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ORIGINAL RESEARCH

An efficient numerical method for a singularly perturbed Fredholm integro-differential equation with integral boundary condition

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Abstract

In this paper, a linear singularly perturbed Fredholm integro-differential initial value problem with integral condition is being considered. On a Shishkin-type mesh, a fitted finite difference approach is applied using a composite trapezoidal rule in both; in the integral part of equation and in the initial condition. The proposed technique acquires a uniform second-order convergence in respect to perturbation parameter. Further provided the numerical results to support the theoretical estimates.

Keywords Finite difference scheme · Fredholm integro-differential equation · Integral boundary condition · Shishkin mesh · Singular perturbation · Uniform convergence

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1 Introduction

Singularly perturbed differential equations are described by a small parameter ε multiplying all or some of the differential equation's highest order terms, as boundary layers are generally present in their solutions. These equations are crucial for sophisticated scientific computations in the twenty-first century. Singularly perturbed problems (SPPs) are used to express a variety of mathematical models, ranging from chemical reactions to problems in mathematical engineering, fluid dynamics, electrical networks, control theory, aerodynamics, biology and neuroscience. Further information on SPPs may be found in the works [18, 26, 27, 29] and their references. Numerical analysis of SPPs has always been difficult because of the solution's boundary layer behavior. Within some thin layers at the inside or boundary of the problem domain, such a problem exhibits fast changes [26, 29]. Standard numerical techniques for resolving such problems are widely recognized for being unstable and failing to produce exact results when the perturbation parameter is small. On account of this, it is critical to design numerical methods for solving problems whose accuracy is independent on parameter value. The references [18, 22, 26, 33, 35, 40] cover a variety of techniques for numerically solving this type differential equations.

Differential equations with integral boundary conditions have also been utilized to describe a variety of processes in the applied sciences, such as subsurface water flow, chemical engineering and heat conduction [11, 21, 28]. Therefore, many authors have studied boundary value problems with integral boundary conditions. Researchers have considered the singularly perturbed cases of these problems. The authors in [9, 10, 25, 36] investigated first-order convergent finite difference schemes on non-uniform meshes for various problems with integral boundary conditions.

Integro-differential equations have emerged in most engineering applications and several fields of sciences. Plasma physics, financial mathematics, epidemic models, population dynamics, biology, artificial neural networks, fluid mechanics, electromagnetic theory, financial mathematics, oceanography and physical processes are among these (see, e.g., [8, 39]). For instance, in [23], the integro-differential equation used to modelling infectious diseases in optimal control strategies for policy decisions and applications in COVID-19 has been expressed as follows:

$$\partial_t S(t,p) = \frac{\mathcal{R}_0}{\mathcal{N}_0} S(t,p) \int\limits_{t-\delta_{IP}-\delta_{CO}}^{t-\delta_{IP}} \int\limits_{\mathcal{P}} \int\limits_{\mathcal{K}} \hat{\gamma}_I(t,p,\tilde{p},\kappa,t-\tau)\,\mu(\tilde{p},\kappa)\,\partial_\tau S(\tau,\tilde{p})\,d_\kappa d_{\tilde{p}} d_\tau,$$
$$S(\tau,p) = S_0(\tau,p),$$

where

- $\mathcal{P} \subset \mathbb{R}^n$, $n \in \mathbb{N}$ is the set of features characterizing dissimilar styles of populations (e.g. sex, age),
- $\mathcal{N}_0 \in \mathbb{N}_{\geq 1}$ the aggregate number of people aforethought,
- $\mathcal{K} \subset \mathbb{R}^n$, $n \in \mathbb{N}$ represent a parametrization of different courses of diseases and $\mu : \mathcal{P} \times \mathbb{R}_{\geq 0}$ the probability of a person with property $\tilde{p} \in \mathcal{P}$ suffering from disease $(t, p, \tilde{p}, \tau) \in \mathbb{R}_{>0} \times \mathcal{P}^2 \times \mathbb{R}_{>0}$.

- $\mathcal{R}_0$ the basic breeding number, i.e. the number of people infected by a single infectious individual in a completely responsive population.
- $\hat{\gamma}_I : \mathbb{R}_{>0} \times \mathcal{P}^2 \times \mathcal{K} \times \mathbb{R} \to \mathcal{R}_{\geq 0}$, with $\|\gamma_I(t, p, \tilde{p}, .)\|_{L^1(0,\infty)} = 1 \, \forall (t, p, \tilde{p}) \in \mathbb{R}_{\geq 0} \times \mathcal{P}^2$, $\tau \to \gamma_I(t, p, \tilde{p}, t - \tau)$ the probability of an infection event between a person with property $\tilde{p}$ infected at time τ infecting a person with property p at time t.
- $S : [-\delta_{IP} - \delta_{CO}, 0] \times \mathcal{P} \to \mathbb{R}$, $(t, p, \tau) \in [0, T] \times \mathcal{P} \times (-\delta_{IP} - \delta_{CO}, 0]$ and S_0 is the initial datum. Further, the Incubation Period has been defined by $\delta_{IP} \in \mathbb{R}_{>0}$, and the infectious (COntagious) period by $\delta_{CO} \in \mathbb{R}_{>0}$.

That's why, many researchers have been pondering the Fredholm integro-differential equations (FIDEs) for a long time. An overview of existence and uniqueness results for the solution of FIDEs can be found in some references such as [1, 19] (see also references therein). Furthermore, researchers employed fitted analytical approaches because of the difficulty of obtaining accurate solutions to these types of problems. Some of these methods are reproducing kernel Hilbert space method [7], Nyström method [38], Touchard polynomials method [2], Tau method [20, 32], Collocation and Kantorovich methods [37], Galerkin method [12, 41, 43], Boole collocation method [14], parameterization method [17], Legendre collocation matrix method[44], variational iteration technique [19]. The increasing interest in recent years is not limited to only FIDEs, but also the numerical solutions of linear and nonlinear Volterra or Volterra-Fredholm integro-differential equations are increasing in popularity. Recently, Turkyilmazoglu presented an effective technique for solving the linear FIDEs and nonlinear Volterra-Fredholm-Hammerstein integro-differential equations based on the Galerkin method [41, 42] (see also references therein).

We consider a singularly perturbed Fredholm integro-differential equation (SPFIDE) with integral boundary condition as follows:

$$Lu := \varepsilon u'(x) + a(x)u(x) + \lambda \int_0^l K(x, s)u(s)ds = f(x), \quad x \in \Omega, \tag{1}$$

$$u(0) = \mu u(l) + \int_0^l c(s)u(s)ds + A, \tag{2}$$

where $\Omega = (0, l]$ $\left(\bar{\Omega} = \Omega \cup \{x = 0\}\right)$. $0 < \varepsilon \leq 1$ is a perturbation parameter. λ, A and $\mu \leq 0$ are given constants. We assume that $a(x) \geq \alpha > 0$, $c(x) \leq 0$, $f(x)$ and $K(x, s)$ are the sufficiently smooth functions satisfying certain regularity conditions to be specified. Under these conditions, the solution $u(x)$ of the problem (1)-(2) has in general initial layer at $x = 0$ for small values of ε. This means that the derivatives of the solution become unbounded for small values of perturbation parameter near $x = 0$.

The above-mentioned papers, related to FIDEs, were dealt mainly with the regular cases (i.e., when the boundary layers are absent). Scientists have also given numerical approaches to singular perturbation situations of FIDEs in recent years. Amiraliyev et al. [3, 5] proposed an exponentially fitted difference method on a uniform mesh for solving first and second-order linear SPFIDEs, demonstrating that the approach is first-order convergent uniformly in ε. Difference schemes of the fitted homogeneous type with an accuracy of $O(N^{-2}\ln N)$ on a piecewise uniform mesh for this type of problems are given in [4, 15]. It should also be noted that in [30, 31], for the numerical solution of singularly perturbed Volterra integro-differential equations, first-order difference schemes on a piecewise uniform mesh are given, followed by Richardson extrapolation to obtain the second order of accuracy.

The aim of this work is to present a homogeneous (non-hybrid) type difference scheme for the numerical solution of SPFIDE with an integral condition. A special technique is necessary to establish the appropriate difference scheme and investigate the error analysis for the numerical solution of such problems. The scheme is built using the integral identity method and suitable quadrature rules, with the remainder terms in integral form. The goal is to develop an ε-uniformly second-order homogeneous finite difference method that produces uniform convergent numerical approximations in order to solve problem (1)-(2).

The content is arranged as follows: Some properties of the solution of (1)-(2) are given in Sect. 2. A finite difference scheme and a special piecewise uniform mesh are presented in Sect. 3. The stability and convergence analysis of this scheme are shown in Sect. 4. The numerical results of two examples to verify the theoretical estimates are presented in Sect. 5. Finally, the work ends with a summary of the conclusions in Sect. 6.

2 Properties of the exact solution

We now present some properties of the solution of (1)-(2), which are needed in later sections for the analysis of the appropriate numerical solution. Here, we will use the following notations:

$$\|g\|_\infty \equiv \|g\|_{\infty,\bar{\Omega}} = \max_{0\le x\le l} |g(x)|, \quad \|g\|_1 \equiv \|g\|_{1,\Omega} = \int\limits_0^l |g(x)|\,dx.$$

Lemma 1 *Assume that $a, f \in C^2[0,l]$ and $\frac{\partial^m K}{\partial x^m} \in C[0,l]^2$, $(m = 0, 1, 2)$. Moreover*

$$|\lambda| < \frac{\alpha}{(|\mu| + \|c\|_1 + 1)\max\limits_{0\le x\le l}\int_0^l |K(x,s)|\,ds}. \tag{3}$$

Then the solution $u(x)$ of the problem (1)-(2) satisfies the bounds

$$\left|u^{(k)}(x)\right| \le C\left\{1 + \frac{1}{\varepsilon^k}e^{-\frac{\alpha x}{\varepsilon}}\right\}, \quad x \in [0,l], \quad k = 0, 1, 2. \tag{4}$$

Proof From (1) we have the following relation for $u(x)$:

$$u(x)=u(0)\,e^{-\frac{1}{\varepsilon}\int\limits_0^x a(\tau)d\tau}+\frac{1}{\varepsilon}\int\limits_0^x f(\xi)\,e^{-\frac{1}{\varepsilon}\int\limits_\xi^x a(\tau)d\tau}d\xi$$
$$-\frac{\lambda}{\varepsilon}\int\limits_0^x e^{-\frac{1}{\varepsilon}\int\limits_\xi^x a(\tau)d\tau}\left(\int\limits_0^l K(\xi,s)\,u(s)\,ds\right)d\xi.$$

By using the boundary condition (2) we get

$$u(0)=\frac{\frac{\mu}{\varepsilon}\int\limits_0^l f(\xi)\,e^{-\frac{1}{\varepsilon}\int\limits_\xi^l a(\tau)d\tau}d\xi+\frac{1}{\varepsilon}\int\limits_0^l c(x)\left[\int\limits_0^x f(\xi)\,e^{-\frac{1}{\varepsilon}\int\limits_\xi^x a(\tau)d\tau}d\xi\right]dx+A}{1-\mu e^{-\frac{1}{\varepsilon}\int\limits_0^l a(\tau)d\tau}-\int\limits_0^l c(x)\,e^{-\frac{1}{\varepsilon}\int\limits_0^x a(\tau)d\tau}dx}$$
$$-\frac{\frac{\mu\lambda}{\varepsilon}\int\limits_0^l e^{-\frac{1}{\varepsilon}\int\limits_\xi^l a(\tau)d\tau}\left(\int\limits_0^l K(\xi,s)\,u(s)\,ds\right)d\xi}{1-\mu e^{-\frac{1}{\varepsilon}\int\limits_0^l a(\tau)d\tau}-\int\limits_0^l c(x)\,e^{-\frac{1}{\varepsilon}\int\limits_0^x a(\tau)d\tau}dx}$$
$$-\frac{\frac{\lambda}{\varepsilon}\int\limits_0^l c(x)\left[\int\limits_0^x e^{-\frac{1}{\varepsilon}\int\limits_\xi^x a(\tau)d\tau}\left(\int\limits_0^l K(\xi,s)\,u(s)\,ds\right)d\xi\right]dx}{1-\mu e^{-\frac{1}{\varepsilon}\int\limits_0^l a(\tau)d\tau}-\int\limits_0^l c(x)\,e^{-\frac{1}{\varepsilon}\int\limits_0^x a(\tau)d\tau}dx}. \tag{5}$$

Since $\mu\le 0$ and $c(x)\le 0$, the denominator is bounded below by one.
Also, we can write the numerator of (5) as

$$\left|\frac{\mu}{\varepsilon}\int\limits_0^l f(\xi)\,e^{-\frac{1}{\varepsilon}\int\limits_\xi^l a(\tau)d\tau}d\xi+\frac{1}{\varepsilon}\int\limits_0^l c(x)\left[\int\limits_0^x f(\xi)\,e^{-\frac{1}{\varepsilon}\int\limits_\xi^x a(\tau)d\tau}d\xi\right]dx\right|$$
$$+\left|\frac{\mu\lambda}{\varepsilon}\int\limits_0^l e^{-\frac{1}{\varepsilon}\int\limits_\xi^l a(\tau)d\tau}\left(\int\limits_0^l K(\xi,s)\,u(s)\,ds\right)d\xi\right|$$
$$+\left|A-\frac{\lambda}{\varepsilon}\int\limits_0^l c(x)\left[\int\limits_0^x e^{-\frac{1}{\varepsilon}\int\limits_\xi^x a(\tau)d\tau}\left(\int\limits_0^l K(\xi,s)\,u(s)\,ds\right)d\xi\right]dx\right|$$

$$\leq |A| + |\mu|\,\alpha^{-1}\,\|f\|_\infty\left(1-e^{-\frac{\alpha l}{\varepsilon}}\right)+\alpha^{-1}\,\|f\|_\infty\int\limits_0^l |c\,(x)|\left(1-e^{-\frac{\alpha x}{\varepsilon}}\right)dx$$

$$+\,|\mu|\,|\lambda|\,\alpha^{-1}\,\|u\|_\infty\left(1-e^{-\frac{\alpha l}{\varepsilon}}\right)\max_{0\leq\xi\leq l}\int\limits_0^l |K\,(\xi,s)|\,ds$$

$$+\,|\lambda|\,\alpha^{-1}\,\|u\|_\infty\int\limits_0^l |c\,(x)|\left(1-e^{-\frac{\alpha l}{\varepsilon}}\right)dx\,\max_{0\leq\xi\leq l}\int\limits_0^l |K\,(\xi,s)|\,ds. \tag{6}$$

Considering (5) and (6) together, we obtain

$$\begin{aligned}|u\,(0)| \leq |A| &+ (|\mu| + \|c\|_1)\,\alpha^{-1}\,\|f\|_\infty \\ &+ (|\mu| + \|c\|_1)\,|\lambda|\,\alpha^{-1}\,\|u\|_\infty\max_{0\leq\xi\leq l}\int\limits_0^l |K\,(\xi,s)|\,ds.\end{aligned} \tag{7}$$

Later on, according to the maximum principle for $L_1u = \varepsilon u'\,(x) + a\,(x)\,u\,(x)$ from (1), we have

$$\|u\|_\infty \leq |u\,(0)| + \alpha^{-1}\,\|f\|_\infty + \alpha^{-1}\,|\lambda|\,\|u\|_\infty\max_{0\leq x\leq l}\int\limits_0^l |K\,(x,s)|\,ds.$$

Now, considering the estimate of (7) instead of $u\,(0)$ in the above inequality by virtue of (3), we acquire

$$\|u\|_\infty \leq \frac{|A| + (|\mu| + \|c\|_1 + 1)\,\alpha^{-1}\,\|f\|_\infty}{1-(|\mu| + \|c\|_1 + 1)\,|\lambda|\,\alpha^{-1}\max\limits_{0\leq x\leq l}\int\limits_0^l |K\,(x,s)|\,ds},$$

which implies the validity of (4) for $k = 0$. The proof of (4) for $k = 1, 2$ can be proved in a similar way as in [3, 4]. □

3 Designing of the numerical method

Let ω_N be any non-uniform mesh on $[0, l]$:

$$\omega_N = \{0 < x_1 < \ldots < x_N = l,\;\; h_i = x_i - x_{i-1}\}$$

and

$$\overline{\omega}_N = \omega_N \cup \{x_0 = 0\}\,,\quad \hbar_i = \frac{h_i + h_{i+1}}{2}.$$

Prior to describing our numerical technique, we present certain notations for the mesh functions. To any mesh function $v(x)$ described on $\overline{\omega}_N$, we utilize

$$v_i = v(x_i), \quad v_{\bar{x},i} = \frac{v_i - v_{i-1}}{h_i}, \quad \|v\|_1 \equiv \|v\|_{1,\omega_N} = \sum_{i=1}^{N} \hbar_i \, |v_i| \, . \tag{8}$$

We construct the numerical method using the identity

$$\chi_i^{-1} h_i^{-1} \int_{x_{i-1}}^{x_i} Lu\varphi_i(x)dx = \chi_i^{-1} h_i^{-1} \int_{x_{i-1}}^{x_i} f(x)\varphi_i(x)dx, \qquad 1 \le i \le N, \tag{9}$$

with the basis functions

$$\varphi_i(x) = e^{-\frac{a_i(x_i - x)}{\varepsilon}}$$

and

$$\chi_i = h_i^{-1} \int_{x_{i-1}}^{x_i} \varphi_i(x)dx = \frac{1 - e^{-a_i \rho_i}}{a_i \rho_i}, \qquad \rho_i = \frac{h_i}{\varepsilon}.$$

We note that the function $\varphi_i(x)$ is the solution of the problem

$$-\varepsilon\varphi'(x) + a_i\varphi(x) = 0, \quad \varphi(x_i) = 1, \qquad x_{i-1} < x < x_i.$$

Using the method of exact difference schemes [6, 13, 24, 45] (see also [34], pp. 207-214), for the differential part from (9), we obtain

$$\begin{aligned} \chi_i^{-1} h_i^{-1} \int_{x_{i-1}}^{x_i} \left[\varepsilon u'(x) + a(x)u(x)\right] \varphi_i(x)dx &= \varepsilon\theta_i u_{\bar{x},i} + a_i u_i \\ &+ \chi_i^{-1} h_i^{-1} \int_{x_{i-1}}^{x_i} [a(x) - a(x_i)] \, u(x)\varphi_i(x)dx \end{aligned} \tag{10}$$

with

$$\theta_i = \frac{a_i \rho_i}{1 - e^{-a_i \rho_i}} e^{-a_i \rho_i}. \tag{11}$$

By Newton interpolation formula with respect to mesh point (x_{i-1}, x_i) we have

$$a(x) - a(x_i) = (x - x_i)a_{\bar{x},i} + \frac{a''\big(\xi_i(x)\big)}{2}(x - x_{i-1})(x - x_i).$$

Therefore we get

$$\chi_i^{-1} h_i^{-1} \int_{x_{i-1}}^{x_i} [a(x) - a(x_i)]\, u(x)\varphi_i(x)dx = a_{\bar{x},i}\, \chi_i^{-1} h_i^{-1} \int_{x_{i-1}}^{x_i} (x - x_i) u(x)\varphi_i(x)dx$$
$$+ \frac{1}{2}\chi_i^{-1} h_i^{-1} \int_{x_{i-1}}^{x_i} a''\big(\xi_i(x)\big)(x - x_{i-1})(x - x_i)u(x)\varphi_i(x)dx. \tag{12}$$

Also using

$$u(x) = u(x_i) - \int_x^{x_i} u'(s)ds$$

in the first term at the right side of (12), we have

$$\chi_i^{-1} h_i^{-1} \int_{x_{i-1}}^{x_i} [a(x) - a(x_i)]\, u(x)\varphi_i(x)dx = \left(a_{\bar{x},i}\, \chi_i^{-1} h_i^{-1} \int_{x_{i-1}}^{x_i} (x - x_i)\varphi_i(x)dx \right) u_i$$
$$+ R_i^{(1)},$$

where

$$R_i^{(1)} = \frac{1}{2}\chi_i^{-1} h_i^{-1} \int_{x_{i-1}}^{x_i} a''\big(\xi_i(x)\big)(x - x_{i-1})(x - x_i)u(x)\varphi_i(x)dx$$
$$- a_{\bar{x},i}\, \chi_i^{-1} h_i^{-1} \int_{x_{i-1}}^{x_i} (x - x_i)\varphi_i(x)\left(\int_x^{x_i} u'(s)ds \right)dx. \tag{13}$$

Simple calculation gives

$$\chi_i^{-1} h_i^{-1} \int_{x_{i-1}}^{x_i} (x - x_i)\varphi_i(x)dx = h_i \delta_i,$$

with

$$\delta_i = \frac{e^{-a_i \rho}}{1 - e^{-a_i \rho}} - \frac{1}{a_i \rho}. \tag{14}$$

It is easy to see that $-1 \le \delta_i \le 0$. So, the identity (10) degrades to

$$\chi_i^{-1} h_i^{-1} \int_{x_{i-1}}^{x_i} \big[\varepsilon u'(x) + a(x)u(x)\big]\, \varphi_i(x)dx = \varepsilon\theta_i u_{\bar{x},i} + \bar{a}_i u_i + R_i^{(1)}, \tag{15}$$

where

$$\bar{a}_i = a_i + a_{\bar{x},i} h_i \delta_i \tag{16}$$

and δ_i is given by (14). Analogously we derive

$$\chi_i^{-1} h_i^{-1} \int\limits_{x_{i-1}}^{x_i} f(x)\varphi_i(x)dx = \bar{f}_i + R_i^{(2)}, \tag{17}$$

where

$$\bar{f}_i = f_i + f_{\bar{x},i} h_i \delta_i, \tag{18}$$

$$R_i^{(2)} = \frac{1}{2}\chi_i^{-1} h_i^{-1} \int\limits_{x_{i-1}}^{x_i} f''\big(\eta_i(x)\big)(x - x_{i-1})(x - x_i)\varphi_i(x)dx. \tag{19}$$

It remains to obtain an approximation for integral term from (1). Using the Taylor expansion

$$K(x,s) = K(x_i,s) + (x - x_i)\frac{\partial}{\partial x}K(x_i,s) + \frac{(x - x_i)^2}{2}\frac{\partial^2}{\partial x^2}K\big(\xi_i(x),s\big),$$

we get

$$\begin{aligned}\chi_i^{-1} h_i^{-1} \lambda \int\limits_{x_{i-1}}^{x_i} \varphi_i(x)\left(\int\limits_0^l K(x,s)u(s)ds\right)dx &= \lambda \int\limits_0^l K(x_i,s)u(s)ds \\ &+ h_i\delta_i\lambda \int\limits_0^l \frac{\partial}{\partial x}K(x_i,s)u(s)ds + R_i^{(3)} \\ &\equiv \lambda \int\limits_0^l \mathcal{K}(x_i,s)u(s)ds + R_i^{(3)},\end{aligned} \tag{20}$$

where

$$\mathcal{K}(x_i,s) = K(x_i,s) + h_i\delta_i \frac{\partial}{\partial x}K(x_i,s), \tag{21}$$

$$R_i^{(3)} = \frac{1}{2}\lambda\chi_i^{-1} h_i^{-1} \int\limits_{x_{i-1}}^{x_i} (x - x_i)^2 \varphi_i(x)\left(\int\limits_0^l \frac{\partial^2}{\partial x^2}K\big(\xi_i(x),s\big)u(s)ds\right)dx. \tag{22}$$

Next, if the first term at the right side of (20) is operated by applying the composite trapezoidal integration rule with the remainder term in the integral form [4], we get

$$\lambda \int\limits_0^l \mathcal{K}(x_i,s)u(s)ds = \lambda \sum_{j=0}^{N} \hbar_j \mathcal{K}_{ij} u_j + R_i^{(4)}, \tag{23}$$

where

$$R_i^{(4)} = \frac{1}{2}\lambda \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} (x_j - \xi)(x_{j-1} - \xi) \frac{d^2}{d\xi^2}\big(\mathcal{K}(x_i, \xi)u(\xi)\big) d\xi \tag{24}$$

and

$$\hbar_i = \frac{h_i + h_{i+1}}{2}, \quad (1 \le i \le N-1), \quad \hbar_0 = \frac{h_1}{2}, \quad \hbar_N = \frac{h_N}{2}.$$

To approximate the boundary condition (2), using again the composite trapezoidal integration rule, we have

$$u_0 = \mu u_N + \sum_{j=0}^{N} \hbar_j c_j u_j + A + r_i, \tag{25}$$

where

$$r_i = \frac{1}{2} \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} (x_j - \xi)(x_{j-1} - \xi) \frac{d^2}{d\xi^2}\big(c(\xi)u(\xi)\big) d\xi. \tag{26}$$

After taking into consideration (15), (17), (20) and (23) in (9) we obtain the following discrete identity for $u(x)$:

$$\varepsilon\theta_i u_{\bar{x},i} + \bar{a}_i u_i + \lambda \sum_{j=0}^{N} \hbar_j \mathcal{K}_{ij} u_j + R_i = \bar{f}_i \tag{27}$$

with remainder term

$$R_i = R_i^{(1)} - R_i^{(2)} + R_i^{(3)} + R_i^{(4)}, \tag{28}$$

where $R_i^{(1)}$, $R_i^{(2)}$, $R_i^{(3)}$, $R_i^{(4)}$ and r_i are defined by (13), (19), (22), (24) and (26) respectively.

Based on (27) we propose the following difference scheme for approximating (1)-(2):

$$L_N y_i := \varepsilon\theta_i y_{\bar{x},i} + \bar{a}_i y_i + \lambda \sum_{j=0}^{N} \hbar_j \mathcal{K}_{ij} y_j = \bar{f}_i, \qquad 1 \le i \le N, \tag{29}$$

$$y_0 = \mu y_N + \sum_{j=0}^{N} \hbar_j c_j y_j + A, \tag{30}$$

where θ_i, $\bar{a}_i$, $\bar{f}_i$ and $\mathcal{K}_{ij}$ are given by (11), (16), (18) and (21) respectively.

To discretize the interval $[0, l]$, we will use the piecewise-uniform Shishkin type mesh. As the problem (1)-(2) has an exponential initial layer in the neighborhood at $x = 0$, we divide $[0, l]$ into two subinterval $[0, \sigma]$ and $[\sigma, l]$. For an even N, a uniform mesh with $N/2$ intervals is placed on each subinterval, where the transition point σ, which separates the fine and coarse portions of ω_N, that is defined as

$$\sigma = \min\left\{\frac{l}{2}, \alpha^{-1}\varepsilon \ln N\right\}.$$

Hence, if we denote by $h^{(1)}$ and $h^{(2)}$ the stepsizes in $[0, \sigma]$ and $[\sigma, l]$ respectively, our piecewise-uniform mesh can be expressed as

$$\overline{\omega}_N = \begin{cases} x_i = ih^{(1)}, & i = 0, 1, ..., \frac{N}{2}; \qquad h^{(1)} = \frac{2\sigma}{N}; \\ x_i = \sigma + \left(i - \frac{N}{2}\right) h^{(2)}, & i = \frac{N}{2} + 1, ..., N; \qquad h^{(2)} = \frac{2(l-\sigma)}{N}. \end{cases}$$

4 The convergence

We proceed to estimate the error of the approximate solution $z_i = y_i - u_i$, $(0 \leq i \leq N)$. From (27) and (29) we have

$$L_N z_i := \varepsilon\theta_i z_{\bar{x},i} + \bar{a}_i z_i + \lambda \sum_{j=0}^{N} \hbar_j \mathcal{K}_{ij} z_j = R_i, \qquad 1 \leq i \leq N, \tag{31}$$

$$z_0 = \mu z_N + \sum_{j=0}^{N} \hbar_j c_j z_j - r_i, \tag{32}$$

where the truncation error functions r_i and R_i is given by (26) and (28).

It should be noted that since $a \in C^2[0, l]$ and $|\delta_i| \leq 1$, then exist a number $\bar{\alpha}$ such that for sufficiently large values of N will be $\bar{a}_i \geq \bar{\alpha} > 0$ (δ_i is defined by (14)).

Lemma 2 *Assume that* $a, f, c \in C^2[0, l]$ *and* $\frac{\partial^m K}{\partial x^m}, \frac{\partial^{m+1} K}{\partial x \partial s^m} \in C^2[0, l]^2$, $(m = 0, 1, 2)$. *Then the truncation error functions R_i and r_i satisfy the estimates*

$$\|R\|_{\infty,\overline{\omega}_N} \leq CN^{-2} \ln N, \tag{33}$$

$$|r| \leq CN^{-2} \ln N. \tag{34}$$

Proof First, we estimate the remainder term r_i. From the explicit expression (26), under the condition of Lemma 1, we obtain

$$|r_i| \leq C \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} (x_j - \xi)(\xi - x_{j-1})(1 + |u'(\xi)| + |u''(\xi)|)d\xi$$

$$\leq C\left(\sum_{j=1}^{N} h_j^3 + \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} (x_j - \xi)(\xi - x_{j-1})\frac{1}{\varepsilon} e^{-\frac{\alpha\xi}{\varepsilon}}\right) d\xi$$

$$+C\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}(x_j-\xi)(\xi-x_{j-1})\frac{1}{\varepsilon^2}e^{-\frac{\alpha\xi}{\varepsilon}}d\xi$$
$$\leq C\left(\sum_{j=1}^{N}h_j^3+\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}(x_j-\xi)(\xi-x_{j-1})\frac{1}{\varepsilon^2}e^{-\frac{\alpha\xi}{\varepsilon}}\right)d\xi. \tag{35}$$

Now we find a convergence error estimate for the first term in the right-side of (35) in our special piecewise-uniform mesh

$$\sum_{j=1}^{N}h_j^3=\frac{N}{2}\left|h^{(1)}\right|^3+\frac{N}{2}\left|h^{(2)}\right|^3=4\sigma^3N^{-2}+4(l-\sigma)^3N^{-2}$$
$$\leq CN^{-2}. \tag{36}$$

Note that the above estimate is valid for values both $\sigma=\frac{l}{2}$ and $\sigma=\alpha^{-1}\varepsilon\ln N$.

For the second two term in the right-side of (35), we find the estimate for the case $\sigma=\frac{l}{2}$. Then it has the form $\frac{l}{2}<\alpha^{-1}\varepsilon\ln N$ and $h^{(1)}=h^{(2)}=lN^{-1}$. Thus we get

$$\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}(x_j-\xi)(\xi-x_{j-1})\frac{1}{\varepsilon^2}e^{\frac{-\alpha\xi}{\varepsilon}}d\xi\leq\frac{\left|h^{(1)}\right|^2}{\varepsilon^2}\int_0^l e^{\frac{-\alpha\xi}{\varepsilon}}d\xi$$
$$\leq\frac{\left|h^{(1)}\right|^2}{\varepsilon}\alpha^{-1}\left(1-e^{\frac{-\alpha l}{\varepsilon}}\right)$$
$$\leq 2\alpha^{-2}lN^{-2}\ln N$$
$$\leq CN^{-2}\ln N,\qquad 1\leq i\leq N. \tag{37}$$

For two term in the right-side of (35), we find the estimate for the case $\sigma=\alpha^{-1}\varepsilon\ln N<\frac{l}{2}$. From this inequality, we can write

$$\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}(x_j-\xi)(\xi-x_{j-1})\frac{1}{\varepsilon^2}e^{\frac{-\alpha\xi}{\varepsilon}}d\xi=\sum_{j=1}^{N/2}\int_{x_{j-1}}^{x_j}(x_j-\xi)(\xi-x_{j-1})\frac{1}{\varepsilon^2}e^{\frac{-\alpha\xi}{\varepsilon}}d\xi$$
$$+\sum_{j=\frac{N}{2}+1}^{N}\int_{x_{j-1}}^{x_j}(x_j-\xi)(\xi-x_{j-1})\frac{1}{\varepsilon^2}e^{\frac{-\alpha\xi}{\varepsilon}}d\xi. \tag{38}$$

For the first term in the right-side of (38), we have

$$\begin{aligned}\sum_{j=1}^{N/2}\int_{x_{j-1}}^{x_j}(x_j-\xi)(\xi-x_{j-1})\frac{1}{\varepsilon^2}e^{\frac{-\alpha\xi}{\varepsilon}}d\xi &= \left|h^{(1)}\right|^2\int_0^{\sigma}\frac{1}{\varepsilon^2}e^{\frac{-\alpha\xi}{\varepsilon}}d\xi\\ &\le \frac{\left|h^{(1)}\right|^2}{\varepsilon}\alpha^{-1}\\ &\le 2l\alpha^{-2}N^{-2}\ln N\\ &\le CN^{-2}\ln N. \end{aligned}\tag{39}$$

For the second term in the right-side of (38), we obtain

$$\begin{aligned}\sum_{j=\frac{N}{2}+1}^{N}\int_{x_{j-1}}^{x_j}(x_j-\xi)(\xi-x_{j-1})\frac{1}{\varepsilon^2}e^{\frac{-\alpha\xi}{\varepsilon}}d\xi &= 2\alpha^{-1}\sum_{j=\frac{N}{2}+1}^{N}\int_{x_{j-1}}^{x_j}\left(x_j-x-\frac{h^{(2)}}{2}\right)\frac{1}{\varepsilon}e^{\frac{-\alpha x}{\varepsilon}}dx\\ &\le 2\alpha^{-1}h^{(2)}\int_{\sigma}^{l}\frac{1}{\varepsilon}e^{\frac{-\alpha x}{\varepsilon}}dx\\ &= 2\alpha^{-2}h^{(2)}\left(e^{\frac{-\alpha\sigma}{\varepsilon}}-e^{\frac{-\alpha l}{\varepsilon}}\right)\\ &\le 2\alpha^{-2}h^{(2)}N^{-1}\\ &\le CN^{-2}. \end{aligned}\tag{40}$$

Therefore, the estimates (36), (37), (39) and (40) along with (35) yield (34).

Further, to confirm (33), we will estimate the remainder terms $R_i^{(1)}$, $R_i^{(2)}$, $R_i^{(3)}$ and $R_i^{(4)}$ separately. For $R_i^{(4)}$, taking into account the boundedness of $\frac{\partial^2 K}{\partial x^2}$, from (24) similar to above, we get

$$\left|R_i^{(4)}\right| \le CN^{-2}\ln N. \tag{41}$$

Next, we will estimate $R_i^{(1)}$. Since $a\in C^2[0,l]$, $|x-x_{i-1}|\le h_i$ and $|x-x_i|\le h_i$, by using Lemma 1, it follows that

$$\begin{aligned}\left|R_i^{(1)}\right| &\le Ch_i^2+\left|a_{\overline{x},i}\delta_i\right|h_i\int_{x_{i-1}}^{x_i}\left|u'(x)\right|dx\\ &\le Ch_i^2+Ch_i\int_{x_{i-1}}^{x_i}\left|u'(x)\right|dx\\ &\le C\left(h_i^2+h_i\int_{x_{i-1}}^{x_i}\frac{1}{\varepsilon}e^{\frac{-\alpha x}{\varepsilon}}dx\right). \end{aligned}\tag{42}$$

We find the estimate for the case $\sigma = \frac{l}{2}$. Then $\frac{l}{2} < \alpha^{-1}\varepsilon \ln N$ and $h^{(1)} = h^{(2)} = lN^{-1}$. Hence we have

$$\begin{aligned} h_i \int\limits_{x_{i-1}}^{x_i} \frac{1}{\varepsilon} e^{\frac{-\alpha x}{\varepsilon}} dx &\leq \frac{\left|h^{(1)}\right|^2}{\varepsilon} \\ &\leq CN^{-2}, \qquad 1 \leq i \leq N. \end{aligned} \tag{43}$$

We now consider the case $\sigma = \alpha^{-1}\varepsilon \ln N < \frac{l}{2}$ in (42) on ω_N. The inequalities

$$\begin{aligned} h_i \int\limits_{x_{i-1}}^{x_i} \frac{1}{\varepsilon} e^{\frac{-\alpha x}{\varepsilon}} dx &\leq \frac{\left|h^{(1)}\right|^2}{\varepsilon} = \left(\frac{2\sigma}{N}\right)^2 \frac{1}{\varepsilon} = \left(\frac{2\alpha^{-1}\varepsilon \ln N}{N}\right)^2 \frac{1}{\varepsilon} = 4\alpha^{-2}\varepsilon N^{-2} \ln^2 N \\ &\leq \frac{l}{2} 4\alpha^{-1} N^{-2} \ln N \\ &\leq CN^{-2} \ln N, \qquad 1 \leq i \leq \frac{N}{2}, \\ h_i \int\limits_{x_{i-1}}^{x_i} \frac{1}{\varepsilon} e^{\frac{-\alpha x}{\varepsilon}} dx &\leq h^{(2)} \alpha^{-1} \left(e^{\frac{-\alpha x_{i-1}}{\varepsilon}} - e^{\frac{-\alpha x_i}{\varepsilon}} \right) = h^{(2)} \alpha^{-1} e^{\frac{-\alpha x_{i-1}}{\varepsilon}} \left(1 - e^{\frac{-\alpha h^{(2)}}{\varepsilon}} \right) \\ &\leq h^{(2)} \alpha^{-1} e^{\frac{-\alpha x_{i-1}}{\varepsilon}} \\ &\leq h^{(2)} \alpha^{-1} N^{-1} \\ &\leq CN^{-2}, \qquad \frac{N}{2} + 1 \leq i \leq N \end{aligned}$$

imply that

$$h_i \int\limits_{x_{i-1}}^{x_i} \frac{1}{\varepsilon} e^{\frac{-\alpha x}{\varepsilon}} dx \leq CN^{-2} \ln N, \qquad 1 \leq i \leq N. \tag{44}$$

Therefore, from (43) and (44), we deduce that

$$\left|R_i^{(1)}\right| \leq CN^{-2} \ln N, \qquad 1 \leq i \leq N. \tag{45}$$

Third, we will estimate $R_i^{(2)}$. Since $f \in C^2[0, l]$, $|x - x_{i-1}| \leq h_i$ and $|x - x_i| \leq h_i$, by using Lemma 1, it follows that

$$\begin{aligned} \left|R_i^{(2)}\right| &= \frac{1}{2} \chi_i^{-1} h_i^{-1} \left| \int\limits_{x_{i-1}}^{x_i} f''\big(\eta_i(x)\big)(x - x_{i-1})(x - x_i)\varphi_i(x) dx \right| \\ &\leq Ch_i^2 \\ &\leq CN^{-2}. \end{aligned} \tag{46}$$

Note that the above estimate is valid for values both $\sigma = \frac{l}{2}$ and $\sigma = \alpha^{-1}\varepsilon \ln N$. Fourth, we will estimate $R_i^{(3)}$. By taking into account the boundedness of $\frac{\partial^2 K}{\partial x^2}$, from (22) it follows that

$$\begin{aligned} \left|R_i^{(3)}\right| &\le \frac{1}{2}\chi_i^{-1}h_i^{-1}\left|\int\limits_{x_{i-1}}^{x_i}(x-x_i)^2\varphi_i(x)\left(\int\limits_0^l \frac{\partial^2}{\partial x^2}K\big(\xi_i(x),s\big)u(s)ds\right)dx\right| \\ &\le Ch_i^2 \\ &\le CN^{-2}. \end{aligned} \tag{47}$$

Note that the above estimate is valid for values both $\sigma = \frac{l}{2}$ and $\sigma = \alpha^{-1}\varepsilon \ln N$. The inequalities (41), (45), (46) and (47) finish the proof of (33). □

Theorem 1 *Let a, c and K satisfy the assumptions from Lemma 2. Moreover*

$$|\lambda| < \frac{\bar{\alpha}}{(|\mu| + \|c\|_1 + 1)\max\limits_{0\le i\le N}\sum\limits_{j=1}^{N}\hbar_j\left|\mathcal{K}_{i,j}\right|}. \tag{48}$$

Then for the solution z of the difference problem (31)-(32) holds the estimate

$$\|z\|_{\infty,\overline{\omega}_N} \le CN^{-2}\ln N.$$

Proof Equation (31) may be rewritten as

$$\varepsilon\theta_i z_{\bar{x},i} + \bar{a}_i z_i = F_i, \quad 1 \le i \le N-1, \tag{49}$$

where

$$F_i = R_i - |\lambda|\sum_{j=0}^{N}\hbar_j\mathcal{K}_{ij}z_j.$$

From (49) we get

$$z_i = \frac{\varepsilon\theta_i}{\varepsilon\theta_i + \bar{a}_i h_i}z_{i-1} + \frac{h_i F_i}{\varepsilon\theta_i + \bar{a}_i h_i}.$$

The solution to the above first-order difference equation will be as follows:

$$z_i = z_0 Q_i + \sum_{k=1}^{i}\phi_k Q_{i-k}, \tag{50}$$

where

$$Q_{i-k} = \begin{cases} 1, & k = i, \\ \prod\limits_{l=k+1}^{i} \frac{\varepsilon\theta_l}{\varepsilon\theta_l + \bar{a}_l h_l}, & 0 \leq k \leq i-1, \end{cases} \qquad \phi_k = \frac{h_i F_i}{\varepsilon\theta_i + \bar{a}_i h_i}.$$

Then, from (32) and (50), we obtain

$$z_0 = \frac{\mu \sum\limits_{k=1}^{N} \frac{h_k F_k}{\varepsilon\theta_k + \bar{a}_k h_k} Q_{N-k} + \sum\limits_{i=1}^{N} \hbar_i c_i \left(\sum\limits_{k=1}^{i} \frac{h_k F_k}{\varepsilon\theta_k + \bar{a}_k h_k} Q_{i-k} \right) - r}{1 - \mu Q_N - \sum\limits_{k=1}^{N} \hbar_k c_k Q_k}. \tag{51}$$

Since, the denominator is bounded below by one and the equality (51) reduces to

$$\begin{aligned} &\left| \mu \sum_{k=1}^{N} \frac{h_k F_k}{\varepsilon\theta_k + \bar{a}_k h_k} Q_{N-k} + \sum_{i=1}^{N} \hbar_i c_i \left(\sum_{k=1}^{i} \frac{h_k F_k}{\varepsilon\theta_k + \bar{a}_k h_k} Q_{i-k} \right) - r \right| \\ &\quad \leq C \left(|\mu| \, \|F\|_{\infty,\omega_N} + \|c\|_1 \, \|F\|_{\infty,\omega_N} + |r| \right). \end{aligned} \tag{52}$$

Considering (51) and (52) together, we have

$$\begin{aligned} |z(0)| &\leq C \left(|\mu| + \|c\|_1 \right) \|R\|_{\infty,\omega_N} \\ &\quad + C \left(\left(|\mu| + \|c\|_1 \right) \bar{\alpha}^{-1} |\lambda| \max_{1 \leq i \leq N} \sum_{j=1}^{N} \hbar_j \left| \mathcal{K}_{ij} \right| \|z\|_{\infty,\omega_N} + |r| \right). \end{aligned} \tag{53}$$

Now, applying discrete maximum principle for (49), we get

$$\|z\|_{\infty,\bar{\omega}_N} \leq |z(0)| + \bar{\alpha}^{-1} \|R\|_{\infty,\omega_N} + |\lambda| \, \alpha^{-1} \max_{1 \leq i \leq N} \sum_{j=1}^{N} \hbar_j \left| \mathcal{K}_{ij} \right| \|z\|_{\infty,\bar{\omega}_N}.$$

Finally, instead of $z(0)$ in the above inequality, considering the estimate of (53), we get

$$\|z\|_\infty \leq \frac{C \left(|\mu| + \|c\|_1 + 1 \right) \alpha^{-1} \|R\|_{\infty,\omega_N} + C |r|}{1 - \left(|\mu| + \|c\|_1 + 1 \right) |\lambda| \, \alpha^{-1} \max\limits_{1 \leq i \leq N} \sum\limits_{j=1}^{N} \hbar_j \left| \mathcal{K}_{ij} \right|}.$$

Therefore

$$\|z\|_\infty \leq C \left(\|R\|_{\infty,\omega_N} + |r| \right).$$

This inequality together with (33) and (34) produces the desired result. □

5 Numerical results

Here, we have considered two specific problems to demonstrate the feasibility of the proposed approach. The following iterative technique will be used.

$$y_i^{(n)} = \frac{\bar{f}_i + \frac{\varepsilon\theta_i}{h_i} y_{i-1}^{(n)} - \lambda \sum\limits_{j=0}^{i-1} \hbar_j \mathcal{K}_{ij} y_j^{(n)} - \lambda \sum\limits_{j=i+1}^{N} \hbar_j \mathcal{K}_{ij} y_j^{(n-1)}}{\frac{\varepsilon\theta_i}{h_i} + \bar{a}_i + \lambda \hbar_i \mathcal{K}_{ii}}, \quad i = 1, 2, ..., N,$$

$$y_0^{(n)} = \mu y_N^{(n-1)} + \sum_{j=0}^{N} \hbar_j c_j y_j^{(n-1)} + A, \quad n = 1, 2, ...,$$

where $y_1^{(0)}, y_2^{(0)}, ..., y_N^{(0)}$ are the given initial iterations.

Example 1 We consider the test problem

$$\begin{aligned} \varepsilon u'(x) + u(x) + \frac{1}{20}\int_0^1 x u(s)\, ds &= -\frac{\varepsilon}{(1+x)^2} + \frac{1}{1+x} + x\varepsilon\left(1 - e^{-\frac{x}{\varepsilon}}\right) \\ &\quad + x\ln(1+x) - \frac{19}{20}x\left[\varepsilon\left(1 - e^{-\frac{x}{\varepsilon}}\right) + \ln(1+x)\right] \\ &\quad + \frac{1}{20}x\left[\varepsilon\left(e^{-\frac{x}{\varepsilon}} - e^{-\frac{1}{\varepsilon}}\right) + \ln\left(\frac{2}{1+x}\right)\right], \quad 0 < x \le 1, \end{aligned}$$

$$u(0) + 2u(1) + \int_0^1 s u(s)\, ds = 4 + \varepsilon^2 + (2 - \varepsilon(1+\varepsilon))\, e^{-\frac{1}{\varepsilon}} - \ln 2.$$

The exact solution of test problem is given by

$$u(x) = e^{-\frac{x}{\varepsilon}} + \frac{1}{1+x}.$$

We define the exact errors as follows:

$$e_\varepsilon^N = \|y - u\|_{\infty, \bar{\omega}_N}.$$

The results of the problem obtained by using different ε and N values for both the present method and solving exact of SPFIDE are given in the following tables 1-6. In addition, in tables, exact errors are shown according to the exact solutions and approximate solutions.

Figs. 1 and 2 represent the solution plots for different values of ε and N in Example 1, according to the table values. The figures clearly show that the exact solution and the approximated solution for Example 1 overlap, thereby showing the aptness of the proposed techniques.

Table 1 The numerical results of Example 1 for $\varepsilon = 2^{-4}$ and $N = 64$

x_i	u_i	y_i	e_i^{64}
0.0081	1.8704	1.8701	0.0003
0.0162	1.7557	1.6543	0.1014
0.0243	1.6541	1.4174	0.2367
0.0486	1.4132	0.9774	0.4358
0.0729	1.2435	0.8890	0.3545
0.1377	0.9894	0.8380	0.1514
0.2187	0.8508	0.7877	0.0631
0.4447	0.6930	0.6778	0.0152
0.6757	0.5968	0.5989	0.0021
0.8605	0.5375	0.5509	0.0134

Table 2 The numerical results of Example 1 for $\varepsilon = 2^{-4}$ and $N = 128$

x_i	u_i	y_i	e_i^{128}
0.0047	1.9229	1.9223	0.0006
0.0094	1.8511	1.7829	0.0682
0.0235	1.6636	1.2688	0.3948
0.0376	1.5117	0.9836	0.5281
0.0799	1.2045	0.8777	0.3268
0.1363	0.9930	0.8375	0.1555
0.2162	0.8537	0.7879	0.0658
0.4450	0.6929	0.6776	0.0153
0.6630	0.6013	0.6028	0.0015
0.8592	0.5379	0.5515	0.0136

Table 3 The numerical results of Example 1 for $\varepsilon = 2^{-4}$ and $N = 256$

x_i	u_i	y_i	e_i^{256}
0.0027	1.9550	1.9549	0.0001
0.0108	1.8306	1.6244	0.2062
0.0162	1.7557	1.3597	0.3960
0.0297	1.5929	0.9796	0.6133
0.0864	1.1714	0.8720	0.2994
0.1620	0.9355	0.8208	0.1147
0.2565	0.8124	0.7659	0.0465
0.4486	0.6911	0.6757	0.0154
0.6730	0.5977	0.5995	0.0018
0.8566	0.5386	0.5518	0.0132

Table 4 The numerical results of Example 1 for $\varepsilon = 2^{-8}$ and $N = 128$

x_i	u_i	y_i	e_i^{128}
0.0006	1.8587	1.7957	0.0630
0.0012	1.7372	1.4661	0.2711
0.0024	1.5429	1.0599	0.4830
0.0056	1.2312	0.9909	0.2403
0.0649	0.9391	0.9379	0.0012
0.1108	0.9003	0.9009	0.0006
0.3097	0.7635	0.7718	0.0083
0.5086	0.6629	0.6785	0.0156
0.7075	0.5857	0.6086	0.0229
0.9064	0.5245	0.5547	0.0302

Table 5 The numerical results of Example 1 for $\varepsilon = 2^{-8}$ and $N = 256$

x_i	u_i	y_i	e_i^{256}
0.0003	1.9167	1.8776	0.0391
0.0010	1.7701	1.3999	0.3702
0.0019	1.6191	1.0523	0.5668
0.0054	1.2446	0.9911	0.2535
0.0100	1.0677	0.9867	0.0810
0.0977	0.9110	0.9109	0.0001
0.3029	0.7675	0.7749	0.0074
0.5005	0.6664	0.6811	0.0147
0.7057	0.5863	0.6083	0.0220
0.9033	0.5254	0.5546	0.0292

Table 6 The numerical results of Example 1 for $\varepsilon = 2^{-8}$ and $N = 512$

x_i	u_i	y_i	e_i^{512}
0.0002	1.9522	1.9292	0.0230
0.0010	1.7638	1.1965	0.5673
0.0014	1.6924	1.0489	0.6435
0.0061	1.2042	0.9904	0.2138
0.0100	1.0675	0.9867	0.0808
0.1004	0.9088	0.9089	0.0001
0.3018	0.7682	0.7760	0.0078
0.5032	0.6652	0.6805	0.0153
0.7008	0.5880	0.6104	0.0224
0.9022	0.5257	0.5555	0.0298

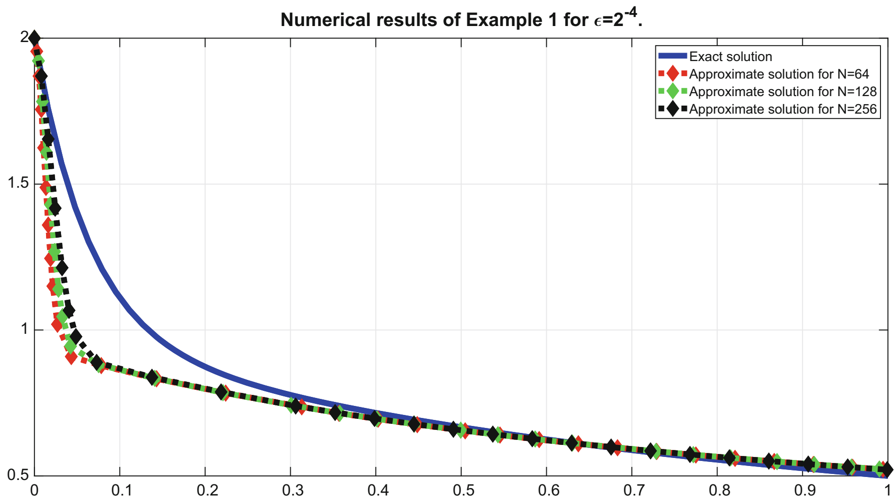

Fig. 1 Numerical results of Example 1 for $\epsilon = 2^{-4}$ and $N = 64, 128, 256$

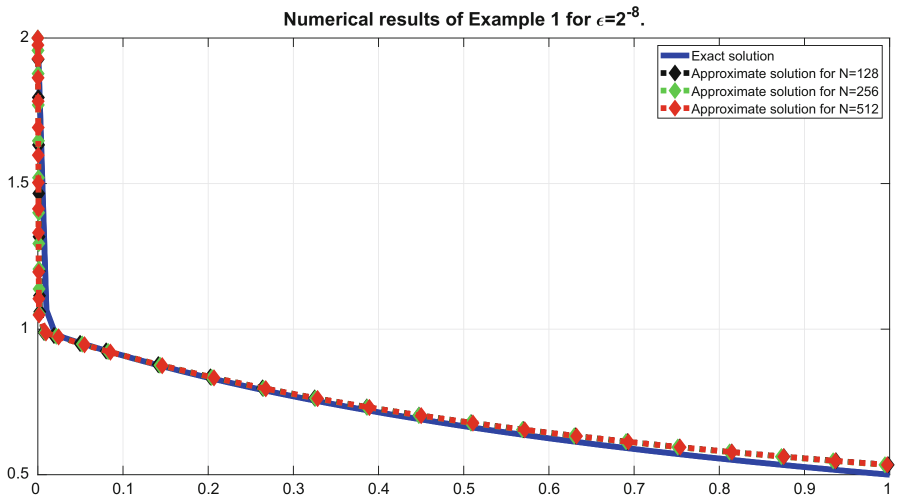

Fig. 2 Numerical results of Example 1 for $\epsilon = 2^{-8}$ and $N = 128, 256, 512$

Example 2 Consider the other problem:

$$\varepsilon u'(x) + \frac{4}{1+x^2} u(x) + \frac{1}{10} \int_0^1 e^{1-xs} u(s)\, ds = 2x + 1, \quad 0 < x \le 1,$$

$$u(0) + 2u(1) + \int_0^1 \sin\left(\frac{\pi s}{2}\right) u(s)\, ds = -2.$$

Table 7 Maximum point-wise errors and the rates of convergence for different vales of ε and N

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^0	0.05368	0.01607	0.00452	0.00117	0.00029
	1.74	1.83	1.95	2.01	
2^{-4}	0.05558	0.01687	0.00481	0.00127	0.00032
	1.72	1.81	1.92	1.99	
2^{-8}	0.05610	0.01703	0.00496	0.00132	0.00034
	1.72	1.78	1.91	1.96	
2^{-12}	0.05544	0.01683	0.00497	0.00135	0.00035
	1.72	1.76	1.88	1.95	
2^{-16}	0.05680	0.01736	0.00516	0.00142	0.00037
	1.71	1.75	1.86	1.94	
e^N	0.05680	0.01736	0.00516	0.00142	0.00037
p^N	1.66	1.74	1.86	1.93	

The exact solution to this problem is unknown. For this reason, we estimate errors and calculate solutions using the double-mesh method, which compares the obtained solution to a solution computed on a mesh that is twice as fine. We introduce the maximum point-wise errors and the computed as

$$e_\varepsilon^N = \max_i |y_i^{\varepsilon,N} - \tilde{y}_{2i}^{\varepsilon,2N}|_{\infty,\overline{\omega}_N}, \quad e^N = \max_\varepsilon e_\varepsilon^N,$$

where $\tilde{y}_i^{\varepsilon,2N}$ is the approximate solution of the respective method on the mesh

$$\tilde{\omega}_{2N} = \{x_{i/2} : i = 0, 1, ..., 2N\}$$

with

$$x_{i+1/2} = \frac{x_i + x_{i+1}}{2} \quad \text{for} \quad i = 0, 1, ..., N-1.$$

We also describe the rates of convergence and computed ε-uniform rate of convergence of the form

$$p_\varepsilon^N = \frac{\ln\left(e_\varepsilon^N / e_\varepsilon^{2N}\right)}{\ln 2}, \quad p^N = \frac{\ln\left(e^N / e^{2N}\right)}{\ln 2}.$$

The values of ε and N for which we resolve the Example 2 are $\varepsilon = 2^0, 2^{-4}, 2^{-8}, 2^{-12}, 2^{-16}$ and $N = 64, 128, 256, 512, 1024$. From Table 7, we observe that the ε-uniform rate of convergence p^N is monotonically increasing towards two, therefore in agreement with the theoretical rate given by Theorem 1.

6 Conclusion

This article comprises a numerical method employed to solve a linear SPFIDE of the form (1)-(2). On a special piecewise uniform mesh, the differential equation is discretized by using a fitted finite difference operator. The composite trapezoidal integration rule with the remainder term in integral form has been used for the integral part in (1) and initial condition (2), yielding uniform second-order convergence. Specific test problems have been performed to assess and test the performance of the numerical scheme. The obtained results can be presented to more complicated FIDEs.

References

1. Abdulghani, M., Hamoud, A., Ghandle, K.: The effective modification of some analytical techniques for Fredholm integro-differential equations. Bulletin of the International Mathematical Virtual Institute **9**, 345–353 (2019)
2. Abdullah, J.T.: Numerical solution for linear Fredholm integro-differential equation using Touchard polynomials. Baghdad Sci. J. **18**(2), 330–337 (2021)
3. Amiraliyev, G.M., Durmaz, M.E., Kudu, M.: Uniform convergence results for singularly perturbed Fredholm integro-differential equation. J. Math. Anal. **9**(6), 55–64 (2018)
4. Amiraliyev, G.M., Durmaz, M.E., Kudu, M.: Fitted second order numerical method for a singularly perturbed Fredholm integro-differential equation Bull. Belg. Math. Soc. - Simon Stevin **27**(1), 71–88 (2020)
5. Amiraliyev, G.M., Durmaz, M.E., Kudu, M.: A numerical method for a second order singularly perturbed Fredholm integro-differential equation. Miskolc Math. Notes **22**(1), 37–48 (2021)
6. Amiraliyev, G.M., Mamedov, Y.D.: Difference schemes on the uniform mesh for singularly perturbed pseudo-parabolic equations. Turkish J. Math. **19**, 207–222 (1995)
7. Arqub, O.A., Al-Smadi, M., Shawagfeh, N.: Solving Fredholm integro-differential equations using reproducing kernel Hilbert space method. Appl. Math. Comput. **219**(17), 8938–8948 (2013)
8. Brunner, H.: Contemporary Computational Mathematics-A Celebration of the 80th Birthday of Ian Sloan. In: Dick, J., et al. (eds.) Numerical Analysis and Computational Solution of Integro-Differential Equations, pp. 205–231. Springer, Cham (2018)
9. Cakir, M.: A numerical study on the difference solution of singularly perturbed semilinear problem with integral boundary condition. Math. Model. Anal. **21**(5), 644–658 (2016)
10. Cakir, M., Arslan, D.: A new numerical approach for a singularly perturbed problem with two integral boundary conditions. Comput. Appl. Math. **40**, 189 (2021)
11. Cannon, J.R.: The solution of the heat equation subject to the specification of energy. Q. Appl. Math. **21**(2), 155–160 (1963)
12. Chen, J., He, M., Huang, Y.: A fast multiscale Galerkin method for solving second order linear Fredholm integro-differential equation with Dirichlet boundary conditions. J. Comput. Appl. Math. **364**, 112352 (2020)
13. Cimen, E., Cakir, M.: A uniform numerical method for solving singularly perturbed Fredholm integro-differential problem. Comput. Appl. Math. **40**, 42 (2021)
14. Dag, H.G., Bicer, K.E.: Boole collocation method based on residual correction for solving linear Fredholm integro-differential equation. Journal of Science and Arts **3**(52), 597–610 (2020)
15. Durmaz, M.E., Amiraliyev, G.M.: A robust numerical method for a singularly perturbed Fredholm integro-differential equation. Mediterr. J. Math. **18**, 1–17 (2021)
16. Durmaz, M.E., Amiraliyev, G.M., Kudu, M.: Numerical solution of a singularly perturbed Fredholm integro differential equation with Robin boundary condition. Turk. J. Math. **46**(1), 207–224 (2022)
17. Dzhumabaev, D.S., Nazarova, K.Z., Uteshova, R.E.: A modification of the parameterization method for a linear boundary value problem for a Fredholm integro-differential equation. Lobachevskii J. Math. **41**, 1791–1800 (2020)
18. Farrell, P.A., Hegarty, A.F., Miller, J.J.H., O'Riordan, E., Shishkin, G.I.: Robust Computational Techniques for Boundary Layers. Chapman Hall/CRC, New York (2000)

19. Hamoud, A.A., Ghadle, K.P.: Usage of the variational iteration technique for solving Fredholm integro-differential equations. J. Comput. Appl. Mech. **50**(2), 303–307 (2019)
20. Hosseini, S.M., Shahmorad, S.: Tau numerical solution of Fredholm integro-differential equations with arbitrary polynomial bases. Appl. Math. Model. **27**(2), 145–154 (2003)
21. Ionkin, N.I.: Solution of a boundary value problem in heat conduction theory with nonlocal boundary conditions. Differ. Equ. **13**, 294–304 (1977)
22. Kadalbajoo, M.K., Gupta, V.: A brief survey on numerical methods for solving singularly perturbed problems. Appl. Math. Comput. **217**, 3641–3716 (2010)
23. Keimer, A., Pflug, L.: Modeling infectious diseases using integro-differential equations: Optimal control strategies for policy decisions and Applications in COVID-19, (2020), https://doi.org/10.13140/RG.2.2.10845.44000
24. Kudu, M., Amirali, I., Amiraliyev, G.M.: A finite-difference method for a singularly perturbed delay integro-differential equation. J. Comput. Appl. Math. **308**, 379–390 (2016)
25. Kudu, M.: A parameter uniform difference scheme for the parameterized singularly perturbed problem with integral boundary condition, Adv. Differ. Equ., 170 (2018)
26. Miller, J.J.H., O'Riordan, E., Shishkin, G.I.: Fitted Numerical Methods for Singular Perturbation Problems: Error Estimates in the Maximum Norm for Linear Problems in One and Two Dimensions, Rev World Scientific, Singapore (2012)
27. Nayfeh, A.H.: Introduction to Perturbation Techniques. Wiley, New York (1993)
28. Nicoud, F., Schönfeld, T.: Integral boundary conditions for unsteady biomedical CFD applications. Int. J. Numer. Methods Fluids **40**, 457–465 (2002)
29. O'Malley, R.E.: Singular Perturbations Methods for Ordinary Differential Equations. Springer, New York (2013)
30. Panda, A., Mohapatra, J., Amirali, I.: A second-order post-processing technique for singularly perturbed Volterra integro-differential equations. Mediterr. J. Math. **18**, 231 (2021)
31. Panda, A., Mohapatra, J., Reddy, N.R.: A comparative study on the numerical solution for singularly perturbed Volterra integro-differential equations. Comput. Math. Model. **32**, 364–375 (2021)
32. Pour-Mahmoud, J., Rahimi-Ardabili, M.Y., Shahmorad, S.: Numerical solution of the system of Fredholm integro-differential equations by the Tau method. Appl. Math. Comput. **168**(1), 465–478 (2005)
33. Roos, H.G., Stynes, M., Tobiska, L.: Robust Numerical Methods for Singularly Perturbed Differential Equations. Springer-Verlag, Berlin Heidelberg (2008)
34. Samarskii, A.A.: The Theory of Difference Schemes. Marcell Dekker, Inc., New York (2001)
35. Shakti, D., Mohapatra, J.: A second order numerical method for a class of parameterized singular perturbation problems on adaptive grid. Nonlinear Eng. **6**(3), 221–228 (2017)
36. Shakti, D., Mohapatra, J.: A uniformly convergent numerical scheme for singularly perturbed differential equation with integral boundary condition arising in neural network. Int. J. Computing Science and Mathematics **10**(4), 340–350 (2019)
37. Tair, B., Guebbai, H., Segni, S., Ghiat, M.: An approximation solution of linear Fredholm integro-differential equation using Collocation and Kantorovich methods. J. Appl. Math, Comput (2021)
38. Tair, B., Guebbai, H., Segni, S., Ghiat, M.: Solving linear Fredholm integro-differential equation by Nyström method. J. Appl. Math. Comput. Mech. **20**(3), 53–64 (2021)
39. Thieme, H.R.: A model for the spatial spread of an epidemic. J. Math. Biol. **4**, 337–351 (1977)
40. Turkyilmazoglu, M.: Analytic approximate solutions of parameterized unperturbed and singularly perturbed boundary value problems. Appl. Math. Model. **35**(8), 3879–3886 (2011)
41. Turkyilmazoglu, M.: An effective approach for numerical solutions of high-order Fredholm integro-differential equations. Appl. Math. Comput. **227**, 384–398 (2014)
42. Turkyilmazoglu, M.: High-order nonlinear Volterra - Fredholm - Hammerstein integro-differential equations and their effective computation. Appl. Math. Comput. **247**, 410–416 (2014)
43. Turkyilmazoglu, M.: Effective computation of exact and analytic approximate solutions to singular nonlinear equations of Lane-Emden-Fowler type. Appl. Math. Model. **37**, 7539–7548 (2013)
44. Yalcinbas, S., Sezer, M., Sorkun, H.H.: Legendre polynomial solutions of high-order linear Fredholm integro-differential equations. Appl. Math. Comput. **210**(2), 334–349 (2009)

45. Yapman, Ö., Amiraliyev, G.M.: Convergence analysis of the homogeneous second order difference method for a singularly perturbed Volterra delay-integro-differential equation. Chaos Solit. Fractals **150**, 111100 (2021)