \ t > 0, \tag{5.1}$$

where $B(y,t)$ denotes $S(y,t)$, $F(y,t)$, and $I(y,t)$. With the help of Eq. (5.1), model (2.3) can be rewritten as the following multi-group form:

$$\begin{cases} \dfrac{dS(y_i,t)}{dt} = k_S \displaystyle\sum_{k=1}^{n} \frac{a}{n} P_{ik}S(y_k,t) + \Lambda_i - \beta_0 F(y_i,t)S(y_i,t)I(y_i,t) - (\mu_i \\ \qquad + k_S)S(y_i,t) + \gamma_i I(y_i,t), \\ \dfrac{dF(y_i,t)}{dt} = k_F \displaystyle\sum_{k=1}^{n} \frac{a}{n} P_{ik}F(y_k,t) + c_i - (\theta_i + k_F)F(y_i,t), \\ \dfrac{dI(y_i,t)}{dt} = k_I \displaystyle\sum_{k=1}^{n} \frac{a}{n} P_{ik}I(y_k,t) + \beta_0 F(y_i,t)S(y_i,t)I(y_i,t) - (\mu_i + \gamma_i \\ \qquad + k_I)I(y_i,t). \end{cases} \tag{5.2}$$

Obviously, the disease-free equilibrium $E_d^0 := (S^0, F^0, 0)$ of model (5.2) satisfy that

$$S^0 = (-A_{S,n})^{-1}\Lambda_{\mathbb{N}} \text{ and } F^0 = (-A_{F,n})^{-1}c_{\mathbb{N}}, \tag{5.3}$$

where

$$\Lambda_{\mathbb{N}} = (\Lambda_1, \Lambda_2, \cdots, \Lambda_n)^\top,\ c_{\mathbb{N}} = (c_1, c_2, \cdots, c_n)^\top, \tag{5.4}$$

$$A_{S,n} = \begin{pmatrix} \frac{ak_S}{n}P_{11} - (\mu_1 + k_S) & \frac{ak_S}{n}P_{12} & \cdots & \frac{ak_S}{n}P_{1n} \\ \frac{ak_S}{n}P_{21} & \frac{ak_S}{n}P_{22} - (\mu_2 + k_S) & \cdots & \frac{ak_S}{n}P_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{ak_S}{n}P_{n1} & \frac{ak_S}{n}P_{n2} & \cdots & \frac{ak_S}{n}P_{nn} - (\mu_n + k_S) \end{pmatrix}, \tag{5.5}$$

and

$$A_{F,n} = \begin{pmatrix} \frac{ak_F}{n}P_{11} - (\theta_1 + k_F) & \frac{ak_F}{n}P_{12} & \cdots & \frac{ak_F}{n}P_{1n} \\ \frac{ak_F}{n}P_{21} & \frac{ak_F}{n}P_{22} - (\mu_2 + k_F) & \cdots & \frac{ak_F}{n}P_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{ak_F}{n}P_{n1} & \frac{ak_F}{n}P_{n2} & \cdots & \frac{ak_F}{n}P_{nn} - (\mu_n + k_F) \end{pmatrix}. \tag{5.6}$$

It is not hard to see that $E_d^0 \to E^0$ as $n \to +\infty$. Let $H_i = \mu_i + \gamma_i + k_I$. The work of [43] shows that for model (5.2) the spectral radius of the next generation matrix $R_n = F_n(-A_{I,n})^{-1}$ is defined as the basic reproduction number $R_{n,0}$, where

$$F_n = \begin{pmatrix} \beta_0 S_1^0 F_1^0 & 0 & \cdots & 0 \\ 0 & \beta_0 S_2^0 F_2^0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_0 S_n^0 F_n^0 \end{pmatrix}, \tag{5.7}$$

and

$$A_{I,n} = \begin{pmatrix} \frac{ak_I}{n}P_{11} - H_1 & \frac{ak_I}{n}P_{12} & \cdots & \frac{ak_I}{n}P_{1n} \\ \frac{ak_I}{n}P_{21} & \frac{ak_I}{n}P_{22} - H_2 & \cdots & \frac{ak_I}{n}P_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{ak_I}{n}P_{n1} & \frac{ak_I}{n}P_{n2} & \cdots & \frac{ak_I}{n}P_{nn} - H_n \end{pmatrix}. \tag{5.8}$$

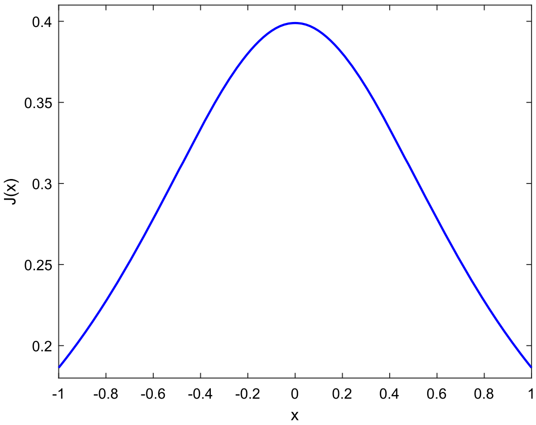
It follows from [31, 44] that $R_{n,0} \to R_0$ as $n \to +\infty$.

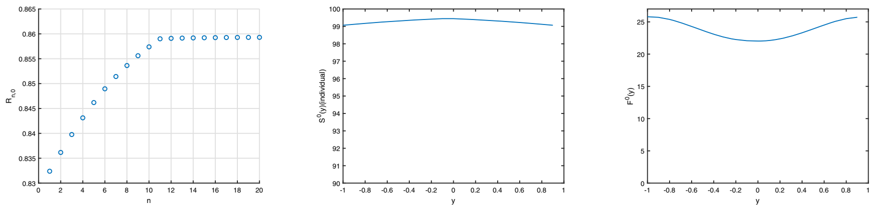
6 Numerical simulation

By using numerical simulations, this section demonstrates the validity of our theoretical results established in the previous sections and explores the relationship between dispersal rate and basic reproduction number. We set $\Omega = [-1, 1]$ and employ the dispersal kernel function

$$P(y) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-\frac{(\pi \arctan y)^2}{8}}, & y \in [-1, 1], \\ 0, & y \in (-\infty, -1) \cup (1, +\infty). \end{cases} \tag{6.1}$$

Fig. 1 Kernel function $P(y)$ defined by Eq. (6.1) on $[-1, 1]$





(a) Evolution of the basic reproduction number $R_{n,0}$

(b) Spatial distribution of susceptible individuals $S^0(y)$ at E^0

(c) Spatial distribution of air pollutants $F^0(y)$ at E^0

Fig. 2 Numerical simulation results for R_0, $S^0(y)$, and $F^0(y)$

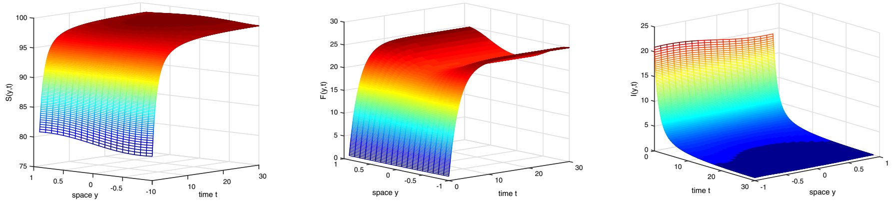
It is not hard to see that $\int_{-1}^{1} P(y)dy = 1$. The diagram for $P(x)$ is shown as Fig. 1.

6.1 Numerical simulation of the threshold dynamics

6.1.1 Disease extinction

Based on [13, 15], we choose the parameters of model (5.2) as $k_S = 1 \times 10^{-2}\ day^{-1}$, $k_F = 4\times10^{-2}\ day^{-1}, k_I = 1\times10^{-2}\ day^{-1}, \Lambda(y) = 0.707\times10^2\ individuals\cdot km^{-1}\cdot day^{-1}$, $\gamma(y) = 2.01 \times 10^{-2}\ day^{-1}$, $\beta = 2.4769 \times 10^{-4}\ individuals \cdot km \cdot day^{-1}$, $\mu(y) = 7.07\times10^{-1}\ day^{-1}, c(y) = 10\ day^{-1}, \theta(y) = 0.4–0.04 \sin(\pi y–1.5)\ day^{-1}$. The initial value of model (5.2) is selected as $S_0(y) = 80+\sin(2y)\ individuals\cdot km^{-1}$, $F_0(y) = 10 - 0.1 \cos(y)$, $I_0(y) = 20 - \sin(2y)\ individuals \cdot km^{-1}$. Through the method introduced in Sect. 5, we can get the following numerical simulations of R_0, $S^0(y)$, and $F^0(y)$.

Subsequently, according to Theorem 4.1, it follows that the disease-free steady state E^0 is globally asymptotically stable. In fact, one can obtain from Figs. 2and 3that the susceptible individuals $S(y, t)$ tends to $S^0(y)$, the air pollutants $F(y, t)$ converges to



(a) Evolution of $S(y, t)$ (b) Evolution of $F(y, t)$ (c) Evolution of $I(y, t)$

Fig. 3 Numerical simulation results for $S(y, t)$, $I(y, t)$, and $F(y, t)$ when $R_0 < 1$

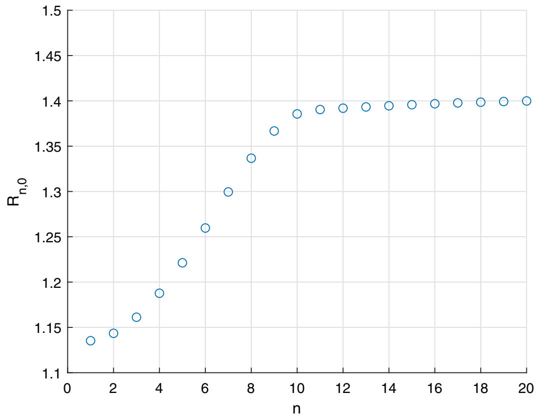


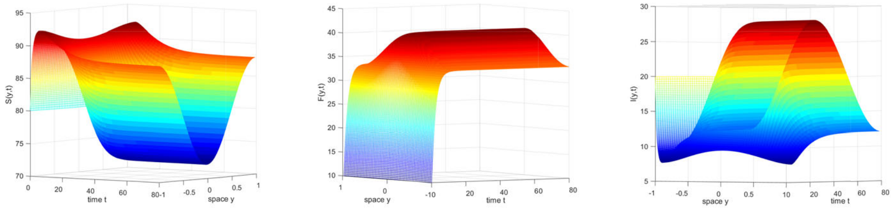
Fig. 4 Evolution of the basic reproduction number

$F^0(y)$, and the infected individuals $I(y, t)$ tends to zero over time. This conclusion is the same as that given by Theorem 4.1.

6.1.2 Disease persistence

With the help of works [13, 15], we choose the parameters of model (5.2) as $k_F = 1 \times 10^{-2}\ day^{-1}$, $k_I = 1 \times 10^{-3}\ day^{-1}$, $k_S = 0\ day^{-1}$, $c(y) = 15\ day^{-1}$, $\theta(y) = 0.4 + 0.04 \sin(\pi y - 1.5)\ day^{-1}$. The initial value of model (5.2) is selected as $S_0(y) = 80 + 0.1 \sin(2y - 1.5)\ individuals \cdot km^{-1}$, $F_0(y) = 10 + 0.1 \cos(y)\ individuals \cdot km^{-1}$, $I_0(y) = 20 - 0.1 \sin(2y - 1.5)\ individuals \cdot km^{-1}$. Here, the values of other parameters are the same as those in Sect. 6.1.1. In virtue of the method introduced in Sect. 5, it follows that $R_0 \approx 1.3999 > 1$ (see Fig. 4).

Subsequently, it follows from Theorem 4.6 that the disease is uniformly persistent. Actually, one can obtain from Fig. 5 that when the time evolves to infinity the air pollutants $F(y, t)$, the susceptible humans $S(y, t)$, and the infected humans $I(y, t)$ converge to the heterogenous steady state. This conclusion is the same as that given by Theorem 4.6.



(a) Evolution of $S(y, t)$ (b) Evolution of $F(y, t)$ (c) Evolution of $I(y, t)$

Fig. 5 Numerical simulation results for $S(y, t)$, $I(y, t)$, and $F(y, t)$ when $R_0 > 1$

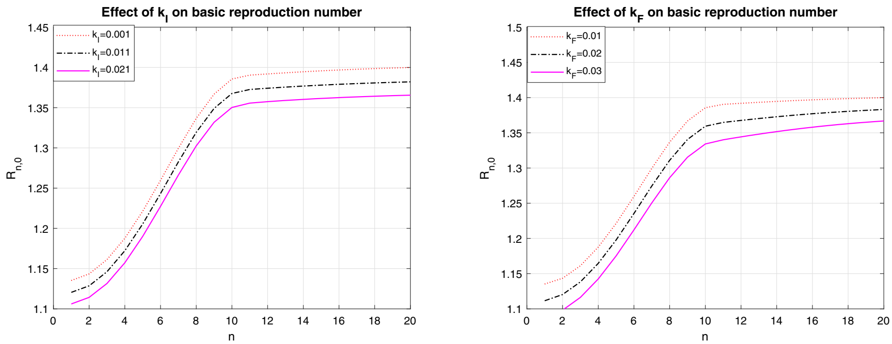


Fig. 6 Influence of dispersal rate to basic reproduction number

6.2 Effect of dispersal rate on basic reproduction number

This subsection studies how dispersal rates impact upon the basic reproduction number. Each parameter value is the same as that in Sect. 6.1.2. It follows from Fig. 6increasing the dispersal rate of infected humans or air pollutants can reduce R_0. From a biological point of view, this may be due to the fact that infected individuals move more slowly, resulting in a longer stay in a particular place, thus making the susceptible individuals more easily infected. For pollutants, with greater dispersal rates, the concentration of the pollutants at a particular location will be smaller, resulting in a lower infection rate at this location. As a result, increasing the dispersion rate of diseased persons or air pollutants is an effective control tool for avoiding disease development.

7 Conclusions

This paper establishes a nonlocal dispersal infectious disease model, in which the transmission rate is closely related to the concentration of pollutants. First, we explore the existence, uniqueness, positivity, and boundedness of the solution of model (2.3) in Theorem 3.1, Lemma 3.1, and Theorem 3.2. Subsequently, we define the basic reproduction number R_0 based on a large number of previous works. The dynamic behaviors of this model are then discussed in Theorem 4.1 and Theorem 4.6, including the global stability of disease-free steady state E^0 and uniform persistence. Biologically, whether an outbreak of epidemics that are affected by air pollution is determined by the basic reproduction number R_0. More specifically, when $R_0 < 1$, the diseases become extinct, however, when $R_0 > 1$, the diseases are persistent. Furthermore, the model is discretized into the n-dimensional space, in which the corresponding threshold value $R_{n,0}$ can be explicitly calculated, and the corresponding threshold value $R_{n,0}$ converges to R_0 as $n \to +\infty$. Finally, by numerical simulation, it is found that increasing the dispersal rate of infected individuals or air pollutants is an effective control tool for avoiding diseases development.

Since time-delay phenomena often occur in the spread of the diseases, it is of interest to study the dynamics and bifurcation problems of a infectious diseases model with air pollution and nonlocal diffusion. This will be our future research direction.

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Declaration

Conflicts of interest The authors have no conflict of interest.

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