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Qualitative analysis of a SIR epidemic model with saturated treatment rate

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Abstract On account of the effect of limited treatment resources on the control of epidemic disease, a saturated removal rate is incorporated into Hethcote's SIR epidemiological model (Hethcote, SIAM Rev. 42:599–653, 2000). Unlike the original model, the model has two endemic equilibria when $R_0 < 1$. Furthermore, under some conditions, both the disease free equilibrium and one of the two endemic equilibria are asymptotically stable, i.e., the model has bistable equilibria. Therefore, disease eradication not only depends on R_0 but also on the initial sizes of all sub-populations. By the Poincaré-Bendixson theorem, Poincaré index, center manifold theorem, Hopf bifurcation theorem and Lyapunov-Lasalle theorem, etc., the existence and asymptotical stability of the equilibria, the existence, stability and direction of Hopf bifurcation are discussed, respectively.

Keywords Poincare index · Bistable equilibria · Center manifold theorem · Hopf bifurcation · Stability

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1 Introduction

Epidemic model can contribute to the design and analysis of epidemiological surveys, suggesting crucial data that should be collected, identifying trends, making general

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forecasts, and estimating the uncertainty in forecasts [6]. Therefore, to add new factors such as different treatment plans, etc., to classical epidemic model and then investigate their effects on the dynamical behaviors has become a focus in applied mathematics. In [2, 3, 7, 10, 15, 17] and references therein, several kinds of removal rates, i.e., the recovery from the infected subpopulation per unit time, have been adopted to characterize different treatment truths of epidemic disease. For example, in [15], W. Wang and S. Ruan suppose a removal rate has the form

$$h(I) = \begin{cases} r, & I > 0, \\ 0, & I = 0, \end{cases} \tag{1}$$

and formulate the following model

$$\begin{cases} \frac{dS}{dt} = A - dS - \lambda SI, \\ \frac{dI}{dt} = \lambda SI - (d+\gamma)I - h(I), \\ \frac{dR}{dt} = \gamma I + h(I) - dR, \end{cases} \tag{2}$$

where S, I, and R denote the numbers of the susceptible, infective and recovered individuals at time t, respectively. Then they discuss the stability of equilibria and prove the model exhibits Bogdanov-Takens bifurcation, Hopf bifurcation and Homoclinic bifurcation. Obviously, the removal rate of (1) is a reasonable approximation to the treatment truth of the disease that human possesses rich treatment sources and better treatment techniques. However, for the new outbreak disease such as SARS, etc., maybe saturated treatment rate is a better alternative. In fact, the treatment rate is small for there is short of effective treatment techniques at the beginning of the outbreak. Then, the treatment rate will be increased with the improving of the hospital's treatment conditions including effective medicines, skillful techniques, etc. At last, for the treatment capacity of any community is limited, the treatment will reach to its maximum if the number of infective individuals is large enough. In view of above description, we suppose the removal rate to be

$$h(I) = \frac{\beta I}{1+\alpha I}, \quad I \geq 0,\ \alpha, \beta > 0. \tag{3}$$

Clearly, the epidemicity of disease is closely related to the stability of the equilibria of mathematical models. In classical models there usually exist a disease-free equilibrium and a unique endemic equilibrium. The stability of the disease free equilibrium is determined by a threshold parameter R_0, known as the basic reproductive number, i.e. the disease free equilibrium is asymptotically stable if $R_0 < 1$, while it is unstable and the unique endemic equilibrium is asymptotically stable if $R_0 > 1$. However, recent studies have showed that some epidemiological models (see [1, 9, 12, 14] for examples) have multiple endemic equilibria, and which can exist simultaneously even R_0 is less than the unity. Furthermore, the disease free equilibrium and one of the endemic equilibria can be stable simultaneity, i.e. there exist bistable equilibria. Therefore, it is not always effective to eliminate the disease by reducing R_0 to the values less than 1. In fact, to eradicate this kind of disease, one not only needs to reduce R_0 but also needs to restrict the initial value of each subpopulation to the domain of attraction of the disease free equilibrium.

In the present paper, considering the saturated removed rate and supposing the removal individuals perpetual immunity, a new SIR epidemic model is formulated. Then, we discuss the existence and asymptotical stability of equilibria and the Hopf bifurcation. It is interesting that out model has bistable equilibria in two cases: $R_0 < 1$ and $\Delta = 0$ (will be defined in the text); $R_0 < 1$ and $\Delta > 0$. The remainder of this paper is arranged as follows: Sect. 2 presents the existence of equilibria, Sect. 3 researches the stability of every equilibrium, the existence, stability and direction of Hopf bifurcation, Sect. 4 is the numerical simulation which verifies our qualitative results, Sect. 5 makes the conclusions.

2 The existence of equilibria

Obviously, incorporating the saturated treatment rate (3) into model (2) leads to

$$\begin{cases} \frac{dS}{dt} = A - dS - \lambda SI, \\ \frac{dI}{dt} = \lambda SI - (d+\gamma)I - \frac{\beta I}{1+\alpha I}, \\ \frac{dR}{dt} = \gamma I + \frac{\beta I}{1+\alpha I} - dR, \end{cases} \tag{4}$$

where A is the recruitment rate of the population, d the natural death rate of the population, γ the natural recovery rate of the infective individuals, λSI the bilinear incidence rate and $\frac{\beta I}{1+\alpha I}$ the removal rate of the infected individuals.

Remark 1 It is easy to see that the treatment rate $h(I) = \frac{\beta I}{1+\alpha I}$ is a continuously differentiable function of I and : $h(0) = 0$, $\frac{dh(I)}{dI} > 0$, $\lim_{I\to\infty} h(I) = \frac{\beta}{\alpha}$, where $\frac{\beta}{\alpha}$ is the maximal treatment capacity of some a community.

Noticing the last equation of (4) is independent of the others, we mainly investigate the dynamic properties of S and I in the present paper. Therefore, only the following subsystem is focused on.

$$\begin{cases} \frac{dS}{dt} = A - dS - \lambda SI, \\ \frac{dI}{dt} = \lambda SI - (d+\gamma)I - \frac{\beta I}{1+\alpha I}. \end{cases} \tag{5}$$

For system (5), we have

Lemma 2 $\mathbb{O} = \{(S, I)|S \geq 0, I \geq 0, S + I \leq \frac{A}{d}\}$ *is a positively invariant region of system* (5).

By the Bendixson-Dulac criterion and Lemma 1, we have

Lemma 3 *If* $\lambda < \frac{d^3}{A^2\alpha}$, *system* (5) *does not have closed orbit in the first quadrant of* (S, I) *plane.*

Proof Let $D=\frac{1+\alpha I}{SI}$, $P=A-dS-\lambda SI$ and $Q=\lambda SI-(d+\gamma)I-\frac{\beta I}{1+\alpha I}$. After computation, we get

$$\frac{\partial DP}{\partial S}+\frac{\partial DQ}{\partial I}=\frac{\lambda\alpha S^2 I-A\alpha I-n\alpha SI-A}{S^2 I}\leq\frac{\lambda\alpha S^2-A\alpha-d}{S^2 I}.$$

Because $\mathbb{O}$ is a positively invariant set of system (5), we restrict (S,I) to the inner of $\mathbb{O}$. Therefore, we have

$$\frac{\partial DP}{\partial S}+\frac{\partial DQ}{\partial I}\leq\frac{\lambda\alpha(\frac{A}{d})^3-A}{S^2 I}.$$

Then, by Bendixson-Dulac criterion, system(5) does not have closed orbit in the first quadrant of (S,I) plane if $\lambda<\frac{d^3}{A^2\alpha}$. □

In the following, we discuss the existence of equilibria. For convenience, we introduce two notations: $R_0=\frac{\lambda A}{d(\beta+n)}$, $R_1=\frac{\lambda A\alpha}{nd\alpha+\lambda(\beta+n)}$, where $n=d+\gamma$ and R_0 is called the basic reproduction number in [6].

Let the right hand side of system (5) be zero. It isn't difficult to obtain that system (5) always has a disease free equilibrium $E_0=(\frac{A}{d},0)$. Then, under the condition of $I\neq 0$, solving algebraic equations by the fixed theorem yields.

Theorem 4

(A) *If* $R_0>1$, *there exists a unique endemic equilibrium* $E_1=(S_1^*,I_1^*)$.
(B_1) *If* $R_0=1$ *and* $R_1>1$, *there exists a unique endemic equilibrium* $E_2=(S_2^*,I_2^*)$.
(B_2) *If* $R_0=1$ *and* $R_1\leq 1$, *no endemic equilibrium exists*.
(C_1) $R_0<1$, $R_1>1$ *and* $\Delta>0$, *there exist two endemic equilibria*: E_1, $E_3=(S_3^*,I_3^*)$.
(C_2) *If* $R_0<1$, $R_1>1$ *and* $\Delta=0$, *there exists a unique endemic equilibrium* $E_4=(S_4^*,I_4^*)$.
(C_3) *If* $R_0<1$, $R_1>1$ *and* $\Delta<0$, *no endemic equilibrium exists*.
(C_4) *If* $R_0<1$ *and* $R_1\leq 1$, *no endemic equilibrium exists*, *where* $I_1^*=\frac{\lambda A\alpha-nd\alpha-(\beta+n)\lambda+\sqrt{\Delta}}{2n\lambda\alpha}$, $I_2^*=\frac{\lambda A\alpha-nd\alpha-(\beta+n)\lambda}{n\lambda\alpha}$, $I_3^*=\frac{\lambda A\alpha-nd\alpha-(\beta+n)\lambda-\sqrt{\Delta}}{2n\lambda\alpha}$, $I_4^*=\frac{\lambda A\alpha-nd\alpha-(\beta+n)\lambda}{2n\lambda\alpha}$, $S_j^*=\frac{A}{d+\lambda I_j^*}$, $\Delta=(-\lambda A\alpha+nd\alpha+(\beta+n)\lambda)^2+4n\lambda\alpha(\lambda A-(\beta+n)d)$, $j=1,2,3,4$.

3 The qualitative results

For the epidemicity of disease is closely related to the stability of the equilibria of mathematical models, we first investigate the stability of the disease free equilibrium E_0. The Jacobian matrix of system (5) at E_0 is

$$M_{E_0}:=\begin{pmatrix}-d & -\frac{\lambda A}{d}\\ 0 & (\beta+n)(R_0-1)\end{pmatrix}.$$

Then, by the Hurwitz criterion (see [8, 11, 18] for the details), we have

Theorem 5 *If $R_0 > 1$, the disease free equilibrium E_0 is unstable, while if $R_0 < 1$ the disease free equilibrium E_0 is asymptotically stable*

Obviously, the matrix M_{E_0} has two real eigenvalues: 0, $-d$ when $R_0 = 1$, i.e. E_0 a non-hyperbolic equilibrium. We study its stability by the center manifold theorem (see [4, 16] for the details). Let $y_1 = S - \frac{A}{d}$, $y_2 = I$, then we have

$$\begin{cases} \frac{dy_1}{dt} = -dy_1 - \frac{\lambda A}{d} y_2 - \lambda y_1 y_2, \\ \frac{dy_2}{dt} = \lambda y_1 y_2 - \frac{\alpha\beta y_2^2}{1+\alpha y_2}. \end{cases} \tag{6}$$

By the following transformation:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & \frac{d}{\beta+n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

system (6) can be transformed into the standard form

$$\begin{cases} \frac{dx_1}{dt} = f_1(x_1, x_2), \\ \frac{dx_2}{dt} = f_2(x_1, x_2), \end{cases} \tag{7}$$

where

$$f_1(x_1, x_2) = -dx_1 - \frac{\lambda(\beta+n)}{d} x_1 x_2 - \frac{\lambda(\beta+n)^2}{d^2} x_2^2,$$

$$f_2(x_1, x_2) = \frac{\lambda(\beta+n)}{d} x_1 x_2 + \frac{(\beta+n)^2(\lambda-\alpha\beta)}{d^2} x_2^2 + \frac{\alpha^2\beta(\beta+n)^3}{d^3(1+\frac{\alpha(\beta+n)}{d} x_2)} x_2^3.$$

By the center manifold theory, there exists a center manifold for system (7), which can be expressed locally by

$$W^c(0) = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 = H(x_2), \|x_2\| < \delta, H(0) = 0, DH(0) = 0, |\delta > 0\}$$

where δ is sufficiently small and DH is the derivative of H with respect to x_2.

To compute the center manifold $W^c(0)$, we assume $H(x_2)$ has the form

$$x_1 = H(x_2) = h_2 x_2^2 + h_3 x_2^3 + h_4 x_2^4 + h_5 x_2^5 + h_6 x_2^6 + \cdots. \tag{8}$$

By the invariance of $W^c(0)$ under the dynamics of (7) yields

$$DH \cdot f_2(x_2, H(x_2)) + dH(x_2) - f_1(x_2, H(x_2)) = 0. \tag{9}$$

Substituting (7) into (9), and then equating coefficients on each power of x_2 to zero yields

$$h_2 = \frac{-\lambda k^2}{d}, \qquad h_3 = \frac{\lambda k^3}{d^2}(-2k\alpha\beta + 2\lambda k + \lambda),$$

$$
\begin{aligned}
h_4 &= \frac{-k^4\lambda}{d^3}((7k+1+6k^2)\lambda^2 - k\alpha\beta(5+12k)\lambda + 2k\alpha^2\beta(3k\beta-d)),\\
h_5 &= \frac{\lambda k^5}{d^4}((24k^3+44k^2+16k+1)\lambda^3 - k\alpha\beta(72k^2+70k+9)\lambda^2\\
&\quad + \alpha^2\beta k(72k^2\beta - 14kd + 26k\beta - 5d)\lambda - 2\alpha^3\beta k(3k\beta-d)(4k\beta-d)),\\
h_6 &= \frac{\lambda k^6}{d^5}(k\alpha\beta(14+237k+480k^3+738k^2)\lambda^3\\
&\quad - (178k^2+30k+1+292k^3+120k^4)\lambda^4\\
&\quad - k\alpha^2\beta(720k^3\beta+600k^2\beta-94k^2d+71k\beta-79kd-9d)\lambda^2\\
&\quad + k\alpha^3\beta(5d^2+480k^3\beta^2-188k^2\beta d+16kd^2-59k\beta d)\lambda\\
&\quad - 2k\alpha^4\beta(5k\beta-d)(3k\beta-d)(4k\beta-d)),
\end{aligned} \tag{10}
$$

where $k = \frac{\beta+n}{d}$.

Then, we get an approximation to H as follows

$$
\begin{aligned}
H &= \frac{-\lambda k^2}{d}x_2^2 + \frac{\lambda k^3}{d^2}(-2k\alpha\beta + 2\lambda k + \lambda)x_2^3 + \frac{-k^4\lambda}{d^3}((7k+1+6k^2)\lambda^2\\
&\quad - k\alpha\beta(5+12k)\lambda + 2k\alpha^2\beta(3k\beta-2d))x_2^4 + \frac{\lambda k^5}{d^4}((24k^3+44k^2+16k+1)\lambda^3\\
&\quad - k\alpha\beta(72k^2+70k+9)\lambda^2 + \alpha^2\beta k(72k^2\beta-14kd+26k\beta-5d)\lambda\\
&\quad - 2\alpha^3\beta k(3k\beta-d)(4k\beta-d))x_2^5 + \frac{\lambda k^6}{d^5}(k\alpha\beta(14+237k+480k^3+738k^2)\lambda^3\\
&\quad - (178k^2+30k+1+292k^3+120k^4)\lambda^4\\
&\quad - k\alpha^2\beta(720k^3\beta+600k^2\beta-94k^2d+71k\beta-79kd-9d)\lambda^2\\
&\quad + k\alpha^3\beta(5d^2+480k^3\beta^2-188k^2\beta d+16kd^2-59k\beta d)\lambda\\
&\quad - 2k\alpha^4\beta(5k\beta-d)(3k\beta-d)(4k\beta-d))x_2^6 + \cdots .
\end{aligned} \tag{11}
$$

Substituting (8) into the second equation of system (9) leads to the vector field reduced to the center manifold

$$
\begin{aligned}
\frac{dz_2}{dt} &= k^2(\lambda-\alpha\beta)x_2^2 + \frac{k^3(\alpha^2\beta d - \lambda^2)}{d}x_2^3 - \frac{k^4(\alpha^3\beta d^2 - 2\lambda^2 k\alpha\beta - 2\lambda^3 k - \lambda^3)}{d^2}x_2^4\\
&\quad - \frac{k^5}{d^3}((k+1)(6k+1)\lambda^4 - \alpha k\beta(12k+5)\lambda^3\\
&\quad + 2\alpha^2 k\beta(3k\beta-d)\lambda^2 - \alpha^4\beta d^3)x_2^5\\
&\quad + \frac{k^6}{d^4}((24k^3+44k^3+16k+1)\lambda^5 - k\alpha\beta(27k^2+70k+9)\lambda^4
\end{aligned}
$$

$$+\alpha^3 k\beta(26k\beta + 72k^2\beta - 14kd - 5d)\lambda^3$$
$$- 2\alpha^3 k\beta(3k\beta - d)(4k\beta - d)\lambda^2 - \alpha^5\beta d^4)x_2^6 + \cdots . \tag{12}$$

Then, by the center manifold theorem, we have the following results about the non-hyperbolic equilibrium E_0.

Theorem 6 *Under the condition of $R_0 = 1$, the disease free equilibrium E_0 is asymptotically stable if $\lambda < \alpha\beta$, and unstable if $\lambda > \alpha\beta$.*

Remark 7 If $R_1 \geq 1$, we have $\lambda < \alpha\beta$. Then, if $R_1 \geq 1$ and $R_0 = 1$, E_0 is aseptically stable.

By the Lyapunov's second method, we have the following result with respect to the globally asymptotically stability of the disease free equilibrium E_0.

Theorem 8 *If $R_0 < 1 - \frac{\beta\alpha A}{d(d+\alpha A)(\beta+n)}$, E_0 is globally asymptotically stable.*

Proof Let $V_2 = \frac{I^2}{2}$, then the total differential of V along the flow of (5) is

$$\left.\frac{dV}{dt}\right|_{(5)} = \left(\lambda S - n - \frac{\beta}{1+\alpha I}\right) I^2.$$

Note that $\mathbb{O}$ is the positively invariant set of system (5), we restrict $(S, I) \in \mathbb{O}$, and which leads to $\lambda S \leq \frac{\lambda A}{d}$. Then, we have

$$\left.\frac{dV}{dt}\right|_{(5)} \leq \left(d(\beta + n)(R_0 - 1) + \frac{\beta\alpha A}{d + \alpha A}\right) I^2.$$

By the Lyapunov-Lasalle theorem [5], E_0 is globally stable if $R_0 < 1 - \frac{\beta\alpha A}{d(d+\alpha A)(\beta+n)}$. □

Next, we study the stability of the endemic equilibrium E_2 under the conditions:

$$R_0 = 1, \qquad R_1 > 1. \tag{13}$$

Clearly, (13) leads to

$$\lambda < \alpha d, \qquad A > \frac{nd}{\lambda}, \qquad \beta = \frac{\lambda A}{d} - n. \tag{14}$$

The Jacobian matrix of system (5) at E_2 has the form

$$M_{E_2} := \begin{pmatrix} -d - \lambda I_2^* & -\lambda S_2^* \\ \lambda I_2^* & \frac{\alpha\beta I_2^*}{(1+\alpha I_2^*)} \end{pmatrix}.$$

It is easy to obtain the characteristic equation of M_{E_2} is

$$a_1\theta^2 + a_2\theta + a_3 = 0, \tag{15}$$

where θ is a complex number and

$$
\begin{aligned}
a_1 &= \alpha nd(\lambda A - nd)(\lambda - \alpha d)^2,\\
a_2 &= b_1 A^2 + b_2 A + b_3,\\
a_3 &= n(\alpha d - \lambda)(-\alpha d\lambda A + \alpha d^2 n + \lambda^2 A)^2,\\
b_1 &= -\lambda^2(\lambda - \alpha d)^3,\\
b_2 &= \lambda nd(\lambda - \alpha d)((\lambda - \alpha d)^2 + \lambda n\alpha),\\
b_3 &= \alpha^2 n^3 \lambda d^3 > 0.
\end{aligned}
$$

By using (13–14), we get

$$a_1 > 0, \qquad a_3 > 0, \qquad b_1 > 0, \qquad b_2 < 0.$$

Then, we obtain

Theorem 9 *E_2 is asymptotically stable if $a_2 > 0$ and unstable if $a_2 < 0$.*

Let $\Delta_1 = b_2^2 - 4b_1b_3$. It is easy to see that equation $a_2 = 0$ has positive root A if and only if $\Delta_1 \geq 0$. Without lose of generality, we suppose $\Delta_1 > 0$. Then, $a_2 = 0$ has two distinct solutions: $A_1 = \frac{-b_2 - \sqrt{\Delta_1}}{2b_1}$, $A_2 = \frac{-b_2 + \sqrt{\Delta_1}}{2b_1}$. Obviously, (15) has a simple pair of pure imaginary roots $\lambda_{1,2} = \pm\omega i$ if $A = A_j$, where $\omega = \sqrt{\frac{a_3}{a_1}} = \sqrt{\frac{(-\alpha d\lambda A + \alpha d^2 n + \lambda^2 A)^2}{\alpha d(\lambda A - nd)(\alpha d - \lambda)}}$, i is the imaginary unit, $j = 1, 2$. It is easy to obtain that $\frac{d\mathfrak{Re}\lambda_{1,2}}{dA}|_{A=A_j} = -\frac{\acute{a}_2}{2a_1}|_{A=A_j}$, where $\acute{a}_2$ denotes the derivative of a_2 with respect to A. Then, we have $\frac{d\mathfrak{Re}\lambda_{1,2}}{dA}|_{A=A_1} > 0$ and $\frac{d\mathfrak{Re}\lambda_{1,2}}{dA}|_{A=A_2} < 0$. By the Hopf bifurcation theorem [4, 16], a Hopf bifurcation occurs at the endemic equilibrium E_2 when A passes through each critical value A_j, $j = 1, 2$.

Next, under the condition of $a_2 = 0$, we verify the stability of the bifurcated periodic orbits which occurs near the positive equilibrium. For the end, we need to compute the index number in the Hopf bifurcation theorem. Let $x_1 = S - S^*$ and $x_2 = I - I_2^*$, system (5) can be transformed into

$$
\begin{cases}
\frac{dx_1}{dt} = -(d + \lambda I_2^*)x_1 - \lambda S_2^* x_2 - \lambda x_1 x_2,\\
\frac{dx_2}{dt} = \lambda I_2^* x_1 + \frac{\alpha\beta I_2^*}{(1+\alpha I_2^*)^2} x_2 + \lambda x_1 x_2 - (\frac{\beta(x_2 + I_2^*)}{1+\alpha(x_2+I_2^*)} - \frac{\beta I_2^*}{1+\alpha I_2^*} - \frac{\beta}{(1+\alpha I_2^*)} x_2).
\end{cases}
\tag{16}
$$

Obviously, M_{E_2} can be re-expressed as

$$
M_{E_2} = \begin{pmatrix} \frac{\lambda A_j n_1}{n_4} & \frac{n_4}{n_1} \\ -\frac{n_3}{n_4} & -\frac{\lambda n n_3}{n_1^2 n_2} \end{pmatrix},
$$

where $n_1 = \lambda - \alpha d < 0$, $n_2 = \lambda A_j - nd > 0$, $n_3 = \alpha d^2 n + \lambda^2 A_j - \alpha d\lambda A_j < 0$, $n_4 = \alpha dn > 0$, $j = 1, 2$.

Let $\lambda_{1,2} = \pm\omega i$ be the pure imaginary roots of M_{E_2} and

$$\begin{pmatrix} x_1(t) \\ z_2(t) \end{pmatrix} = T \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \quad \text{with } T = \begin{pmatrix} \frac{n_4}{n_1} & 0 \\ -\frac{\lambda A n_1}{n_4} & \omega \end{pmatrix},$$

then system (16) becomes

$$\begin{cases} \frac{dy_1}{dt} = \omega y_2 + f_1(y_1, y_2), \\ \frac{dy_2}{dt} = -\omega y_1 + f_2(y_1, y_2), \end{cases}$$

where

$$f_1(y_1, y_2) = -\frac{\lambda n_4}{n_1}\left(-\frac{\lambda A n_1}{n_4} y_1 + \omega y_2\right) y_2,$$

$$f_2(y_1, y_2) = \frac{\lambda n_4}{n_1}\left(-\frac{\lambda A n_1}{n_4} y_1 + \omega y_2\right) y_2 - \left(\frac{\beta((-\frac{\lambda A_j n_1}{n_4} y_1 + \omega y_2) + I_2^*)}{1 + \alpha((-\frac{\lambda A_j n_1}{n_4} y_1 + \omega y_2) + I_2^*)} - \frac{\beta I_2^*}{1 + \alpha I_2^*} - \frac{\beta}{(1 + \alpha I_2^*)^2}\left(-\frac{\lambda A_j n_1}{n_4} y_1 + \omega y_2\right)\right).$$

For convenience, we rewrite f_1, f_2 as f^1, f^2, respectively. Therefore, we can compute the index K in [4] as follows

$$\begin{aligned} K &= \frac{1}{16}\left[f^1_{x_1x_1x_1} + f^1_{x_1x_2x_2} + f^2_{x_1x_1x_2} + f^2_{x_2x_2x_2}\right] + \frac{1}{16\omega}\left[f^1_{x_1x_2}(f^1_{x_1x_1} + f^1_{x_2x_2}) \right. \\ &\quad \left. - f^2_{x_1x_2}(f^2_{x_1x_1} + f^2_{x_2x_2}) - f^1_{x_1x_1} f^2_{x_1x_1} + f^1_{x_2x_2} f^2_{x_2x_2}\right] \\ &= -\frac{3\beta\alpha^2\omega(\lambda^2 A_j^2 n_1^2 + \omega^2 n_4^2)}{8(1 + \alpha I_2^*)^4 n_4^2} - \frac{1}{16\omega}\left[-\frac{2\lambda^3\omega A_j n_1 n_4}{n_2^2} \right. \\ &\quad + \frac{2\lambda^2 A_j n_1}{n_2}\left(\frac{2\beta\lambda^2 A_j^2 n_1^2\alpha}{n_4^2(1 + \alpha I_2^*)^2} - \frac{2\lambda^2 A_j n_1}{n_2} - \frac{2\beta I_2^*\alpha^2\lambda^2 A_j^2 n_1^2}{(1 + \alpha I_2^*)^3 m_4^2}\right) \\ &\quad + \left(\frac{\lambda n_4\omega}{n_2} - \frac{2\beta\lambda A_j n_1\alpha\omega}{n_4(1 + \alpha I_2^*)} + \frac{2\beta I_2^*\alpha^2\lambda A_j n_1\omega}{(1 + \alpha I_2^*)^3 n_4}\right) \\ &\quad \times \left(\frac{2\beta\lambda^2 A_j^2 n_1^2\alpha}{n_4^2(1 + \alpha I_2^*)^2} - \frac{2\lambda^2 A_j n_1}{n_2} - \frac{2\beta I_2^*\alpha^2\lambda^2 A_j^2 n_1^2}{(1 + \alpha I_2^*)^3 n_4^2} \right. \\ &\quad \left.\left. + \frac{2\beta\omega^2\alpha}{(1 + \alpha I_2^*)^2 - \frac{2\beta I_2^*\alpha^2\omega^2}{(1+\alpha I_2^*)^3}}\right)\right]. \end{aligned}$$

By using the Hopf bifurcation theorem, we obtain the following results about the periodic solutions bifurcated at E_2.

Theorem 10 *If $K < 0$, the periodic solution is stable, while if $K > 0$, the periodic solution is unstable. The case $K < 0$ is referred to as a supercritical bifurcation, and the case $K > 0$ is referred to as a subcritical bifurcation.*

In the following, the dynamic property of E_4 is focused on under the conditions:

$$R_0 < 1, \qquad R_1 > 1, \qquad \Delta = 0. \tag{17}$$

The Jacobian matrix of system (5) at E_4 is

$$M_{E_4} := \begin{pmatrix} -d - \lambda I_4^* & -\lambda S_4^* \\ \lambda I_4^* & \frac{\alpha\beta I_4^*}{(1+\alpha I_4^*)^2} \end{pmatrix}.$$

By using the relationship $(I_4^*)^2 = \frac{(n(\alpha d+\lambda)+\lambda\beta-\lambda A\alpha)I_4^*+(\beta+n)d-\lambda A}{\lambda n\alpha}$, we get the characteristic equation of M_{E_4} is as follows

$$\theta^2 + p\theta + q = 0, \tag{18}$$

where

$$\begin{aligned} p = & -\frac{1}{(1+\alpha I_4^*)^2 n^2}\big[(-\lambda n A\alpha - \lambda\beta^2 + 2\lambda\beta A\alpha + A\alpha^2 nd - \lambda A^2\alpha^2 + n^2\alpha\beta)I_4^* \\ & + (-\lambda A^2\alpha + \lambda\beta A - \beta^2 d + d\alpha An - \lambda An + A\alpha d\beta)\big], \\ q = & \frac{I_4^*}{(1+\alpha I_4^*)^2}\big[(\alpha(n(d\alpha+\lambda) + \lambda\beta - \lambda A\alpha) - \alpha\beta\lambda + \lambda\alpha^2 + 2\lambda nd)I_4^* \\ & + \alpha((\beta+n)d - \lambda A) + \lambda(n+\alpha) - \alpha\beta d\big] \end{aligned} \tag{19}$$

and θ is a complex number.

Noticing $R_1 > 1$, we have $A > \frac{nd\alpha+(\beta+n)\lambda}{\alpha\lambda}$. For convenience, let $A = \frac{nd\alpha+(\beta+n)\lambda}{\alpha\lambda} + \epsilon$, where $\epsilon > 0$. By means of $\Delta = 0$, we obtain

$$\begin{aligned} p &= \frac{(\alpha\epsilon+2n)^3(\alpha\epsilon+2(\beta+n))d - 2\epsilon\alpha\beta n((\alpha\epsilon+2n)^2+4\beta n)}{4n^2(1+\alpha I_4^*)^2(\alpha^2\epsilon+4\beta n+4n^2+4n\alpha\epsilon)}, \\ q &= \frac{2\alpha\beta d I_4^*(\alpha-\beta)(d\epsilon+2n)}{(1+\alpha I_4^*)^2(4\beta n+(\alpha\epsilon+2n)^2+4n\beta)}, \\ \lambda &= \frac{4nd\alpha\beta}{\alpha^2\epsilon^2+4n\beta+4n^2+4n\epsilon\alpha}, \qquad I_4^* = \frac{\epsilon}{2n}. \end{aligned}$$

Let $d^* = \frac{2\epsilon\alpha\beta n((\alpha\epsilon+2n)^2+4\beta n)}{(\alpha\epsilon+2n)^3(\alpha\epsilon+2(\beta+n))}$. Then, under the condition of (17), we arrive at

Theorem 11

(1) *If* $\alpha > \beta$ *and* $d > d^*$, E_4 *is an asymptotically stable node or focus.*
(2) *If* $\alpha > \beta$ *and* $d < d^*$, E_4 *is an unstable node or focus.*
(3) *If* $\alpha = \beta$, *or* $d = d^*$, $\alpha > \beta$, E_4 *is a non-hyperbolic equilibrium.*
(4) *If* $\alpha < \beta$, E_4 *is a saddle.*

Obviously, (18) has a negative root $-p$ except 0 if

$$\alpha = \beta, \qquad d > d^*. \tag{20}$$

In the sequel, we discuss the stability of E_4 under the assumptions of (20). For the purpose, we first transfer E_4 to the origin by $x_1 = S - S_4^*, x_2 = I - I_4^*$. Therefore, we get

$$\begin{cases} \frac{dx_1}{dt} = -\frac{d(2(\beta+n)+\epsilon\beta)(2n+\beta\epsilon)}{(\beta\epsilon+2n)^2+4n\epsilon\beta}x_1 - \frac{n(2(\beta+n)+\beta\epsilon)}{2n+\beta\epsilon}x_2 + f_1(x_1,x_2), \\ \frac{dx_2}{dt} = \frac{2d\beta^2\epsilon}{(\beta\epsilon+2n)^2+4n\beta}x_1 + \frac{2\beta\epsilon n}{(2n+\epsilon\beta)^2}x_2 + f_2(x_1,x_2), \end{cases} \tag{21}$$

where $f_1 = \overline{a}_1 x_1 x_2,\ f_2 = -\overline{a}_1 x_1 x_2 + b_2 x_2^2 + b_3 x_2^3 + b_4 x_2^4 + b_5 x_2^5 + O(x_2^6)$, $\overline{a}_1 = -\frac{4nd\beta^2}{(\beta\epsilon+2n)^2+4n\beta}$, $\overline{k} = \frac{2\beta n}{2n+\beta\epsilon}$, $b_j = (-1)^j \frac{\overline{k}^j}{\beta}$, $j = 2, 3, 4, 5$.

Secondly, we define

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = S \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix},$$

where $s_{11} = -n((\beta\epsilon + 2n)^2 + 4\beta n)$, $s_{12} = -((\beta\epsilon + 2n)^2 + 2\beta(\beta\epsilon + 2n))$, $s_{21} = d(2n + \beta\epsilon)$, $s_{22} = 2\epsilon\beta^2$.

It isn't difficult to get the inverse of S as follows

$$S^{-1} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix},$$

where $t_{11} = \frac{2\epsilon\beta^2}{\det(S)}$, $t_{12} = \frac{(2n+\epsilon\beta)(2(n+\beta)+\epsilon\beta)}{\det(S)}$, $t_{21} = \frac{-d(2n+\epsilon\beta)^2}{\det(S)}$, $t_{22} = \frac{-n(4n^2+4n\epsilon\beta+\epsilon^2\beta^2+4n\beta)}{\det(S)}$.

Corresponding, system (21) becomes

$$\begin{pmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -p \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} F_1(y_1,y_2) \\ F_2(y_1,y_2) \end{pmatrix}, \tag{22}$$

where

$$\begin{aligned} F_1 = {} & \overline{a}_1 t_{11}(s_{11}y_1 + s_{12}y_2)(s_{21}y_1 + s_{22}y_2) + t_{12}(-\overline{a}_1(s_{21}y_1 + s_{22}y_2)^2 \\ & + b_2(s_{21}y_1 + s_{22}y_2)^2 + b_3(s_{21}y_1 + s_{22}y_2)^3 + b_4(s_{21}y_1 + s_{22}y_2)^4 \\ & + b_4(s_{21}y_1 + s_{22}y_2)^5) + O(\|y\|^6), \end{aligned}$$

$$\begin{aligned}F_2 = {} & \overline{a}_1 t_{21}(s_{11}y_1 + s_{12}y_2)(s_{21}y_1 + s_{22}y_2) + t_{22}(-\overline{a}_1(s_{21}y_1 + s_{22}y_2)^2 \\ & + b_2(s_{21}y_1 + s_{22}y_2)^2 + b_3(s_{21}y_1 + s_{22}y_2)^3 + b_4(s_{21}y_1 + s_{22}y_2)^4 \\ & + b_4(s_{21}y_1 + s_{22}y_2)^5) + O(\|y\|^6), \\ y = {} & (y_1, y_2)^T \in \mathbb{R}^2.\end{aligned}$$

By the existence theorem in the center manifold theory, there exists a center manifold for system (24), which can be expressed locally as follows

$$W^c(0) = \{(y_1, y_2) \in \mathbb{R}^2 | y_2 = \overline{H}(y_1), \|y_1\| < \delta, \overline{H}(0) = 0, D\overline{H}(0) = 0, |\delta > 0\}$$

where δ is sufficiently small and $D\overline{H}$ is the derivative of $\overline{H}$ with respect to y_1.

Obviously, the first task is to compute the center manifold $W^c(0)$. For the purpose, let $\overline{H}(y_1)$ be

$$y_2 = \overline{H}(y_1) = \overline{h}_2 y_1^2 + \overline{h}_3 y_1^3 + \overline{h}_4 y_1^4 + \cdots . \tag{23}$$

By the invariance of $W^c(0)$ under the dynamics of (24), the center manifold satisfies

$$Dh \cdot F_1(y_1, \overline{H}(y_1)) + p\overline{H}(y_1) - F_2(y_1, \overline{H}(y_1)) = 0. \tag{24}$$

Balance equation (24) with respect to x_1 yields

$$\begin{aligned}\overline{h}_2 &= \frac{s_{21}}{\det(S)}(t_{22}b_2 s_{21} - t_{22}\overline{a}_1 s_{21} + t_{21}\overline{a}_1 s_{11}), \\ \overline{h}_3 &= \frac{s_{21}}{p^2}(c_1 s_{21}^3 + c_2 s_{21}^2 + c_3 s_{21} + c_4)\end{aligned}$$

with

$$\begin{aligned}c_1 &= -2t_{22}t_{11}(\overline{a}_1 - b_2)^2, c_4 = -s_{22}s_{21}^2\overline{a}_1^2 t_{21}^2, \\ c_3 &= -t_{21}\overline{a}_1 s_{11}(2\overline{a}_1 t_{11}s_{11} - \overline{a}_1 s_{21}t_{21} + 3\overline{a}_1 s_{22}t_{22} - 3s_{22}t_{22}b_2), \\ c_2 &= 2(\overline{a}_1 - b_2)^2 s_{22}t_{22}^2 + t_{22}(b_3 p - 2b_2\overline{a}_1 s_{11}t_{11} - \overline{a}_1^2 s_{12}t_{12} + 2\overline{a}_1^2 s_{11}t_{11} + b_2 s_{12}s_{21}) \\ & \quad + 2\overline{a}_1 t_{21}t_{12}(\overline{a}_1 - b_2).\end{aligned}$$

Substituting (23) into the first equation of system (24) leads to

$$\frac{dy_1}{dt} = d_2 y_1^2 + d_3 y_1^3 + O(y_1^4)$$

with

$$\begin{aligned}d_2 &= \frac{s_{21}}{\det(S)}(s_{21}(-(\det(S)t_{12})\overline{a}_1 s_{21} + (\det(S)t_{12})b_2 s_{21} + (\det(S)t_{11})\overline{a}_1 s_{11})), \\ d_3 &= \frac{s_{21}}{p}((-2t_{12}b_2^2 t_{22} + 4t_{12}b_2 t_{22}\overline{a}_1 - 2t_{12}\overline{a}_1^2 t_{22})s_{21}^3 \\ & \quad + (-4t_{22}^2\overline{a}_1 s_{22}b_2 - 2t_{11}\overline{a}_1 s_{11}t_{22}b_2 + 2t_{11}\overline{a}_1^2 s_{11}t_{22} + t_{22}b_3 p + t_{21}\overline{a}_1 s_{12}t_{22}b_2\end{aligned}$$

$$- t_{21}\overline{a}_1^2 s_{12} t_{22} + 2t_{22}^2\overline{a}_1^2 s_{22} + 2t_{22}^2 b_2^2 s_{22} - 2t_{12} b_2 t_{21}\overline{a}_1 s_{11} + 2t_{12}\overline{a}_1^2 t_{21} s_{11}) s_{21}^2$$
$$+ (-2t_{11}\overline{a}_1^2 s_{11}^2 t_{21} - 3t_{21}\overline{a}_1^2 s_{11} s_{22} t_{22} + t_{21}^2\overline{a}_1^2 s_{12} s_{11}$$
$$+ 3t_{21}\overline{a}_1 s_{11} s_{22} t_{22} b_2) s_{21} + s_{22} s_{11}^2 \overline{a}_1^2 t_{21}^2).$$

Noticing $\overline{a}_1 < 0, b_2 > 0, s_{21} > 0, s_{11} < 0, \det(S) t_{12} > 0, H t_{11} > 0$, we have $d_2 \det(S) > 0$. Then, by the center manifold theorem, we arrive at

Theorem 12 *Under the conditions of* (17) *and* (20), *the endemic equilibrium* E_4 *is asymptotically stable if* $\det(S) < 0$, *and unstable if* $\det(S) > 0$.

Suppose both E_1 and E_3 are hyperbolic equilibria. In the following, under the conditions of

$$R_0 < 1, \qquad R_1 > 1, \qquad \Delta > 0 \tag{25}$$

we investigate the stability of E_1 and E_3 by the technique presented in [1, 12] related to the Poincare index (see [13, 18] for the detail).

By Theorem 3.1, the disease free equilibrium E_0 is asymptotically stale if $R_0 < 1$. Suppose Ω is a small neighborhood of E_0, and which lies in the domain of attraction of E_0. Let $\mathbb{U} = \mathbb{O} \bigcup \overline{\Omega}$, where Ω is the closure of Ω. Then $\mathbb{U}$ is a positively invariant set of system (5). Let $\mathfrak{L}$ be the boundary of $\mathbb{U}$. It follows from the Theorem 2.1 of [12] that the Poincare index of $\mathfrak{L}$ related to the vector field F is 1, i.e.

$$a - s + r = 1, \tag{26}$$

where F represents the right hand sides of the equations in system (5), and a, s and r are the number of attractors (nodes), saddles and repellers (sources), respectively (see [13] for the detail). Since there are three hyperbolic equilibria in $\mathbb{U}$, we have

$$a + s + r \leq 3. \tag{27}$$

Combining (26) with (27), we get $s \leq 1$.

If $s = 0$, (26) leads to $r = 0$, and further system (5) only have the disease free equilibrium E_0. Obviously, it is impossible if the condition (17) holds.

If $s = 1$, (26) leads to $a + r = 2$. Then, by using Theorem 3.1 and (27), we have that $a \geq 1$, and further have $r \leq 1$. Clearly, $a = 2$ if $r = 0$, while $a = 1$ if $r = 1$. The first case implies the disease free equilibrium E_0 and one of the two endemic equilibria are asymptotically stable, and the other endemic equilibrium is unstable, i.e., system (5) exhibits bi-stability. The second case means that both of the two endemic equilibria are unstable. Summarizing the above discussion yields

Theorem 13 *Under the conditions of* (25), *either one of the two endemic equilibria* (*i.e.* E_1, E_3) *is asymptotically stable and the other endemic equilibrium is unstable or both of them are unstable.*

Finally, we discuss the stability of the endemic equilibrium E_1 under the condition of $R_0 > 1$. Clearly, the Jacobian matrix of system (5) at E_1 has the form

$$M_{E_1} := \begin{pmatrix} -d - \lambda I_1^* & -\frac{n+n\alpha I_1^*+\beta}{1+\alpha I_1^*} \\ \lambda I_1^* & \frac{\alpha\beta I_1^*}{(1+\alpha I_1^*)^2} \end{pmatrix}.$$

Let $p = d + \lambda I_1^* - \frac{\alpha\beta I_1^*}{(1+\alpha I_1^*)^2}$, $q = \frac{\lambda^2 A}{d+\lambda I_1^*} q_1(I_1^*)$, where $q_1(I_1^*) = 1 - \frac{\alpha\beta(d+\lambda I_1^*)^2}{\lambda^2 A(1+\alpha I_1^*)^2}$. Clearly, q_1 is decrease with respect to I_1^* when $\lambda > \alpha d$ and increase when $\lambda < \alpha d$. Note that $p \geq d - \frac{\beta}{4} + \lambda I_1^*$ and $q_1(0) = 1 - \frac{\alpha\beta d^2}{\lambda^2 A}$, then, under the conditions of $R_0 > 1$ and $d > \frac{\beta}{4}$, we have

Theorem 14 *If one of the following is satisfied*: (1) *I* $\lambda \geq \alpha d$ *and* $A \leq \frac{\beta}{\alpha}$, (2) $\lambda < \alpha d$ *and* $A \leq \frac{(\beta+n)^2}{\alpha\beta}$, *then* E_1 *is asymptotically stable.*

Proof (1) By using $R_0 > 1$, we get $q_1(0) > \frac{\lambda d(\beta+n)-\alpha\beta d^2}{\lambda A}$. If $\lambda \geq \alpha d$, then $q_1(0) \geq \frac{nd^2\alpha}{\lambda^2 A} > 0$. Then $q > 0$ if $\lim_{I_1^*\to\infty} q_1(I_1^*) = 1 - \frac{\beta}{A\alpha} > 0$. So, E_1 is asymptotically stable if $\lambda \geq \alpha d$ and $A \leq \frac{\beta}{\alpha}$.

(2) Obviously, $R_0 > 1$ leads to $d < \frac{\lambda A}{\beta+n}$. Then, we have $q_1(0)$ yields $q_1(0) > 1 - \frac{A\alpha\beta}{(\beta+n)^2}$ and further have $q > 0$ if $A \leq \frac{(\beta+n)^2}{\alpha\beta}$. □

By Bendixion-Poincaré theorem, it isn't difficult to obtain

Theorem 15 *If* $p > 0$ *and* $q > 0$, *and further* $\lambda < \frac{d^3}{\alpha A^2}$, *the endemic equilibrium* E_1 *is globally asymptotically stable*, *while if* $p < 0$ *and* $q > 0$ *it is unstable and there at least exists one periodic orbit.*

4 Numerical simulation

In this section, to understand our results more intuitively, some numerical simulations are carried out. Six figures are employed to exhibit the dynamic behaviors of system (5), such as the stability of the endemic equilibria, the existences of the periodic orbit and the bistable equilibria, respectively.

Firstly, let $R_0 > 1$. Then there exist a unstable disease free equilibrium E_0 and a unique endemic equilibrium E_1. Unlike the traditional epidemic model, the stability of the unique endemic equilibrium E_1 is not invariant. It can be asymptotically stable and also can be unstable. In Fig. 1, E_1 is asymptotically stable. However, in Fig. 2, it is unstable and at least one periodic orbit occurs.

Secondly, let $R_0 = 1$. Then there exists a unique endemic equilibrium E_2 if $R_1 > 1$, and its stability depends on the parameters which is sketched in Figs. 3–4, respectively.

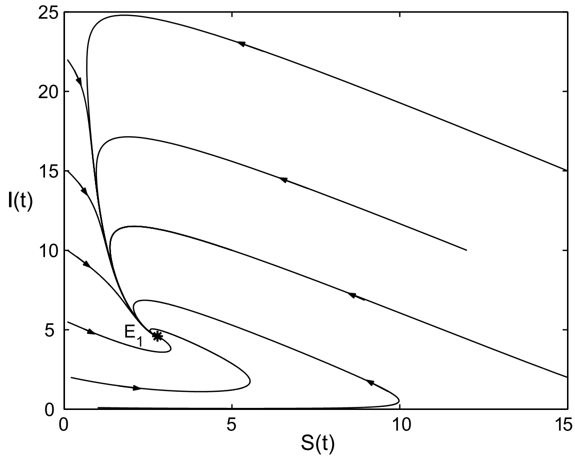

Fig. 1 $R_0 > 1$, $A = 7.65$, $d = 0.5$, $\lambda = 0.5$, $n = 0.75$, $\beta = 2$, $\alpha = 0.5$. E_1 is asymptotically stable

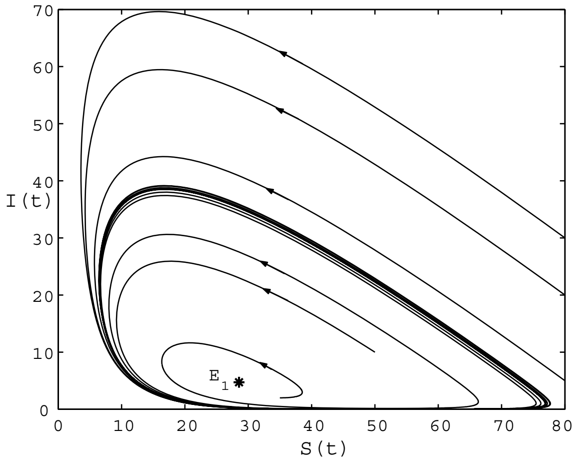

Fig. 2 $R_0 > 1$, $A = 76.5$, $d = 0.5$, $\lambda = 0.5$, $n = 7.5$, $\beta = 20$, $\alpha = 0.5$. E_1 is unstable, and there at least exists a periodical orbit in the first quadrant of (S, I) plane

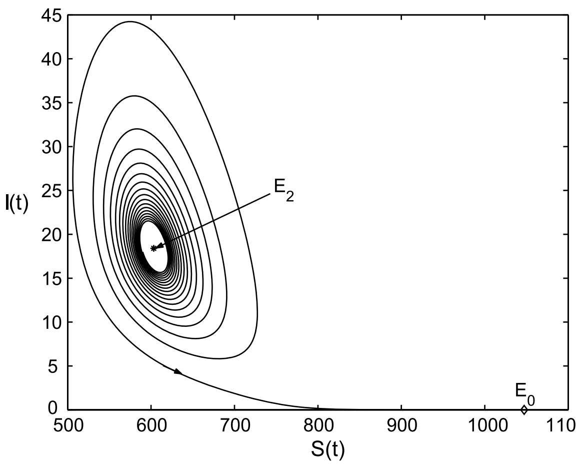

Fig. 3 $R_0 = 1$, $R_1 > 1$, $A = 523.6304348$, $d = 0.5$, $\lambda = 0.02$, $n = 11.1$, $\beta = 9.8452174$, $\alpha = 0.5$. E_2 is unstable, and the trajectories around it approach to the disease free equilibrium E_0

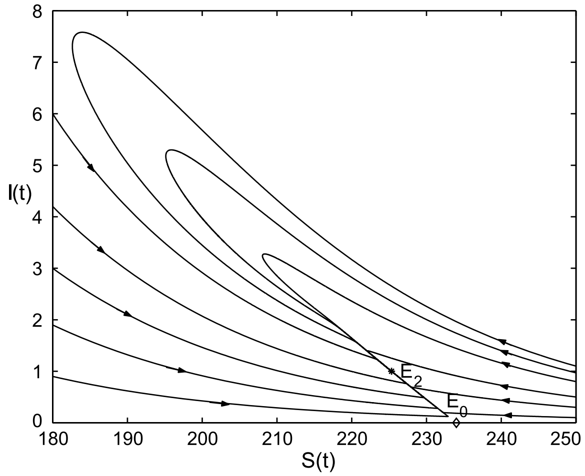

Fig. 4 $R_0 = 1$, $R_1 > 1$, $A = 1.17$, $d = 0.005$, $\lambda = 0.0001923076923$, $n = 0.01$, $\beta = 0.035$, $\alpha = 0.05$. E_2 is asymptotically stable

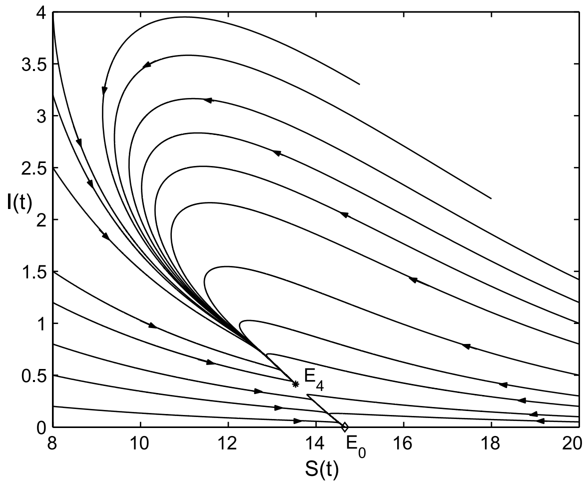

Fig. 5 $R_0 < 1$, $R_1 > 1$, $A = 14.65685425$, $d = 1$, $\lambda = 0.2$, $n = 2$, $\beta = 1$, $\alpha = 0.35$, $\Delta = 0$. Both the unique endemic equilibrium E_4 and the disease free equilibrium E_0 are asymptotically stable

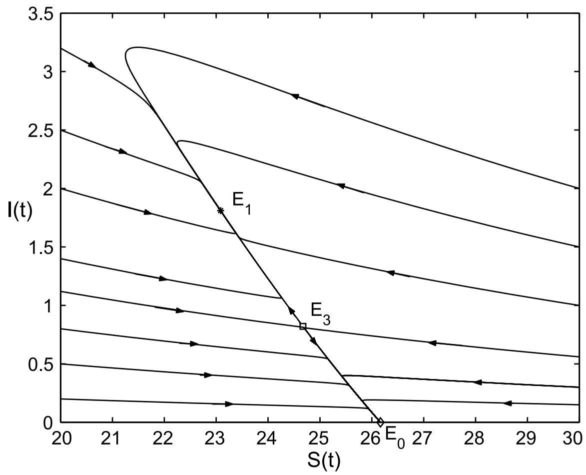

Fig. 6 $R_0 < 1$, $R_1 > 1$, $A = 392.5043354$, $d = 15$, $\lambda = 1.10624999$, $n = 15.2$, $\beta = 14.08052805$, $\alpha = 0.2$, $\Delta > 0$. Their exist two endemic equilibria E_1 and E_3 except the disease free equilibrium E_0. It is easy to see that E_1 and E_0 are bistable equilibria, while E_3 is a saddle

Finally, let $R_0 < 1$ and $R_1 > 1$. By Theorem 2.1, the existence of the endemic equilibrium depends on the sign of Δ, i.e. system (5) has two endemic equilibria E_1, E_3 if $\Delta > 0$ and a unique endemic equilibrium E_4 if $\Delta = 0$. The dynamic behaviors of the endemic equilibria are showed in Figs. 5–6, respectively.

5 Conclusions

In the present paper, considering the perpetual immunity of the removed individuals and the saturated treatment rate, a SIR epidemic model is formulated, and which characterizes the effect of limited treatment capacity on the spread of epidemic disease. Unlike the traditional SIR model, multiple endemic equilibria co-exist in system (5), and it has bistable equilibria. Therefore, it is not always effective to reduce the basic reproduction number R_0 to the value less than the unity in the control and elimination of epidemic disease. In the same time, we have to control each sub-population to a given domain, and in which the disease free equilibrium E_0 is asymptotically stable.

By using the Hopf bifurcation theorem, Lyapunov-Lasalle theorem, Poincaré index, Poincaré-Bendixson, center manifold theorem and fixed point theorem, we analyze the existence and stability of endemic equilibria, the existence and direction of Hopf bifurcation. Finally, numerical simulation are presented, and which intuitively show the existence of bistable equilibria and the stability of each equilibrium.

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