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Existence results for second-order system with impulse effects via variational methods

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Abstract In this paper, we investigate the positive solutions of a second-order system with impulse effects. By using critical point theory the existence result of positive solutions is obtained.

Keywords Second-order system · Impulsive effects · Variational methods · Mountain Pass Theorem

Mathematics Subject Classification (2000) 34B15 · 34B37 · 58E30

1 Introduction

In recent years, a great deal of work has been done in the study of the second-order boundary value problems with impulses, by which a number of physical, biological phenomena are described, please refer to [1, 4–8].

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There are two most common techniques to study boundary value problem with impulse: fixed point theorems in cones [3] are applied to get interesting results, see for example, [1, 6, 7]; The method of lower and upper solutions with monotone iterative technique has been used, see [4, 8].

Lin and Jiang [7] have studied the problem

$$\begin{cases} -x'' = f(t,x), & t \neq t_k,\ t \in [0,T], \\ -\Delta x'|_{t=t_k} = I_k(x(t_k)), & k = 1,2,\ldots,m, \\ x(0) = x(1) = 0. \end{cases} \tag{1.1}$$

By using fixed point index in cones, the existence of positive solutions is obtained.

On the other hand, many researchers have used variational methods to study the existence of solutions for boundary value problems (or systems), please refer to [2, 9–12, 14, 15]. Periodic problems and Dirichlet boundary value problems without impulses have been studied by critical point theorems.

However, to our best knowledge, few authors have studied the existence of positive solutions for second-order system with impulse effects by using variational methods. As a result the goal of this paper is to fill the gap in this area.

We, in this paper, study the existence of positive solutions for the following second-order system with impulse effects

$$\begin{cases} \frac{d}{dt}(\Phi_p(\dot{x}(t))) + \nabla F(t,x(t)) = 0, & t \in [0,T], \\ -\Delta\Phi_p(\dot{x}(t_i)) = \nabla I_i(x(t_i)), & i = 1,2,\ldots,l, \\ x(0) = x(T) = 0, \end{cases} \tag{1.2}$$

where $T > 0, x \in R^N, p > 1,\ 0 = t_0 < t_1 < \cdots < t_l < t_{l+1} = T,\ \Delta\Phi_p(\dot{x}(t_i)) = \Phi_p(\dot{x}(t_i^+)) - \Phi_p(\dot{x}(t_i^-))$, where $\dot{x}(t_i^+)$(respectively $\dot{x}(t_i^-)$) denotes the right limit (respectively left limit) of $\dot{x}(t)$ at $t = t_i$.

$$\Phi_p(x) = |x|^{p-2}x = \left(\sqrt{\sum_{i=1}^N x_i^2}\right)^{p-2} \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}.$$

Throughout this paper, we assume that $\nabla F(t,x) = \frac{\partial}{\partial x}F(t,x)$, $\nabla I_i(x) = (\frac{\partial I_i}{\partial x_1}, \ldots, \frac{\partial I_i}{\partial x_N})$, $\nabla F(t,x) : [0,T] \times (R^+)^N \to (R^+)^N$, $R^+ = [0,+\infty)$, $\nabla F(t,0) \not\equiv 0$, $\nabla I_i \in C((R^+)^N, (R^+)^N)$. Obviously, for $N = 1, p = 2, T = 1$, system (1.2) reduced to (1.1).

Our aim of this paper is to apply critical point theory to system (1.2) and prove the existence of at least one positive solution. The character of this paper is as follows: First, the problem that we deal with is a system with a p-Laplacian, which is different from those in the literature, [1, 5–8]; Secondly, we transfer the solutions of (1.2) into the critical points of some functional, which is called the variational framework. By applying critical point theory to the functional, we prove the existence of positive solutions for system (1.2).

In this paper, we assume that the following conditions hold:

(C1) $|\nabla F(t,x)| = \circ(|x|^{p-1}), |\nabla I_i(x)| = \circ(|x|^{p-1})$ as $|x| \to 0$ uniformly in $t \in [0,T], i = 1,2,\dots,l$;

(C2) there exist $\mu > p$, such that

$$0 < \mu F(t,x) \le (\nabla F(t,x),x), \qquad \mu I_i(x) \le (\nabla I_i(x),x),$$
$$i = 1,2,\dots,l,\ (t,x) \in [0,T] \times (R^+)^N.$$

For the remainder of this section, we present Mountain Pass Theorem which will be needed in Sects. 2 and 3.

Lemma 1.1 [3] *Let E be a Banach space and $\varphi \in C^1(E,R)$ satisfy Palais-Smale condition. Assume there exist $x_0, x_1 \in E$, and a bounded open neighborhood Ω of x_0 such that $x_1 \setminus \overline{\Omega}$ and*

$$\max\{\varphi(x_0),\varphi(x_1)\} < \inf_{x\in\partial\Omega} \varphi(x).$$

Let

$$\Gamma = \{h \mid h : [0,1] \to E \text{ is continuous and } h(0) = x_0, h(1) = x_1\}$$

and

$$c = \inf_{h\in\Gamma} \max_{s\in[0,1]} \varphi(h(s)).$$

Then c is a critical value of φ, that is, there exists $x^* \in E$ such that $\varphi'(x^*) = \Theta$ and $\varphi(x^*) = c$, where $c > \max\{\varphi(x_0),\varphi(x_1)\}$.

2 Related lemmas

To begin with, we introduce some notations. Here, and in the sequel, we assume that, $|x| = \sqrt{\sum_{i=1}^N x_i^2}$, $x = (x_1,x_2,\dots,x_N)$, $(x,y) = \sum_{i=1}^N x_i y_i$, $x, y \in R^N$. The function $x(t) \ge 0$ denotes $x_i(t) \ge 0$, $t \in [0,T]$, $i = 1,\dots,N$.

Define the space

$$X = \{x \in W^{1,p}([0,T]) : x(0) = x(T) = 0\}$$

equipped with the norm $\|x\|_X = (\int_0^T |\dot{x}(t)|^p dt)^{\frac{1}{p}}$, which is clearly equivalent to the usual one. Obviously, X is a reflexive Banach space. We denote $\|x\|_\infty := \max_{x\in[0,T]} |x(t)|$ to be the norm in $C^0([0,T])$. Moreover, $\|x\|_{L^r}$ stands for the norm in $L^r([0,T])$, $r \in [1,+\infty]$. Clearly, there exists a positive constant K satisfying

$$\|x\|_\infty \le K\|x\|_X \quad \text{for } x \in X. \tag{2.1}$$

Definition 2.1 A function $x \in X$ is said to be a classical solution of system (1.2) if x satisfies equation in (1.2) for all $t \in [0,T] \backslash \{t_1,\dots,t_l\}$ and impulsive condition and boundary condition of (1.2). Moreover, x is said to be a positive classical solution of system (1.2) if $x(t) \ge 0$, $x(t) \not\equiv 0$, $t \in [0,T]$.

Lemma 2.1 *For $x \in X$. Let $x^{\pm} = \max\{\pm x, 0\}$, then the following six properties hold*:

(i) $x \in X \Rightarrow x^+, x^- \in X$;
(ii) $x = x^+ - x^-$;
(iii) $\|x^+\|_X \le \|x\|_X$;
(iv) *if* (x_n) *uniformly converges to* x *in* $C([0,T])$, *then* (x_n^+) *uniformly converges to* x^+ *in* $C([0,T])$;
(v) $(x^+(t), x^-(t)) = 0$, $((\dot{x^+})(t), (\dot{x^-})(t)) = 0$ *for* $t \in [0,T]$;
(vi) $(\Phi_p(x), x^+) = |x^+|^p$, $(\Phi_p(x), x^-) = -|x^-|^p$.

Lemma 2.2 *If $x \in C([0,T])$ is a classical solution of system*

$$\begin{cases} \frac{d}{dt}(\Phi_p(\dot{x}(t))) + \nabla F(t, x^+(t)) = 0, & t \ne t_i,\ t \in [0,T], \\ -\Delta\Phi_p(\dot{x}(t_i)) = \nabla I_i(x^+(t_i)), & i = 1, 2, \dots, l, \\ x(0) = x(T) = 0, \end{cases} \tag{2.2}$$

then $x(t) \ge 0$, $x(t) \not\equiv 0, t \in [0,T]$ and hence it is a positive classical solution of system (1.2).

Proof If $x \in C([0,T])$ is a classical solution of system (2.2), by Lemma 2.1 we have

$$\begin{aligned} 0 &= \int_0^T \left(\frac{d}{dt}(\Phi_p(\dot{x}(t))) + \nabla F(t, x^+(t)), x^-(t)\right)dt \\ &= \sum_{i=0}^{l}(\Phi_p(\dot{x}(t)), x^-(t))|_{t=t_i}^{t_{i+1}} - \int_0^T (\Phi_p(\dot{x}(t)), (\dot{x^-})(t))dt \\ &\quad + \int_0^T (\nabla F(t, x^+(t)), x^-(t))dt \\ &= -\sum_{i=1}^{l}(\Delta\Phi_p(\dot{x}(t_i)), x(t_i)) - \int_0^T (\Phi_p(\dot{x}(t)), (\dot{x^-})(t))dt \\ &\quad + \int_0^T (\nabla F(t, x^+(t)), x^-(t))dt \\ &\ge -\sum_{i=1}^{l}(\Delta\Phi_p(\dot{x}(t_i)), x^-(t_i)) + \int_0^T |(\dot{x^-})(t)|^p dt \\ &= \sum_{i=1}^{l}(\nabla I_i(x^+(t_i)), x^-(t_i)) + \int_0^T |(\dot{x^-})(t)|^p dt \ge 0, \end{aligned} \tag{2.3}$$

so $x^-(t) = 0$ for $t \in [0,T]$, that is $x(t) \ge 0$ for $t \in [0,T]$. If $x(t) \equiv 0$ for $t \in [0,T]$, the fact $\nabla F(t,0) \not\equiv 0$ for $t \in [0,T]$ gives a contradiction. □

Remark 2.1 By Lemma 2.2, in order to find the positive classical solutions of system (1.2) it suffices to get classical solutions of (2.2).

For each $x \in X$, put

$$\varphi(x) := \int_0^T \left[\frac{1}{p} |\dot{x}(t)|^p - (F(t, x^+(t)) - (\nabla F(t, 0), x^-(t))) \right] dt - \sum_{i=1}^{l} [I_i(x(t_i)) - (\nabla I_i(0), x^-(t_i))]. \tag{2.4}$$

Clearly, φ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $x \in X$ is the functional $\varphi'(x) \in X^*$, given by

$$\langle \varphi'(x), v \rangle = \int_0^T [(\Phi_p(\dot{x}(t)), \dot{v}(t)) - (\nabla F(t, x^+(t)), v(t))] dt - \sum_{i=1}^{l} (\nabla I_i(x^+(t_i)), v(t_i)) \tag{2.5}$$

for every $v \in X$.

Lemma 2.3 *If the function $x \in X$ is a critical point of the function φ, then $x \in X$ is a solution of system* (2.2).

Proof If $x \in X$ is critical point of the function φ, then $\langle \varphi'(x), v \rangle = 0$ for any $v \in X$. By integration by parts, one has

$$\begin{aligned}
\langle \varphi'(x), v \rangle &= \int_0^T [(\Phi_p(\dot{x}(t)), \dot{v}(t)) - (\nabla F(t, x^+(t)), v(t))] dt \\
&\quad - \sum_{i=1}^{l} (\nabla I_i(x^+(t_i)), v(t_i)) \\
&= \sum_{i=0}^{l} (\Phi_p(\dot{x}(t)), v(t))|_{t=t_i}^{t_{i+1}} \\
&\quad - \int_0^T \left(\frac{d}{dt}(\Phi_p(\dot{x}(t))) + \nabla F(t, x^+(t)), v(t) \right) dt \\
&\quad - \sum_{i=1}^{l} (\nabla I_i(x^+(t_i)), v(t_i)) \\
&= -\sum_{i=1}^{l} (\Delta \Phi_p(\dot{x}(t_i)) + \nabla I_i(x^+(t_i)), v(t_i)) \\
&\quad - \int_0^T \left(\frac{d}{dt}(\Phi_p(\dot{x}(t))) + \nabla F(t, x^+(t)), v(t) \right) dt.
\end{aligned}$$

So

$$\sum_{i=1}^{l}(\Delta\Phi_p(\dot{x}(t_i))+\nabla I_i(x^+(t_i)),v(t_i))$$
$$+\int_0^T\left(\frac{d}{dt}(\Phi_p(\dot{x}(t)))+\nabla F(t,x^+(t)),v(t)\right)dt=0 \tag{2.6}$$

for all $v\in X$. Without loss of generality, we assume that $v\in C_0^\infty(t_i^+,t_{i+1})$, $v(t)\equiv 0$ for $t\in[0,t_i]\cup[t_{i+1},T]$. Then by fundamental lemma of variational methods,

$$\frac{d}{dt}(\Phi_p(\dot{x}(t)))+\nabla F(t,x^+(t))=0,\quad t\in(t_i,t_{i+1}).$$

That is, the equation of system (2.2) holds. Thus by (2.6),

$$\sum_{i=1}^{l}(\Delta\Phi_p(\dot{x}(t_i))+\nabla I_i(x^+(t_i)),v(t_i))=0$$

holds for all $v\in X$. Since $v\in X$ is arbitrary, one has $-\Delta\Phi_p(\dot{x}(t_i))=\nabla I_i(x^+(t_i))$, $i=1,2,\ldots,l$. Besides $x\in X$ means that $x(0)=x(T)=0$. Therefore, x is a solution of system (2.2). □

Lemma 2.4 *Suppose that* (C2) *holds. Then the functional* φ *satisfies Palais-Smale condition, i.e., every sequence* $\{x_n\}$ *in* X *satisfying* $\varphi(x_n)$ *is bounded and* $\varphi'(x_n)\to 0$ *has a convergent subsequence.*

Proof First we prove that (x_n) is a bounded sequence in X. By Lemma 2.1(vi) and (2.5) we have

$$\begin{aligned}\langle\varphi'(x_n),x_n^-\rangle &= \int_0^T[(\Phi_p(\dot{x}_n(t)),\dot{x}_n^-(t))-(\nabla F(t,x_n^+(t)),x_n^-(t))]dt\\ &\quad-(\nabla I_i(x_n^+(t_i)),x_n^-(t_i))\\ &=-\int_0^T|(\dot{x}_n^-)(t)|^p dt-\int_0^T(\nabla F(t,x_n^+(t)),x_n^-(t))dt\\ &\quad-(\nabla I_i(x_n^+(t_i)),x_n^-(t_i))\\ &\le-\|x_n^-\|_X^p.\end{aligned} \tag{2.7}$$

Set $w_n^-=\frac{x_n^-}{\|x_n^-\|_X}$. Dividing $\|x_n^-\|_X$ on the both sides of the above inequality, we have

$$\|x_n^-\|_X^{p-1}\le-\langle\varphi'(x_n),w_n^-\rangle\to 0\quad\text{as } n\to\infty.$$

So $x_n^-\to 0$ in X. Now we shall show that (x_n^+) is bounded.

By (2.4), (2.5), one has

$$\begin{aligned}\frac{\mu}{p}\|x_n\|_X^p - \|x_n^+\|_X^p = \mu\varphi(x_n) - \langle\varphi'(x_n), x_n^+\rangle \\ + \mu\int_0^T [F(t, x_n^+(t)) - (\nabla F(t,0), x_n^-(t))]dt \\ - \int_0^T (\nabla F(t, x_n^+(t)), x_n^+(t))dt + \mu\sum_{i=1}^{l}[I_i(x(t_i)) \\ - (\nabla I_i(0), x_n^-(t_i))] \\ - \sum_{i=1}^{l}(\nabla I_i(x_n^+(t_i)), x_n^+(t_i)).\end{aligned}$$

From (C2) and Lemma 2.1, it follows that

$$\begin{aligned}\left(\frac{\mu}{p} - 1\right)\|x_n^+\|_X^p \le \mu\varphi(x_n) - \langle\varphi'(x_n), x_n^+\rangle - \mu\int_0^T (\nabla F(t,0), x_n^-(t))dt \\ - \mu\sum_{i=1}^{l}(\nabla I_i(0), x_n^-(t_i)) \\ \le \mu\varphi(x_n) - \langle\varphi'(x_n), x_n^+\rangle. \qquad (2.8)\end{aligned}$$

Suppose that (x_n^+) is unbounded. Passing to a subsequence, we may assume if necessary, that $\|x_n^+\|_X \to +\infty$ as $n \to +\infty$. Dividing the both sides of (2.8) by $\|x_n^+\|_X^p$, denoting $w_n^+ = \frac{x_n^+}{\|x_n^+\|_X}$, we have

$$\frac{\mu}{p} - 1 \le \frac{\mu\varphi(x_n)}{\|x_n^+\|_X^p} - \frac{\langle\varphi'(x_n), w_n^+\rangle}{\|x_n^+\|_X^{p-1}}.$$

Noticing that $\varphi(x_n)$ is bounded and $\varphi'(x_n) \to 0$ as $n \to +\infty$. Let $n \to +\infty$, we have $\frac{\mu}{p} - 1 \le 0$, a contradiction. So (x_n) is bounded in X.

From the reflexivity of X, we may extract a weakly convergent subsequence, that for simplicity, we call (x_n), $x_n \rightharpoonup x$. Following we will show that (x_n) strongly converges to x. By (2.5) one has

$$\begin{aligned}&\langle\varphi'(x_n) - \varphi'(x), x_n - x\rangle \\ &= \int_0^T (\Phi_p(\dot{x_n}(t)) - \Phi_p(\dot{x}(t)), \dot{x_n}(t) - \dot{x}(t))dt \\ &\quad - \int_0^T (\nabla F(t, x_n^+(t)) - \nabla F(t, x^+(t)), x_n(t) - x(t))dt \\ &\quad - \sum_{i=1}^{l}(\nabla I_i(x_n^+(t_i)) - \nabla I_i(x^+(t_i)), x_n(t_i) - x(t_i)). \qquad (2.9)\end{aligned}$$

Since $x_n \rightharpoonup x$ in X, we have (x_n) uniformly converges to x in $C([0,T])$. So

$$\int_0^T (\nabla F(t, x_n^+(t)) - \nabla F(t, x^+(t)), x_n(t) - x(t))dt + \sum_{i=1}^{l} (\nabla I_i(x_n^+(t_i)) - \nabla I_i(x^+(t_i)), x_n(t_i) - x(t_i)) \to 0 \tag{2.10}$$

as $n \to +\infty$. By $\varphi'(x_n) \to 0$ and $x_n \rightharpoonup x$, we have

$$\langle \varphi'(x_n) - \varphi'(x), x_n - x \rangle \to 0 \quad \text{as } n \to +\infty. \tag{2.11}$$

By (2.2) of [13], there exist $c_i > 0, i = 0, 1, \dots, l$ such that

$$\begin{aligned}\int_0^T & (\Phi_p(\dot{x_n}(t)) - \Phi_p(\dot{x}(t)), \dot{x}_n(t) - \dot{x}(t))dt \\ &= \sum_{i=0}^{l} \int_{t_i}^{t_{i+1}} (\Phi_p(\dot{x_n}(t)) - \Phi_p(\dot{x}(t)), \dot{x}_n(t) - \dot{x}(t))dt \\ &\ge \sum_{i=0}^{l} c_i \int_{t_i}^{t_{i+1}} |\dot{x_n}(t) - \dot{x}(t)|^p dt \ge \min\{c_i : i = 0, 1, \dots, l\} \int_0^T |\dot{x_n}(t) - \dot{x}(t)|^p dt \\ &= \min\{c_i : i = 0, 1, \dots, l\} \|x_n - x\|_X^p, \quad \text{for } p \ge 2. \end{aligned} \tag{2.12}$$

So for $p \ge 2$, (2.9)–(2.12) yield that $\|x_n - x\|_X \to 0$ as $n \to \infty$.

By (2.2) of [13], there exist $d_i > 0, i = 0, 1, \dots, l$ such that

$$\begin{aligned}\int_0^T & (\Phi_p(\dot{x_n}(t)) - \Phi_p(\dot{x}(t)), \dot{x}_n(t) - \dot{x}(t))dt \\ &= \sum_{i=1}^{l} \int_{t_i}^{t_{i+1}} (\Phi_p(\dot{x_n}(t)) - \Phi_p(\dot{x}(t)), \dot{x}_n(t) - \dot{x}(t))dt \\ &\ge \sum_{i=1}^{l} d_i \int_{t_i}^{t_{i+1}} \frac{|\dot{x_n}(t) - \dot{x}(t)|^2}{(|\dot{x_n}(t)| + |\dot{x}(t)|)^{2-p}} dt \quad \text{for } 1 < p < 2. \end{aligned} \tag{2.13}$$

For $1 < p < 2$, by Hölder's inequality, we have for $x, y \in X$,

$$\begin{aligned}\int_{t_i}^{t_{i+1}} & |\dot{x}(t) - \dot{y}(t)|^p dt \\ &\le \left(\int_{t_i}^{t_{i+1}} \frac{|\dot{x}(t) - \dot{y}(t)|^2}{(|\dot{x}(t)| + |\dot{y}(t)|)^{2-p}} dt \right)^{\frac{p}{2}} \times \left(\int_{t_i}^{t_{i+1}} (|\dot{x}(t)| + |\dot{y}(t)|)^p \right)^{\frac{2-p}{2}} \\ &\le \left(\int_{t_i}^{t_{i+1}} \frac{|\dot{x}(t) - \dot{y}(t)|^2}{(|\dot{x}(t)| + |\dot{y}(t)|)^{2-p}} dt \right)^{\frac{p}{2}} 2^{\frac{(p-1)(2-p)}{2}} \end{aligned}$$

$$\times \left[\int_{t_i}^{t_{i+1}} (|\dot{x}(t)|^p + |\dot{y}(t)|^p) dt\right]^{\frac{2-p}{2}}. \tag{2.14}$$

So (2.13), (2.14) yields

$$\begin{aligned}
&\int_0^T (\Phi_p(\dot{x_n}(t)) - \Phi_p(\dot{x}(t)), \dot{x}_n(t) - \dot{x}(t)) dt \\
&\quad \geq \sum_{i=0}^{l} d_i \left\{ \frac{\int_{t_i}^{t_{i+1}} |\dot{x}_n(t) - \dot{x}(t)|^p dt}{2^{\frac{(p-1)(2-p)}{2}} [\int_{t_i}^{t_{i+1}} (|\dot{x}_n(t)|^p + |\dot{x}(t)|^p) dt]^{\frac{2-p}{2}}} \right\}^{\frac{2}{p}} \\
&\quad = \sum_{i=0}^{l} d_i \frac{(\int_{t_i}^{t_{i+1}} |\dot{x}_n(t) - \dot{x}(t)|^p dt)^{\frac{2}{p}}}{2^{\frac{(p-1)(2-p)}{p}} [\int_{t_i}^{t_{i+1}} (|\dot{x}_n(t)|^p + |\dot{x}(t)|^p) dt]^{\frac{2-p}{p}}} \\
&\quad \geq \frac{\min\{d_i : i = 0, 2, \ldots, l\} \sum_{i=0}^{l} (\int_{t_i}^{t_{i+1}} |\dot{x}_n(t) - \dot{x}(t)|^p dt)^{\frac{2}{p}}}{2^{\frac{(p-1)(2-p)}{p}} \sum_{i=0}^{l} [\int_{t_i}^{t_{i+1}} (|\dot{x}_n(t)|^p + |\dot{x}(t)|^p) dt]^{\frac{2-p}{p}}}.
\end{aligned} \tag{2.15}$$

By the basic inequality $(\sum_{i=1}^{n} |a_i|)^p \leq \max\{n^{p-1}, 1\} \sum_{i=1}^{n} |a_i|^p$,

$$\begin{aligned}
\sum_{i=0}^{l} \left(\int_{t_i}^{t_{i+1}} |\dot{x}_n(t) - \dot{x}(t)|^p dt \right)^{\frac{2}{p}} &\geq \frac{(\sum_{i=0}^{l} \int_{t_i}^{t_{i+1}} |\dot{x}_n(t) - \dot{x}(t)|^p dt)^{\frac{2}{p}}}{\max\{(l+1)^{\frac{2}{p}-1}, 1\}} \\
&= \frac{\|x_n - x\|_X^2}{\max\{(l+1)^{\frac{2}{p}-1}, 1\}}.
\end{aligned} \tag{2.16}$$

From (2.15), (2.16) one has

$$\begin{aligned}
&\int_0^T (\Phi_p(\dot{x_n}(t)) - \Phi_p(\dot{x}(t)), \dot{x}_n(t) - \dot{x}(t)) dt \\
&\quad \geq \frac{M \|x_n - x\|_X^2}{\sum_{i=0}^{l} [\int_{t_i}^{t_{i+1}} (|\dot{x}_n(t)|^p + |\dot{x}(t)|^p) dt]^{\frac{2-p}{p}}}.
\end{aligned} \tag{2.17}$$

Thus (2.9)–(2.11), (2.17) yield $\|x_n - x\|_X \to 0$ in X, that is (x_n) strongly converges to x in X. □

3 Main results

Theorem 3.1 *Suppose that* (C1) (C2) *hold. Then system* (1.2) *has at least one positive solution.*

Proof We use Lemma 1.1 to prove the existence of a nontrivial critical point of φ. We already know that $\varphi(0) = 0$ and φ satisfies the Palais-Smale condition. Hence, it suffices to prove that φ satisfies the following conditions:

(a) there are constants $\overline{\alpha}$ and $\rho > 0$ such that $\varphi|_{\partial B_\rho \cap Y} \geq \overline{\alpha}$;
(b) there is $q_0 \in X \setminus B_\rho$ such that $\varphi(q_0) \leq 0$.

By (C1), for all $\varepsilon > 0$, there is a $\delta > 0$ such that $|F(t,x)| \leq \varepsilon |x|^p$, $|I_i(x)| \leq \varepsilon |x|^p$ whenever $|x| \leq \delta$. Let $\rho = \delta$. By (2.1),

$$\int_0^T F(t, x^+(t))dt \leq \varepsilon \int_0^T |x^+(t)|^p dt \leq \varepsilon T \|x\|_\infty^p \leq \varepsilon T K^p \|x\|_X^p$$

and

$$\left|\sum_{i=1}^l I_i(x^+(t_i))\right| \leq \varepsilon \sum_{i=1}^l |x^+(t_i)|^p \leq \varepsilon l K^p \|x\|_X^p.$$

Therefore, for $\|x\|_X = \rho$, we have

$$\begin{aligned}
\varphi(x) &= \int_0^T \left[\frac{1}{p}|\dot{x}(t)|^p - (F(t, x^+(t)) - (\nabla F(t,0), x^-(t)))\right] dt \\
&\quad - \sum_{i=1}^l [I_i(x(t_i)) - (\nabla I_i(0), x^-(t_i))] \\
&\geq \frac{1}{p}\|x\|_X^p - \varepsilon T K^p \|x\|_X^p - l\varepsilon K^p \|x\|_X^p \\
&= \left[\frac{1}{p} - \varepsilon T K^p - l\varepsilon K^p\right] \|x\|_X^p.
\end{aligned}$$

It suffices to choose $\varepsilon = \frac{1}{2p(T+l)K^p}$ to get

$$\varphi(x) \geq \frac{\rho^p}{2p} = \alpha > 0.$$

Consider

$$\begin{aligned}
\varphi(\sigma x) &= \int_0^T \left[\frac{1}{p}\sigma^p |\dot{x}(t)|^p - (F(t, \sigma x^+(t)) - \sigma(\nabla F(t,0), x^-(t)))\right] dt \\
&\quad - \sum_{i=1}^l [I_i(\sigma x(t_i)) - \sigma(\nabla I_i(0), x^-(t_i))]
\end{aligned}$$

for all $\sigma \in R$. By (C2), there exists a continuous function $\alpha_1(t) > 0$ such that $F(t,x) \geq \alpha_1(t)|x|^\mu$, $I_i(x) \geq \beta_1 |x|^\mu$ for all $|x| \geq 1$. Let $q \in X$ be satisfying $|q| \geq 1$ on an open and nonempty interval $I \subset [0,T]$. For any $\sigma \geq 1$, one has

$$\begin{aligned}
\varphi(\sigma q) &= \frac{\sigma^p}{p} \int_0^T |\dot{q}(t)|^p dt - \int_0^T [F(t, \sigma q^+(t)) - (\nabla F(t,0), \sigma q^-(t))] dt \\
&\quad - \sum_{i=1}^l [I_i(\sigma q^+(t_i)) - (\nabla I_i(0), \sigma q^-(t_i))]
\end{aligned}$$

$$\leq \frac{\sigma^p}{p}\|q\|_X^p - \int_I \alpha_1(t)|\sigma q^+(t)|^\mu dt + \int_0^T (\nabla F(t,0), \sigma q^-(t))dt$$
$$- \sum_{i=1}^{l}[\beta_1|\sigma q^+(t_i)|^\mu - (\nabla I_i(0), \sigma q^-(t_i))].$$

Since $\mu > p$, we can find $\sigma \geq 1$ such that $\|\sigma q\|_X \geq R > \rho$ and $\varphi(\sigma q) \leq 0 = \varphi(0)$. Therefore, φ satisfies (a), (b). Applying Lemma 1.1 to φ, there exists y^* such that $\varphi'(y^*) = \Theta$, $\varphi(y^*) = c > \max\{\varphi(0), \varphi(\lambda_0 y_0)\} = 0$. Clearly $y^*(k) \not\equiv 0$ for all $k \in [0, T+1]$. Lemma 2.2 means that system (1.2) has at least one positive solution. □

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