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ORIGINAL RESEARCH

Dynamics and control of delayed rumor propagation through social networks

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Abstract

Investigation of rumor spread dynamics and its control in social networking sites (SNS) has become important as it may cause some serious negative effects on our society. Here we aim to study the rumor spread mechanism and the influential factors using epidemic like model. We have divided the total population into three groups, namely, ignorant, spreader and aware. We have used delay differential equations to describe the dynamics of rumor spread process and studied the stability of the steady-state solutions using the threshold value of influence which is analogous to the basic reproduction number in disease model. Global stability of rumor prevailing state has been proved by using Lyapunov function. An optimal control system is set up using media awareness campaign to minimize the spreader population and the corresponding cost. Hopf bifurcation analyses with respect to time delay and the transmission rate of rumors are discussed here both analytically and numerically. Moreover, we have derived the stability region of the system corresponding to change of transmission rate and delay values.

Keywords Social networking sites (SNS) · Logistic growth · Threshold value of influence · Hopf bifurcation · Optimal control

Mathematics Subject Classification 34D23 · 49J15 · 70K50 · 91D30

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1 Introduction

Unlimited use of internet made social networking sites (SNS) a popular mode of expression which provides a useful environment among its users [1,2], especially in young generation [3]. Nowadays SNS like Facebook, Twitter, Instagram are used by a sizable section of population. It is estimated that, over 49% of all individuals have accessed social media worldwide by the middle of the year 2020 [4]. A rumor is basically a " circulating story of questionable veracity, which is apparently credible but hard to verify, and produces sufficient skepticism and/or anxiety so as to motivate finding out the actual truth" [5]. Rumors may become harmful to both individual and the society [6]. Before the advent of social media, people used to share news with their own social circle, where every individual is personally known to them. Nowadays, with advent and popular use of SNS, rumors are now spreading instantly and becoming a far more widespread agent [7,8]. For example, during the ongoing pandemic COVID-19, a piece of baseless news that, the infection could be transmitted by consuming chicken was circulated on SNS. As a result, people started to believe in it and the wholesale price of chicken had fallen by nearly 70% [9]. Rumors regarding fluctuation in international share market indices like Dow Jones, Nikkei, Sensex are capable of spreading huge panic all over the world. For instance, in the year 2015, a Scottish trader's tweets caused the stock price of two organizations to plummet approximately by 28 percent [10]. The share-market trader was charged with securities fraud by the United States' Securities and Exchange Commission. The various negative fallouts of rumor propagation demand that we understand the mechanism of transmission of rumor in details. Nowadays, along with social science, mathematical and computing tools like network theory, graph theory and nonlinear dynamics are used to explain the progression of rumors in SNS [11]. Moreno [12] used nodes in a complex network instead of account holders to investigate rumor spreading dynamics. Kawachi [13] studied the impact of age-dependent rumor transmission rate in an age structured deterministic model.

Mathematical modeling in epidemiology using nonlinear dynamical system has played a significant role to understand the mechanism and to obtain insight on control of diseases [14,16–19]. Nonlinear dynamics is often used to study age-dependent disease models with nonlinear incidence rate [20–22]. Apart from epidemiology, dynamical behavior of various age-structured model like drug abuse model, alcoholism model are studied with the help of nonlinear dynamics [23,24]. In ecology, different functional interactions in prey-predator model with competition and herd behavior are investigated using mathematical modeling [25–28].

In 1964, Daley and Kendall [29] proposed a mathematical model on sociology dividing the total population into three sub-groups namely, ignorant population, who have not heard about the rumor, spreader, who spread the rumor and stiflers, who lose their interest in the rumor and cease to spread the rumor. Wang et al. investigated models on rumor propagation with the analogy of Susceptible-Infected-Recovered (SIR) model [30]. Also Hu et al.[31] and Dhar et al.[32] studied dynamics of rumor propagation model similar to disease model. Jain et al. [33] discussed the stochastic effect on rumor propagation dynamics.

Time delay is a real phenomenon in social networking. Zhu et al. [34] studied the effect of delay on rumor propagation model. In [35] the influence delay of thinkers is included and optimal control of rumor is analyzed. Optimal control is also used in rumor propagation model of [11,36] with controls such as media awareness and official media coverage. The effect of media on spread of infectious disease, process of game communication has been discussed with the help of mathematical modeling and analysis using nonlinear dynamics [37],[38], [39].

From the previous examples, it is clear that the impact of circulation of unverified news on the change of the share prices is critical. We can recall that, ‘White house having been bombed, injuring Barack Obama’, had been tweeted by AP’s Twitter account by hackers. Associated Press(AP) is known as one of the ultimate sources for independent reporting from around the world. Consequently, it shook the US stock markets within minutes [40]. This incident is an example of potentiality of rumors to harm the economy. To control this kind of situations we need proper awareness campaigning strategy. Respective governments usually take steps against rumor propagation by broadcasting the fact with the help of media. So, the aim is to achieve maximum impact using limited public fund. Hence optimal control using Pontryagin’s Maximum Principle [41] has been applied to find out optimal media announcement schedule with minimum cost.

However, the majority of previous researches have not considered the logistic growth for spreader population. It is more realistic in the dynamics of rumor propagation, as rumor spreads fast initially, but after some time the transmission rate slows down when it becomes difficult to find new ignorant who has not heard of it. Therefore, consideration of the logistic growth for spreader population in rumor propagation model is important. We have calculated the threshold value of influence for rumor, that is, the condition when a rumor prevails and when it disappears eventually. Although, some of the previous studies have discussed optimal control strategy to control rumor, optimal control of rumor in difficult situations by awareness campaign through media is a very new perspective. Media is the easiest way to reach all the users on SNS. So, optimal control of media effort is the most effective approach to curb a rumor on SNS. Numerical simulations demonstrate the level of execution of the control required to eradicate disturbances caused by rumor. In this paper, we have investigated the effect of the delay. When a controversial topic flashes on timeline repeatedly, people get confused. Then they take time to think about its authenticity and after that they propagate the news. This kind of time lag is very important to reflect the reality more effectively. This sort of time delay, one ignorant usually takes to get convinced and become a spreader, is not discussed before.

The paper is arranged as follows: Model formulation is done in the Sect. 2. Existence and conditions of stability of equilibria are deduced in Sect. 3. In the same section, the global stability of the endemic equilibrium has been discussed. In Sect. 4, associated optimal control system for the non-delayed model has been set up. Analysis of Hopf bifurcation with respect to the time delay and the transmission rate of rumor have been performed in Sect. 5. The Sect. 6 is devoted to numerical simulation to support the analytical work. Finally, in Sect. 7, some concluding remarks are given.

2 Model formulation

Here we modified Daley-Kendall model [29] on rumor propagation by introducing logistic growth in the spreader population and a delay in the interaction term between the spreader and the ignorant class in SNS. News, that appears on SNS, can be true or partially true or completely fake and their questionable veracity breed rumors. Although some of the users are aware of the fake news, others are unaware of it. Rest of the users get initially confused and later, after seeing that news repeatedly on their news-feed, start believing it.

Considering these characters of users in SNS, we divide the total population $N(t)$ into three sub-populations as: $x(t)$ denotes the ignorant class who are unaware of the rumor yet, analogically related to the susceptible population in SIR model $y(t)$ stands for the spreader class, like the infected population in SIR model, $z(t)$ is the aware class who ignore the rumor, so does not spread it, like immune population in SIR model.

We consider that the spreader population follows logistic growth curve. Logistic growth happens when the growth rate slows down as the population tends to reach a maximum sustainable value K, where total environmental resources are limited. In logistic growth, this maximum sustainable population is called the carrying capacity. Initially the rumor spreads exponentially. But it slows down after achieving the maximum population size K, because there fewer new ignorant are found to become spreader as most of them either have joined the spreader or aware class as time advances. Moreover the total population is bounded, which is proved later. The rate of increase of the aware population reduces the growth rate of the spreader population and the people loses their interest about the rumor as time goes on. Therefore after some time rumor spread does not increase more and it becomes asymptotic to a constant value K. With this notion we have taken the logistic form $y\left(1-\frac{y}{K}\right)$ for spreader population with carrying capacity K.

According to [42], false rumors on SNS usually take more time to be proved as a false one. Sometimes people improvise news in different ways which creates more confusion. Considering these, certain time is needed by the ignorant population after they come across a rumor,to ascertain if to believe it or not. The time, required to think about the authenticity of the rumor before spreading it, is defined as delay due to thinking process, τ. We exclude those who do not go through thinking process, rather instantaneously forward any news they find on their newsfeed. To express this phenomenon mathematically, We take the delay τ in the interaction term between the ignorant and the spreader class as the former takes some time to get convinced by the rumor and become spreader.

At first we consider some assumptions:

- Recruitment of new user at time t is a whole number b.
- The new users do not believe in the news that flashes on their timelines with probability p.
- A spreader individual never comes back to ignorant class, rather can only go to the aware class.
- Once an individual gets aware, the person never comes back to any of the rest of the classes.

- There is no interaction between the aware class and the ignorant class.
- We exclude those spreaders who are somewhat biased for different reasons, for example, commercial, political or other societal issues, who intentionally forward rumors of their preferences.

Taking into account the above assumptions, we can formulate the model as follows:

$$\begin{aligned}
\frac{dx}{dt} &= (1-p)b - \frac{\beta x(t-\tau)y}{1+\alpha y} - \mu x \\
\frac{dy}{dt} &= y\left(1-\frac{y}{K}\right)\frac{\beta x(t-\tau)y}{1+\alpha y} - \eta yz - \mu y \\
\frac{dz}{dt} &= pb + \eta yz - \mu z
\end{aligned} \tag{2.1}$$

Here, $N(t) = x(t) + y(t) + z(t)$ is the total number of accounts active on a particular SNS at time t. When an ignorant individual interacts with a spreader one, becomes a spreader with transmission rate β. A spreader one becomes an aware individual at a rate η after coming in contact with an aware individual. All the parameters $p, b, \beta, \alpha, \eta, \mu, K$ are considered as positive constants and their definitions are given in the Table 1.

By putting $\tau = 0$, we get the following non-delayed system.

$$\begin{aligned}
\frac{dx}{dt} &= (1-p)b - \frac{\beta xy}{1+\alpha y} - \mu x \\
\frac{dy}{dt} &= \frac{\beta xy}{1+\alpha y} + y\left(1-\frac{y}{K}\right) - \eta yz - \mu y \\
\frac{dz}{dt} &= pb + \eta yz - \mu z.
\end{aligned} \tag{2.2}$$

2.1 Non-negativity and boundedness

The non-negativity and boundedness of the considered system (2.1) are necessary to show that the model is realistic.

Table 1 Parameter description

Parameter	Parameter definition
p	Probability that one newcomer does not believe in the rumor.
b	Number of new account holders on SNS.
β	Transmission rate of rumor.
μ	The rate at which the population becomes inactive related to the particular event.
η	The rate of one spreader becoming aware after coming in contact with the aware population.
α	Saturation constant.
K	The carrying capacity of the spreader class.

Theorem 1 *The solution set of the system (2.1) is always non-negative with the initial conditions* $x(0) \geq (1-p)b$, $y(0) \geq 0$, $z(0) \geq pb$ *and* $N_0 = x_0 + y_0 + z_0$.

Proof Integrating the first equation of the system (2.1) we obtain,

$$x(t) = e^{\int (\frac{(1-p)b}{x} - \frac{\beta y}{1+\alpha y} - \mu)dt} \geq 0.$$

Similarly, from the second and third equation of the system we get

$$y(t) = e^{\int (\frac{\beta x}{1+\alpha y} + (1-\frac{y}{K}) - \eta z - \mu)dt} \geq 0 \text{ and } z(t) = e^{\int (\frac{pb}{z} - \eta y - \mu)dt} \geq 0, \text{ respectively.}$$

□

Theorem 2 *The solution set of the system (2.1) is bounded.*

Proof Here we already have $x(t) \geq 0$, $y(t) \geq 0$, $z(t) \geq 0$ and adding all equations of the system (2.1) we get

$$\frac{dN}{dt} = b + g(y) - \mu N(t) \tag{2.3}$$

where,

$$g(y) = y(1 - \frac{y}{K}).$$

Differentiating $g(y)$ we get,

$$g'(y) = 1 - \frac{2y}{K}.$$

Then $g(y)$ attains its maximum at $y = \frac{K}{2}$ and maximum value is $\frac{K}{4}$. Using maximum value for $g(y)$ which is $\frac{K}{4}$, we can write

$$\frac{dN}{N(t) - \frac{b+\frac{K}{4}}{\mu}} \leq -\mu dt.$$

Now integrating both sides,

where k_1 is any integrating constant and taking limit as $t \to \infty$, we get

$$\limsup_{t\to\infty} N(t) \leq \lim_{t\to\infty} \left[\frac{b + \frac{K}{4}}{\mu} - k_1 e^{-\mu t} \right].$$

Therefore, we obtain

$$N(t) \leq \frac{b + \frac{K}{4}}{\mu}. \tag{2.4}$$

So, from (2.4), set of solution $(x(t), y(t), z(t))$ is bounded, since $x(t) \leq N(t)$, $y(t) \leq N(t)$, $z(t) \leq N(t)$. □

3 Equilibrium points and stability analysis

3.1 Threshold value of influence for the non-delayed system

The basic reproduction number for a disease model, $\mathcal{R}_0$, is defined as the expected number of secondary cases produced by a single (typical) infection in a completely susceptible population [43]. It is important to note that $\mathcal{R}_0$ is a dimensionless number. Mathematically it gives the threshold value for the stability of disease-free state.

In this article $\mathcal{R}_0$ gives the threshold value for stability of rumor-free state. It measures the potential influence of rumor on ignorant population. Therefore $\mathcal{R}_0$ is represented here as the threshold value of influence that determines whether or not the rumor will continue to spread among ignorant class. Here, $\mathcal{R}_0$ describes the expected number of persons in a completely ignorant class, who have just become spreaders through contact with one spreader. $\mathcal{R}_0$ greater than one implies that rumor will continue to affect the population if no external influences intervene. $\mathcal{R}_0$ less than one implies that the spreader population will lead to extinction.
Now, we shall calculate $\mathcal{R}_0 = \rho(GH^{-1})$ using next generation matrix method, where are G, H are Jacobians of $\mathcal{F}$, $\mathcal{V}$ at $\bar{E}$ respectively. For system (2.2), $\mathcal{F}$, $\mathcal{V}$ are given by

$$\mathcal{F} = \frac{\beta x y}{(1+\alpha y)} + y(1 - \frac{y}{K})$$
$$\mathcal{V} = \eta y z + \mu y.$$

Calculating the Jacobians G,H of $\mathcal{F}$, $\mathcal{V}$ at $\bar{E}$ we obtain

$$G = \left.\frac{\partial \mathcal{F}}{\partial y}\right|_{\bar{E}} = \beta \frac{(1-p)b}{\mu} + 1 \tag{3.1}$$
$$H = \left.\frac{\partial \mathcal{V}}{\partial y}\right|_{\bar{E}} = \eta \frac{pb}{\mu} + \mu. \tag{3.2}$$

Where G gives the matrix of rate of secondary spreader growth. We obtain the next generation matrix GH^{-1}. Then we have $\mathcal{R}_0 = \rho(GH^{-1})$, the largest eigen value of GH^{-1}. So, the threshold value of influence for the model(2.2) is

$$\mathcal{R}_0 = \frac{\beta(1-p)b + \mu}{\eta pb + \mu^2}. \tag{3.3}$$

We know that $\mathcal{R}_0$ plays a significant role in case of highly contagious disease models. Here we conceptualize $\mathcal{R}_0$ as rumor eradicating factor within SNS. Now $\mathcal{R}_0$ depends on β, α, μ, η, p, b. So it is really difficult to find that particular set of parameters for which we shall get that crucial value.

3.2 Equilibria of the system

There exist two equlibria of the model (2.1)

(i) Rumor Free Equilibrium (RFE), $\bar{E}$ =($\bar{x}$, 0, $\bar{z}$) where $\bar{x} = \frac{(1-p)b}{\mu}$ and $\bar{z} = \frac{pb}{\mu}$, which means rumor wipes out eventually.
(ii) Rumor Existing or prevailing Equilibrium(REE), ($\hat{x}$, $\hat{y}$, $\hat{z}$), with $\hat{y} \neq 0$, which means rumor will continue to spread across the SNS.

At the interior equilibrium point, we get

$$\begin{aligned} (1-p)b - \frac{\beta xy}{1+\alpha y} - \mu x = 0 \\ \frac{\beta xy}{1+\alpha y} + y(1-\frac{y}{K}) - \eta yz - \mu y = 0 \\ pb + \eta yz - \mu z = 0. \end{aligned} \tag{3.4}$$

Solving 1st and 3rd equation of (3.4) we get

$$\hat{x} = \frac{(1-p)b}{[\frac{\beta\hat{y}}{1+\alpha\hat{y}} + \mu]} \text{ and } \hat{z} = \frac{pb}{\mu - \eta\hat{y}}.$$

Substituting the values of $\hat{x}$ and $\hat{z}$ in the 2nd equation of the system (3.4) we get the following cubic equation in $\hat{y}$.

$$C_1\hat{y^3} - C_2\hat{y}^2 + C_3\hat{y} + C_4 = 0 \tag{3.5}$$

where

$$C_1 = \frac{\eta}{K}(\beta + \alpha\mu) \tag{3.6}$$

$$C_2 = (1-\mu)\eta(\beta+\alpha\mu) + \frac{\mu}{K}(\beta+\alpha\mu-\eta) \tag{3.7}$$

$$C_3 = \mu(1-\mu)(\beta+\alpha\mu-\eta) - \frac{\mu^2}{K} - (\beta(1-p)b + \eta pb(\beta+\alpha\mu)) \tag{3.8}$$

$$C_4 = \mu^2[(\eta pb + \mu^2) - (\beta(1-p)b + \mu)]. \tag{3.9}$$

This fixed point exists only when the equation (3.5) has a positive root. Clearly, C_1 and C_2 are positive. Here we make one assumption

$$\mathbf{H_1} : (\eta pb + \mu^2) - (\beta(1-p)b + \mu) < 0 \text{ or } \mathcal{R}_0 > 1.$$

If $\mathbf{H_1}$ holds then the following cases appear.

1A) C_3 is negative, which means there is exactly one positive root.
1B) C_3 is positive, which implies there are either three roots or exactly one positive root.

And when H_1 does not hold, there may be following two cases:

2A) C_3 is negative, which means either two positive real roots or no real root by Descarte's rule of sign.

2B) C_3 is positive, which again means either two positive real roots or no real root by Descarte's rule of sign.

If we get one positive real root of equation (3.5), we find the value of $\hat{y}$, then obtain $\hat{x}$ and $\hat{z}$. Hence the following lemma directly follows:

Lemma 1 *If* $\mathbf{H_1}$ *holds i.e. when* $\mathcal{R}_0 > 1$ *then the system (2.1) has at least one positive rumor endemic equilibrium.*

3.3 Local stability analysis

Theorem 3 *If* $\mathcal{R}_0 < 1$ *the RFE is locally asymptotically stable and unstable if* $\mathcal{R}_0 > 1$.

Proof If $\mathcal{R}_0 < 1$ then $(\beta(1-p)b + \mu - (\eta pb + \mu^2)) < 0$.
Jacobian matrix for RFE is given by

$$J_0 = \begin{pmatrix} -\mu & \frac{-\beta(1-p)b}{\mu} & 0 \\ 0 & \frac{\beta(1-p)b}{\mu} + 1 - \eta\frac{pb}{\mu} - \mu & 0 \\ 0 & \eta\frac{pb}{\mu} & -\mu \end{pmatrix} \tag{3.10}$$

The characteristic equation about $\bar{E}$ is given by

$$(\lambda - \mu)^2\{\lambda + (\eta\bar{z} + \mu) - (\beta\bar{x} + 1)\} = 0$$

where $\bar{x} = \frac{(1-p)b}{\mu}$, $\bar{y} = 0$, $\bar{z} = \frac{pb}{\mu}$. Then we get $\lambda_1 = \lambda_2 = -\mu$ and $\lambda_3 = [(\beta(1-p)b + \mu] - [\eta pb + \mu^2]$. So, if $\mathbf{H_1}$ holds then all the eigen values of the corresponding Jacobian matrix (3.10) are negative. If $\mathcal{R}_0 > 1$ then $\lambda_3 = (\beta\bar{x}+1)-(\eta\bar{z}+\mu)$ becomes positive, therefore system becomes unstable. Hence the theorem. □

Note: At $\mathcal{R}_0 = 1$ the system experiences transcritical bifurcation and RFE of the system (2.2) becomes unstable.

Next, we will discuss the local stability analysis of the system (2.1) about the endemic equilibrium.

The Jacobian matrix at REE for the system (2.1) is given by

$$\hat{J} = \begin{pmatrix} \frac{-\beta\hat{y}}{1+\alpha\hat{y}}e^{-\lambda\tau} - \mu & \frac{-\beta\hat{x}}{(1+\alpha\hat{y})^2} & 0 \\ \frac{\beta\hat{y}}{1+\alpha\hat{y}}e^{-\lambda\tau} & \frac{\beta\hat{x}}{(1+\alpha\hat{y})^2} + 1 - \frac{2\hat{y}}{K} - \eta\hat{z} - \mu & -\eta\hat{y} \\ 0 & \eta\hat{z} & \eta\hat{y} - \mu \end{pmatrix}. \tag{3.11}$$

The characteristic polynomial for REE is given by

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 + Ae^{-\lambda\tau}(\lambda^2 - c_1\lambda + b_1) = 0 \tag{3.12}$$

where

$$A = \frac{\beta \hat{y}}{1+\alpha\hat{y}},\ P_0 = \frac{\hat{y}}{1+\alpha\hat{y}},\ Q_0 = \frac{\hat{x}}{(1+\alpha\hat{y})^2}$$
$$a = \eta\hat{y},\ b' = 1 - 2\frac{\hat{y}}{K},\ Q = \beta Q_0,$$
$$a_1 = 3\mu + \eta\hat{z} - a - b' - Q$$
$$a_2 = 3\mu^2 + 2\mu\eta\hat{z} + ab' - 2(a+b')\mu - 2Q\mu - Qa$$
$$a_3 = \mu^3 - \mu^2(a + b' - \eta\hat{z}) + \mu ab' - Q\mu(\mu - a)$$
$$b_1 = \mu^2 - \mu(a+b') + ab' + \eta\hat{z}(\mu - a)$$
$$c_1 = a + b' - \eta\hat{z} - 2\mu.$$

When $\tau = 0$, the characteristic equation for system (2.2) becomes

$$\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 = 0 \tag{3.13}$$

where

$$p_1 = A + a_1$$
$$p_2 = a_2 - Ac_1$$
$$p_3 = a_3 + Ab_1.$$

By Routh-Hurwitz criterion, we get the following result.

Theorem 4 *The REE of the non-delayed system (2.2) will be locally asymptotically stable if all the roots of the equation(3.13) are with negative real parts, that is, when* $p_i > 0$ *for* $i = 1, 3$ *and* $p_1 p_2 > p_3$.

3.4 Global stability analysis of REE

To study the global stability of REE of the system (2.2), let us consider the Lyapunov function

$$V(t) = [x - \hat{x} + y - \hat{y} + z - \hat{z}]^2$$

so that, $V(t) \geq 0$ in the solution space

$$\Gamma = \{(x, y, z) \in \mathbb{R}^3 : x(t) \geq 0, y(t) \geq 0, z(t) \geq 0, N(t) \leq \frac{b + \frac{K}{4}}{\mu}\}$$

and

$$V(t)|_{\hat{x},\hat{y},\hat{z}} = 0,\ V(t) > 0 \text{ whenever } \{x \neq \hat{x}, y \neq \hat{y}, z \neq \hat{z}\}.$$

Then

$$\dot{V}(t) = 2[(x - \hat{x}) + (y - \hat{y}) + (z - \hat{z})](\dot{x} + \dot{y} + \dot{z})$$
$$= 2[(x - \hat{x}) + (y - \hat{y}) + (z - \hat{z})][b + y(1 - \frac{y}{K}) - \mu(x + y + z)].$$

Now from (3.4),

$$(\hat{x}+\hat{y}+\hat{z})=\frac{1}{\mu}\left(b+\hat{y}(1-\frac{\hat{y}}{K})\right). \tag{3.14}$$

Also, in our solution space Γ, we have

$$0 \le y\left(1-\frac{y}{K}\right) \le \frac{K}{4}$$

and from (2.4), we have $N(t) \le \frac{b+\frac{K}{4}}{\mu}$ and $N(t) \ge 0$.

$$\dot{V}(t) = 2[N(t)-(\hat{x}+\hat{y}+\hat{z})][b+y(1-\frac{y}{K})-\mu N(t)] \tag{3.15}$$

$$= \frac{2}{\mu}[\mu N(t)-(b+\hat{y}(1-\frac{\hat{y}}{K}))][b+y(1-\frac{y}{K})-\mu N(t)] \tag{3.16}$$

$$\le \frac{2}{\mu}[\mu N(t)-(b+\hat{y}(1-\frac{\hat{y}}{K}))][b+\frac{K}{4}-\mu N(t)] \tag{3.17}$$

Now, $\hat{y}$ is a positive a root of the equation (3.5). Then if $\mathcal{R}_0 > 1$ and $C_3 < 0$ in (3.5), the equation has exactly one positive root. Using theory of equations for (3.5), we obtain

$$\hat{y}\left(1-\frac{\hat{y}}{K}\right) \ge \frac{C_2}{C_1}\left(1-\frac{C_2}{KC_1}\right) = K_2.$$

Then from inequality (3.17), we have

$$\dot{V}(t) \le -\frac{2}{\mu}[b+K_2-\mu N(t)]^2 \le 0$$

whenever $\frac{K}{4} \le K_2$ and $\dot{V}(t) \le 0$ in Γ. Therefore, the system (2.2) is globally stable around REE by Lyapunov's global stability theorem. So, we can state the following theorem.

Theorem 5 *If $\mathcal{R}_0 > 1$ the model (2.2) is globally stable around REE in the solution space*

$$\Gamma = \left\{(x,y,z)\in\mathbb{R}^3 : x(t)\ge 0,\, y(t)\ge 0,\, z(t)\ge 0,\, N(t) \le \frac{b+\frac{K}{4}}{\mu}\right\}$$

if $\frac{K}{4} \le \frac{C_2}{C_1}(1-\frac{C_2}{KC_1})$ with $C_3 < 0$, where $C_i, i = 1, 2, 3, 4$ are given by (3.6-3.8).

If the endemic state becomes globally stable, then the rumor continues to persist in a large scale as time goes on. Sometimes, SNS release unverified news even before it appears on mainstream media. As in 2018, when many people in India refused to take the 10 rupee coin due to false news, a great deal of ambiguity was created in marketplaces. The news about the validity of 10 rupee coin[44], was broadcast via media, made them get to know the truth. As a result the situation gets stabilized. Clearly, a responsible media awareness campaign on SNS is beneficial to resolve these kind of situations. In the next section we investigate optimum use of media awareness campaign to control rumor.

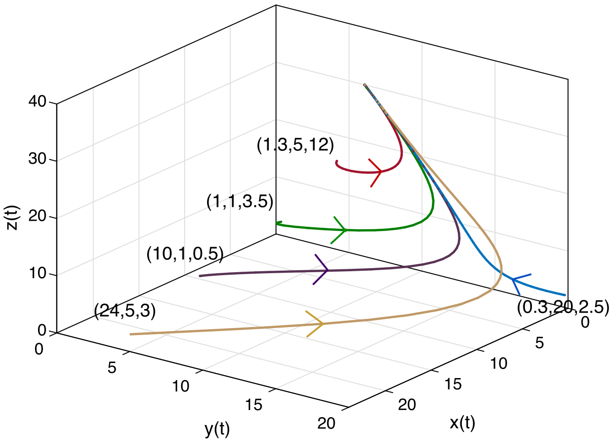


Fig. 1 The system converges to same point with different initial values

4 Optimal control by media intervention

The mainstream media like newspapers, radio and television has great impact on public's thoughts, understanding and crisis management [37,45]. To resolve any ambiguous situation caused by rumor the primary impact of media coverage is to promote science education, broadcasting the true fact with proper explanation so that the probability of ignorant individuals of believing in rumors reduces. When a rumor breaks out, awareness programs like publishing the authoritative information in newspaper, blogs, websites of responsible agency like AFP, Reuters, any press conference of responsible government clarifying the truth discourage spreaders to spread it further. According to the point of view of an administrator in crisis, the following optimal control model is formulated to reduce the number of spreaders on SNS by media awareness program as well as the total cost associated with the control. The cost function includes all negative impacts in the society and loss of economy as well as the cost of media intervention control, which we need to minimize. When an emergency occurs, media awareness program is broadcast to curb rumor. Control is applied for finite time period and is removed when number of spreaders is reduced enough to bring back the stability of the system. In this situation, every individual irrespective of their affiliation to the spreader or aware class suffers actively or passively. So, we count cost for every user, when the media intervention control is applied. Let u be the media control.
Here we assume some facts :

- We consider the cost function of the control as $u^2(t)(x(t) + y(t) + z(t))$ since control of media effort works for all of the users. Although our aim to minimize the spreader population, awareness campaign through media cannot be applied for any specific group instead of whole population.
- The ignorant population directly enters to the aware class with the rate α_1, whereas the spreader population transfers to aware class with the rate α_2. It is evident that α_1 is greater than α_2.

$$\frac{dx}{dt} = (1-p)b - \frac{\beta xy}{1+\alpha y} - \mu x - \alpha_1 ux$$
$$\frac{dy}{dt} = y\left(1-\frac{y}{K}\right) + \frac{\beta xy}{1+\alpha y} - \eta yz - \mu y - \alpha_2 uy \tag{4.1}$$
$$\frac{dz}{dt} = pb + \eta yz - \mu z + (\alpha_1 x + \alpha_2 y)u.$$

with the initial condition $x(0) \geq 0, y(0) \geq 0, z(0) \geq 0$. Here our objective is to minimize the total number of spreaders and the corresponding cost. So,

$$I(x, y, z, u) = \min_{u\in\hat{\Theta}} \int_0^T [B_0 y(t) + B_1 u^2(t)(x(t) + y(t) + z(t))]dt \tag{4.2}$$

where

$$\hat{\Theta} = \{u : u \in L^2[0, T], 0 \leq u \leq 1\}.$$

By [46], to find the optimal solution, we find the Hamiltonian of our optimal control problem (4.1) as given by

$$H = [B_0 y + B_1 u^2(x+y+z)] + \lambda_1((1-p)b - \frac{\beta xy}{(1+\alpha y)} - \mu x - \alpha_2 ux)$$
$$+ \lambda_2(\frac{\beta xy}{(1+\alpha y)} - \eta yz - \mu y + y(1-\frac{y}{K}) - \alpha_3 uy)$$
$$+ \lambda_3[pb + \eta yz - \mu z + u(\alpha_2 x + \alpha_3 y)] \tag{4.3}$$

where $\lambda_i, i = 1, 2, 3$ are adjoint variables.

4.1 Existence of optimal control

Theorem 6 *For the system (4.1) and objective function (4.2) there exists an optimal control $u^* \in \hat{\Theta}$ for which*

$$I(x^*, y^*, z^*, u^*) = \min_{u\in\hat{\Theta}} \int_0^T [B_0 y + B_1 u^2(x+y+z)]dt.$$

Proof As all the state variables and co-state variables are non-negative and so is the control variable u. Also the control space $\hat{\Theta}$ is convex and closed as the state space Ω. Then by [47] the proof is straightforward. □

4.2 Analysis of the control

Here we find the set of necessary conditions for our optimal control problem with the help of Pontryagin's Maximum Principle [41].

Theorem 7 *If u^* is the optimal control variable which optimizes (4.2) for system (4.1), with optimal state variables x^*, y^*, z^* and optimal co-state variable $\bar{\lambda}_i$, $i = 1, 2, 3$ where $\bar{\lambda}_i$ satisfies the following system;*

$$\begin{cases} \frac{d\lambda_1}{dt} = -B_1u^2 + \lambda_1(\frac{\beta y}{1+\alpha y} + \mu + \alpha_2\mu) - \lambda_2(\frac{\beta y}{1+\alpha y}) - \lambda_3\alpha_2 u \\ \frac{d\lambda_2}{dt} = -B_0 - B_1u^2 + \lambda_1\frac{\beta x}{(1+\alpha y)^2} - \lambda_2(\frac{\beta x}{(1+\alpha y)^2} - \eta z - \mu - \alpha_3 u + 1 - \frac{2y}{K}) - \lambda_3(\eta z + \alpha_3 y) \\ \frac{d\lambda_3}{dt} = -B_1u^2 + \lambda_2\eta y - \lambda_3(\eta y - \mu) \end{cases} \tag{4.4}$$

with transversality condition

$$\lambda_i(T) = 0 \ \textit{for} \ i = 1, 2, 3. \tag{4.5}$$

Then we obtain

$$u^* = \min\left\{\max\left\{\frac{\alpha_2 x(\lambda_1 - \lambda_3) + \alpha_3 y(\lambda_2 - \lambda_3)}{2B_1(x+y+z)}, 0\right\}, 1\right\}. \tag{4.6}$$

Proof Using Pontryagin's Maximum Principle on the Hamiltonian (4.3), we obtain the mathematical expression of the adjoint system (4.4), which is

$$\begin{cases} \frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial x} = -B_1u^2 + \lambda_1(\frac{\beta y}{1+\alpha y} + \mu + \alpha_2\mu) - \lambda_2(\frac{\beta y}{1+\alpha y}) - \lambda_3\alpha_2 u \\ \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial y} = -B_0 - B_1u^2 + \lambda_1\frac{\beta x}{(1+\alpha y)^2} - \lambda_2(\frac{\beta x}{(1+\alpha y)^2} - \eta z - \mu - \alpha_3 u + 1 - \frac{2y}{K}) - \lambda_3(\eta z + \alpha_3 y) \\ \frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial z} = -B_1u^2 + \lambda_2\eta y - \lambda_3(\eta y - \mu) \end{cases}$$

From the optimality condition $\frac{\partial H}{\partial u}|_{(x=x^*, y=y^*, z=z^*)} = 0$, we obtain

$$u^* = \frac{\alpha_2 x(\lambda_1 - \lambda_3) + \alpha_3 y(\lambda_2 - \lambda_3)}{2B_1(x+y+z)}. \tag{4.7}$$

By definition, the highest and the lowest value of control are 1 and 0 respectively. That is, if $u^* \leq 0$ then $u^* = 0$ and if $u^* \geq 1$ then $u^* = 1$. So, for u^* we get the optimum value of $I(x, y, z, u)$ for the problem (4.1). Hence the theorem. □

5 Effect of the delay

Let us consider the delayed system (2.1) to investigate the effect of the delay on the behavior of the system. The system 2.1 is stable around REE only when all the eigen values are with negative real part and loses its stability when a pair of purely imaginary eigen values traverses from left towards right across the imaginary axis. Now to determine whether the real part of the eigen values are negative or not, we apply the technique of Ruan and Wei [48]. To check if there is any Hopf point at $\tau = \tau_0$, let us put $\lambda = i\omega$ in the equation (3.12) and we get,

$$-i\omega^3 - a_1\omega^2 + ia_2\omega + a_3 = A[\cos(\omega\tau) - i\sin(\omega\tau)][\omega^2 + ic_1\omega - b_1] \tag{5.1}$$

$$(a_3 - a_1\omega^2) + i(a_2\omega - \omega^3) = A[\cos(\omega\tau) - i\sin(\omega\tau)][ic_1\omega - (b_1 - \omega^2)]. \quad (5.2)$$

Now equating real and imaginary part of the equation 5.2 and then squaring and adding we get

$$A^2[(b_1 - \omega^2) + c_1^2\omega^2] = [a_3 - a_1\omega^2]^2 + \omega^2(\omega^2 - a_2)^2. \quad (5.3)$$

Let us put $\omega^2 = z$ in the equation (5.3) and we get the following equation

$$z^3 + ({a_1}^2 - 2a_2)z^2 - z[A^2(1 - {c_1}^2) - a_2^2 - 2a_1a_3] + {a_3}^2 - A^2b_1 = 0. \quad (5.4)$$

Clearly, if ${a_3}^2 - A^2b_1 < 0$. So, the equation (5.4) has at least one positive root, for which we get $\omega = \pm\sqrt{z}$. These calculations ensure that there exists a pair of purely imaginary roots. Now from equation (5.3) we get

$$\cos(\omega\tau) = \frac{(a_1\omega^2 - a_3)(b_1 - \omega^2) + \omega(a_2 - \omega^2)c_1\omega}{A[(b_1 - \omega^2)^2 + c_1^2\omega^2]} \quad (5.5)$$

$$\tau = \frac{1}{\omega}\arccos\left[\frac{(a_1\omega^2 - a_3)(b_1 - \omega^2) + \omega^2(a_2 - \omega^2)c_1}{A[(b_1 - \omega^2)^2 + c_1^2\omega^2]}\right] + 2k\pi, \quad k = 1, 2, 3, \ldots. \quad (5.6)$$

So, for one value of ω we get a sequence τ_k. Now $\pm i\omega_0$ is a pair of purely imaginary roots of (3.12) with $\tau = \tau_j$. Clearly, the sequence τ_j is increasing and $\lim_{j\to\infty}\tau_j = \infty$. Then we can define $\tau_0 = \min \tau_j$.

5.1 Hopf bifurcation analysis with respect to time delay and transmission rate

To show that, the system(2.1) undergoes a Hopf bifurcation at τ_0, the following theorem is used.

Theorem 8 *Let $\lambda(\tau) = \gamma(\tau) + i\omega(\tau)$ be the root of equation (3.12) near $\tau = \tau_0$ satisfying $\gamma(\tau_0) = 0$ and $\omega(\tau_0) = \omega_0$. To show that there is a Hopf bifurcation with respect to τ_0, the following transversality condition should hold.*

$$Re\left[\frac{d\lambda(\tau)}{d\tau}\right]^{-1}_{\tau=\tau_0} \neq 0 \quad (5.7)$$

Proof Let

$$R(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 \quad (5.8)$$

$$I(\lambda) = A[\lambda^2 - c_1\lambda + b_1]. \quad (5.9)$$

Then the equation (3.12) can be rewritten in the form

$$R(\lambda) + e^{-\lambda\tau}I(\lambda) = 0. \quad (5.10)$$

Now differentiating (5.10) with respect to τ we get

$$R'(\lambda)\frac{d\lambda}{d\tau} - [-\lambda e^{-\lambda\tau} I(\lambda) + e^{(-\lambda\tau)} I'(\lambda)\frac{d\lambda}{d\tau} - \tau e^{(-\lambda\tau)} I(\lambda)\frac{d\lambda}{d\tau}] = 0.$$

Then we get

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(3\lambda^2 + 2\lambda a_1 + a_2) + [-\tau A(\lambda^2 - c_1\lambda + b_1) + A(2\lambda - c_1)]e^{-\lambda\tau}}{A\lambda(\lambda^2 - c_1\lambda + b_1)e^{-\lambda\tau}}.$$

Now, calculating $\left(\frac{d\lambda}{d\tau}\right)^{-1}$ at $\tau = \tau_0$, i.e. $\lambda = i\omega_0$ we get

$$\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_0} = \frac{(-3\omega_0{}^2 + 2i\omega_0 a_1 + a_2) + [-\tau A(-\omega_0{}^2 - c_1 i\omega_0 + b_1) + A(2i\omega_0 - c_1)]e^{-i\omega_0\tau}}{A\lambda(-\omega_0{}^2 - c_1 i\omega_0 + b_1)e^{-i\omega_0\tau}}. \tag{5.11}$$

Distinguishing real and imaginary part we can rewrite equation (5.11) in the following form

$$\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_0} = \frac{1}{A\omega_0}\frac{\begin{array}{c}[(a_2 - 3\omega_0{}^2) - A\tau_0(b_1 - \omega_0{}^2)\cos(\omega_0\tau_0) - A\omega_0(2 + c_1\tau_0)\sin(\omega_0\tau_0)] \\ + i[2a_1\omega_0 + \omega_0(2 + c_1\tau_0)\cos(\omega_0\tau_0) - A\tau_0(b_1 - \omega_0{}^2)\sin(\omega_0\tau_0)]\end{array}}{\begin{array}{c}[-(b_1 - \omega_0{}^2)\sin(\omega_0\tau_0) + c_1\omega_0\cos(\omega_0\tau_0)] \\ + i[(b_1 - \omega_0{}^2)\cos(\omega_0\tau_0) - c_1\omega_0\cos(\omega_0\tau_0)]\end{array}} \tag{5.12}$$

or we can write

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{(\tau=\tau_0)} = \frac{1}{A\omega_0}\frac{d_2 + id_3}{d_4 + id_5} \tag{5.13}$$

where

$$\begin{aligned} d_2 =& [(a_2 - 3\omega_0{}^2) - A\tau_0(b_1 - \omega_0{}^2)\cos(\omega_0\tau_0) - A\omega_0(2 + c_1\tau_0)\sin(\omega_0\tau_0)] \\ d_3 =& [2a_1\omega_0 + \omega_0(2 + c_1\tau_0)\cos(\omega_0\tau_0) - A\tau_0(b_1 - \omega_0{}^2)\sin(\omega_0\tau_0)] \\ d_4 =& [-(b_1 - \omega_0{}^2)\sin(\omega_0\tau_0) + c_1\omega_0\cos(\omega_0\tau_0)] \\ d_5 =& [(b_1 - \omega_0{}^2)\cos(\omega_0\tau_0) - c_1\omega_0\cos(\omega_0\tau_0)]. \end{aligned}$$

Equation (5.13) implies that

$$Re\left[\frac{d\lambda(\tau)}{d\tau}\right]^{-1}_{\tau=\tau_0} = \frac{1}{A\omega_0}\frac{d_2 d_4 - d_3 d_5}{d_4{}^2 + d_5{}^2}. \tag{5.14}$$

Therefore, if $d_2 d_4 - d_3 d_5 \neq 0$, the transversality condition for Hopf bifurcation holds and the system (2.1) bifurcates from fixed point REE at time delay $\tau = \tau_0$. □

Lastly, from Theorem 4 and Theorem 8 we can conclude the following statement.

Theorem 9 *When $\mathcal{R}_0 > 1$ then for system(2.1)* *(i) the endemic equilibrium is locally asymptotically stable (LAS) for all* $\tau \in [0, \tau_0]$

(ii) for $\tau > \tau_0$*, if the transversality condition(5.7) holds, the system experiences a Hopf bifurcation at* τ_0 *i.e. the endemic equilibrium* $(\hat{x}, \hat{y}, \hat{z})$ *becomes unstable, where,*

$$\tau_0 = \frac{1}{\omega_0}\arccos\left[\frac{(a_1\omega_0^2 - a_3)(b_1 - \omega_0{}^2) + \omega_0^2(a_2 - \omega_0{}^2)c_1}{A[(b_1 - \omega_0{}^2)^2 + c_1^2\omega_0{}^2]}\right]. \tag{5.15}$$

Next, we find that the dynamical behavior of the system (2.1) is sensitive with respect to the transmission rate β. Here we shall fix the delay at $\tau_1 (< \tau_0)$ and then if the parameter β is varied, we can find that the system experiences a bifurcation about REE. Here we find the critical value of β, say β_c, for which the pair of purely imaginary root $\lambda = \pm i\omega_c$ appears. Now, from equation (5.5)

$$\begin{aligned}\cos(\omega_c\tau_1)A[(b_1 - \omega_c^2)^2 + c_1^2\omega_c{}^2] = (\alpha_1' - \beta_c Q_0)\omega_c^2 - \alpha_3' + \beta_c Q_0 d_1)(b_1 - \omega_c^2)\\ + \omega_c(\alpha_2' - \beta_c Q_0 e_1 - \omega_c^2)c_1\omega_c\end{aligned} \tag{5.16}$$

$$\beta_c = \frac{(\alpha_1'\omega_c{}^2 - \alpha_3')(b_1 - \omega_c^2) + c_1\omega_c{}^2(\alpha_2' - \omega_c{}^2) - \cos(\omega_c\tau_1)A[(b_1 - \omega_c^2)^2 + c_1^2\omega_c^2]}{Q_0[(\omega_c{}^2 - d_1)(b_1 - \omega_c^2) + e_1c_1\omega_c{}^2]} \tag{5.17}$$

where

$$\begin{aligned}\alpha_1' &= a_1 + \beta Q_0,\ \alpha_2' = a_2 + \beta Q_0 e_1,\ \alpha_3' = a_3 + \beta Q_0 d_1\\ d_1 &= \mu^2 - \mu\eta\hat{y},\ e_1 = 2\mu + \eta\hat{y}.\end{aligned}$$

Now to check the transversality condition for Hopf bifurcation at β_c, we need to prove the following lemma.

Lemma 2 *Let* $\lambda(\beta) = \gamma(\beta) + i\omega(\beta)$ *be the root of equation (3.12) near* $\beta = \beta_c$ *satisfying* $\gamma(\beta_c) = 0$ *and* $\omega(\beta_c) = \omega_c$*. To show that there is a Hopf bifurcation with respect to* β *at* β_c*, the following transversality condition should hold.*

$$Re\left[\frac{d\lambda(\beta)}{d\beta}^{-1}\right]_{\beta=\beta_c} \neq 0 \tag{5.18}$$

Proof Let

$$\begin{aligned}R_1(\lambda) &= \lambda^3 + \alpha_1'\lambda^2 + \alpha_2'\lambda + \alpha_3'\\ R_2(\lambda) &= Q_0(\lambda^2 + e_1\lambda + d_1)\\ I_1(\lambda) &= [\lambda^2 - c_1\lambda + b_1].\end{aligned}$$

Then the equation (3.12) can be rewritten in the form

$$R_1(\lambda) - \beta R_2(\lambda) + \beta P_0 e^{(-\lambda\tau_1)} I_1(\lambda) = 0. \tag{5.19}$$

Now differentiating (5.19) with respect to β we get

$$R_1'(\lambda)\frac{d\lambda}{d\beta} - R_2(\lambda) - \beta R_2'(\lambda)\frac{d\lambda}{d\beta} + P_0 e^{-\lambda\tau_1} I_1(\lambda) + \beta e^{-\lambda\tau_1}[I_1'(\lambda) - \tau_1 I_1(\lambda)]\frac{d\lambda}{d\beta} = 0.$$

$$\left(\frac{d\lambda}{d\beta}\right)^{-1} = \frac{R_1'(\lambda) - \beta R_2'(\lambda) + \beta P_0 e^{-\lambda\tau_1}[I_1'(\lambda) - \tau_1 I_1(\lambda)]}{R_2(\lambda) - P_0 e^{-\lambda\tau_1} I_1(\lambda)} \tag{5.20}$$

Now to check the value of $\frac{d\lambda}{d\beta}$ at $\beta = \beta_c$, we substitute λ by $i\omega_c$ in (5.20) and we get

$$\left.\left(\frac{d\lambda}{d\beta}\right)\right|_{\beta_c} = \frac{R_1'(i\omega_c) - \beta R_2'(i\omega_c) + \beta P_0 e^{-i\omega_c\tau_1}[I_1'(i\omega_c) - \tau_1 I_1(i\omega_c)]}{R_2(i\omega_c) - P_0 e^{-i\omega_c\tau_1} I_1(i\omega_c)} \tag{5.21}$$

$$= \frac{\begin{array}{c}(a_2 - 3\omega_c{}^2) + \beta P_0[\cos(\omega_c\tau_1)(\tau_1\omega_c{}^2 - \tau_1 b_1 - c_1) + \omega_c(2 + c_1\tau_1)\sin(\omega_c\tau_1)] \\ - i[\beta P_0(\sin(\omega_c\tau_1)(\tau_1\omega_c{}^2 - \tau_1 b_1 - c_1) - \omega_c(2 + c_1\tau_1)\cos(\omega_c\tau_1)) - 2\omega_c a_1 + -2\beta Q_0\omega_c]\end{array}}{\begin{array}{c}Q_0(d_1 - \omega_c{}^2) - P_0(b_1 - \omega_c{}^2)\cos(\omega_c\tau_1) + P_0\omega_c c_1\sin(\omega_c\tau_1) \\ + i[Q_0 e_1\omega_c + P_0(b_1 - \omega_c{}^2)\sin(\omega_c\tau_1) + P_0\omega_c c_1\cos(\omega_c\tau_1)]\end{array}} \tag{5.22}$$

$$= \frac{s_1 - is_2}{s_3 + is_4} \tag{5.23}$$

where

$$\begin{aligned}
s_1 =& (a_2 - 3\omega_c{}^2) + \beta P_0[\cos(\omega_c\tau_1)(\tau_1\omega_c{}^2 - \tau_1 b_1 - c_1) + \omega_c(2 + c_1\tau_1)\sin(\omega_c\tau_1)] \\
s_2 =& -i[\beta P_0(\sin(\omega_c\tau_1)(\tau_1\omega_c{}^2 - \tau_1 b_1 - c_1) - \omega_c(2 + c_1\tau_1)\cos(\omega_c\tau_1)) - 2\omega_c a_1 + -2\beta Q_0\omega_c] \\
s_3 =& Q_0(d_1 - \omega_c{}^2) - P_0(b_1 - \omega_c{}^2)\cos(\omega_c\tau_1) + P_0\omega_c c_1\sin(\omega_c\tau_1) \\
s_4 =& [Q_0 e_1\omega_c + P_0(b_1 - \omega_c{}^2)\sin(\omega_c\tau_1) + P_0\omega_c c_1\cos(\omega_c\tau_1)].
\end{aligned}$$

Equation (5.23) implies

$$Re\left[\left(\frac{d\lambda(\beta)}{d\beta}\right)^{-1}\right]_{\beta=\beta_c} = \frac{s_1 s_3 + s_2 s_4}{s_3{}^2 + s_4{}^2}. \tag{5.24}$$

Therefore, from equation (5.23), $Re\left[\left(\frac{d\lambda(\beta)}{d\beta}\right)^{-1}\right]_{\beta=\beta_c} \neq 0$ or transversality condition for the occurrence of Hopf bifurcation of system (2.1) holds at $\beta = \beta_c$ if $(s_1 s_3 + s_2 s_4) \neq 0$, which completes the proof. □

Now, from the above discussion we can conclude the following theorem.

Theorem 10 *When the time delay is fixed at $\tau = \tau_1$ with $0 < \tau_1 < \tau_0$, where τ_0, the critical value for the parameter τ is given by equation (5.15), then for system(2.1)*

(i) the endemic equilibrium is LAS for all $\beta \in [0, \beta_c]$

(ii) for $\beta > \beta_c$, if the transversality condition(5.18) holds, the system experiences a Hopf bifurcation at β_c i.e. the endemic equilibrium $(\hat{x}, \hat{y}, \hat{z})$ becomes unstable, where β_c is given by

$$\beta_c = \frac{(\alpha_1'\omega_c{}^2 - \alpha_3')(b_1 - \omega_c^2) + c_1\omega_c{}^2(\alpha_2' - \omega_c{}^2) - \cos(\omega_c\tau_1)A[(b_1 - \omega_c^2)^2 + c_1^2\omega_c^2]}{Q_0[(\omega_c{}^2 - d_1)(b_1 - \omega_c^2) + e_1 c_1\omega_c{}^2]}.$$

6 Numerical simulation and discussions

In this section we demonstrate some important numerical results to verify our analytical findings using Matlab. The values of parameters for the delayed system is given in Table 2. First we present plots for the non-delayed system to explain the pervasiveness of rumor.

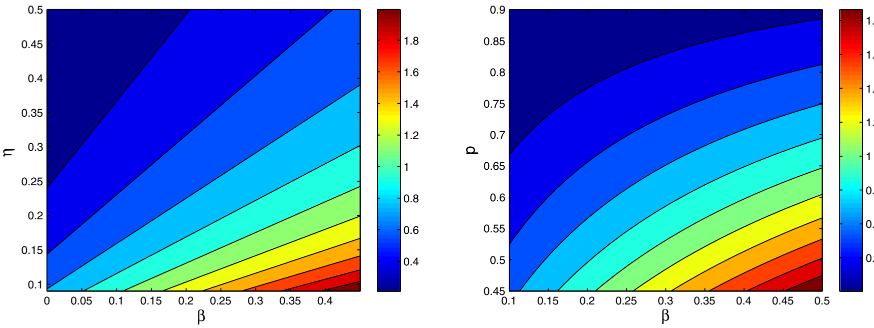
6.1 Threshold value of Influence and local stability analysis for the non-delayed system

$\mathcal{R}_0$ is one of the most significant parametric expressions, that indicates when a rumor will be completely wiped out or when it will prevail for the non-delayed system (2.2). In Fig. 2, we describe how the threshold value of influence, $\mathcal{R}_0$ changes as β and η increases. From Fig. 2a, we can say that β is more effective in increasing the $\mathcal{R}_0$ than η to decrease the $\mathcal{R}_0$. Figure 2b reflects the non-linear effect of the parameter p on $\mathcal{R}_0$. Again we observe that β is more effective in increasing $\mathcal{R}_0$ than whatever p is for decreasing $\mathcal{R}_0$.

Figure 3a demonstrates when $\mathcal{R}_0 < 1$, trajectories with different initial values converge to a point $y = 0$, RFE and it is locally stable. When $\mathcal{R}_0 > 1$, both the equilibrium points exist and only the REE is locally stable and trajectories from different initial points converge to a point $y \neq 0$ (Fig. 3b), rumor prevails as time advances.

Table 2 Values of parameters

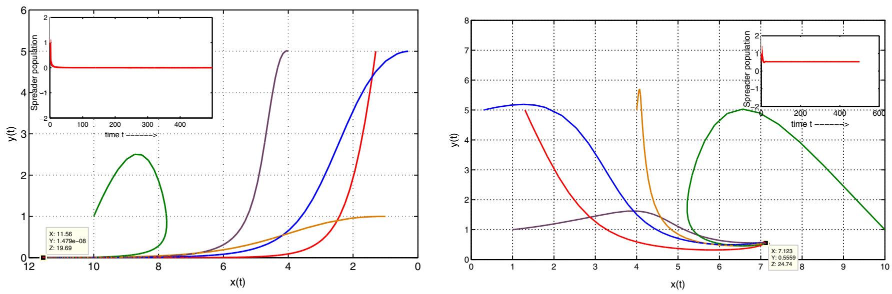
Parameter set	p	b	β	α	K	η	μ	τ_0
P_1	0.3	10	0.55	0.7	20	0.0162	0.4	2.95
P_2	0.3	10	0.56	0.6	20	0.0182	0.4	2.63
P_3	0.35	10	0.76	0.7	20	0.0182	0.5	2.35



(a) β and η is varied and the fixed parameter values are $p = 0.67$, $b = 7$, $\mu = 0.56$.

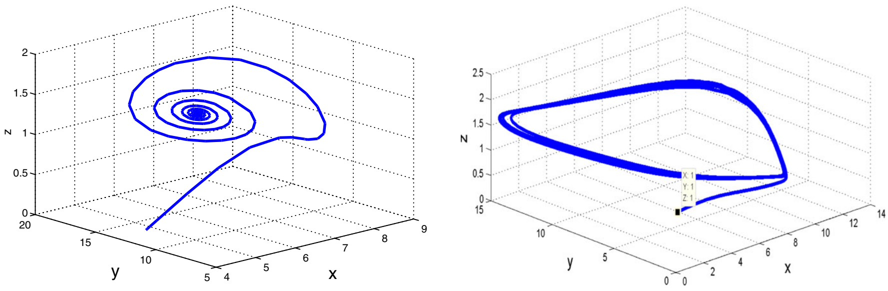
(b) β and p is varied and the fixed parameter values are $\eta = 0.3$, $b = 10$, $\mu = 0.5$.

Fig. 2 Contour plots demonstrating the nature of progress of $\mathcal{R}_0$ for variation in pair of parameters



(a) $\mathcal{R}_0 = 0.92175$, $\beta = 0.3$, $\eta = 0.23$. **(b)** $\mathcal{R}_0 = 2.355$, $\beta = 0.5$, $\eta = 0.13$.

Fig. 3 $2D$ phase portraits of trajectories from different initial points converge to a point on x-y plane, showing the switch of stability of RFE to REE, as $\mathcal{R}_0$ crosses 1. The rest of the parameters are $p = 0.63$, $b = 10$, $K = 30$, $\mu = 0.32$, $\alpha = 0.73$. The inset of each figures show time evaluation of Spreader population which becomes stable, for 3a at $y = 0$ and for 3b $y \neq 0$



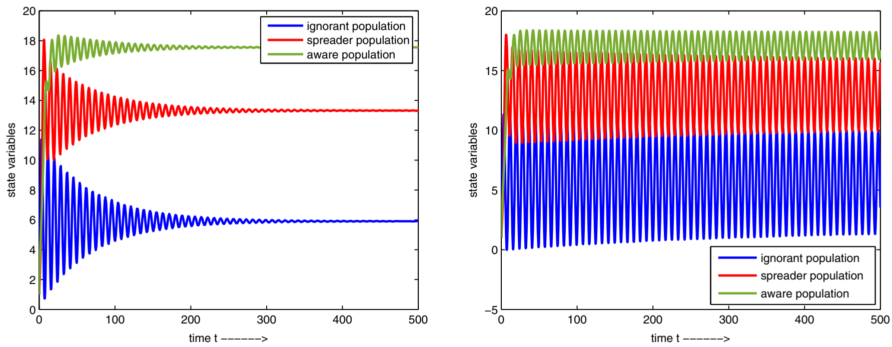
(a) Stable at $\tau = 2.74$ **(b)** Unstable at $\tau = 3.5$

Fig. 4 The stable fixed point is replaced by stable limit cycle as the time delay increases with parameter value set P_1 of Table 2

6.2 Hopf bifurcation analyses for the delayed system

In this part, we illustrate the numerical results to validate our theoretical findings regarding the occurrence of Hopf bifurcation with respect to time delay and transmission rate of rumor. Time delay has great impact in the stability of system (2.1). In Fig. 4 for a slight change in the value of delay, the stable spiral vanishes and a limit cycle appears in the phase portrait. For the parameter set P_2 in Table 2 as τ crosses the critical value $\tau_0 = 2.63$ the stable nature of the state variables is changed to oscillatory in Fig. 5. Figure 6 depicts the occurrence of Hopf bifurcation with respect to time delay.

The transmission rate of rumor β is highly sensitive. In Fig. 7 we see the change in stability of the solution trajectories state due to slight increase in β with fixed time delay τ_1 less than its critical value τ_0. Figure 8 shows that the system (2.1) experiences Hopf bifurcation as β crosses a certain value. It is also observed that the critical value of τ_0 increases with decreasing value of β , that is, if β changes from 0.8 to 0.74, τ_0 changes from 2.23 to 2.52 with parameter set P_3. Figure 9 demonstrate how the critical value



(a) Stable at $\tau = 2.42$ **(b)** Unstable at $\tau = 2.72$

Fig. 5 Solution trajectory shows oscillation as time delay crosses the threshold value with parameter values set P_2 of Table 2

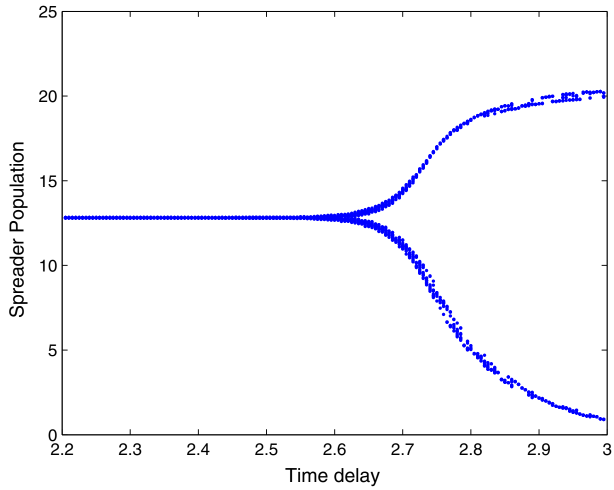
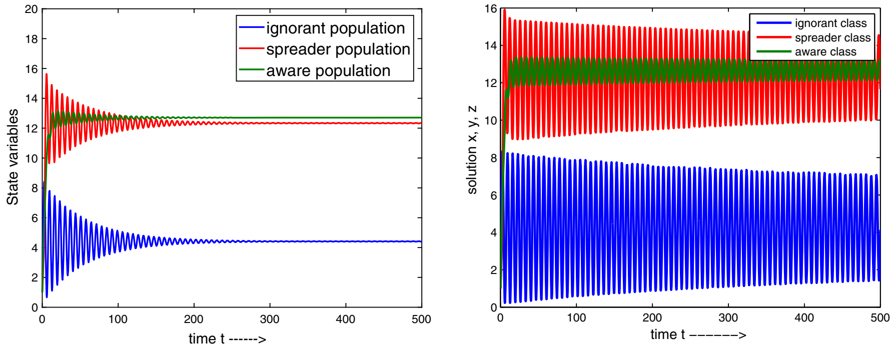


Fig. 6 Bifurcation diagram with respect to time delay

of the bifurcating parameter β changes for change in delay values. Similarly, from the same figure, we can observe change in critical value of the bifurcating parameter τ, as β is changes.

Sometimes a different kind of situation occurs where the original information or some selective parts of it is used tactfully to present a different perception [49]. As a result the true event becomes imprecise with time. According to [50], SNS has great impact in manipulating users, their political views, perception, judgment. But it needs time to manipulate the perception of a large number of people. This characterization is explained here by introducing another time delay τ_2 in the interaction term between spreader and ignorant class in model (2.1) along with the delay due to thinking process of ignorant class presented as $\frac{\beta x(t-\tau)y(t-\tau_2)}{(1+\alpha y(t-\tau_2))}$. Here we consider that the spreader class got the original information τ_2 time ago and spreaders take that τ_2 time to improvise



(a) Stable at $\beta = 0.76$. **(b)** Unstable at $\beta = 0.8$.

Fig. 7 The system loses its stability as β increases, where $\tau_1 = 2.23$ with rest of the parameters from set P_3 in Table 2

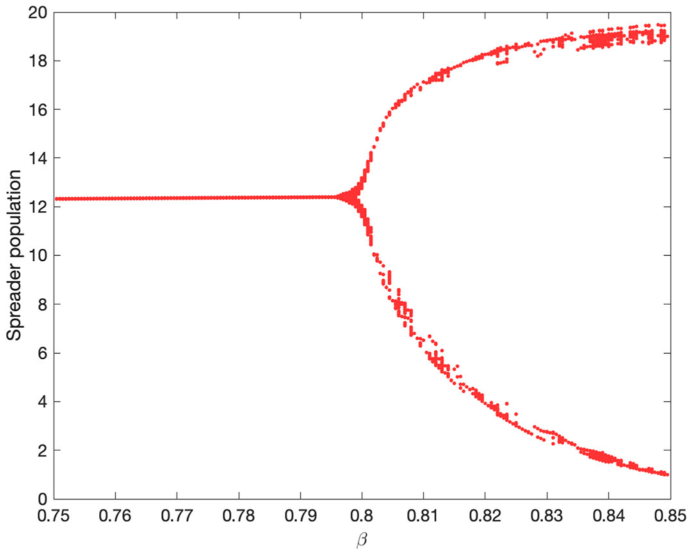


Fig. 8 Bifurcation diagram with respect to transmission rate β

the real information on their own, then start to interact with ignorant individuals with the twisted information.

From Fig. 10, we see that the system loses its stability when time delay τ_2 crosses a certain value along with fixed value of previous delay $\tau = 2.35$. This scenario may be explained as the spreader class successfully fabricates or twists the real data in certain amount of time and stability of the system is lost when spreaders start to interact with the twisted data. Moreover, spreader population dominates others in later case (see Fig. 10b) and in first i.e. in Fig. 101 aware class dominates with stable nature as time advances.

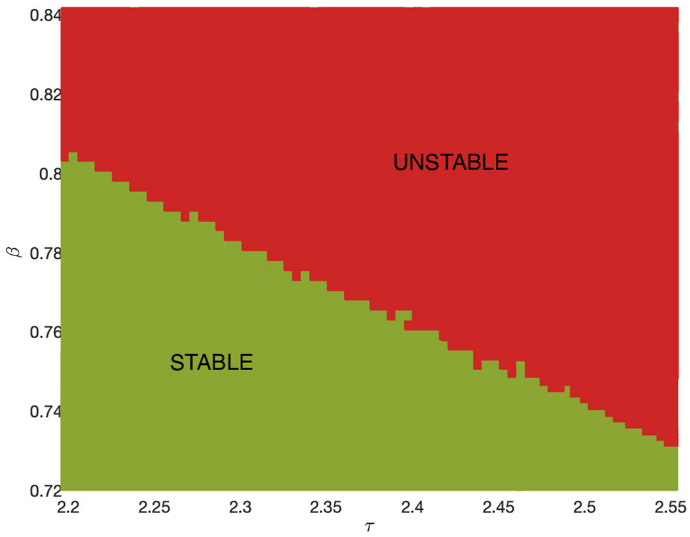
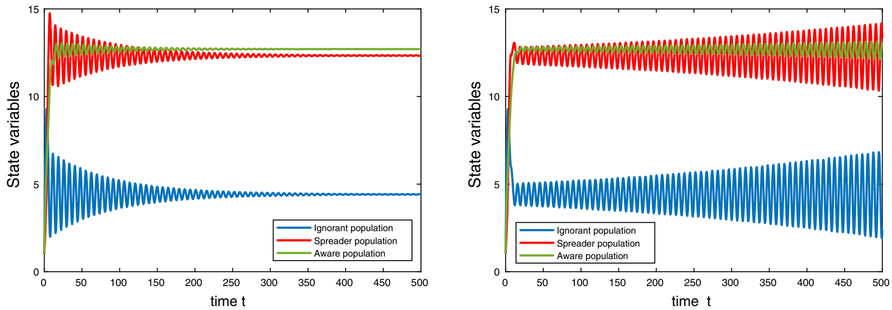


Fig. 9 The nature of stability of system (2.1) due to change in τ and β for parameter set P_3



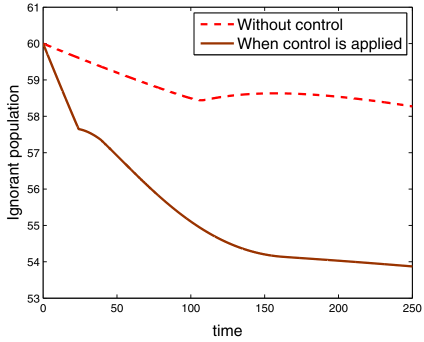
(a) Time series when $\tau_2 = 2.23$ and $\tau = 2.23$.

(b) Time series when $\tau_2 = 7.3$ and $\tau = 2.23$.

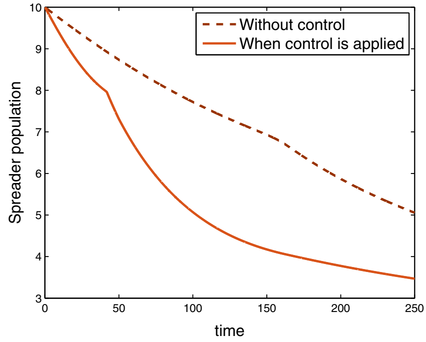
Fig. 10 The stable time series of state variables is changed as τ_2 crosses a certain value with parameter set P_3 of Table 2

6.3 Simulation for optimal control of media effort

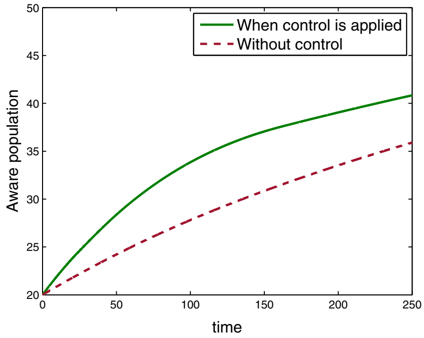
Here we solve the optimal control problem numerically by "forward-backward sweep" method [51] to interpret the analytical results obtained in section 4. The main objective is to minimize the revenue function defined in (4.2) to control a widespread rumor by awareness campaign through media. The system (4.1), made of three ordinary differential equations for the state variables is solved by forward fourth-order Runge-Kutta Method and the adjoint system (4.4) is solved by backward fourth-order Runge-Kutta method with the transversality condition (4.5) and optimal condition (4.6). For simulation, we consider the parameter set $P_4(p = 0.4, b = 35, \beta = 0.7, \alpha = 5, K = 30, \mu = 0.3, \eta = 0.09, \alpha_1 = 0.6, \alpha_2 = 0.44)$ with initial condition $(x_0, y_0, z_0) =$



(a) Ignorant population trajectory.



(b) Spreader population trajectory.



(c) Aware population trajectory.

Fig. 11 Solution trajectories of the system (4.1) with and without application of control for parameter set P_4

(60, 10, 20). We choose the positive weights (B_0, B_1) of equation (4.2) as (0.82, 0.02). The time range of application of control is [0, 250].

Figure 11b represents the spreader population before and after applying control. It is evident that after applying control by media awareness campaign, the spreader population decreases more rapidly. Also the growth rate of aware population becomes higher when control is applied (Fig. 11c). After application of control, number of ignorant people decreases a lot, as they come to know the truth with the help of media (Fig. 11a). Figure 12 represents the optimal control function to minimize total cost. For optimal outcome, the control needs to be applied at the highest rate for a brief period initially, followed by gradual semi-linear decrease. In this way, the requirement of control ceases some time before the terminal time. But the cost function before application of control is higher than the cost function associated with control, as we have assigned higher cost to the presence of spreader population than the control (see Fig. 13).

Next we consider two sets of parameters P_5 ($p = 0.35$; $b = 35$; $\beta = 0.85, \alpha = 5, K = 30, \mu = 0.3, \eta = 0.09, \alpha_2 = 0.6, \alpha_3 = 0.44$) and P_6 ($p = 0.35, b = 35, \beta = 0.5, \alpha = 5, K = 30, \mu = 0.3, \eta = 0.09, \alpha_2 = 0.6, \alpha_3 = 0.44$) and compare their corresponding cost and control graph. In the following Fig. 14 we can see that

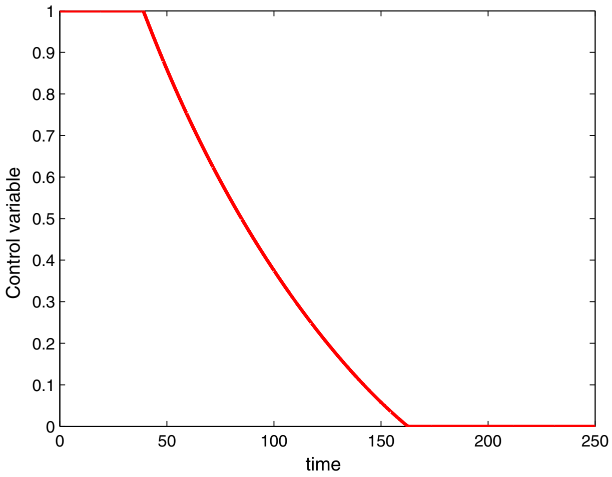


Fig. 12 Control function of media optimal control of media effort

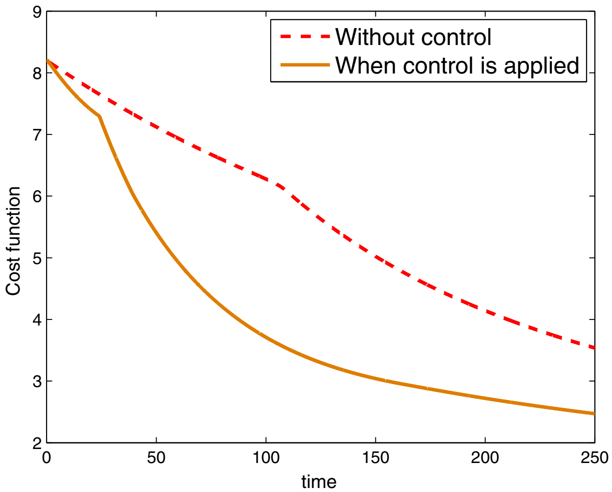


Fig. 13 Cost function with and without control

as transmission rate β increases, control needs to be applied for more time to reduce spreader population to the minimum possible level.

When transmission rate increases a bit, control needs to be applied for longer time duration. Cost function is as usual higher than without control case (see Fig. 15). It is clear form the figure that the increase in cost is higher in absence of control. This implies that we count huge loss for higher transmission rate in Fig. 15b compared to Fig. 15a. But in presence of control, we pay the extra cost only for the extra time of application of media intervention control and finally end up with better result (see Fig. 16).

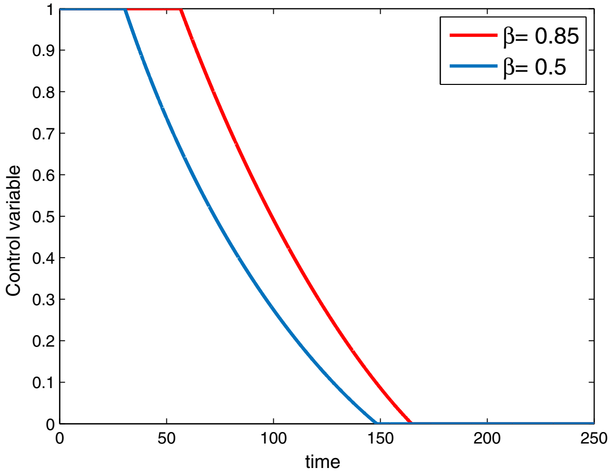
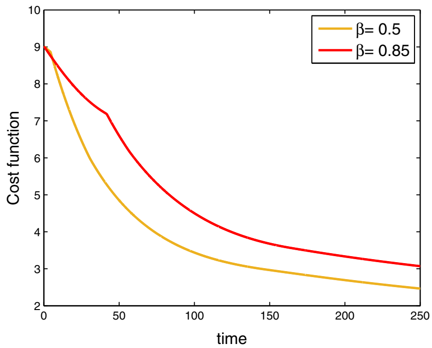
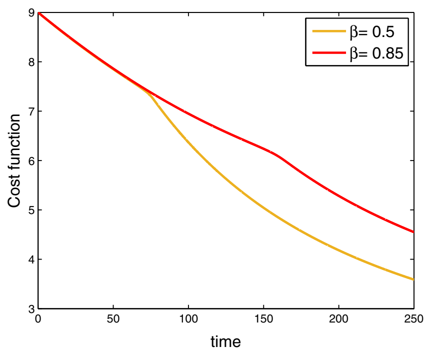


Fig. 14 Application of control with different transmission rate

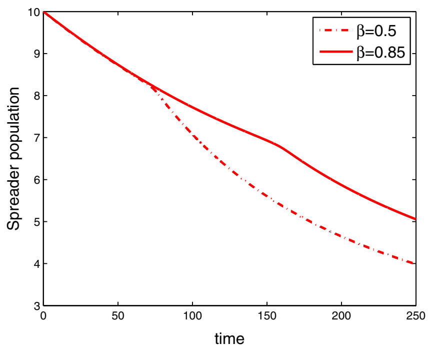


(a) Cost with different transmission rate in presence ofcontrol.

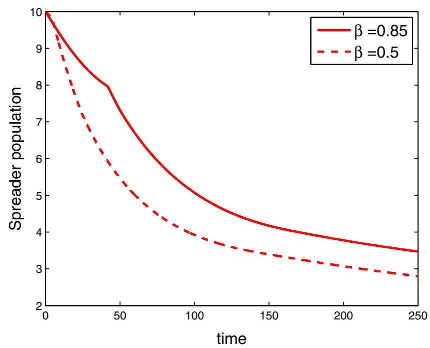


(b) Cost with different transmission rate without control.

Fig. 15 Cost function increase notably less in presence of control than the case of without control



(a) When there is no control.



(b) After application of control.

Fig. 16 Density of spreader class with and without control for different values of β

7 Conclusion

Rumor propagation has increased in a tremendous rate with the increasing popularity of SNS in our society. If an event occurs, users in SNS share information regarding that event. This can sometimes mislead the judgment of the social netizen and also can cause social disturbance for the time being. Considering the fact that a person needs time to make a judgment after they get to know a rumor, here we studied a rumor propagation model with time delay τ in the interaction term between spreader and ignorant class. We derived $\mathcal{R}_0$, the threshold value of influence and used it to investigate the stability of RFE. We also derived the conditions for local and global stability of endemic state. Here, we noticed that when the threshold value of influence $\mathcal{R}_0 < 1$, the RFE is stable which means the rumor will eventually wipe out from SNS. When $\mathcal{R}_0 > 1$, the endemic state appears and REE is the only stable equilibrium. All the trajectories about this point converge to it for $0 \leq \tau < \tau_0$. But when the time delay crosses τ_0, the system bifurcates from the fixed point, and the state becomes unstable. We have also noticed that for greater values of transmission rate, system bifurcates at lesser value of time delay (see Fig. 9) which can be interpreted as the situation where a news spreads faster, people take lesser time to think about its genuineness. It may happen under the impression that if a large number of people spread a particular news, then it may be considered true. When the spreader population becomes large enough, that is, the spreader density is higher in comparison to the ignorant or the aware class, we can consider this as a viral scenario. Moreover, we have numerically analyzed the situation where the spreaders take time to twist or fabricate a news by introducing another delay τ_2 and noticed that stability disappears as τ_2 increases along with fixed $\tau < \tau_0$ (see Fig. 10). The spreader population contains both active spreaders, who deliberately spread a fake news and passive ones, who share it out of anxiety or confusion. In this paper, we have considered them as a whole and we are leaving this classification of spreader class for our further work.

In this article we considered an optimal control strategy to reduce the disturbances or panic in the society caused by a widespread rumor transmission (viral scenario). As we discussed earlier, our aim is to reduce the number of spreaders as well as the corresponding cost. Here we choose awareness campaign through mainstream media like newspaper, TV or websites broadcasting statements from responsible agency like AFP, Reuters etc. because it is the easiest way to reach all the netizens in SNS. We have seen that control is applied for the longer time for the higher transmission rate. So we can say that the extra cost is only for the extra time of execution of control. It is clear that the number of spreaders reduces drastically even if β is too high after applying the media intervention control and we reach our goal. When control is applied, the cost function remarkably reduces (see Fig. 13). Even when the transmission rate is high, the cost function reduces more after application of control with better result (see Fig. 16). So the media campaigning strategy is very cost-effective even if the control is applied to all account holders irrespective of their attitude towards the rumor.

From the above discussion, we can say that the awareness campaign through media on SNS is really effective to control the imbalanced situations that may appear due to transmission of fake news. The present model can be enhanced using different age-structured users or a separate state for social media influencers and may offer some advanced ideas about dynamics of social networking, which is ever growing in today's world.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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