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ORIGINAL RESEARCH

Further development of F-index for fuzzy graph and its application in Indian railway crime

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Abstract

From the mid-nineteenth century, the railway network has been the most important mode of conveying people and goods in India. 22.15 million passengers used this network and 3.32 million metric tons of goods were also shipped daily from 2019 to 2020. The national rail network comprised 126,366 km of track over a route of 67,368 km and 7,325 stations. It is the fourth-largest national railway network globally after the United States, Russia and China. But with the passage, they pose a threat to the general public while travelling, being the instances of crimes rising fourfold in rails. The ongoing railway crime has become a cause of concern for the common passenger now. In this article, F-index for fuzzy graphs is used to analyze the railway crimes in India and compared with three other topological indices. F-index for fuzzy graphs and the first Zagreb index for fuzzy graphs provide similar results whereas F-index for fuzzy graphs provides better realistic results than F-index for crisp graphs and first Zagreb index for crisp graphs to detect the crime in Indian railway. Also, this index is studied for several operations such as Cartesian product, composition, union and join of two fuzzy graphs. Some interesting relations of F-index are established during fuzzy graph transformations. Using those transformation, it is shown that n-vertex star has maximum F-index among the n-vertex trees. Also, maximal n-vertex unicyclic fuzzy graph having r cycle is determined with respect to F-index.

Keywords Fuzzy graph · Topological index · Forgotten topological index

Mathematics Subject Classification 05C09, 05C72

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1 Introduction

1.1 Research background

Nowadays, due to the vast applications of fuzzy graph theory, many researchers are working on topological indices (TIs) on fuzzy graphs. In the year 1965, the concept of uncertainty in a classical set was firstly introduced by Zadeh [45] and called it fuzzy set. In 1975, inspired by this, Rosenfeld [37] introduced the fuzzyness for a graph and called the graph is the fuzzy graph (FG). Generalized the neutrosophic planer graph [23] and link prediction in the neutrosophic graph [24] are studied by Mahapatra et al. A telecommunication system based on fuzzy graphs [40] and a social network system based on fuzzy graphs [41] is studied by Samanta and Pal. Bipolar fuzzy graph [3], fuzzy soft graph [4], m-polar fuzzy graph [5], generalized m-polar fuzzy graph [43], bipolar fuzzy soft hypergraph [42] and measures of connectivity in rough fuzzy network [6] are studied by Akram et al. Rashmanlou et al. [38] studied the product of bipolar fuzzy graphs with their degree. In 2016, Rashmanlou and Borzooei studied vague graph in [39]. The degree of a vertex in a FG is also discussed in [33] and also discussed strong degree, strong neighbour of a FG. In [5, 32, 33], one can see for more generalization of a FG and related results.

In the field of mathematical chemistry, molecular topology and chemical graph theory, TIs are molecular descriptors that are calculated on the molecular graph of a chemical compound. These TIs are numerical quantities of a graph that describe its topology. Vertex represents an atom and an edge represents the bond between two atoms in a molecular graph. In 1947, the Wiener index (WI) was introduced by Wiener, which is used to calculate the boiling point of paraffins. Zagreb index (ZI) is degree-based TI established by Gutman and Trinajstic in 1972 [15] and used to calculate π-electron energy of a conjugate system. In 2015, Fortula and Gutman defined another degree-based topological index called forgotten topological index (F-index) [14]. Amin and Nayeem studied the F-index and F-coindex of subdivision graph and line graph in [7]. In 2017, Abdo et al. [1] studied the extremal trees (crisp) with respect to F-index. Extremal unicyclic and bicyclic graphs (crisp) with respect to F-index is determined by Akhtar et al. De et al. [13] studied F-index for some crisp graph operations.

In 2019, Binu et al. [8, 10, 11] introduced connectivity index, average connectivity index, connectivity status and cyclic connectivity index of a FG. Motivated from these articles, recently, Binu et al. [9] introduced Wiener index of a FG and Islam and Pal also studied Wiener index for FG in [16]. In 2021, Islam and Pal also introduced hyper-Wiener index [19], hyper-connectivity index [20], first Zagreb index [17] and F-index [18] for fuzzy graphs. In [22], Mahapatra et al. provided the RSM index in fuzzy graph. Poulik et al. studied Cretain index [34], Wiener absolute index [35] and Randic index [36] for bipolar fuzzy graphs. In [21] Kalathian et al. also introduced so many index for fuzzy graphs.

Table 1 Review of related works

Year	Author	Work
1947	Wiener [44]	He has first introduced the topological index “Wiener index” for crisp graphs.
1965	Zadeh [45]	He has first introduced the concept of fuzzy set.
1972	Gutman and Trinajstic [15]	They have introduced another topological index “Zagreb index” for crisp graphs.
1975	Rosenfeld [37]	He has introduced the fuzzy graph.
2015	Fortula and Gutman [14]	They have first introduced F-index for crisp graphs.
2016	De et al. [14]	They have studied the F-index of some graph (crisp) operations.
2017	Abdo et al. [1]	They have find maximal trees with respect to the F-index for crisp graph.
2017	Akhtar et al. [2]	They have find maximal unicyclic graph with respect to the F-index for crisp graph.
2018	Amin et al. [7]	They have studied F-index and F-coindex of the line graphs of the subdivision graphs.
2019	Binu et al. [8]	They have introduced the connectivity index for fuzzy graphs.
2019	Mondal et al. [26, 27]	They have introduced some neighbourhood degree based topological indices for crisp graphs.
2020	Binu et al. [8]	They have introduced the Wiener index for fuzzy graphs.
2020	Kalathian et al. [21]	They have studied some topological indices for fuzzy graphs.
2020	Poulik and Ghorai [34]	They have introduced the certain index for bipolar fuzzy graphs.
2020	Maji and Ghorai [25]	They have studied F-index for *k*th generalized transformation graphs (crisp).
2021	Poulik and Ghorai [35]	They have introduced the Wiener absolute index for bipolar fuzzy graphs.
2021	Poulik et al. [36]	They have introduced the Randic index for bipolar fuzzy graphs.
2021	Mondal et al. [29]	They have studied neighbourhood Zagreb index for product graph.
2021	Islam and Pal [17]	They have studied the first Zagreb index for fuzzy graphs.
2021	Islam and Pal [18]	They have studied the F-index for fuzzy graphs.
2021	Islam and Pal [19]	They have studied the hyper-Wiener index for fuzzy graphs.
2021	Islam and Pal [20]	They have studied the hyper-connectivity index for fuzzy graphs.
	This paper	We have partially generalized the results of [1, 2, 13] and provided an realistic application and compared with other three topological indices.

1.2 Motivation

Topological indices have an important role in molecular chemistry, chemical graph theory, spectral graph theory, network theory, etc. Zagreb index (ZI) is one such TIs which is degree-based TI and established by Gutman and Trinajstic in 1972 [15] and these TIs are used to calculate π-electron energy of a conjugate system. Followed by this topological index, in 2015, Fortula and Gutman defined another degree-based topological index called forgotten topological index (F-index) [14] and they proved that first ZI and F-index have the almost same entropy, predictive ability and acentric factor and F-index obtain correlation coefficients greater than 0.95. So this TI is very much useful in molecular chemistry, spectral graph theory, network theory, and several fields of mathematics and chemistry. Some neighbourhood degree-based topological indices are introduced and studied their correlations between the physico-chemical properties of some chemical compounds by Mondal et al. [26, 27]. In [29] they also studied neighbourhood Zagreb index of product graph and analyzed QSPR of some novel degree-based topological descriptors [30]. In [28, 31], they also studied molecular descriptors of some chemicals that prevent COVID-19. But, those indices are defined in the crisp graph only. Recently, in 2021, Islam and Pal [18] introduced and studied F-index for fuzzy graphs. Also, they introduced so many interesting results on F-index for various fuzzy graphs like path, cycle, star, complete fuzzy graph, etc. Motivated by this article, we start research on further developing the F-index for fuzzy graphs. F-index is used not only for application but also for theoretical purposes. Abdo et al. determined the extremal n-vertex trees (Crisp) with respect to the F-index and Akhtar et al. [2] determined the extremal n-vertex unicyclic and bicyclic graphs with respect to the F-index. Still, those results for the fuzzy graph have been unsolved till now. In this article, We established that n-vertex star has maximal F-index among the class of n-vertex trees (fuzzy) and determined the n-vertex unicyclic fuzzy graph with cycle length r having maximal F-index, which are partially generalization of the article [1] and [2], respectively. De et al. [13] studied F-index of some graph (crisp) operations. Motivated by this article, we also studied this index of some fuzzy graph operations (Cartesian product, composition, join and union of fuzzy graphs).

1.3 Significance and objective of the article

Topological indices for a crisp graph has huge applications in real life. Also, laboratory testing of chemicals to understand their different properties are costly. To overcome this, many topological indices have been presented in theoretical chemistry. But sometimes it is seen that most real-life problems cannot be modelled using crisp graphs. Those topological indices are needed to define in a fuzzy graph for these circumstances. In 2021, Islam and Pal [18] introduced and studied F-index for fuzzy graphs. In this paper, some bounds on F-index are provided for several fuzzy graph operations such as Cartesian product, composition, union and join of two fuzzy graphs. Some interesting relations of the F-index are established during fuzzy graph transformations. Using those transformations, it is shown that n-vertex star has maximal F-index among the class of n-vertex trees (fuzzy) and determined the n-vertex unicyclic fuzzy

Table 2 The list of abbreviation

Abbreviation	Meaning
FS	Fuzzy set
FSS	Fuzzy subset
FG	Fuzzy graph
FSG	Fuzzy subgraph
PFSG	Partial fuzzy subgraph
CFG	Complete fuzzy graph
TI	Topological index
ZI	Zagreb index
F-index	Forgotten Topological Index
MV	Membership value

graph with cycle length r having maximal F-index, which are partially generalization of the article [1] and [2], respectively. At the end of this article, railway crime in India is studied, and F-index is used to analyze the crimes rate in the railway in India and compared with three other topological indices. Also, F-index for fuzzy graphs and the first Zagreb index for fuzzy graphs provide similar results and F-index for fuzzy graphs provides better realistic results than F-index for crisp graphs and first Zagreb index for crisp graphs to detect the crime in Indian railway.

1.4 Framework of the article

The article's structure is as follows: Section 2 provides some basic definitions that are essential to developing our main results. In Sect. 3, some bounds on F-index are provided for several fuzzy graph operations such as Cartesian product, composition, union and join of two fuzzy graphs. In Sect. 4, some interesting relations of the F-index are established during fuzzy graph transformations. Using those transformation, it is also shown that, $n-$vertex star hs maximum F-index among the n-vertex trees. Also, maximal n-vertex unicyclic fuzzy graph with r cycle is determined with respect to this index. In Sect. 5, this index is used to analyze the crimes in railway in India and compared with three other topological indices. Also, it is shown that the F-index for fuzzy graphs and the first Zagreb index for fuzzy graphs provide similar results and F-index for fuzzy graphs provides better realistic results than F-index for crisp graphs and first Zagreb index for crisp graphs to detect the crime in Indian railway.

2 Preliminaries

In this portion, some basic definitions are provided which are essential to develop our results are given, most of them one can be found in [32, 33].

Let X be a universal set. A FS S on X is a mapping $\mu : X \to [0, 1]$. Here μ is called the membership function of the FS S. Generally a FS is denoted by $S = (x, \mu)$.

Definition 1 Let $X(\neq \phi)$ be a given finite set. The FG is a triplet, $G = (V, E)$, where V is nonempty finite subset of X with $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$ satisfying $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$, where $\wedge$ represents the minimum.

The set V is the set of vertices and $E := \{(x, y) : \mu(x, y) > 0\}$ is the set of edge of the FG. $\sigma(x)$ represents the vertex MV of x and $\mu(x, y)$ represents the edge MV of (x, y) (or simply xy).

Definition 2 Let $v \in V$ then degree of v is denoted by $d_G(v)$ or simply $d(v)$ and defined as $d(v) = \sum_{x\in V} \mu(xv)$.

Let $\Delta(G)$ or Δ be the maximum degree of G and defined as $\Delta = \vee_{v\in V} d(v)$ and $\delta(G)$ or δ be the minimum degree of G and defined as $\delta = \wedge_{v\in V} d(v)$. The total degree of G is denoted by $T(G)$ or simply T, i.e. $T = \sum_{v\in V} d(v) = 2\sum_{uv\in E} \mu(uv) \leq 2\cdot m$, where $m =$ no. of edges.

Throughout this article, we consider, $G_1 = (V_1, E_1)$ with n_1-vertices, m_1-edges and $G_2 = (V_2, E_2)$ with n_2-vertices, m_2-edges be two FGs and $\Delta_1 = \Delta(G_1)$, $\Delta_2 = \Delta(G_2)$, $\delta_1 = \delta(G_1)$, $\delta_2 = \delta(G_2)$.

Cartesian product of two FGs are defined below:

Definition 3 [32] Cartesian product of G_1 and G_2 is a FG $G_1 \times G_2 = (V, E)$, where $V = V_1 \times V_2$, $(u, v), (u_1, v_1), (u_2, v_2) \in V$, $\sigma(u, v) = \wedge\{\sigma_1(u), \sigma_2(v)\}$ and

$$\mu((u_1, v_1), (u_2, v_2)) = \begin{cases} \wedge\{\sigma_1(u_1), \mu_2(v_1, v_2)\} & \text{if } u_1 = u_2, \\ & \quad v_1 v_2 \in E_2 \\ \wedge\{\sigma_2(v_1), \mu_1(u_1, u_2)\} & \text{if } u_1 u_2 \in E_1, \\ & \quad v_1 = v_2 \\ 0 \quad \text{otherwise.} \end{cases}$$

Composition of two FGs are defined below:

Definition 4 [32] Let $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ be two FGs. Then the composition of G_1 and G_2 is a FG $G_1[G_2] = (V, E)$, where $V = V_1 \times V_2$, $(u, v), (u_1, v_1), (u_2, v_2) \in V$, $\sigma(u, v) = \wedge\{\sigma_1(u), \sigma_2(v)\}$ and

$$\mu((u_1, v_1), (u_2, v_2)) = \begin{cases} \wedge\{\sigma_1(u_1), \mu_2(v_1, v_2)\} & \text{if } u_1 = u_2 \text{ and } v_1 v_2 \in E_2 \\ \wedge\{\sigma_2(v_1), \sigma_2(v_2), \mu_1(u_1, u_2)\} & \text{if } u_1 u_2 \in E_1 \\ 0 \quad \text{otherwise.} \end{cases}$$

Join of two FGs are defined below:

Definition 5 [32] Let $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ be two FGs. Then the join of G_1 and G_2 is a FG $G_1 + G_2 = (V, E)$, where $V = V_1 \cup V_2$, $u, v, u_1, v_1 \in V$,

$$\sigma(u) = \begin{cases} \sigma_1(u), & \text{if } u \in V_1 \\ \sigma_2(u), & \text{if } u \in V_2 \end{cases}$$

and

$$\mu(u,v) = \begin{cases} \wedge\{\sigma_1(u), \sigma_2(v)\} \text{ if } u \in V_1 \text{ and } v \in V_2 \\ \mu_1(u,v) \text{ if } uv \in E_1 \\ \mu_2(u,v) \text{ if } uv \in E_2 \\ 0 \quad \text{otherwise.} \end{cases}$$

Union of two FGs are defined below:

Definition 6 [32] Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two FGs. Then the union of G_1 and G_2 is a FG $G_1 \cup G_2 = (V, E)$, where $V = V_1 \cup V_2$, $u, v \in V$,

$$\sigma(u) = \begin{cases} \sigma_1(u), & \text{if } u \in V_1 \backslash V_2 \\ \sigma_2(u), & \text{if } u \in V_2 \backslash V_1 \\ \max\{\sigma_1(u), \sigma_2(u)\}, & \text{if } u \in V_1 \cap V_2 \end{cases}$$

and

$$\mu(u,v) = \begin{cases} \mu_1(u,v) \text{ if } uv \in E_1 \backslash E_2 \\ \mu_2(u,v) \text{ if } uv \in E_2 \backslash E_1 \\ \vee\{\sigma_1(u,v), \sigma_2(u,v)\} \text{ if } (u,v) \in E_1 \cap E_2 \\ 0 \quad \text{otherwise.} \end{cases}$$

Topological indices has an important role in molecular chemistry, chemical graph theory, spectral graph theory, network theory, etc. Zagreb index (ZI) is one such TIs which is degree-based TI and established by Gutman and Trinajstic in 1972 [15] and this TIs are used to calculate π-electron energy of a conjugate system. The first and second ZI for crisp graph is defined as follows:

Definition 7 [15] Let $G = (V, E)$ be a crisp graph. Then the first ZI of the graph G is denoted by $M_1(G)$ and is defined by:

$$M_1(G) = \sum_{v \in V} d^2(v).$$

Definition 8 [15] Let $G = (V, E)$ be a crisp graph. Then the second ZI of the G is denoted by $M_2(G)$ and is defined by:

$$M_2(G) = \sum_{uv \in E} d(u)d(v).$$

This two TIs are degree based TIs. Followed by this two topological indices, in 2015, Fortula and Gutman defined another degree based topological indices called forgotten topological index (F-index) [14] and they proved that first ZI and F-index have the almost same entropy, predictive ability and acentric factor and F-index obtain correlation coefficients greater than 0.95. So this TI is very much useful in molecular chemistry and used in spectral graph theory, network theory, and several field of mathematics. F-index for crisp graph is defined as follows:

Definition 9 [14] Suppose $G = (V, E)$ be a graph (crisp). F-index of G is indicated by $F(G)$ and is defined by:

$$F(G) = \sum_{v \in V} d^3(v).$$

Those indices are defined in crisp graph only. In 2020, Kalathian et al. [21] studied first and second ZI for a FG as follows:

Definition 10 [21] Suppose $G = (V, E, \mu)$ be a FG. Then first ZI of the FG G is indicated by $M(G)$ and is defined by:

$$M(G) = \sum_{v \in V} \sigma(v) d^2(v).$$

Definition 11 [21] Suppose $G = (V, E)$ be a FG. Then the second ZI of the FG G is indicated by $ZF_2(G)$ and is defined by:

$$ZF_2(G) = \sum_{uv \in E} \sigma(u) d(u) \sigma(v) d(v).$$

In 2021, the definition of the first ZI for a FG was modified by Islam and Pal [17].

Definition 12 [17] Suppose $G = (V, E)$ be a FG. Then first ZI of the FG G is indicated by $ZF_1(G)$ and is defined by:

$$ZF_1(G) = \sum_{v \in V} [\sigma(v) d(v)]^2.$$

In 2021, Islam and Pal [18] introduced F-index for a FG.

Definition 13 [18] Suppose $G = (V, E)$ be a FG. Then F-index of the FG G is indicated by $FF(G)$ and is defined by:

$$FF(G) = \sum_{v \in V} [\sigma(v) d(v)]^3.$$

In [18], Islam and Pal provided the upper and lower bounds of F-index for connected n-vertex fuzzy graph with m-edges.

Theorem 1 *[18]* $\frac{n\delta^6}{m^3} \leq FF(G) \leq n\Delta^3$.

In [18], Islam and Pal provided a relation of F-index between fuzzy graph and its partial fuzzy subgraph.

Proposition 1 *[18] Let H be a PFSG of a FG G. Then $FF(H) \leq FF(G)$.*

Here, relation between F-index and first ZI are studied.

3 Bounds of F-index during fuzzy graph operations

In this section, some bounds of F-index are discussed during FG operations.

Theorem 2 $FF(G_1 \times G_2) \leq n_2 FF(G_1) + n_1 FF(G_2) + 3n_1n_2\Delta_1\Delta_2(\Delta_1 + \Delta_2)$.

Proof As $G_1 \times G_2$ is CP of G_1 and G_2, then for $(u, v), (u_1, v_1) \in V, \sigma(u, v) = \wedge\{\sigma_1(u), \sigma_2(v)\}$ and

$$\mu((u, v), (u_1, v_1)) = \begin{cases} \wedge\{\sigma_1(u), \mu_2(v, v_1)\} & \text{if } u = u_1 \text{ and } vv_1 \in E_2 \\ \wedge\{\sigma_2(v), \mu_1(u, u_1)\} & \text{if } uu_1 \in E_1 \text{ and } v = v_1 \\ 0 & \text{otherwise.} \end{cases}$$

Then degree of the vertex (u, v) is given below:

$$\begin{aligned} d_{G_1 \times G_2}(u, v) &= \sum_{v_i v \in E_2} \mu\{(u, v), (u, v_i)\} + \sum_{u_i u \in E_1} \mu\{(u, v), (u_i, v)\} \\ &= \sum_{v_i v \in E_2} \wedge\{\sigma_1(u), \mu_2(v, v_i)\} + \sum_{u_i u \in E_1} \wedge\{\sigma_2(v), \mu_1(u, u_i)\} \\ &\leq \sum_{v_i v \in E_2} \mu_2(v, v_i) + \sum_{u_i u \in E_1} \mu_1(u, u_i) \\ &= d_{G_1}(u) + d_{G_2}(v). \end{aligned}$$

Then F-index of $G_1 \times G_2$ is:

$$\begin{aligned} ZF_1(G_1 \times G_2) &= \sum_{(u,v) \in V} [\sigma(u, v) d_{G_1 \times G_2}(u, v)]^3 \\ &\leq \sum_{(u,v) \in V} \left[\wedge\{\sigma_1(u), \sigma_2(v)\}\{d_{G_1}(u) + d_{G_2}(v)\}\right]^3 \\ &\leq \sum_{(u,v) \in V} \left[\sigma_1(u) d_{G_1}(u) + \sigma_2(v) d_{G_2}(v)\right]^3 \\ &= \sum_{(u,v) \in V} [\sigma_1(u) d_{G_1}(u)]^3 + \sum_{(u,v) \in V} \left[\sigma_2(v) d_{G_2}(v)\right]^3 \end{aligned}$$

$$+3\sum_{(u,v)\in V}\sigma_1(u)\sigma_2(v)d_{G_1}(u)d_{G_2}(v)[\sigma_1(u)d_{G_1}(u)+\sigma_2(v)d_{G_2}(v)]$$
$$\leq \sum_{v\in V_2}FF(G_1)+\sum_{u\in V_1}FF(G_2)+3\sum_{(u,v)\in V}\Delta_1\Delta_2(\Delta_1+\Delta_2)$$
$$=n_2FF(G_1)+n_1FF(G_2)+3n_1n_2\Delta_1\Delta_2(\Delta_1+\Delta_2).$$

Hence the result. □

Corollary 1 *(i)* $FF(G_1\times G_2)\leq n_1n_2(\Delta_1+\Delta_2)^3$, *(ii)* $FF(G_1\times G_2)\leq n_1n_2(n_1+n_2-2)^3$, *(iii)* $FF(G_1\times G_2)\leq 8m_1^3n_1^3n_2+8m_2^3n_2^3n_1+2n_1n_2(n_1-1)(n_2-1)(n_1+n_2-2)$.

Proof (i) From theoremrem 2 and theoremrem 1, the following inequalities are holds:

$$\begin{aligned}FF(G_1\times G_2)&\leq n_2FF(G_1)+n_1FF(G_2)+3n_1n_2\Delta_1\Delta_2(\Delta_1+\Delta_2)\\&\leq n_1n_2\Delta_1^3+n_1n_2\Delta_2^3+3n_1n_2\Delta_1\Delta_2(\Delta_1+\Delta_2)\\&=n_1n_2(\Delta_1+\Delta_2)^3.\end{aligned}$$

(ii) Using (i) and the facts $\Delta_1\leq n_1-1$ and $\Delta_2\leq n_2-1$, the result follow.

(iii) From theoremrem 2, Proposition 1 and the facts $\Delta_1\leq n_1-1$ and $\Delta_2\leq n_2-1$, the required inequality hold. □

Theorem 3 $FF(G_1[G_2])\leq n_2^4FF(G_1)+n_1FF(G_2)+3n_1n_2^2\Delta_1\Delta_2(\Delta_1+\Delta_2)$.

Proof As $G_1[G_2]$ is composition graph of G_1 and G_2, then for $(u,v),(u_1,v_1)\in V$, $\sigma(u,v)=\wedge\{\sigma_1(u),\sigma_2(v)\}$ and

$$\mu((u,v),(u_1,v_1))=\begin{cases}\wedge\{\sigma_1(u),\mu_2(v,v_1)\} & \text{if } u=u_1 \text{ and } vv_1\in E_2\\ \wedge\{\sigma_2(v),\sigma_2(v_1),\mu_1(u,u_1)\} & \text{if } uu_1\in E_1\\ 0 & \text{otherwise.}\end{cases}$$

Then,

$$\begin{aligned}d_{G_1[G_2]}(u,v)&=\sum_{v_iv\in E_2}\mu\{(u,v),(u,v_i)\}+\sum_{u_iu\in E_1,v_j\in V_2}\mu\{(u,v),(u_i,v_j)\}\\&=\sum_{v_iv\in E_2}\wedge\{\sigma_1(u),\mu_2(v,v_i)\}+\sum_{u_iu\in E_1,v_j\in V_2}\wedge\{\sigma_2(v),\sigma_2(v_j)\mu_1(u,u_i)\}\\&\leq\sum_{v_iv\in E_2}\mu_2(v,v_i)+\sum_{u_iu\in E_1,v_j\in V_2}\mu_1(u,u_i)\\&=n_2d_{G_1}(u)+d_{G_2}(v).\end{aligned}$$

Then F-index of $G_1[G_2]$ is:

$$FF(G_1[G_2])=\sum_{(u,v)\in V}[\sigma(u,v)d_{G_1[G_2]}(u,v)]^3$$

$$
\begin{aligned}
&\leq \sum_{(u,v)\in V} [\wedge\{\sigma_1(u),\sigma_2(v)\}\{n_2 d_{G_1}(u)+d_{G_2}(v)\}]^3\\
&\leq \sum_{(u,v)\in V} [n_2\sigma_1(u)d_{G_1}(u)+\sigma_2(v)d_{G_2}(v)]^3\\
&= n_3^2 \sum_{(u,v)\in V} [\sigma_1(u)d_{G_1}(u)]^3 + \sum_{(u,v)\in V} \left[\sigma_2(v)d_{G_2}(v)\right]^2\\
&\quad + 3n_2 \sum_{(u,v)\in V} \sigma_1(u)\sigma_2(v)d_{G_1}(u)d_{G_2}(v)[n_2\sigma_1(u)d_{G_1}(u)+\sigma_2(v)d_{G_2}(v)]\\
&\leq n_3^2 \sum_{v\in V_2} FF(G_1) + \sum_{u\in V_1} FF(G_2) + 3n_2 \sum_{(u,v)\in V} \Delta_1\Delta_2(n_2\Delta_1+\Delta_2)\\
&= n_2^4 FF(G_1) + n_1 FF(G_2) + 3n_1n_2^2\Delta_1\Delta_2(n_2\Delta_1+\Delta_2).
\end{aligned}
$$

Hence the result. □

Corollary 2 *(i)* $FF(G_1[G_2]) \leq n_1n_2^4(n_1-1)^3 + n_1n_2(n_2-1)^3 + 3n_1n_2^2(n_1-1)(n_2-1)(n_1n_2-1)$ *and*

(ii) $FF(G_1[G_2]) \leq 8n_1^3n_2^4m_1^3 + 8n_1n_2^3m_2^3 + n_1n_2(n_2-1)^3 + 3n_1n_2^2(n_1-1)(n_2-1)(n_1n_2-1)$.

Theorem 4

$$
\begin{aligned}
FF(G_1)+FF(G_2) \leq FF(G_1+G_2) &\leq FF(G_1)+FF(G_2)+n_1n_2(n_1^2+n_2^2)\\
&+3n_1n_2(\Delta_1^2+\Delta_2^2+n_2\Delta_1+n_1\Delta_2).
\end{aligned}
$$

Proof As G_1+G_2 is join graph of G_1 and G_2, then for $u, v, u_1, v_1 \in V$,

$$
\sigma(u) = \begin{cases} \sigma_1(u), & u \in V_1\\ \sigma_2(u), & u \in V_2 \end{cases}
$$

and

$$
\mu(u,v) = \begin{cases} \wedge\{\sigma_1(u),\sigma_2(v)\} & u \in V_1 \text{ and } v \in V_2\\ \mu_1(u,v) & uv \in E_1\\ \mu_2(u,v) & uv \in E_2\\ 0 & \text{otherwise.} \end{cases}
$$

Then,

$$
d_{G_1+G_2}(u) = \begin{cases} d_{G_1}(u) + \sum_{v\in V_2} \wedge\{\sigma_1(u),\sigma_2(v)\}, & u \in V_1\\ d_{G_2}(u) + \sum_{v\in V_1} \wedge\{\sigma_2(u),\sigma_1(v)\}, & u \in V_2 \end{cases} \tag{1}
$$

$$
\geq \begin{cases} d_{G_1}(u), & u \in V_1\\ d_{G_2}(u), & u \in V_2. \end{cases} \tag{2}
$$

Again, from Eq. (1), the following hold:

$$d_{G_1+G_2}(u) \leq \begin{cases} d_{G_1}(u) + \sum_{v\in V_2} \sigma_1(u), & u \in V_1 \\ d_{G_2}(u) + \sum_{v\in V_1} \sigma_2(u), & u \in V_2 \end{cases} \tag{3}$$

$$\leq \begin{cases} d_{G_1}(u) + n_2\sigma_1(u), & u \in V_1 \\ d_{G_2}(u) + n_1\sigma_2(u), & u \in V_2 \end{cases} \tag{4}$$

Then F-index of $G_1 + G_2$ is:

$$\begin{aligned} FF(G_1+G_2) &= \sum_{u\in V}[\sigma(u)d_{G_1+G_2}(u)]^3 \\ &= \sum_{u\in V_1}[\sigma_1(u)d_{G_1+G_2}(u)]^3 + \sum_{u\in V_2}\left[\sigma_2(u)d_{G_1+G_2}(u)\right]^3 \\ &\geq \sum_{u\in V_1}[\sigma_1(u)d_{G_1}(u)]^3 + \sum_{u\in V_2}\left[\sigma_2(u)d_{G_2}(u)\right]^3 \qquad \text{[by (2)]} \\ &= FF(G_1) + FF(G_2). \end{aligned}$$

Now,

$$\begin{aligned} ZF_1(G_1+G_2) &= \sum_{u\in V}\left[\sigma(u)d_{G_1+G_2}(u)\right]^3 \\ &= \sum_{u\in V_1}\left[\sigma_1(u)d_{G_1+G_2}(u)\right]^3 + \sum_{u\in V_2}\left[\sigma_2(u)d_{G_1+G_2}(u)\right]^3 \\ &\leq \sum_{u\in V_1}\left[\sigma_1(u)\{d_{G_1}(u) + n_2\sigma_1(u)\}\right]^3 \\ &\quad + \sum_{u\in V_2}\left[\sigma_2(u)\{d_{G_2}(u) + n_2\sigma_2(u)\}\right]^3 \qquad \text{[by (4)]} \\ &= K_1 + K_2, \end{aligned}$$

where $K_1 = \sum_{u\in V_1}[\sigma_1(u)\{d_{G_1}(u) + n_2\sigma_1(u)\}]^3$ and $K_2 = \sum_{u\in V_2}[\sigma_2(u)\{d_{G_2}(u) + n_2\sigma_2(u)\}]^3$ Now

$$\begin{aligned} K_1 &= \sum_{u\in V_1}\left[\sigma_1(u)\{d_{G_1}(u) + n_2\sigma_1(u)\}\right]^3 \\ &= \sum_{u\in V_1}[\sigma_1(u)d_{G_1}(u)]^3 + n_2^3\sum_{u\in V_1}[\sigma_1(u)]^6 \\ &\quad + 3n_2\sum_{u\in V_1}\sigma_1^4(u)d_{G_1}(u)\left[d_{G_1}(u) + n_2\sigma_1(u)\right] \end{aligned}$$

$$\leq FF(G_1) + n_1 n_2^3 + 3n_1 n_2 \Delta_1(\Delta_1 + n_2).$$

Similarly, $K_2 \leq FF(G_2) + n_1^3 n_2 + 3n_1 n_2 \Delta_2(\Delta_2 + n_1)$. Therefore,

$$\begin{aligned} FF(G_1 + G_2) &\leq K_1 + K_2 \\ &\leq FF(G_1) + FF(G_2) + n_1 n_2 (n_1^2 + n_2^2) \\ &\quad + 3n_1 n_2 (\Delta_1^2 + \Delta_2^2 + n_2 \Delta_1 + n_1 \Delta_2). \end{aligned}$$

Hence the result. □

Corollary 3 *(i)* $FF(G_1 + G_2) \leq n_1(n_1 - 1)^3 + n_2(n_2 - 1)^3 + n_1 n_2 (n_1^2 + n_2^2) + 3n_1 n_2[(n_1 - 1)^2 + (n_2 - 1)^2 + 2n_1 n_2 - n_1 - n_2]$ *and*
(ii) $FF(G_1 + G_2) \leq 8n_1^3 m_1^3 + 8n_2^3 m_2^3 + n_1 n_2 (n_1^2 + n_2^2) + 3n_1 n_2[(n_1 - 1)^2 + (n_2 - 1)^2 + 2n_1 n_2 - n_1 - n_2]$.

Theorem 5 $FF(G_1 \cup G_2) \geq FF(G_1) + FF(G_2) - k\Delta^3$, *where* $k = |V_1 \cap V_2|$, $\Delta = \max\{\Delta_1, \Delta_2\}$.

Proof As $G_1 \cup G_2$ is union of G_1 and G_2, then for any $u, v \in V$,

$$\sigma(u) = \begin{cases} \sigma_1(u), \text{ if } u \in V_1 \backslash V_2 \\ \sigma_2(u), \text{ if } u \in V_2 \backslash V_1 \\ \vee\{\sigma_1(u), \sigma_2(u)\}, \text{ if } u \in V_1 \cap V_2 \end{cases}$$

$$\text{and } \mu(uv) = \begin{cases} \mu_1(uv), \text{ if } uv \in E_1 \backslash E_2 \\ \mu_2(uv), \text{ if } uv \in E_2 \backslash E_1 \\ \vee\{\mu_1(uv), \mu_2(uv)\}, \text{ if } uv \in E_1 \cap E_2 \\ 0 \quad \text{otherwise.} \end{cases}$$

Then the degree of a vertex u is given below:

$$d_{G_1 \cup G_2}(u) \geq \begin{cases} d_1(u), \text{ if } u \in V_1 \backslash V_2 \\ d_2(u), \text{ if } u \in V_2 \backslash V_1 \\ d_1(u) \text{ or } d_2(u), \text{ if } u \in V_1 \cap V_2. \end{cases}$$

Hence,

$$\begin{aligned} FF(G_1 \cup G_2) &= \sum_{u \in V} [\sigma(u) d_{G_1 \cup G_2}(u)]^3 \\ &= \sum_{u \in V_1} [\sigma_1(u) d_{G_1}(u)]^3 + \sum_{u \in V_2} [\sigma_2(u) d_{G_2}(u)]^3 \\ &\quad - \sum_{u \in V_1 \cap V_2} [\sigma(u) d_{G_1 \cup G_2}(u)]^3 \end{aligned}$$

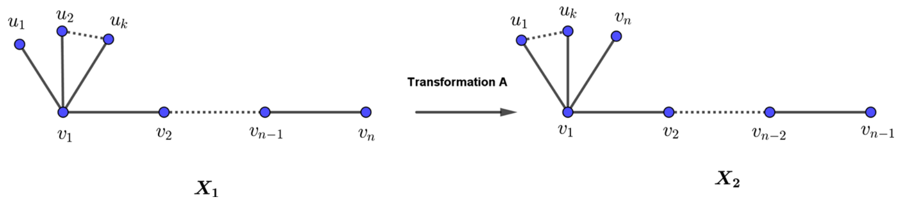

Fig. 1 Transformation A

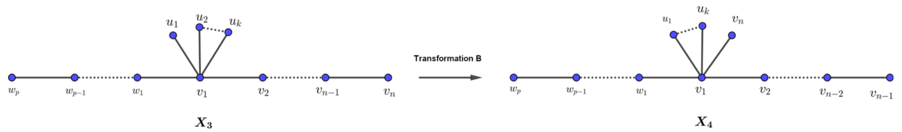

Fig. 2 Transformation B

$$\geq FF(G_1) + FF(G_2) - k\Delta^3.$$

□

4 F-index for fuzzy graph transformation

Transformation A: Let X_1 be a fuzzy graph shown in Fig. 1. Now the fuzzy graph X_2 is obtained by X_1 whose vertex set is same as the vertex set X_1 and edge set is $E(X_2) = E(X_1) \cup \{v_1 v_n\} \backslash \{v_{n-1} v_n\}$ with $\sigma_{X_2}(x) = \sigma_{X_1}(x) = \sigma(x), \forall x \in V(X_2)$, and $xy \in E(X_2)$

$$\mu_{X_2}(xy) = \begin{cases} \mu_{X_2}(xy) \text{ if } xy \in E(X_1) \\ \sigma(v_1) \wedge \mu(v_{n-1} v_n) \text{ if } xy = v_{n-1} v_n. \end{cases}$$

Theorem 6 *If $\sigma(v_1) \geq \sigma(v_i)$ for $i = n-1, n$, then $FF(X_2) \geq FF(X_1)$.*

Proof

$$\begin{aligned} FF(X_2) - FF(X_1) &= \sigma^3(v_1)[d^3_{X_2}(v_1) - d^3_{X_1}(v_1)] \\ &\quad + \sigma^3(v_{n-1}) \left[d^3_{X_2}(v_{n-1}) - d^3_{X_1}(v_{n-1}) \right] \\ &= \sigma^3(v_1)[(d_{X_1}(v_1) + \mu(v_{n-1}v_n))^3 - d^3_{X_1}(v_1)] \\ &\quad + \sigma^3(v_{n-1}) \left[(d_{X_2}(v_{n-1}) - \mu(v_{n-1}v_n))^3 - d^3_{X_1}(v_{n-1}) \right] \\ &\geq 0. \end{aligned}$$

□

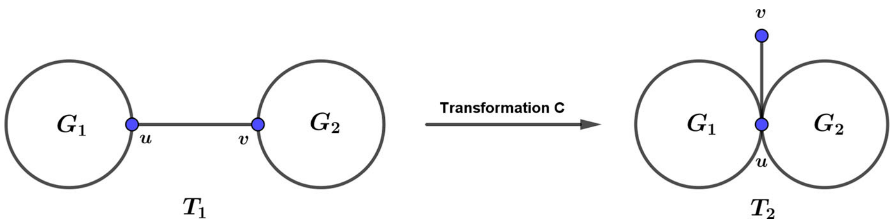

Fig. 3 Transformation C

Transformation B: Let X_3 be a fuzzy graph shown in Fig. 2. Now the fuzzy graph X_4 is obtained by X_3 whose vertex set is same as the vertex set X_3 and edge set is $E(X_4) = E(X_3) \cup \{v_1 v_n\} \backslash \{v_{n-1} v_n\}$ with $\sigma_{X_4}(x) = \sigma_{X_3}(x) = \sigma(x), \forall x \in V(X_4)$, and $xy \in E(X_4)$

$$\mu_{X_4}(xy) = \begin{cases} \mu_{X_4}(xy) \text{ if } xy \in E(X_3) \\ \sigma(v_1) \wedge \mu(v_{n-1}v_n) \text{ if } xy = v_{n-1}v_n. \end{cases}$$

Theorem 7 *If $\sigma(v_1) \geq \sigma(v_i)$ for $i = n-1, n$, then $FF(X_4) \geq FF(X_3)$.*

Proof Similar as the proof of Theorem 6. □

Theorem 8 *The n-vertex star (S) with each vertex and edge has membership value 1 has maximum F-index among the $n-$vertex tree (fuzzy).*

Proof Suppose T be any n-vertex tree (fuzzy). Then, by repeated applying of Theorem 6, 7 and by the Proposition 1, one can easily get, $FF(T) \leq FF(S)$.

In this section, some fuzzy graph transformation is defined and studied F-index for those transformation.

Transformation C: Let T_1 be a fuzzy graph shown in Fig. 3. Now the fuzzy graph T_2 is obtained by T_1 whose vertex set is the same as the vertex set T_1 and edge set is

$$E(T_2) = E(G_1) \cup \{uv\} \cup \{uy : vy \in E(G_2)\} \cup \{xy : xy \in E(G_2), x, y \neq v\}$$

with $\sigma_{T_2}(x) = \sigma_{T_1}(x) = \sigma(x)$, $v \in V(T_2)$ and $xy \in E(T_2)$

$$\mu_{T_2}(xy) = \begin{cases} \mu_{T_1}(xy), & xy \in E(G_1) \\ \mu_{T_1}(uv) = \mu, & xy = uv \\ \sigma(u) \wedge \mu_{T_1}(vy), & xy = uy \\ \mu_{T_1}(xy), & xy \in E(G_2), x, y \neq v. \end{cases}$$

□

Theorem 9 *If $\sigma(u) \geq \sigma(v)$, then $FF(T_2) \geq FF(T_1)$.*

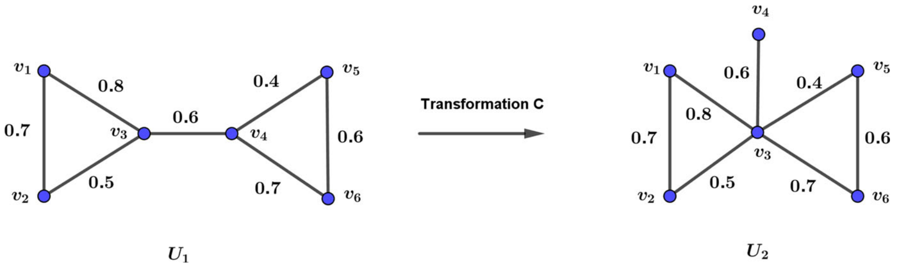

Fig. 4 $FF(U_3) \leq FF(U_4)$

Proof

$$\begin{aligned} FF(T_2) - FF(T_1) &= \sigma^3(u)d^3_{T_2}(u) + \sigma^3(v)d^3_{T_2}(v) - \sigma^3(u)d^3_{T_1}(u) - \sigma^3(v)d^3_{T_1}(v) \\ &= \sigma^3(u)\left[d_{G_1}(u) + d_{G_2}(v) + \mu\right]^3 + \sigma^3(v)\mu^3 \\ &\quad - \sigma^3(u)\left[d_{G_1}(u) + \mu\right]^3 - \sigma^3(v)\left[d_{G_2}(v) + \mu\right]^3 \\ &= \sigma^3(u)[\{d_{G_1}(u) + \mu\}^3 + 3d_{G_2}(v)\{d_{G_1}(u) + \mu\}^2 \\ &\quad + 3d^2_{G_2}(v)\{d_{G_1}(u) + \mu\} + d^3_{G_2}(v)] + \sigma^3(v)\mu^3 - \sigma^3(u)[d_{G_1}(u) \\ &\quad + \mu]^3 - \sigma^3(v)[d^3_{G_2}(v) + 3\mu d^2_{G_2}(v) + 3\mu^2 d_{G_2}(v) + \mu^3] \\ &= \left[\sigma^3(u) - \sigma^3(v)\right]\left[d^3_{G_2}(v) + 3\mu d^2_{G_2}(v) + 3\mu^2 d_{G_2}(v)\right] \\ &\quad + \sigma^3(u)\left[3d^2_{G_1}(u)d_{G_2}(v) + 6\mu d_{G_1}(u)d_{G_2}(v) + 3d_{G_1}(u)d^2_{G_2}(v)\right]. \end{aligned}$$

As $\sigma(u) \geq \sigma(v)$, the result follows. □

Example 1 Let U_1 be a fuzzy graph shown in Fig. 4 whose each vertex has membership value 1 and edge membership value has shown in Fig. 4. Then U_2 is obtained by U_1 by using the Transformation C. Now, F-index of U_1 and U_2 are given below:

$$\begin{aligned} FF(U_1) &= \sum_{u \in V(U_1)} [\sigma(u)d(u)]^3 \\ &= 1.5^3 + 1.2^3 + 1.9^3 + 1.7^3 + 1.0^3 + 1.3^3 \\ &= 20.072. \\ FF(U_2) &= \sum_{u \in V(U_2)} [\sigma(u)d(u)]^3 \\ &= 1.5^3 + 1.2^3 + 3.0^3 + 0.6^3 + 1.0^3 + 1.3^3 \\ &= 35.516. \end{aligned}$$

Hence, $FF(U_1) \leq FF(U_2)$. Therefore, the Theorem 9 is valid for the fuzzy graph U_1 and U_2.

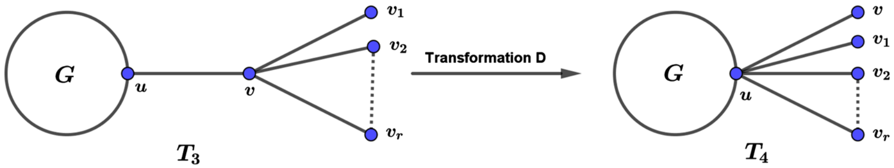

Fig. 5 Transformation D

Transformation D: Let T_3 be a fuzzy graph shown in Fig. 5. Now the fuzzy graph T_4 is obtained by T_3 whose vertex set is the same as the vertex set T_3 and edge set is

$$E(T_4) = E(G) \cup \{uv\} \cup \{uv_i : i = 1, 2, \cdots, r\}$$

with $\sigma_{T_4}(x) = \sigma_{T_3}(x) = \sigma(x)$, $v \in V(T_4)$ and $xy \in E(T_4)$

$$\mu_{T_4}(xy) = \begin{cases} \mu_{T_3}(xy), & xy \in E(G) \\ \mu_{T_3}(uv) = \mu, & xy = uv \\ \sigma(u) \wedge \mu_{T_3}(vv_i) = \mu_i, & xy = uv_i. \end{cases}$$

Theorem 10 *If $\sigma(u) \geq \sigma(v)$, then $FF(T_4) \geq FF(T_3)$.*

Proof

$$\begin{aligned} FF(T_4) - FF(T_3) &= \sigma^3(u)d^3_{T_4}(u) + \sigma^3(v)d^3_{T_4}(v) - \sigma^3(u)d^3_{T_3}(u) - \sigma^3(v)d^3_{T_3}(v) \\ &= \sigma^3(u)\left[d_G(u) + \mu + \sum_{i=1}^{r}\mu_i z\right]^3 + \sigma^3(v)\mu^3 \\ &\quad - \sigma^3(u)\left[d_G(u) + \mu\right]^3 - \sigma^3(v)\left[\mu + \sum_{i=1}^{r}\mu_i\right]^3 \\ &= \sigma^3(u)[\{d_G(u) + \mu\}^3 + 3d_G(u)(\sum_{i=1}^{r}\mu_i)^2 + 3\mu(\sum_{i=1}^{r}\mu_i)^2 + 3\mu^2 \\ &\quad (\sum_{i=1}^{r}\mu_i) + 3\{d^2_G(u) + 2\mu d_G(u)\}(\sum_{i=1}^{r}\mu_i) + (\sum_{i=1}^{r}\mu_i)^3] + \sigma^3(v)\mu^3 \\ &\quad - \sigma^3(u)\left[d_G(u) + \mu\right]^3 - \sigma^3(v)\left[\mu^3 + 3\mu(\sum_{i=1}^{r}\mu_i)^2 + 3\mu^2(\sum_{i=1}^{r}\mu_i)\right] \\ &= \left[\sigma^3(u) - \sigma^3(v)\right]\left[3\mu(\sum_{i=1}^{r}\mu_i)^2 + 3\mu^2(\sum_{i=1}^{r}\mu_i)\right] + \sigma^3(u) \\ &\quad \left[3d_G(u)(\sum_{i=1}^{r}\mu_i)^2 + 3\{d^2_G(u) + 2\mu d_G(u)\}(\sum_{i=1}^{r}\mu_i) + (\sum_{i=1}^{r}\mu_i)^3\right]. \end{aligned}$$

As $\sigma(u) \geq \sigma(v)$, the result follows. □

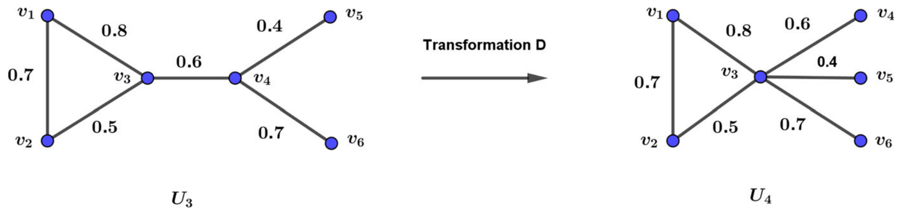

Fig. 6 $FF(U_3) \leq FF(U_4)$

Example 2 Let U_3 be a fuzzy graph shown in Fig. 6 whose each vertex has membership is 1 and edge membership value is shown in Fig. 6. Then U_4 is obtained by U_3 by using the Transformation D. Now, F-index of U_3 and U_4 are given below:

$$\begin{aligned} FF(U_3) &= \sum_{u \in V(U_3)} [\sigma(u)d(u)]^3 \\ &= 1.5^3 + 1.2^3 + 1.9^3 + 1.7^3 + 0.4^3 + 0.7^3 \\ &= 17.282. \\ FF(U_4) &= \sum_{u \in V(U_4)} [\sigma(u)d(u)]^3 \\ &= 1.5^3 + 1.2^3 + 3.0^3 + 0.6^3 + 0.4^3 + 0.7^3 \\ &= 32.726. \end{aligned}$$

Hence, $FF(U_3) \leq FF(U_4)$. Therefore, the Theorem 10 is valid for the fuzzy graph U_3 and U_4.

Transformation E: Let T_5 be a fuzzy graph shown in Fig. 7. Now the fuzzy graph T_6 is obtained by T_5 whose vertex set is the same as the vertex set T_5 and edge set is

$$E(T_6) = E(G) \cup \{uu_i : i = 1, 2, \cdots, s\} \cup \{uv_i : i = 1, 2, \cdots, t\}$$

with $\sigma_{T_6}(x) = \sigma_{T_5}(x) = \sigma(x)$, $v \in V(T_6)$ and $xy \in E(T_6)$

$$\mu_{T_6}(xy) = \begin{cases} \mu_{T_5}(xy), & xy \in E(G) \\ \mu_{T_5}(uu_1) = \mu_i, & xy = uu_i \\ \sigma(u) \wedge \mu_{T_5}(vv_i) = G_i, & xy = uv_i. \end{cases}$$

Theorem 11 *If* $\sigma(u) \geq \sigma(v)$ *and* $d_G(u) \geq d_G(v)$*, then* $FF(T_6) \geq FF(T_5)$.

Proof

$$FF(T_6) - FF(T_5) = \sigma^3(u)d_{T_6}^3(u) + \sigma^3(v)d_{T_6}^3(v) - \sigma^3(u)d_{T_5}^3(u) - \sigma^3(v)d_{T_5}^3(v)$$

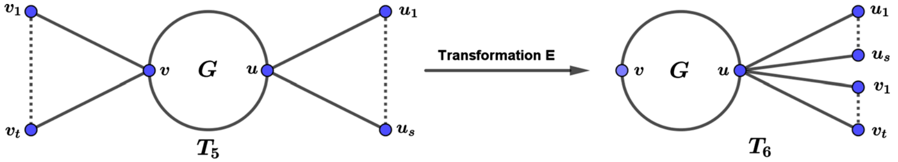

Fig. 7 Transformation E

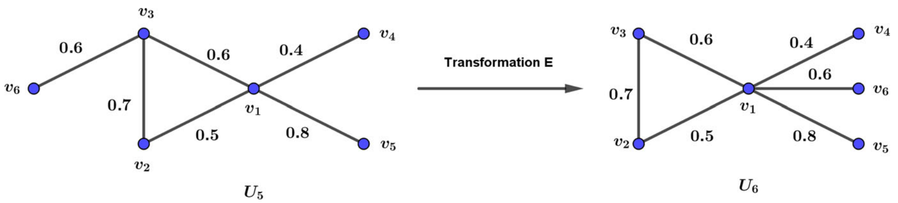

Fig. 8 $FF(U_5) \leq FF(U_6)$

$$
\begin{aligned}
&= \sigma^3(u)\left[d_G(u) + \sum_{i=1}^{s}\mu_i + \sum_{i=1}^{t}\gamma_i\right]^3 + \sigma^3(v)d_G^3(v) \\
&\quad - \sigma^3(u)\left[d_G(u) + \sum_{i=1}^{s}\mu_i\right]^3 - \sigma^3(v)\left[d_G(v) + \sum_{i=1}^{t}\gamma_i\right]^3 \\
&= \sigma^3(u)[\{d_G(u) + \sum_{i=1}^{s}\mu_i\}^3 + \{\sum_{i=1}^{t}\gamma_i\}^3 + 3\{d_G(u) \\
&\quad + \sum_{i=1}^{s}\mu_i\}^2\sum_{i=1}^{t}\gamma_i + 3\{d_G(u) + \sum_{i=1}^{s}\mu_i\}\{\sum_{i=1}^{t}\gamma_i\}^2 + \{\sum_{i=1}^{t}\gamma_i\}^3] \\
&\quad + \sigma^3(v)d_G^3(v) - \sigma^3(u)\left[d_G(u) + \sum_{i=1}^{s}\mu_i\right]^3 - \sigma^3(v)[d_G^3(v) \\
&\quad + 3d_G^2(v)\sum_{i=1}^{t}\gamma_i + 3d_G(v)\{\sum_{i=1}^{t}\gamma_i\}^2 + \{\sum_{i=1}^{t}\gamma_i\}^3].
\end{aligned}
$$

As $\sigma(u) \geq \sigma(v)$ and $d_G(u) \geq d_G(v)$, the result follows. □

Example 3 Let U_5 be a fuzzy graph shown in Fig. 8 whose each vertex has membership is 1 and edge membership value is shown in Fig. 8. Then U_6 is obtained by U_5 by using the Transformation E. Now, F-index of U_5 and U_6 are given below:

$$
\begin{aligned}
FF(U_5) &= \sum_{u \in V(U_5)} [\sigma(u)d(u)]^3 \\
&= 2.3^3 + 1.2^3 + 1.9^3 + 0.4^3 + 0.8^3 + 0.6^3 \\
&= 21.546.
\end{aligned}
$$

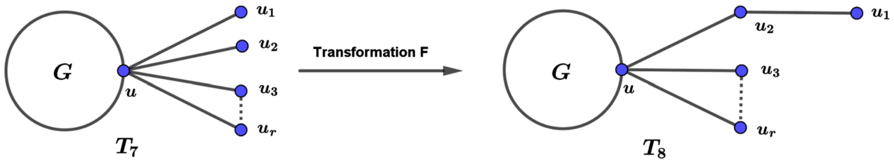

Fig. 9 Transformation F

$$
\begin{aligned}
FF(U_6) &= \sum_{u \in V(U_6)} [\sigma(u)d(u)]^3 \\
&= 2.9^3 + 1.2^3 + 1.3^3 + 0.4^3 + 0.8^3 + 0.6^3 \\
&= 29.106.
\end{aligned}
$$

Hence, $FF(U_5) \leq FF(U_6)$. Therefore, the Theorem 11 is valid for the fuzzy graph U_5 and U_6.

Transformation F: Let T_7 be a fuzzy graph shown in Fig. 9. Now the fuzzy graph T_8 is obtained by T_7 whose vertex set is the same as the vertex set T_7 and edge set is

$$
E(T_8) = E(G) \cup \{uu_i : i = 1, 2, \cdots, r\} \cup \{u_1u_2\}
$$

with $\sigma_{T_8}(x) = \sigma_{T_7}(x) = \sigma(x)$, $v \in V(T_8)$ and $xy \in E(T_8)$

$$
\mu_{T_8}(xy) = \begin{cases} \mu_{T_7}(xy), & xy \in E(G) \\ \mu_{T_7}(uu_i) = \mu_i, & i = 2, 3, \cdots, r \\ \sigma(u_2) \wedge \mu_{T_7}(u_1u_2) = \mu_1 & xy = u_1u_2. \end{cases}
$$

Theorem 12 *If $\sigma(u_1) \leq \sigma(u_2) \leq \sigma(u)$, then $FF(T_7) \geq FF(T_8)$.*

Proof

$$
\begin{aligned}
FF(T_7) - FF(T_8) &= \sigma^3(u)d^3_{T_7}(u) + \sigma^3(u_2)d^3_{T_7}(u_2) - \sigma^3(u)d^3_{T_8}(u) - \sigma^3(u_2)d^3_{T_8}(u_2) \\
&= \sigma^3(u)\left[d_G(u) + \sum_{i=1}^{r}\mu_i\right]^3 - \sigma^3(u)\left[d_G(u) + \sum_{i=2}^{r}\mu_i\right]^3 \\
&\quad + \sigma^3(u_2)\mu_2^3 - \sigma^3(u_2)(\mu_1 + \mu_2)^3 \\
&= \sigma^3(u)[\{d_G(u) + \sum_{i=2}^{r}\mu_i\}^3 + \mu_1^3 + 3\mu_1^2\{d_G(u) + \sum_{i=2}^{r}\mu_i\} \\
&\quad + 3\mu_1\{d_G(u) + \sum_{i=2}^{r}\mu_i\}^2] - \sigma^3(u)\left[d_G(u) + \sum_{i=2}^{r}\mu_i\right]^3 + \sigma^3(u_2)\mu_2^3 \\
&\quad - \sigma^3(u_2)\left[\mu_1^3 + \mu_2^3 + 3\mu_1^2\mu_2 + 3\mu_1\mu_2^2\right]^3
\end{aligned}
$$

As $\sigma(u) \geq \sigma(u_2)$, the result follows. □

Fig. 10 Transformation G

Transformation G: Let T_9 be a fuzzy graph shown in Fig. 10. Now the fuzzy graph T_{10} is obtained by T_9 whose vertex set is the same as the vertex set T_9 and edge set is

$$E(T_{10}) = E(G) \cup \{uu_i : i = 1, 2, \cdots, r\} \cup \{u_1 v_1\}$$

with $\sigma_{T_{10}}(x) = \sigma_{T_9}(x) = \sigma(x)$, $v \in V(T_{10})$ and $xy \in E(T_{10})$

$$\mu_{T_{10}}(xy) = \begin{cases} \mu_{T_9}(xy), & xy \in E(G) \\ \mu_{T_9}(uu_i) = \mu_i, & i = 2, 3, \cdots, r \\ \sigma(u_1) \wedge \mu_{T_9}(vv_1) = \gamma_1 & xy = u_1 v_1. \end{cases}$$

Theorem 13 *If $\sigma(v) \geq \sigma(u_1)$ and $d_G(v) \geq \mu_1$, then $FF(T_9) \geq FF(T_{10})$.*

Proof

$$\begin{aligned} FF(T_9) - FF(T_{10}) &= \sigma^3(u_1)d^3_{T_9}(u_1) + \sigma^3(v)d^3_{T_9}(v) \\ &\quad - \sigma^3(u_1)d^3_{T_{10}}(u_1) - \sigma^3(v)d^3_{T_{10}}(v) \\ &= \sigma^3(u_1)\mu_1^3 + \sigma^3(v)\left[d^3_G(v) + \gamma_1^3 + 3\gamma_1 d^2_G(v) + 3\gamma_1^2 d_G(v)\right] \\ &\quad - \sigma^3(u_1)\left[\mu_1^3 + \gamma_1^3 + 3\mu_1^2\gamma_1 + 3\mu_1\gamma_1^2\right] - \sigma^3(v)d^3_G(v) \\ &= \sigma^3(v)\left[\gamma_1^3 + 3\gamma_1 d^2_G(v) + 3\gamma_1^2 d_G(v)\right] \\ &\quad - \sigma^3(u_1)\left[\gamma_1^3 + 3\mu_1^2\gamma_1 + 3\mu_1\gamma_1^2\right] \end{aligned}$$

As $\sigma(v) \geq \sigma(u_1)$ and $d_G(v) \geq \mu_1$, the result follows. □

Theorem 14 *Let $\mathbb{U}(n, r)$ be the unicyclic fuzzy graph with n vertices and the length of the cycle is r. Suppose $U \in \mathbb{U}(n, r)$ such that $(n - r)$ pendant vertices are joined at a fixed vertex and each vertex and edge has membership value 1. Then U has maximum F-index among the $\mathbb{U}(n, r)$.*

Proof Suppose, $U \in \mathbb{U}_{n,r}(V, \sigma, E_m)$ be any unicyclic graph. Then by using Theorem 8 and repeated applying of Theorem 10, 11, 12 and the Proposition 1, one can easily get, $FF(U) \leq FF(\mathbb{U}(n, r))$. □

5 Application of F-index for fuzzy graphs to detect crime in Indian railways

From the mid-nineteenth century, railways network is the most important mode of the conveyance of people and goods in India. About 22.15 million passengers used this

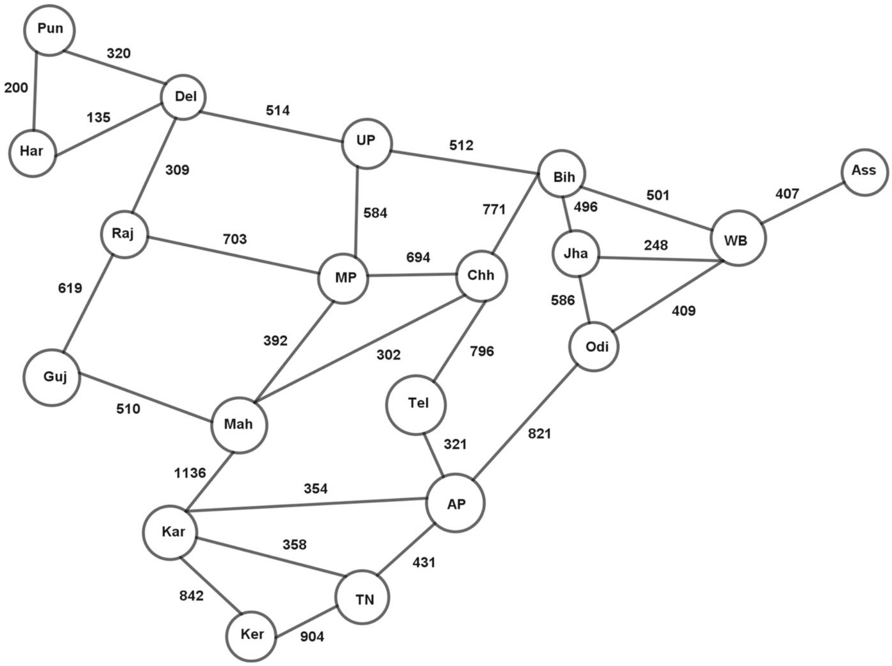

Fig. 11 Railways network in India

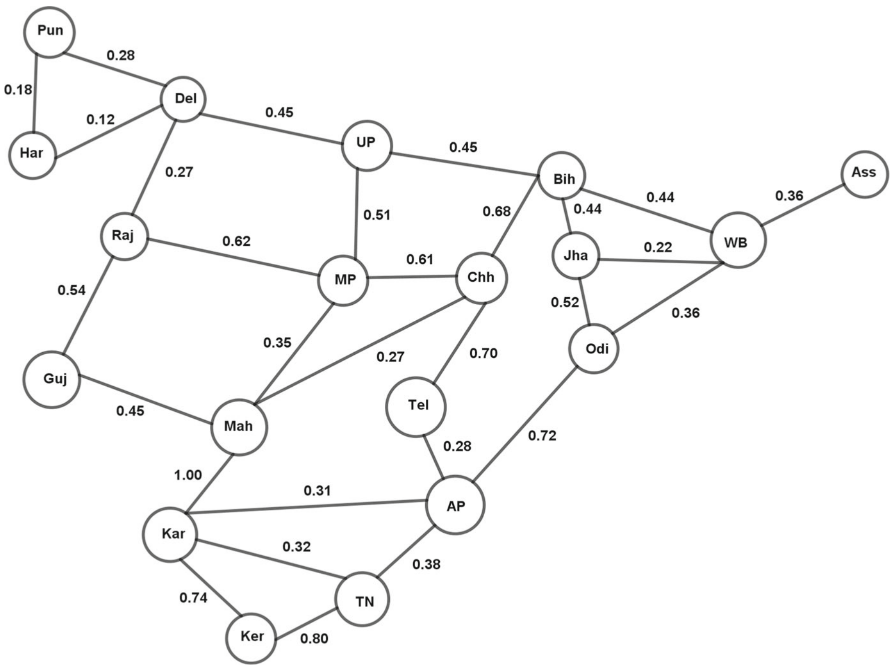

Fig. 12 Fuzzy graph representation of the railway network in India

network and 3.32 million metric tons of good was also shipped by this network daily in the year 2019 to 2020. The national rail network comprised 126,366 km of track over a route of 67,368 km and 7,325 stations. It is the fourth-largest national railway network in the world after United States, Russia and China. But with the passage they are posing as a threat for the general public at large while traveling, being the instances of crimes rising fourfold in rails. The ongoing railway crime has become a cause of concern for the common passenger now. So the objective of this section is to find out the railway where the most crime occurs.

Now, we consider the railways network graph in Fig. 11, where an state is considered as a vertex and if there is rail communication between the neighboring states then an edge is consider between the states. The railways crime share percentage and population share percentage in the year 2019 (last updated data) of each state are listed in the table 3 and railway distance between the main station of the neighboring states are listed in Table 4. Now the vertex membership value of a state S is denoted by $\sigma(\mathrm{S})$ and is calculated by the formula:

$$\wedge\left\{1, \frac{\text{Railways crime share percentage of the state S}}{\text{Population share percentage of the state S}}\right\}.$$

Clearly, $\sigma(\mathrm{S}) \in [0, 1]$ and if $\sigma(\mathrm{S}_1) \geq \sigma(\mathrm{S}_2)$ then the number of railway crimes of S_1 is higher than the number of railway crimes of S_2 per one lakh populations. If there is an edge between the states S_1 and S_2, the edge membership value is denoted by $\mu(\mathrm{S}_1\mathrm{S}_2)$and is calculated by the formula:

$$\wedge \{\sigma(\mathrm{S}_1), \sigma(\mathrm{S}_2)\} \times \frac{\text{Railway distance between the main station of the states } \mathrm{S}_1 \text{and } \mathrm{S}_2}{\text{Maximum railway distance between the main station of the neighboring states}}.$$

The vertex membership values are listed in 3 and the edge membership values are listed in 5. Now, the fuzzy graph representation of the railway network in India is shown in Fig. 12. Now degree of each vertex is listed in Table 3 and which is calculated by the formula:

$$d(S) = \sum_{SS_i \in E} \mu(SS_i).$$

Now F-index of the fuzzy graph shown in Fig. 12 is calculated by the formula:

$$FF(G) = \sum_{S \in V} [\sigma(S)d(S)]^3 .$$

Using this formula and Table 3, one can easily determined the value of F-index for the fuzzy graph representation of the railway network in India shown in Fig. 12 and the value is 4.395103.

Now a fuzzy subgraph $G_{S_1 S_2}$ is constructed from the fuzzy graph shown in Fig. 12 by deleting an edge $\mathrm{S}_1\mathrm{S}_2$. Again by similar manner one can calculate the F-index

Table 3 Vertex membership value of each state

State Name	Vertex Name	Share of railway crime percentage	Share of population percentage	Vertex membership value	Degree
Andhra Pradesh	AP	1.8	3.93	0.46	0.7427
Assam	Ass	0.6	2.6	0.23	0.0827
Bihar	Bih	5.4	9.1	0.59	0.7003
Chhattisgarh	Chh	0.8	2.15	0.37	0.8395
Gujarat	Guj	6.4	4.66	1.00	0.6887
Haryana	Har	1.9	2.06	0.92	0.1818
Jharkhand	Jha	0.7	2.81	0.25	0.2917
Karnataka	Kar	1.9	4.93	0.39	0.9126
Kerala	Ker	1.1	2.6	0.42	0.6257
Madhya Pradesh	MP	6.2	6.22	1.00	1.0993
Maharashtra	Mah	45.5	8.98	1.00	1.2824
Odisha	Odi	1.5	3.38	0.44	0.5433
Punjab	Pun	0.9	2.2	0.41	0.1873
Rajasthan	Raj	2.6	5.91	0.44	0.6316
Tamil Nadu	TN	4.9	5.68	0.86	0.6317
Telangana	Tel	1.5	2.87	0.52	0.3892
Uttar Pradesh	UP	8.6	17.35	0.50	0.7025
West Bengal	WB	1.9	7.26	0.26	0.3466
Delhi	Del	5.6	1.36	1.00	0.5707

Table 4 Railway distance between the main station of the neighboring states

Edge Name	Distance (K.M.)	Edge Name	Distance (K.M.)	Edge Name	Distance (K.M.)	Edge Name	Distance (K.M.)
AP-Kar	354	Bih-WB	501	Jha-Odi	586	MP-UP	584
AP-Odi	821	Chh-MP	694	Jha-WB	248	Odi-WB	409
AP-TN	431	Chh-Mah	302	Kar-Ker	842	Pun-Del	320
AP-Tel	321	Chh-Tel	796	Kar-Mah	1136	Raj-Del	309
Ass-WB	407	Guj-Mah	510	Kar-TN	358	UP-Del	514
Bih-Chh	771	Guj-Raj	619	Ker-TN	904		
Bih-Jha	496	Har-Pun	200	MP-Mah	392		
Bih-UP	512	Har-Del	136	MP-Raj	703		

Table 5 Edge membership value

Edge Name	Edge membership value	Edge Name	Edge membership value	Edge Name	Edge membership value	Edge Name	Edge membership value
AP-Kar	0.12	Bih-WB	0.11	Jha-Odi	0.13	MP-UP	0.26
AP-Odi	0.32	Chh-MP	0.23	Jha-WB	0.05	Odi-WB	0.09
AP-TN	0.17	Chh-Mah	0.10	Kar-Ker	0.29	Pun-Del	0.12
AP-Tel	0.13	Chh-Tel	0.26	Kar-Mah	0.39	Raj-Del	0.12
Ass-WB	0.08	Guj-Mah	0.45	Kar-TN	0.12	UP-Del	0.22
Bih-Chh	0.25	Guj-Raj	0.24	Ker-TN	0.34		
Bih-Jha	0.11	Har-Pun	0.07	MP-Mah	0.34		
Bih-UP	0.23	Har-Del	0.11	MP-Raj	0.27		

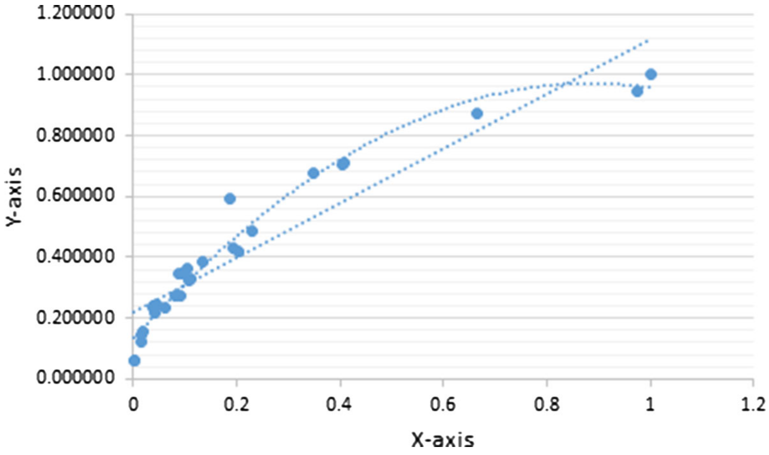

Fig. 13 Linear and parabolic curves fitting of the scores obtained by first Zagreb index for fuzzy graphs with F-index for fuzzy graphs

of the fuzzy subgraph $G_{S_1S_2}$. In Table 6, F-index for all single edge deleted fuzzy subgraph is listed.

Now F-index of an edge S_1S_2 is defined by the formula:

$$FF(S_1S_2) = FF(G) - FF(G_{S_1S_2}).$$

F-index of each edges is listed in Table 7.

Now score of an edge S_1S_2 is defined by the formula:

$$Sc(S_1S_2) = \left[\frac{FF(S_1S_2)}{\vee_{e\in E} FF(e)}\right]^{\frac{1}{3}}.$$

In Table 8, score of each edge is listed.

More score value of an edge reduces the safety. Note that, score of the edge MP-Mah is highest imply most railway crimes occurs per population in the railway in India. Also, most crime free railway per population in India is Ass-WB railway.

5.1 Comparative analysis

For the comparison, other topological indices: first Zagreb index for crisp graphs, F-index for crisp graphs, first Zagreb index for fuzzy graphs are considered to determine the railway in India where most crime occurs. Note that, the topological indices defined only for the crisp graph, does not depend on the nature of the vertex or nature of its neighbour vertices, i.e. those topological indices do not depend on the number of crimes in any state and its neighbour. Hence, the topological indices for crisp graphs cannot predict the actual crime for any railways. Therefore, fuzziness is required to predict such consideration. Here, F-index for fuzzy graphs depends not only on the number of crimes of each state but also on the total number of crimes of neighbour

Table 6 F-index of edge deleted fuzzy subgraph

Deleted edge Name	F-index	Deleted edge Name	F-index	Deleted edge Name	F-index	Deleted edge Name	F-index
AP-Kar	4.363754	Bih-WB	4.364785	Jha-Odi	4.386987	MP-UP	3.639827
AP-Odi	4.350047	Chh-MP	3.720248	Jha-WB	4.394626	Odi-WB	4.388577
AP-TN	4.272960	Chh-Mah	3.936549	Kar-Ker	4.350038	Pun-Del	4.303114
AP-Tel	4.371971	Chh-Tel	4.366505	Kar-Mah	2.96176	Raj-Del	4.290927
Ass-WB	4.39468	Guj-Mah	2.552289	Kar-TN	4.302679	UP-Del	4.221174
Bih-Chh	4.322216	Guj-Raj	4.142606	Ker-TN	4.233469		
Bih-Jha	4.366216	Har-Pun	4.391078	MP-Mah	2.220772		
Bih-UP	4.316898	Har-Del	4.303003	MP-Raj	3.622059		

Table 7 F-index of edges

Edge Name	F-index	Edge Name	F-index	Edge Name	F-index	Edge Name	F-index
AP-Kar	0.031349	Bih-WB	0.030318	Jha-Odi	0.008116	MP-UP	0.755276
AP-Odi	0.045056	Chh-MP	0.674855	Jha-WB	0.000477	Odi-WB	0.006526
AP-TN	0.122143	Chh-Mah	0.458554	Kar-Ker	0.045065	Pun-Del	0.091989
AP-Tel	0.023132	Chh-Tel	0.028598	Kar-Mah	1.433343	Raj-Del	0.104176
Ass-WB	0.000423	Guj-Mah	1.842814	Kar-TN	0.092424	UP-Del	0.173929
Bih-Chh	0.072887	Guj-Raj	0.252497	Ker-TN	0.161634		
Bih-Jha	0.028887	Har-Pun	0.004025	MP-Mah	2.174331		
Bih-UP	0.078205	Har-Del	0.092100	MP-Raj	0.773044		

Table 8 Score of edges for different topological indices

Edge Name	First Zagreb index for crisp graphs	F-index for crisp graphs	First Zagreb index for fuzzy graphs	F-index for fuzzy graphs
AP-Kar	1.0000	1.0000	0.0466	0.2434
AP-Odi	0.9258	0.9113	0.0902	0.2747
AP-TN	0.9258	0.9113	0.1352	0.3830
AP-Tel	0.8452	0.8409	0.0427	0.2199
Ass-WB	0.7559	0.8008	0.0027	0.0579
Bih-Chh	1.0000	1.0000	0.1082	0.3224
Bih-Jha	0.9258	0.9113	0.1130	0.2368
Bih-UP	0.9258	0.9113	0.0404	0.3301
Bih-WB	1.0000	1.0000	0.3491	0.2407
Chh-MP	1.0000	1.0000	0.1888	0.6771
Chh-Mah	1.0000	1.0000	0.0628	0.5952
Chh-Tel	0.8452	0.8409	0.9764	0.2360
Guj-Mah	0.8452	0.8409	0.2288	0.9464
Guj-Raj	0.7559	0.7056	0.0154	0.4879
Har-Pun	0.6547	0.5741	0.0975	0.1228
Har-Del	0.8452	0.8409	0.0957	0.3486
Jha-Odi	0.8452	0.8008	0.0200	0.1551
Jha-WB	0.9258	0.9113	0.0030	0.0603
Kar-Ker	0.8452	0.8409	0.0822	0.2747
Kar-Mah	1.0000	1.0000	0.6648	0.8703
Kar-TN	0.9258	0.9113	0.0965	0.3490
Ker-TN	0.7559	0.7056	0.2045	0.4205
MP-Mah	1.0000	1.0000	1.0000	1.0000
MP-Raj	0.9258	0.9113	0.4096	0.7084
MP-UP	0.9258	0.9113	0.4057	0.7030
Odi-WB	0.9258	0.9113	0.0158	0.1443
Pun-Del	0.8452	0.8409	0.0882	0.3484
Raj-Del	0.9258	0.9113	0.1063	0.3632
UP-Del	0.9258	0.9113	0.1944	0.4309

states for each state. So this index always provides realistic results compared to the other existing indices. The score of each edge for those indices is shown in Table 8. From Table 8, each index provides the edge “MP-Mah" has the highest score, which implies the railway crime is highest on that railway track in India. But the first two indices, defined for crisp graphs, also provide the highest score value for the edges: AP-Kar, Bih-Chh, Bih-WB, Chh-MP, Chh-Mah, Kar-Mah. If the decision-maker used the indices for a crisp graph, they could not distinguish among those edges. But, the other two indices, which are defined for fuzzy graphs, provide different scores for different edges. Hence, from this point of view, the topological indices defined in crisp graphs also need to be introduced in fuzzy graphs. We also fit linear and parabolic curves between the score values obtained by the first Zagreb index for fuzzy graphs and F-index for the fuzzy graphs (See Fig. 13). Here $X-$axis represents the value of the score obtained using the first Zagreb index for fuzzy graphs, and $Y-$axis represents the value of the score obtained using the F-index for fuzzy graphs. The relation between those scores are shown below:

$$\text{(Linear fitting)} Y = 0.8995X + 0.2176,\ R = 0.9338 \text{and}$$
$$\text{(Parabolic fitting)} Y = 1.0789X^2 + 1.9045X + 0.1319,\ R = 0.9841.$$

From the correlation coefficient value (R), it is clear that the score obtained from F-index for fuzzy graphs is closely related to the score obtained from the first Zagreb index for fuzzy graphs.

6 Conclusion

F-index has a vital role in the chemical graph, spectral graph, network theory, molecular chemistry, FG theory, etc. In this article, F-index is studied for several operations such as Cartesian product, composition, union and join of two fuzzy graphs. Some exciting relations of the F-index are established during fuzzy graph transformations. Using those transformations shows that n-vertex star has a maximum F-index among the class of n-vertex trees. Also, maximal n-vertex unicyclic fuzzy graph having r cycle is determined with respect to F-index. At the end of the article, this index analyses the crime in Indian railways and compares it with three other topological indices. Also, it is shown that F-index for fuzzy graphs and the first Zagreb index for fuzzy graphs provide similar results. F-index for fuzzy graphs provides better realistic results than F-index for crisp graphs and first Zagreb index for crisp graphs to detect the crime in Indian railway. The future scopes of the research are:

(i) In this article, it is shown that n-vertex star has a maximum F-index among the class of n-vertex trees. But, for which $n-$vertex tree (fuzzy) has the minimum F-index?
(ii) Also, maximal n-vertex unicyclic fuzzy graph with r cycle is determined with respect to F-index. But, a minimal n-vertex unicyclic fuzzy graph with r cycle is not determined with respect to F-index.
(iii) In this article, some good relations are established for the Cartesian product, composition, union and join of two fuzzy graphs. But, we cannot provide the exact values of those fuzzy graph operations.

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Declarations

Conflict of interest Potential conflict of interest was reported by the authors.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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