A novel steady-state productivity equation for horizontal wells in bottom water drive gas reservoirs

Zhang Liehui[1]*, Zhao Yulong[1] and Liu Zhibin[2]

[1] State Key Laboratory of Oil and Gas Reservoir Geology and Exploitation, Southwest Petroleum University, Chengdu, Sichuan 610500, China

[2] School of Science, Southwest Petroleum University, Chengdu, Sichuan 610500, China

Abstract: It is known that there is a discrepancy between field data and the results predicted from the previous equations derived by simplifying three-dimensional (3-D) flow into two-dimensions (2-D). This paper presents a new steady-state productivity equation for horizontal wells in bottom water drive gas reservoirs. Firstly, the fundamental solution to the 3-D steady-state Laplace equation is derived with the philosophy of source and the Green function for a horizontal well located at the center of the laterally infinite gas reservoir. Then, using the fundamental solution and the Simpson integral formula, the average pseudo-pressure equation and the steady-state productivity equation are achieved for the horizontal section. Two case-studies are given in the paper, the results calculated from the newly-derived formula are very close to the numerical simulation performed with the Canadian software CMG and the real production data, indicating that the new formula can be used to predict the steady-state productivity of such horizontal gas wells.

Key words: Horizontal well, point-source function, bottom water driver gas reservoir, steady-state productivity

1 Introduction

Prediction of the steady-state productivity of a horizontal well is particularly important during the period of reservoir production, as more horizontal wells are being used in single-layer reservoirs, low-permeability reservoirs, fractured reservoirs, multilayered reservoirs stratigraphically screened by an unconformity or tectonically screened by faults, and bottom water drive reservoirs. Several methods, including the analytical method, conformal transformation method, potential superposition, equivalent flow resistance method, and the point source function, were proposed to calculate the horizontal well productivity. Merkulov (1958) firstly presented an analytical equation to calculate the horizontal well productivity. Borisow (1984) reported a theoretical equation to calculate the steady-state oil production from a horizontal well; but the report did not include the derivation of the equation. Later, Giger et al (1984; 1985) and Karcher and Giger (1986) reported reservoir engineering aspects of horizontal drilling and developed a concept of replacement ratio, *FR*, which indicates the number of vertical wells required to produce at the same rate as that of a single horizontal well. Reiss (1987) proposed a productivity-index equation for horizontal wells. Thereafter Joshi (1988)

derived an equation to calculate the productivity of steady-state horizontal wells using potential-fluid theory. Babu and Odeh (1989; 1991) reduced a complex equation to an easy-to-use equation for calculating the productivity of horizontal wells, requiring that the drainage volume be approximately box-shaped, and all the boundaries of the drainage volume be closed. Renard and Dupuy (1991) took the skin factor into consideration and derived the flow efficiency equation for horizontal wells in anisotropic reservoirs. Larsen (1996) studied the pseudosteady-state productivity index and skin factor for multilateral, branched and extended wells. Helmy and Wattenbarger (1998) presented a simple equation to calculate the productivity of a horizontal well when producing at a constant bottomhole flowing pressure or at a constant rate in bounded reservoirs. Chen et al (2000) developed a deliverability model by calculating the productivity and pressure of each branch, and combining them to predict the performance of multilateral wells. Through considering the non-Darcy flow effect and mechanical skin effect in Babu and Odeh's model, Billiter et al (2001) applied the analytical methods and Monte Carlo simulation to yield the equation of dimensionless inflow-performance-relationship (IPR) curves. Anklam and Wiggins (2005) obtained the wellbore pressure along the well and the steady-state productivity with the consideration of the mechanical properties of fluid flow in the wellbore. Lu et al (1993; 2001; 2002; 2003) achieved the productivity equations for horizontal wells under different boundary conditions. These equations are based on the point

source function.

Most of the results above are mainly for oil reservoirs, and there are only a few studies reported on the productivity of horizontal wells in gas reservoirs, especially for the bottom-water-drive gas reservoirs. In this paper, we employed a similar method to that described by Lu (2001; 2002) and derived the steady-state productivity formula for a horizontal well located at the center of a laterally infinite, bottom water-drive gas reservoir.

2 The horizontal well model

2.1 Assumptions

Fig. 1 presents the configuration of a horizontal well drilled in a bottom water drive gas reservoir, which is assumed to be laterally infinite. To make the problem more tractable, the following assumptions are made: 1) The porous medium is homogeneous and anisotropic with constant permeability k_x, k_y, and k_z and porosity of   . 2) The reservoir is horizontal and of uniform thickness of H. 3) The initial reservoir pressure p_i is constant; the upper boundary is closed to flow while the lower boundary is at constant-pressure. 4) The single-phase fluid is compressible and flows from the reservoir to the well at a constant rate, Q_{sc}.
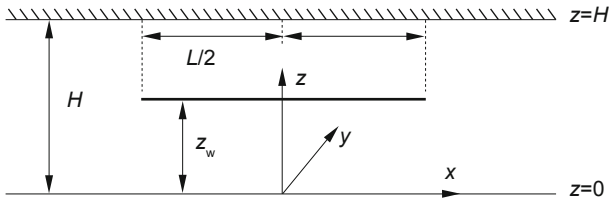

Fig. 1 Schematic of a horizontal well in a bottom-water-drive gas reservoir

2.2 Steady-state equation

According to the theory of source and the Green function (Carslaw and Jaeger, 1959; Gringarten and Ramey, 1973), and substituting the quality of the point source $\rho_{sc}q_{sc}$ at the point $(x_o, 0, z_w)$ into the flow equation, we have

$$-\left[\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z}\right] \qquad (1)$$
$$= \rho_{sc}q_{sc}\delta(x-x_o)\delta(y)\delta(z-z_w)$$

where (x, y, z) is the observation point, $(x_o, 0, z_w)$ is a point in the source point region, and x, y, z are the coordinates, m; q_{sc} is the point convergence intensity (flow rate), m^3/s, ρ and ρ_{sc} are the fluid density under the reservoir and standard conditions, respectively, g/m^3, v_x, v_y, v_z are the fluid velocity in the x, y, z directions respectively, m/s; z_w is the distance between the horizontal well and the bottom of the reservoir; $\delta(x-x_o)$, $\delta(y)$, $\delta(z-z_w)$ are Dirac functions.

Darcy equation:

$$v_x = -\frac{k_x}{\mu}\frac{\partial p}{\partial x}, \quad v_y = -\frac{k_y}{\mu}\frac{\partial p}{\partial y}, \quad v_z = -\frac{k_z}{\mu}\frac{\partial p}{\partial z} \qquad (2)$$

State equation:
$$pM = \rho ZRT \qquad (3)$$

Substituting Eqs. (2) and (3) into Eq. (1) gives:

$$k_x\frac{\partial(\frac{p}{\mu Z}\frac{\partial p}{\partial x})}{\partial x} + k_y\frac{\partial(\frac{p}{Z\mu}\frac{\partial p}{\partial y})}{\partial y} + k_z\frac{\partial(\frac{p}{Z\mu}\frac{\partial p}{\partial z})}{\partial z} \qquad (4)$$
$$= \frac{T}{T_{sc}}p_{sc}q_{sc}\delta(x-x_o)\delta(y)\delta(z-z_w)$$

where p is the reservoir pressure, MPa; T is the reservoir temperature, K; p_{sc} is the standard pressure, $p_{sc}=0.1$ MPa; T_{sc} is the standard temperature, $T_{sc}=273$ K; k is the permeability, D; μ is the gas viscosity, mPa·s; Z is the compressibility factor for gas, fraction; R is the universal gas constant, $R=8.314$ J/(K·mol); M is the molar mass of natural gas, mol/kg.

Here, the pseudo-pressure $m(p)$ is:

$$m(p) = 2\int_{p_o}^{p}\frac{p}{\mu Z}\mathrm{d}p \qquad (5)$$

where p_o is the reference pressure, MPa.

Introducing Eq. (5) into Eq. (4) yields:

$$k_x\frac{\partial^2 m}{\partial x^2} + k_y\frac{\partial^2 m}{\partial y^2} + k_z\frac{\partial^2 m}{\partial z^2}$$
$$= 2\frac{T}{T_{sc}}p_{sc}q_{sc}\delta(x-x_o)\delta(y)\delta(z-z_w) \qquad (6)$$

2.3 Dimensionless steady-state equation

Various dimensionless groups are defined as:

$$\bar{k} = (k_xk_yk_z)^{1/3}, \quad m_D = \frac{\pi\bar{k}LT_{sc}}{q_{sc}p_{sc}T}(m_i - m)$$

$$x_D = \frac{x}{L}\left(\frac{\bar{k}}{k_x}\right)^{1/2}, \quad y_D = \frac{y}{L}\left(\frac{\bar{k}}{k_y}\right)^{1/2}, \quad z_D = \frac{z}{L}\left(\frac{\bar{k}}{k_z}\right)^{1/2}$$

$$H_D = \frac{H}{L}\left(\frac{\bar{k}}{k_z}\right)^{1/2}, \quad L_D = \left(\frac{\bar{k}}{k_y}\right)^{1/2}, \quad z_{wD} = \frac{z_w}{L}\left(\frac{\bar{k}}{k_z}\right)^{1/2}$$

$$\beta = \sqrt{\frac{k_h}{k_v}}, \quad R_{wD} = \left(\beta^{1/2} + \beta^{-1/2}\right)\frac{R_w}{2L}$$

Then, we have

$$k_x\frac{\partial^2 m}{\partial x^2} = -k_x\frac{\partial}{\partial x}\left[\frac{q_{sc}p_{sc}T}{\pi\bar{k}L^2T_{sc}}\left(\frac{\bar{k}}{k_x}\right)^{1/2}\frac{\partial m_D}{\partial x_D}\right]$$
$$= -\frac{q_{sc}p_{sc}T}{\pi L^3T_{sc}}\frac{\partial^2 m_D}{\partial x_D^2} \qquad (7)$$

Similarly

$$k_y\frac{\partial^2 m}{\partial y^2} = -\frac{q_{sc}p_{sc}T}{\pi L^3T_{sc}}\frac{\partial^2 m_D}{\partial y_D^2} \qquad (8)$$

$$k_z\frac{\partial^2 m}{\partial z^2} = -\frac{q_{sc}p_{sc}T}{\pi L^3T_{sc}}\frac{\partial^2 m_D}{\partial z_D^2} \qquad (9)$$

where m_i is the pseudo-pressure under the reservoir pressure; H is the effect reservoir thickness, m; L is the well length, m; r_w is the wellbore radius, m; $\bar{k}$ is the effective permeability.

Because $\delta(cx)=(x)/c$ (where c is a positive constant (Zwillinger, 1996)) we have

$$2\frac{T}{T_{sc}}p_{sc}q_{sc}\delta(x-x_o)\delta(y)\delta(z-z_w)$$
$$=\frac{2T}{T_{sc}}p_{sc}q_{sc}\frac{\bar{k}^{3/2}}{L^3\left(k_xk_yk_z\right)^{1/2}}\delta(x_D-x_{oD})\delta(y_D)\delta(z_D-z_{wD}) \tag{10}$$

where x_{oD} and z_{oD} are the dimensionless coordinates of the point source in the x and z directions, respectively.

Hence, combining Eqs. (7)-(9) with Eq. (6) yields:

$$\frac{q_{sc}p_{sc}T}{\pi L^3T_{sc}}\left(\frac{\partial^2 m_D}{\partial x_D{}^2}+\frac{\partial^2 m_D}{\partial y_D{}^2}+\frac{\partial^2 m_D}{\partial z_D{}^2}\right)$$
$$=-\frac{2T}{T_{sc}L^3}p_{sc}q_{sc}\frac{(\bar{k})^{3/2}}{\left(k_xk_yk_z\right)^{1/2}}\delta(x_D-x_{0D})\delta(y_D)\delta(z_D-z_{wD}) \tag{11}$$

Eq. (11) can be simplified as:

$$\frac{\partial^2 m_D}{\partial x_D{}^2}+\frac{\partial^2 m_D}{\partial y_D{}^2}+\frac{\partial^2 m_D}{\partial z_D{}^2}$$
$$=-2\pi\delta(x_D-x_{oD})\delta(y_D)\delta(z_D-z_{wD}) \tag{12}$$

Here, it has to be clarified if the point convergence density (flow rate) of the point $(x_o, 0, z_w)$ is q_{sc}, then the total productivity of the horizontal well is (Lu, 2001):

$$Q_{sc}=86.4q_{sc}L_D \tag{13}$$

where Q_{sc} is the total productivity of the horizontal well, $10^4 m^3/day$

2.4 Point source solution

The horizontal well produces from a bottom-water-drive gas reservoir. The upper and lower boundary conditions can be expressed in the form of pseudo-pressure:

$$m_D\big|_{z_D=0}=0,\quad \frac{\partial m_D}{\partial z_D}\bigg|_{z_D=H_D}=0 \tag{14}$$

From Eq. (14), the characteristic function of the differential equation with these boundaries is $z(\beta_n, z_D)=\sin\frac{(2n-1)\pi z_D}{2H_D}$, and its characteristic value is $\beta_n=\frac{(2n-1)\pi}{2H_D}$, $(n=1, 2, 3, \cdots, +\infty)$.

Applying the following finite Fourier sine transform to z_D (Kong, 1999), given

$$U(x_D,y_D,\beta_n)=\int_0^{H_D}m_D(x_D,y_D,z_D)\sin\frac{(2n-1)\pi z_D}{2H_D}dz_D \tag{15}$$

So from Eq. (12), we have

$$-\int_0^{H_D}2\pi\delta(x_D-x_{oD})\delta(y_D)\delta(z_D-z_{wD})\sin\frac{(2n-1)\pi z_D}{2H_D}dz_D$$
$$=\int_0^{H_D}\left(\frac{\partial^2 m_D}{\partial x_D{}^2}+\frac{\partial^2 m_D}{\partial y_D{}^2}+\frac{\partial^2 m_D}{\partial z_D{}^2}\right)\sin\frac{(2n-1)\pi z_D}{2H_D}dz_D$$
$$=\int_0^{H_D}\left(\frac{\partial^2 m_D}{\partial x_D{}^2}+\frac{\partial^2 m_D}{\partial y_D{}^2}\right)\sin\frac{(2n-1)\pi z_D}{2H_D}dz_D$$
$$+\int_0^{H_D}\frac{\partial^2 m_D}{\partial z_D{}^2}\sin\frac{(2n-1)\pi z_D}{2H_D}dz_D \tag{16}$$

Combining Eq. (15) with Eq. (16) yields:

$$\frac{\partial^2 U}{\partial x_D{}^2}+\frac{\partial^2 U}{\partial y_D{}^2}-\beta_n{}^2U=-2\pi\delta(x_D-x_{oD})\delta(y_D)\sin\left(\beta_n z_{wD}\right) \tag{17}$$

From the fundamental solution to the modified Helmholtz function (Zwillinger, 1996), we obtain the fundamental solution to Eq. (17):

$$U(x_D,y_D,\beta_n)=\sin\left(\beta_n z_{wD}\right)K_0(R_D\beta_n) \tag{18}$$

with

$$R_D=\sqrt{(x_D-x_{oD})^2+y_D{}^2}$$

According to the inverse Fourier sine transform, there is

$$m_D(x_D,y_D,z_D)$$
$$=\sum_{n=1}^{\infty}\frac{\sin\frac{(2n-1)\pi z_D}{2H_D}}{\int_0^{H_D}\sin^2\frac{(2n-1)\pi z_D}{2H_D}dz_D}U(x_D,y_D,\beta_n) \tag{19}$$
$$=\frac{2}{H_D}\sum_{n=1}^{\infty}\sin\left(\beta_n z_D\right)\sin\left(\beta_n z_{wD}\right)K_0(R_D\beta_n)$$

The dimensionless pseudo-pressure of the point source at the point $(x_{oD}, 0, z_{wD})$ can be achieved by simplifying Eq. (19):

$$m_D(x_D,y_D,z_D)=\frac{2}{H_D}\sum_{n=1}^{\infty}\sin\beta_n z_D\sin\beta_n z_{wD}K_0(R_D\beta_n) \tag{20}$$

3 Steady-state productivity equation

3.1 Dimensionless wellbore pseudo-pressure

The dimensionless wellbore radius is:

$$R_{wD}=\sqrt{(x_{wD}-x_{0D})^2+y_{wD}{}^2}$$

where x_{wD} and y_{wD} are the dimensionless coordinates in the x and y directions at the wellbore, respectively.

Note that $L \gg R_{wD}$, then $y_{wD}\approx 0$, so

$$R_{wD}\approx|x_{wD}-x_{oD}| \tag{21}$$

If the point $(x_w, 0, z_w)$ is on the horizontal line source of

the wellbore and the boundary conditions satisfy Eq. (14), then, in Eq. (20) we can assume

$$z_{\mathrm{D}} = z_{\mathrm{wD}} + R_{\mathrm{wD}} \tag{22}$$

Let x_{oD} be the element of integration along the horizontal section in the interval $(-L_{\mathrm{D}}/2, L_{\mathrm{D}}/2)$, the dimensionless pseudo-pressure of the horizontal well at the point $(x_{\mathrm{wD}}, 0, z_{\mathrm{wD}})$ can be achieved by the superposition of the potential field:

$$m_{\mathrm{wD}} = \int_{-L_{\mathrm{D}}/2}^{L_{\mathrm{D}}/2} m_{\mathrm{D}} \mathrm{d}x_{\mathrm{oD}} \tag{23}$$

Combining Eq. (20) with Eq. (23) gives:

$$m_{\mathrm{wD}} = \int_{-L_{\mathrm{D}}/2}^{L_{\mathrm{D}}/2} \frac{2}{H_{\mathrm{D}}} \sum_{n=1}^{\infty} \sin(\beta_n z_{\mathrm{D}}) \sin(\beta_n z_{\mathrm{wD}}) K_0(R_{\mathrm{wD}} \beta_n) \mathrm{d}x_{\mathrm{oD}} \tag{24}$$

Combining Eqs. (21) and (22) with Eq. (24) gives:

$$m_{\mathrm{wD}} = \int_{-L_{\mathrm{D}}/2}^{L_{\mathrm{D}}/2} \frac{2}{H_{\mathrm{D}}} \sum_{n=1}^{\infty} \begin{matrix} \sin\left[\beta_n\left(z_{\mathrm{wD}} + R_{\mathrm{wD}}\right)\right] \times \\ \sin(\beta_n z_{\mathrm{wD}}) K_0(|x_{\mathrm{wD}} - x_{\mathrm{oD}}|\beta_n) \end{matrix} \mathrm{d}x_{\mathrm{oD}} \tag{25}$$

1) If the target points are the two end points of the horizontal well, $x_{\mathrm{wD}} = \pm L_{\mathrm{D}}/2$, then we have

$$m_{\mathrm{wD}}^{\mathrm{E}} = \int_{-L_{\mathrm{D}}/2}^{L_{\mathrm{D}}/2} \frac{2}{H_{\mathrm{D}}} \sum_{n=1}^{\infty} \begin{matrix} \sin\left[\beta_n\left(z_{\mathrm{wD}} + R_{\mathrm{wD}}\right)\right] \times \\ \sin(\beta_n z_{\mathrm{wD}}) K_0\left(\left|\dfrac{L_{\mathrm{D}}}{2} - x_{\mathrm{oD}}\right|\beta_n\right) \end{matrix} \mathrm{d}x_{\mathrm{oD}} \tag{26}$$

The right side of Eq. (26) can be written as follows:

$$\int_{-L_{\mathrm{D}}/2}^{L_{\mathrm{D}}/2} \frac{2}{H_{\mathrm{D}}} \sum_{n=1}^{\infty} \begin{matrix} \sin\left[\beta_n\left(z_{\mathrm{wD}} + R_{\mathrm{wD}}\right)\right] \times \\ \sin(\beta_n z_{\mathrm{wD}}) K_0\left(\left|\dfrac{L_{\mathrm{D}}}{2} - x_{\mathrm{oD}}\right|\beta_n\right) \end{matrix} \mathrm{d}x_{\mathrm{oD}} \tag{27}$$

$$= \frac{2}{H_{\mathrm{D}}} \sum_{n=1}^{\infty} \int_{-L_{\mathrm{D}}/2}^{L_{\mathrm{D}}/2} \begin{matrix} \sin\left[\beta_n\left(z_{\mathrm{wD}} + R_{\mathrm{wD}}\right)\right] \times \\ \sin(\beta_n z_{\mathrm{wD}}) K_0\left(\left|\dfrac{L_{\mathrm{D}}}{2} - x_{\mathrm{oD}}\right|\beta_n\right) \end{matrix} \mathrm{d}x_{\mathrm{oD}}$$

Let

$$A_n^{\mathrm{E}} = \int_{-L_{\mathrm{D}}/2}^{L_{\mathrm{D}}/2} K_0(|L_{\mathrm{D}}/2 - x_{\mathrm{oD}}|\beta_n) \mathrm{d}x_{\mathrm{oD}}$$

$$B_n = \sin\left[\beta_n\left(z_{\mathrm{wD}} + R_{\mathrm{wD}}\right)\right] \sin(\beta_n z_{\mathrm{wD}})$$

Then

$$\frac{2}{H_{\mathrm{D}}} \int_{-L_{\mathrm{D}}/2}^{L_{\mathrm{D}}/2} \sum_{n=1}^{\infty} \begin{matrix} \sin\left[\beta_n\left(z_{\mathrm{wD}} + R_{\mathrm{wD}}\right)\right] \times \\ \sin(\beta_n z_{\mathrm{wD}}) K_0\left(\left|\dfrac{L_{\mathrm{D}}}{2} - x_{\mathrm{oD}}\right|\beta_n\right) \end{matrix} \mathrm{d}x_{\mathrm{oD}} \tag{28}$$

$$= \frac{2}{H_{\mathrm{D}}} \sum_{n=1}^{\infty} A_n^{\mathrm{E}} B_n$$

Using the similar derivation of Appendix D in the literature (Lu, 2001), we obtain $A_n^{\mathrm{E}} = \dfrac{H_{\mathrm{D}}}{2n-1}$. Then Eq. (28) can be rewritten as follows:

$$\frac{2}{H_{\mathrm{D}}} \sum_{n=1}^{\infty} A_n^{\mathrm{E}} B_n = \frac{2}{H_{\mathrm{D}}} \sum_{n=1}^{\infty} \frac{H_{\mathrm{D}}}{2n-1} B_n$$

$$= 2 \sum_{n=1}^{\infty} \frac{\sin\left[\beta_n\left(z_{\mathrm{wD}} + R_{\mathrm{wD}}\right)\right] \sin(\beta_n z_{\mathrm{wD}})}{2n-1} \tag{29}$$

Using the following expression

$$\sin\left[\beta_n\left(z_{\mathrm{wD}} + R_{\mathrm{wD}}\right)\right] \sin(\beta_n z_{\mathrm{wD}})$$

$$= \frac{1}{2}\left[\cos(\beta_n R_{\mathrm{wD}}) - \cos\left[\beta_n\left(2z_{\mathrm{wD}} + R_{\mathrm{wD}}\right)\right]\right]$$

Eq. (29) can be simplified as:

$$\frac{2}{H_{\mathrm{D}}} \sum_{n=1}^{\infty} A_n^{\mathrm{E}} B_n = \sum_{n=1}^{\infty} \frac{\cos(\beta_n R_{\mathrm{wD}}) - \cos\left[\beta_n\left(2z_{\mathrm{wD}} + R_{\mathrm{wD}}\right)\right]}{2n-1} \tag{30}$$

When $x \neq 0$, we have (Gradshteyn and Ryzhik, 2007)

$$\sum_{n=1}^{\infty} \frac{\cos\left[(2n-1)x\right]}{2n-1} = \frac{1}{2}\ln\left[\cot(\frac{x}{2})\right] \tag{31}$$

Because $\beta_n = \dfrac{(2n-1)\pi}{2H_{\mathrm{D}}}$, so Eq. (30) and Eq. (31) can be combined to give:

$$\frac{2}{H_{\mathrm{D}}} \sum_{n=1}^{\infty} A_n^{\mathrm{E}} B_n$$

$$= \sum_{n=1}^{\infty} \frac{\cos(\beta_n R_{\mathrm{wD}})}{2n-1} - \sum_{n=1}^{\infty} \frac{\cos\left[\beta_n\left(2z_{\mathrm{wD}} + R_{\mathrm{wD}}\right)\right]}{2n-1} \tag{32}$$

$$= \frac{1}{2}\left[\ln\cot\left(\frac{\pi R_{\mathrm{wD}}}{4H_{\mathrm{D}}}\right) - \ln\cot\left(\frac{\pi\left(2z_{\mathrm{wD}} + R_{\mathrm{wD}}\right)}{4H_{\mathrm{D}}}\right)\right]$$

Incorporating Eq. (26) into Eq. (32) yields the dimensionless pseudo-pressure at the end points of the horizontal well:

$$m_{\mathrm{wD}}^{\mathrm{E}} = \frac{1}{2}\ln\left[\ln\cot\left(\frac{\pi R_{\mathrm{wD}}}{4H_{\mathrm{D}}}\right) - \ln\cot\left(\frac{\pi\left(2z_{\mathrm{wD}} + R_{\mathrm{wD}}\right)}{4H_{\mathrm{D}}}\right)\right] \tag{33}$$

Because the dimensionless distance of the horizontal well to the bottom boundary is larger than the dimensionless wellbore radius ($z > R_{\mathrm{wD}}$), so we have $2z_{\mathrm{wD}} + R_{\mathrm{wD}} \approx 2z_{\mathrm{wD}}$. Eq. (33) can be simplified as:

$$m_{\mathrm{wD}}^{\mathrm{E}} = \frac{1}{2}\left\{\ln\left[\cot(\frac{\pi R_{\mathrm{wD}}}{4H_{\mathrm{D}}})\right] - \ln\left[\cot(\frac{\pi z_{\mathrm{wD}}}{2H_{\mathrm{D}}})\right]\right\} \tag{34}$$

2) If the target point is the midpoint of the horizontal well, $x_{\mathrm{oD}} = 0$, we have

$$m_{\mathrm{wD}}^{\mathrm{M}} = \int_{-L_{\mathrm{D}}/2}^{L_{\mathrm{D}}/2} \frac{2}{H_{\mathrm{D}}} \sum_{n=1}^{\infty} \sin\beta_n z_{\mathrm{D}} \sin\beta_n z_{\mathrm{wD}} K_0(|0 - x_{\mathrm{oD}}|\beta_n) \mathrm{d}x_{\mathrm{oD}} \tag{35}$$

The dimensionless pseudo-pressure at the midpoint of the horizontal well is achieved using the method similar to the two end points above:

$$m_{wD}^{M} = \ln\left[\cot(\frac{\pi R_{wD}}{4H_D})\right] - \ln\left[\cot(\frac{\pi z_{wD}}{2H_D})\right] \quad (36)$$

According to Simpson's quadrature rule (Lu, 2001), we obtain the average dimensionless pseudo-pressure along the horizontal section:

$$m_{wD}^{A} \approx \frac{1}{6}(2m_{wD}^{E} + 4m_{wD}^{M})$$

$$= \frac{5}{6}\left\{\ln\left[\cot(\frac{\pi R_{wD}}{4H_D})\right] - \ln\left[\cot(\frac{\pi z_{wD}}{2H_D})\right]\right\} \quad (37)$$

3.2 Horizontal well productivity

The expression for dimensionless pseudo-pressure gives:

$$m_{wD}^{A} = \frac{\pi \bar{k} L T_{sc}}{q_{sc} p_{sc} T}(m_i - m_w^A) \quad (38)$$

Combining Eq. (37) with Eq. (38), we obtain:

$$m_{wD}^{A} = \frac{5}{6}\left\{\ln\left[\cot(\frac{\pi R_{wD}}{4H_D})\right] - \ln\left[\cot(\frac{\pi z_{wD}}{2H_D})\right]\right\}$$

$$= \frac{\pi \bar{k} L T_{sc}}{q_{sc} p_{sc} T}(m_i - m_w^A) \quad (39)$$

Then the productivity equation for a horizontal well in a laterally infinite gas reservoir with a bottom water drive can be obtained:

$$q_{sc} = \frac{\pi \bar{k} L T_{sc}}{p_{sc} T} \frac{(m_i - m_w^A)}{\frac{5}{6}\left\{\ln\left[\cot(\frac{\pi R_{wD}}{4H_D})\right] - \ln\left[\cot(\frac{\pi z_{wD}}{2H_D})\right]\right\}} \quad (40)$$

Note that $-\ln\cot x = \ln\tan x$ and when x is infinitely small, $\tan x \sim x$, then we have:

$$q_{sc} = \frac{\pi \bar{k} L T_{sc}}{p_{sc} T} \frac{(m_i - m_w^A)}{\frac{5}{6}\left(\ln\frac{4H_D}{\pi R_{wD}} + \ln\tan\frac{\pi z_{wD}}{2H_D}\right)} \quad (41)$$

According to the definition of pseudo-pressure, $m(p) = 2\int_{p_o}^{p}\frac{p}{\mu Z}dp$, we have

$$m_i = 2\int_{p_o}^{p_i}\frac{p}{\mu Z}dp, \quad m_w^A = 2\int_{p_o}^{p_w^A}\frac{p}{\mu Z}dp$$

1) When $p > 21$ MPa, $\frac{p}{\mu Z} = \text{Const}$, so

$$m(p) = 2\int_{p_o}^{p}\frac{p}{\mu Z}dp = 2\frac{p}{\mu Z}(p - p_o), \text{ then}$$

$$m_i - m_w^A = 2\int_{p_o}^{p_i}\frac{p}{\mu Z}dp - 2\int_{p_o}^{p_w^A}\frac{p}{\mu Z}dp = \frac{2\bar{p}}{\mu Z}\left(p_i - p_w^A\right)$$

2) When 13 MPa $< p < 21$ MPa, the pseudo-pressure should not be simplified.

3) When $p < 13$ MPa, $\mu Z = \text{Const}$, so

$$m(p) = 2\int_{p_o}^{p}\frac{p}{\mu Z}dp = \frac{1}{\bar{\mu Z}}\left(p^2 - p_o^2\right), \text{ then}$$

$$m_i - m_w^A = 2\int_{p_o}^{p_i}\frac{p}{\mu Z}dp - 2\int_{p_o}^{p_w^A}\frac{p}{\mu Z}dp = \frac{\left(p_i\right)^2 - \left(p_w^A\right)^2}{\bar{\mu Z}}$$

Because $Q_{sc} = 86.4 q_{sc} L_D$, the steady-state productivity equation for a horizontal well in a bottom-water-drive gas reservoir can be written in the form of pressure, pseudo-pressure, and square pressure:

$$Q_{sc} = \frac{542.87\sqrt{k_h k_v} L T_{sc}}{p_{sc} T} \frac{\bar{p}\left(p_i - p_w^A\right)}{\bar{\mu Z}\frac{5}{6}\left(\ln\frac{4H_D}{\pi R_{wD}} + \ln\tan\frac{\pi z_{wD}}{2H_D}\right)}$$

$$\text{for } p \geq 21 \text{ MPa (high-pressure)} \quad (42)$$

$$Q_{sc} = \frac{271.435\sqrt{k_h k_v} L T_{sc}}{p_{sc} T} \frac{(m_i - m_w^A)}{\frac{5}{6}\left\{\ln\frac{4H_D}{\pi R_{wD}} + \ln\tan\frac{\pi z_{wD}}{2H_D}\right\}}$$

$$\text{for } 13 \text{ MPa} < p < 21 \text{ MPa} \quad (43)$$

$$Q_{sc} = \frac{271.435\sqrt{k_h k_v} L T_{sc}}{p_{sc} T} \frac{\left(p_i\right)^2 - \left(p_w^A\right)^2}{\bar{\mu Z}\frac{5}{6}\left(\ln\frac{4H_D}{\pi R_{wD}} + \ln\tan\frac{\pi z_{wD}}{2H_D}\right)}$$

$$\text{for } p \leq 13 \text{ MPa (low-pressure)} \quad (44)$$

4 Applications

Example 1 A horizontal well was located at the center of a bottom-water-drive gas reservoir. The reservoir properties and well data are listed in Table 1. When the initial reservoir pressure and temperature were 14 MPa and 95 ℃, the gas viscosity and Z-factor were 0.025 mPa·s and 0.96, respectively; when the initial reservoir pressure and temperature were 38 MPa and 95 ℃, the gas viscosity and Z-factor were 0.035 mPa·s and 1.053, respectively.

Table 1 Reservoir and well data for Example 1

Parameter	Value
Horizontal well length L, m	360
Well radius r_w, m	0.10
The distance of the horizontal well to the lower boundary z_w, m	18
Reservoir porosity	0.15
Horizontal permeability k_h, mD	50
Vertical permeability k_v, mD	0.5
Formation thickness H, m	20
Reservoir temperature, ℃	95

The productivity of the horizontal well was calculated from Eqs. (42)-(44) and the Canadian CMG simulation software, respectively. Figs. 2 and 3 show the curves calculated from Eqs. (42)- (44) proposed in this study are very close to the curves from the CMG simulation software, indicating that the new steady-state productivity equation for horizontal wells proposed in this paper has potential practical application.
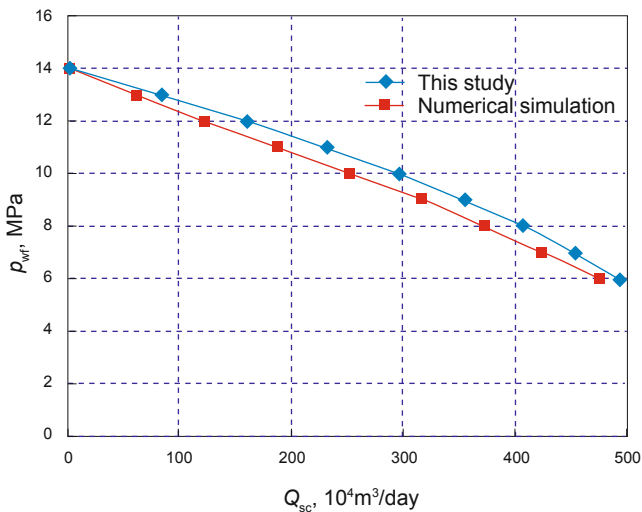

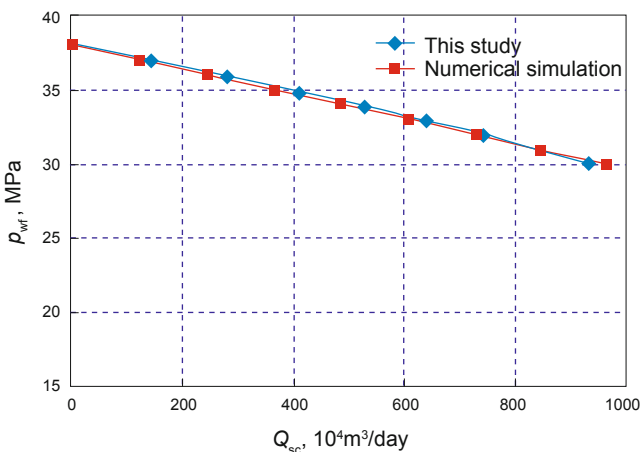**Fig. 2** Horizontal well productivity calculated under p_i=14 MPa

Fig. 3 Horizontal well productivity calculated under p_i=38 MPa

Example 2 A horizontal well was drilled in the Moxi reservoir, which located at Sichuan Province of southwest China.

The gas reservoir was a bottom-water-drive reservoir, with the initial reservoir pressure and temperature of 20.1 MPa and 80 ºC respectively. The gas viscosity and Z-factor were 0.038 mPa·s and 0.99, respectively, under initial reservoir conditions. The pay thickness of the reservoir was about 4.2 m. The horizontal section of the wellbore, nearly 3 m away from the lower boundary, was 516 m long, with a wellbore radius of 0.140 m. Well test analysis indicates that the horizontal permeability in the reservoir was about 1 mD while the vertical permeability approximately 0.1 mD. Table

2 is a summary of the actual well productivity and the well productivity calculated from the formulae proposed in this paper.

Table 2 A summary of the actual and calculated productivity of the horizontal well in the Moxi reservoir

Date mm/yyyy	Shut-in wellhead pressure, MPa	Bottom hole pressure, MPa	Actual well productivity 10^4 m^3/day	Calculated well productivity 10^4 m^3/day
12/2002	20.1	20.1	0	0
1/2003	20.1	19.1	9.4	10.3
3/2003	18.5	17.5	9.2	8.6
5/2003	15.5	14.3	7.2	8.5
7/2003	14.4	13.2	7.1	7.9

5 Conclusions

Using the idea of source and the Green function, the fundamental, 3D steady-state solution to the Laplace equation for horizontal wells in bottom-water-drive gas reservoirs was obtained, and then a novel productivity equation for horizontal wells producing at different pressures was derived. Two field case-studies were conducted and the comparisons between the results obtained from the new formula and the actual productivity and CMG simulation results show that the new equation can be used to predict the steady-state productivity of horizontal gas wells.

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