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RESEARCH

Abelian and consta-Abelian polyadic codes over affine algebras with a finite commutative chain coefficient ring

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Abstract
This paper studies Abelian and consta-Abelian polyadic codes over rings defined as affine algebras over chain rings. For this purpose, we use the classical construction via splittings and multipliers of the underlying Abelian group. We also derive some results on the structure of the associated polyadic codes and the number of codes under these conditions.

Keywords Polycyclic codes · Consta-Abelian codes · Finite chain ring · Affine algebras

Mathematics Subject Classification (2010) 94B05

1 Introduction

Polyadic codes were first introduced in [1]. There is a rich literature on these type of codes, see for example [2, 10, 12] and the references therein. They generalize quadratic residue codes, duadic codes, triadic codes and m-adic residue codes. In the case of codes over rings, in [4] there were considered quadratic residue codes over the non-chain ring $\mathbb{F}_p[u]/\langle u^m - u\rangle$ and, the same authors in [5] considered the case of codes over $\mathbb{F}_q[u]/\langle f(u)\rangle$, where q is a prime power, and f, is a polynomial which splits into distinct linear factors over $\mathbb{F}_q$. In [9], m-adic residue codes over $\mathbb{F}_p[u]/\langle u^2 - u\rangle$ were described using their idempotent generators. All of these approaches have in common that the base ring is a univariate affine algebra with a finite field as the coefficient ring. Recently M. Goyal and M. Raka [6, 7] have extended

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some of these ideas to codes over base rings that are multivariate affine algebras with a finite field as coefficient ring, but in that work the authors assume that the polynomials defining the algebra completely split into linear factors on the base field.

In this paper, we extend those previous results in a twofold way: first, we answer some questions settled in [12], namely the generalization of polyadic Abelian codes to the case of chain rings, and secondly we take as base rings a class of serial rings defined by polynomials that do not entirely split into linear factors. To our knowledge, serial rings have been considered only in the case where the defining polynomials split completely into linear factors over finite fields [6, 7]. Note that general linear codes over this type of rings, namely affine algebras with a finite commutative chain coefficient ring were already studied in [16] and a concrete example of those ambient spaces was studied in [8], that is closely related to the construction in [6, 7]. But in [16] only general linear codes were studied and there was no advantage taken of the underlying group structure as in the present paper.

The outline of the paper is as follows. In Section 2, we introduce some preliminaries on finite chain rings and serial polynomial rings over them and their idempotents. Section 3 is devoted to codes over those types of rings, we will take particular care of the structure of constacyclic codes and multiconstacyclic codes. In Section 4, we define Abelian and consta-Abelian polyadic codes over chain rings via splittings and multipliers. Sections 5 and 6 are the core part of the paper where we study Abelian and consta-Abelian polyadic codes over affine algebra rings with a finite commutative chain coefficient ring. We finish with some conclusions in Section 7.

List of Symbols

The following list describes several symbols that will be later used within the body of the document.

- R Finite chain ring with maximal ideal $\mathfrak{m}$
- $\mathbb{F}_q$ Finite field $R/\mathfrak{m}$ with q elements.
- $\bar{\cdot}$ Natural ring homomorphism from R to $\mathbb{F}_q$ given by $r \mapsto \bar{r} = r + \mathfrak{m}$
- $\mathcal{R}$ Serial ring $\mathcal{R} = R[X_1, \ldots, X_s]/I$ where $I = \langle t_1(X_1), \ldots, t_s(X_s)\rangle$, $t_i(X_i) \in R[X_i]$ $(i = 1, \ldots, s)$ are monic polynomials such that each $\bar{t}_i(X_i) \in \mathbb{F}_q[X_i]$ is a square-free polynomial.
- $\mathcal{C}$ Cyclotomic classes associated with the ideal I.
- $e_C, C \in \mathcal{C}$ Primitive orthogonal idempotents elements of the ring $\mathcal{R}$.
- $\mathcal{K}$ Linear code over $\mathcal{R}$
- $\mathcal{K}_C$ Projection of the linear code over $\mathcal{K}$ provided by the idempotent e_C.
- A Abelian group $A = \prod_{i=1}^{\delta} Z_i$ where, for each i, Z_i is a cyclic group and $r = \prod_{i=1}^{\delta} r_i$ where r_i is the size of the cyclic component Z_i and $\gcd(r_i, q) = 1$ for each $i = 1, \ldots, \delta$.
- I_A Ideal $\langle Y_1^{r_1} - 1, \ldots, Y_\delta^{r_\delta} - 1\rangle \subset R[Y_1, \ldots, Y_\delta]$ associated to the abelian group A.
- $\mathcal{C}_A$ Set of the cyclotomic classes associated to I_A.
- $A_\star = (A, \star)$ Group given by the component-wise multiplication $\star$ in A derived from the multiplication in the components $Z_i \simeq \mathbb{Z}/\mathbb{Z}_{r_i}$.
- $A_\star^*$ Group of units in $A_\star$.
- $u_\star, u \in A_\star^*$ For $u = (u_1, \ldots, u_\delta) \in A_\star^*$, it defines an action $u_\star$ over A given by $a = (a_1, \ldots, a_\delta) \mapsto u_\star(a) = (u_1 a_1, \ldots, u_\delta a_\delta)$ for all a in A .

$\mathcal{S}$	For an integer $m \geq 2$, the m-splitting $\mathcal{S} = (S_\infty, S_0, S_1, ..., S_{m-1})$ of A.
K_i	Even like polyadic codes given by the ideals $I_{(S'_\infty \cap S_i)^c}$ over the chain ring R.
$\widehat{K_i}$	Even like polyadic codes $I_{S'_\infty \cap S_i}$ over the chain ring R.
L_i	Odd like polyadic codes $I_{S_\infty \cap S_i}$ over the chain ring R.
$\widehat{L_i}$	Odd like polyadic codes $I_{(S_\infty \cap S_i)^c}$ over the chain ring R.
$\widetilde{K}$	Ideal $K.I_{S_\infty}$
E_i	Idempotent generator of K_i.
E'_i	Idempotent generator of $\widehat{K_i}$.
D_i	Idempotent generator of L_i.
D'_i	Idempotent generator of $\widehat{L_i}$.
A_i	The set $\{A_i\}_{i=1}^m$ is a partition of the classes in $\mathcal{C}$.
θ_{A_i}	Sum of idempotents $e_{C_{A_i}}$.
$\mathcal{D}_j$	Odd like idempotent generator of polyadic code over serial ring $\mathcal{R}$.
$\mathcal{E}_j$	Even like idempotent generator of polyadic code over serial ring $\mathcal{R}$.
$\mathcal{P}_j, \widehat{\mathcal{P}_j}$	Odd-like pairs of polyadic code over serial ring $\mathcal{R}$.
$\mathcal{Q}_j, \widehat{\mathcal{Q}_j}$	Even-like pairs of polyadic code over serial ring $\mathcal{R}$.
$\mathrm{Rep}(n)$	Repetition code of length n.

2 Preliminaries

In this section, we will fix our notation and recall some basic facts about finite chain rings (see for example [17] for a complete account on finite rings) and serial polynomial rings over a chain ring (see [13]). In this paper, all rings will be associative, commutative, and with identity. A ring R is called a local ring if it has a unique maximal ideal. A local ring is a chain ring if its lattice of ideals is a chain under inclusion. In this case, since the ideals are linearly ordered by inclusion, the ring is also called uniserial. It can be shown [3, Proposition 2.1] that R is a finite commutative chain ring if and only if R is a local ring and its maximal ideal is principal. We will denote by $a \in R$ a fixed generator of the maximal ideal $\mathfrak{m}$, and let t be its nilpotency index, thus the ideals of R are $\mathfrak{m}^i = \langle a^i \rangle$ for $i = 0, \dots, t$. Also, we will denote the residue field of R by $\mathbb{F}_q = R/\mathfrak{m}$, where $q = p^l$, for a prime number p. We will denote the polynomial ring in the indeterminates X_i, $i = 1, \dots s$ with coefficients in R by $R[X_1, \dots, X_s]$. We can extend the natural ring homomorphism $\bar{\cdot}$ from R to $\mathbb{F}_q$ given by $r \mapsto \bar{r} = r + \mathfrak{m}$ to the polynomial rings $R[X_1, \dots, X_s]$ and $\mathbb{F}_q[X_1, \dots, X_s]$ just by applying $\bar{\cdot}$ on each coefficient of the polynomial. Let $t_i(X_i) \in R[X_i]$ $(i = 1, \dots, s)$ be monic polynomials such that each $\bar{t}_i(X_i) \in \mathbb{F}_q[X_i]$ is a square-free polynomial. During this paper, we are interested in codes over the following alphabet

$$\mathcal{R} = R[X_1, \dots, X_s]/I \text{ where } I = \langle t_1(X_1), \dots, t_s(X_s) \rangle. \tag{1}$$

This type of rings include as a particular case the coding alphabet considered in [6–8]. In [13] the ideals of $\mathcal{R}$ have been described explicitly, they have been also studied in [18, 19] for the case where the ring R is the finite field $\mathbb{F}_q$. In the finite field case, the square-free condition on the polynomials $t_i(X_i)$ is known as the "semisimple condition" because of the structure of the ring $\mathcal{R}$ (it can be decomposed as a direct sum of simple ideals). In the general case, the square-free condition on the polynomials $t_i(X_i)$ leads to a decomposition of the ring $\mathcal{R}$ as a direct sum of finite chain rings, and therefore it is a serial ring [19]. In the remaining part of the preliminaries, we will follow [13] to explicitly show decomposition in terms of primitive idempotents.

Let H_i with $i = 1, \ldots, s$ be the set of roots of $\bar{t}_i(X_i)$ in a suitable extension of $\mathbb{F}_q$ where the polynomial splits in linear factors. For each $\nu \in \mathcal{H} = \prod_{i=1}^{s} H_i$ we define the class of ν as $C(\nu) = \{(\nu_1^{q^j}, \ldots, \nu_s^{q^j}) \mid j \in \mathbb{N}\}$. We will denote the set of all the classes as $\mathcal{C} = \mathcal{C}(t_1, \ldots, t_s)$, the elements of $\mathcal{C}$ form a partition of $\mathcal{H}$ and for any ideal $I \lhd \mathcal{R}/\mathfrak{m}$ the set of the common zeros of the elements in I is a union of classes. Also the size of each class is given by $|C(\nu)| = \text{l.c.m.}(d_1, \ldots, d_s) = [\mathbb{F}_q(\nu_1, \ldots, \nu_s) : \mathbb{F}_q]$ where d_i is the degree of the irreducible polynomial of ν_i over $\mathbb{F}_q$.

For all $i = 1, \ldots, s$ and a class C, let $p_{C,i}(X_i)$ denote the polynomial $\text{Irr}(\nu_i, \mathbb{F}_q)$ and $(\nu_1, \ldots, \nu_s) \in C$. Also, for all $i = 2, \ldots, s$ we consider the polynomials $b_{C,i}(X_i) = \text{Irr}(\nu_i, \mathbb{F}_q(\nu_1, \ldots, \nu_{i-1})) \in \mathbb{F}_q(\nu_1, \ldots, \nu_{i-1})[X_i]$ and $\tilde{b}_{C,i}(X_i) = \frac{p_{C,i}(Xi)}{b_{C,i}(Xi)}$. Note that the polynomials above are independent of which the element ν is chosen in each class C for their definition and that $b_{C,i}(X_i)$ and $\tilde{b}_{C,i}(X_i)$ are coprime polynomials. Then, define the multivariable polynomials $w_{C,i}(X_1, \ldots, X_i)$, and $\pi_{C,i}(X_1, \ldots, X_i)$ obtained from $b_{C,i}(X_i)$ and $\tilde{b}_{C,i}(X_i)$ respectively by substituting ν_i by X_i. One has that

$$\mathbb{F}_q[X_1, \ldots, X_n]/\langle p_{C,1}, w_{C,2}, \ldots, w_{C,n}\rangle \simeq \mathbb{F}_q(\nu_1, \ldots, \nu_n),$$

and we denote the Hensel lifts to R of the polynomials $p_{C,i}$, $w_{C,i}$ and $\pi_{C,i}$ by $q_{C,i}$, $z_{C,i}$ and $\sigma_{C,i}$ respectively. If we denote by $I_C = \langle q_{C,1}, z_{C,2}, \ldots, z_{C,s}\rangle$ then the ring $T_C = R[X_1, \ldots, X_s]/I_C$ is a chain ring with maximal ideal $\mathfrak{M} = \langle a, q_{C,1}, z_{C,2}, \ldots, z_{C,s}\rangle + I_C$ and $T_C/\mathfrak{M} \simeq \mathbb{F}_q(\nu_1, \ldots, \nu_s)$ (see [13, Remark 4, & Lemma 3.5]). Now consider the polynomial

$$h_C(X_1, X_2, ..., X_s) = \prod_{i=1}^{s} \frac{t_i(X_i)}{q_{\mathcal{C},i}(X_i)} \prod_{i=1}^{s} \sigma_{q_{\mathcal{C},i}}(X_2, \ldots, X_i),$$

then $I_C + I = \text{Ann}(\langle h_C + I\rangle)$, $\langle h_C + I\rangle \simeq R[X_1, \ldots, X_s]/I_C$ and $\mathcal{R} \simeq \bigoplus_{C \in \mathcal{C}} \langle h_C + I\rangle$ (for a proof see [13, Proposition 3.7, Lemma 3.8 & Theorem 3.9]). This decomposition of the ring $\mathcal{R}$ is equivalent to the existence of primitive orthogonal idempotents elements $e_C \in \mathcal{R}$ where $C \in \mathcal{C}$ such that $1_{\mathcal{R}} = \sum_{C \in \mathcal{C}} e_C$ and $e_C\mathcal{R} \simeq \langle h_C + I\rangle$, i.e. there exists a polynomial g_C such that the idempotent e_C is the element $g_C h_C + I$ and $g_C h_C + I_C = 1 + I_C$. Any ideal of $\mathcal{R}$ is principally generated by $G + I$ where $G = \sum_{i=0}^{t-1} a^i G_i$ and G_i is a sum of primitive idempotents e_C described before (see [13, Corollary 3.12]).

Remark 1 This decomposition includes the cases given in [6, 8]. In [6], $R = \mathbb{F}_q$, there are two variables and t_1, t_2 split completely into linear factors over $\mathbb{F}_q[X]$ whereas, in [8] there is one polynomial in $R = \mathbb{F}_q$ whose roots are all the elements in the field.

3 Structure of codes over $\mathcal{R}$

A *linear code* $\mathcal{K}$ *of length* n *over the ring* $\mathcal{R}$ as an $\mathcal{R}$-submodule of $\mathcal{R}^n$. The *Euclidean dual* of $\mathcal{K}$ will be denoted by $\mathcal{K}^{\perp}$ and it is given by the set $\{\mathbf{x} \in \mathcal{R}^n \mid \mathbf{x} \cdot \mathbf{k} = 0 \text{ for all } \mathbf{k} \in \mathcal{K}\}$, where $\cdot$ is the Euclidean inner product in $\mathcal{R}^n$. Note that, since $\mathcal{R} = \sum_{C \in \mathcal{C}} e_C\mathcal{R}$, for each $\mathbf{x} \in \mathcal{R}^n$ we can define the projection of $\mathbf{x}$ by $e_{C'}$ as $\mathbf{x}_{C'} = (x_{1,C'}, \ldots, x_{n,C'}) \in R^n$ where $x_i = \sum_{C \in \mathcal{C}} x_{i,C} e_C$ and $x_{i,C} \in R$ for $i = 1, \ldots, n$. Indeed, $\mathbf{x}_{C'} = \mathbf{x} \cdot e_{C'}$ for each $C \in \mathcal{C}$ and for a given linear code $\mathcal{K}$ of length n over the ring $\mathcal{R}$ we can define the following codes

$$\mathcal{K}_C = \{\mathbf{x}_C \mid \mathbf{x} \in \mathcal{K}\}, \tag{2}$$

where C ranges in the set of classes in $\mathcal{C}$. It is clear that $\mathcal{K}_C$ is an R-linear code and $\mathcal{K} = \bigoplus_{C\in\mathcal{C}} \mathcal{K}_C e_C$. Moreover, if $\mathcal{K} = \bigoplus_{C\in\mathcal{C}} \mathcal{K}_C e_C$ then $\mathcal{K}^\perp = \bigoplus_{C\in\mathcal{C}} \mathcal{K}_C^\perp e_C$ (note that we slightly abuse the notation since first orthogonality is in $\mathcal{R}$ and the second one in R).

Example 2 (Toy Example) Consider the serial ring defined over $\mathbb{Z}_4$ by the ideal $I = \langle x_1^2 + x_1 + 1, x_2^2 - x_2\rangle$ as

$$\mathcal{R} = \mathbb{Z}_4[x_1, x_2]/\langle x_1^2 + x_1 + 1, x_2^2 - x_2\rangle.$$

Note that the polynomial $x_1^2 + x_1 + 1$ is irreducible over $\mathbb{F}_2$ and has three roots over $\mathbb{F}_8$. The set of these roots is $H_1 = \{\nu, \nu^2, \nu^3\} \subset \mathbb{F}_8$. On the other hand, the polynomial $x_2^2 - x_2$ has two roots and the set of roots is $H_2 = \{0, 1\} \subset \mathbb{F}_2$. Thus, the set $\mathcal{H} = H_1 \times H_2 = \{\{\nu, \nu^2, \nu^3\}\times\{0\}, \{\nu, \nu^2, \nu^3\}\times\{1\}\}$ has two classes. If we compute the orthogonal idempotents associated with each class we have

$$e_{\mathcal{C}_1}(x_1, x_2) = x_2, \text{ and } e_{\mathcal{C}_2}(x_1, x_2) = 3x_2 + 1.$$

It is clear that $e_{\mathcal{C}_i}^2 = e_{\mathcal{C}_i} \bmod I$ for $i = 1, 2$, $e_{\mathcal{C}_1} + e_{\mathcal{C}_2} = 1 \bmod I$ and that $e_{\mathcal{C}_1} \cdot e_{\mathcal{C}_2} = 0 \bmod I$. In this case, since it is a toy example, this could also be noticed directly from the fact that $\mathbb{Z}_4[x_1]/\langle x_1^2 + x_1 + 1\rangle = GR(4, 3)$, the Galois ring of characteristic 4 and 4^3 elements, therefore $\mathcal{R} = GR(4, 3)[x_2]/\langle x_2^2 - x_2\rangle$.

For example, if we consider a linear code on the module $\mathcal{K}$ in $\mathcal{R}^2$ generated by $G = (x_1 + x_2 x_1^2, x_1 + x_2 + 1)$ then we can decompose G in terms of the idempotents $e_{C_1} = x_2$ and $e_{C_2} = 3x_2 + 1$ as

$$((x_1 + x_1^2)e_{C_1} + x_1 e_{C_2}, (x_1 + 2)e_{C_1} + (x_1 + 1)e_{C_2}).$$

Henceforth, by the orthogonality of the idempotents we have the decomposition stated in (2).

3.1 Constacyclic codes over $\mathcal{R}$

A *λ-constacyclic code $\mathcal{K}$ of length n over $\mathcal{R}$* can be regarded as an ideal of $\mathcal{R}[x]/\langle x^n - \lambda\rangle$ where λ is a unit in $\mathcal{R}$ (if $\lambda = 1$ is called a cyclic code). It is clear that if $\mathcal{K}$ is a λ-constacyclic code of length n then $\mathcal{K}^\perp$ is a λ^{-1}-constacyclic code of length n. Note that, as stated above, $\lambda = \sum_{C\in\mathcal{C}} \lambda_C \cdot e_C$ where $\lambda_C \in R$ and λ is a unit if λ_C is a unit in R for each $C \in \mathcal{C}$. Henceforth, $\mathcal{K}$ is λ-constacyclic code of length n in $\mathcal{R}$ if $\mathcal{K}_C$ is λ_C-constacyclic code of length n in R for each $C \in \mathcal{C}$. Thus the following result can be proven in the same fashion as [6, Theorem 3].

Proposition 3 *If $\mathcal{K} = \bigoplus_{C\in\mathcal{C}} \mathcal{K}_C e_C$ is a λ-constacyclic code of length n over $\mathcal{R}$, then $\mathcal{K}^\perp$ is a λ^{-1}-constacyclic code of length n over $\mathcal{R}$ where $\mathcal{K}^\perp = \bigoplus_{C\in\mathcal{C}} \mathcal{K}_C^\perp e_C$ and $\lambda_C^{-1} = \lambda_C^\perp$. Moreover, for $\mathcal{K}$ to be self-dual it is necessary that $\lambda = \sum_{C\in\mathcal{C}} \pm e_C$, i.e. $\lambda^2 = 1_{\mathcal{R}}$.*

The following lemma that characterizes constacyclic codes over chain rings, it can be found in [18] or in a more general way that can be also used for the multivariable case in the language of Canonical Sets of Generators in [14].

Lemma 4 ([18]) *A non-zero λ_C-constacyclic code $\mathcal{K}_C$ over the chain ring R with maximal ideal $\mathfrak{m} = \langle a\rangle$ and nilpotency index t has a generating set in standard form*

$$S = \{a^{b_0} g_{b_0}, a^{b_1} g_{b_1}, \ldots, a^{b_u} g_{b_u}\} \tag{3}$$

such that $\mathcal{K}_C = \langle S\rangle \lhd R[X]/\langle X^n - \lambda_C\rangle$ and

1. $0 \leq b_0 < b_1 < \ldots < b_u < t$,
2. g_{b_i} *is a monic polynomial in* $R[x]$ *for* $i = 0, \ldots, u$,
3. $\deg g_{b_i} > \deg g_{b_{i+1}}$ *for* $i = 1, \ldots, u-1$,
4. $g_{b_u} | g_{b_{u-1}} | \ldots | g_{b_0} | X^n - \lambda_C$.

Moreover, if $d_i = \deg g_{b_i}$ *for* $i = 1, \ldots, u$, *then* $|\mathcal{K}_{\lambda_C}| = |R/\mathfrak{m}|^{\sum_{i=0}^{u}(s-b_i)(d_{i-1}-d_i)}$ *and the code is principal as ideal*

$$\mathcal{K}_C = \left\langle G_C = \sum_{i=0}^{u} a^{b_i} g_{b_i} \right\rangle \lhd R[X]/\langle X^n - \lambda_C \rangle.$$

Taking into account the previous lemma, one can characterize in a polynomial way the class of constacyclic codes over $\mathcal{R}$.

Proposition 5 *Let* $\mathcal{K} = \bigoplus_{C \in \mathcal{C}} \mathcal{K}_C e_C$ *be a* λ*-constacyclic of length* n *over* $\mathcal{R}$ *and suppose that the* λ_C*-constacyclic codes* $\mathcal{K}_C$ *are generated by* $G_C(x)$ *defined as in the previous Lemma for each* $C \in \mathcal{C}$. *Then there exists a polynomial*

$$\mathcal{G}(X) = \sum_{C \in \mathcal{C}} G_C e_C$$

in $\mathcal{R}[X]$ *such that* $\mathcal{K} = \left\langle \mathcal{G} \right\rangle \lhd \mathcal{R}[X]/\langle X^n - \lambda \rangle$, *and* $|\mathcal{K}| = \prod_{C \in \mathcal{C}} |\mathcal{K}_C|$.

Proof It is straightforward from Lemma 4. □

3.2 Consta-Abelian codes over $\mathcal{R}$

Given an Abelian group A of size r, one can write it as $A = \prod_{i=1}^{\delta} Z_i$ where, for each i, Z_i is a cyclic group and $r = \prod_{i=1}^{\delta} r_i$, where r_i is the size of the cyclic component Z_i.

Definition 6 *An Abelian code over the ring* $\mathcal{R}$ with underlying group $A = \prod_{i=1}^{\delta} Z_i$ is an ideal of the ring $\mathcal{R}[Y_1, \ldots, Y_\delta]/I_A$, where $I_A = \langle Y_1^{r_1} - 1, \ldots, Y_\delta^{r_\delta} - 1 \rangle \subset \mathcal{R}[Y_1, \ldots, Y_\delta]$. Consider now the ambient space

$$\mathcal{R}_{A,\boldsymbol{\lambda}} = \mathcal{R}[Y_1, \ldots, Y_\delta]/I_{A,\boldsymbol{\lambda}} = \mathcal{R}[Y_1, \ldots, Y_\delta]/\langle Y_1^{r_1} - \lambda_1, \ldots, Y_\delta^{r_\delta} - \lambda_\delta \rangle, \tag{4}$$

where the element λ_i is an invertible element in $\mathcal{R}$ for each $i = 1, \ldots, \delta$. An ideal in $\mathcal{R}_{A,\boldsymbol{\lambda}}$ is called a $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_\delta)$-*consta-Abelian code* with underlying group A.

We could state a similar result as Lemma 4 in the case of Abelian codes over finite chain rings in terms of Canonical Set of Generators [13, 14] but, as it was pointed in [15, Corollary 1], Abelian codes are principal if the length of the code is coprime with the characteristic of the chain ring, thus we will restrict ourselves to that case stated in [13, Sections 4 and 5]. In the case of consta-Abelian, that is not the general case (see [15, Example 1] where it is shown that negacyclic codes over $\mathbb{Z}_4$ defined by a multiple root polynomial can be seen as principal ideals). Since our purpose is defining polyadic codes using splittings of the roots we will restrict our focus on simple root codes. In other words, in both cases (Abelian and consta-Abelian), we will assume that $\gcd(r_i, q) = 1$ for each $i = 1, \ldots, \delta$. The following result is a straightforward generalization of Proposition 3.

Proposition 7 *If $\mathcal{K} = \bigoplus_{C\in\mathcal{C}} \mathcal{K}_C e_C$ is the decomposition in (2) of a $\boldsymbol{\lambda}$ consta-Abelian code with underlying group A over $\mathcal{R}$, then $\mathcal{K}^{\perp}$ is a $\lambda^{-1} = (\lambda_1^{-1}, \ldots, \lambda_\delta^{-1})$ consta-Abelian code with underlying group A over $\mathcal{R}$, where $\mathcal{K}^{\perp} = \bigoplus_{C\in\mathcal{C}} \mathcal{K}_C^{\perp} e_C$ and $\lambda_C^{-1} = (\lambda_{1,C}^{-1}, \ldots, \lambda_{\delta,C}^{-1}) = \lambda_C^{\perp}$. Moreover, for $\mathcal{K}$ to be self-dual it is necessary that $\lambda^2 = (\lambda_1^2, \ldots, \lambda_\delta^2) = \mathbf{1}_{\mathcal{R}_{A,\lambda}}$.*

A particular version of Theorem 3.13 in [13] (and [13, Corollary 3.14] for devising the sizes of the ideals) suited to our setting provides us the following analogous results to Lemma 4 and Proposition 5.

Lemma 8 *A non-zero $\boldsymbol{\lambda}_C$ consta-Abelian code $\mathcal{K}_C$ over the chain ring R with maximal ideal $\mathfrak{m} = \langle a\rangle$ and nilpotency index s has a generating set in standard form*

$$S = \{a^{b_0} G_{b_0}, a^{b_1} G_{b_1}, \ldots, a^{b_u} G_{b_u}\} \subset R[Y_1, \ldots, Y_\delta] \tag{5}$$

such that $\mathcal{K}_C = \langle S\rangle \triangleleft R[Y_1, \ldots, Y_\delta]$ and

1. *$0 \leq b_0 < b_1 < \ldots < b_u < t$,*
2. *G_{b_i} is a monic polynomial in $R[Y_1, \ldots, Y_\delta]$ for $i = 1, \ldots, u$.*

Moreover, $|\mathcal{K}_{\lambda_C}| = |R/\mathfrak{m}|^{\sum_{i=0}^{t-1}(t-i)N_i}$ (where N_i is the number of zeros in $H_1 \times \cdots \times \times H_r$ of $\bar{G}_{b_i}$) and the code is principal as ideal

$$\mathcal{K}_C = \left\langle G_C = \sum_{j=0}^{u} a^{b_i} G_{b_i} \right\rangle \triangleleft R[Y_1, \ldots, Y_\delta]/\langle Y_1^{r_1} - \lambda_1, \ldots, Y_\delta^{r_\delta} - \lambda_\delta\rangle.$$

Proposition 9 *Let $\mathcal{K} = \bigoplus_{C\in\mathcal{C}} \mathcal{K}_C e_C$ be a $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_\delta)$-consta-Abelian code of length n with underlying group A over $\mathcal{R}$ and suppose that the λ_C-constacyclic code $\mathcal{K}_C$ are generated by $G_C(x)$ defined in the previous Lemma for each $C \in \mathcal{C}$. Then there exists a polynomial $\mathcal{G}(X) = \sum_{C\in\mathcal{C}} G_C e_C$ such that $\mathcal{K} = \left\langle \mathcal{G}\right\rangle \triangleleft \mathcal{R}_{A,\lambda}$ and $|\mathcal{K}| = \prod_{C\in\mathcal{C}} |\mathcal{K}_{\lambda_C}|$.*

4 Polyadic codes over chain ring

4.1 Splittings and multipliers

Let A be a finite Abelian group of size r with the previous decomposition $A = \prod_{i=1}^{\delta} Z_i$ as a product of cyclic groups, $r = \prod_{i=1}^{\delta} r_i$ where $r_i = |Z_i|$ and $\gcd(r_i, q) = 1$ for each $i = 1, \ldots, \delta$. We will associate to A the ideal $I_A = \langle Y_1^{r_1} - 1, \ldots, Y_\delta^{r_\delta} - 1\rangle \subset R[Y_1, \ldots, Y_\delta]$, and it is clear that $R[A] \simeq R[Y_1, \ldots, Y_\delta]/I_A$. We will denote by $\mathcal{C}_A$ the set of the cyclotomic classes associated with the ideal I_A as described in Section 2.

Now, we will define a *splitting* of A following the notation in [12]. For that, we will consider the commutative group $A_\star = (A, \star)$ given by the component-wise multiplication $\star$ in A arising from the multiplication in each of the components $Z_i \simeq \mathbb{Z}/\mathbb{Z}_{r_i}$, $i = 1, \ldots, \delta$. $A_\star^*$ will denote the group of units of $A_\star$. Any $u = (u_1, \ldots, u_\delta) \in A_\star^*$ defines an action $u_\star$ over A given by $a = (a_1, \ldots, a_\delta) \mapsto u_\star(a) = (u_1 a_1, \ldots, u_\delta a_\delta)$ for all a in A . We can apply $u_\star(a)$ to C a union of cyclotomic classes in $\mathcal{C}_A$, defining $u_\star(C)$ as the union of the exponents of the images of the elements $u_\star(a)$ where $a \in A$ is associated to an element in C. We call this application $u_\star$ a *multiplier* on the Abelian group A.

Definition 10 For a positive integer $m \geq 2$ and a nonempty set $S_\infty \subset A$, an $m-$ splitting of A is a m-tuple $\mathcal{S} = (S_\infty, S_0, S_1, ..., S_{m-1})$ which satisfies the following conditions:

1. Each set $S_\infty, S_0, S_1, ..., S_{m-1}$ is a union of cyclotomic classes in $\mathcal{C}_A$,
2. The sets $S_\infty, S_0, S_1, ..., S_{m-1}$ are disjoint and form a partition of $\mathcal{C}_A$,
3. There exists $u \in A_\star^*$ such that $u_*(S_\infty) = S_\infty$ and $u_\star(S_i) = S_{i+1}$ where $u_\star$ is a multiplier.

Note that in the literature, splittings are usually defined over cyclotomic cosets of modular integers. In our case, there is a one-to-one correspondence between cyclotomic cosets and the equivalence classes defined over the roots of the polynomials so our definition is equivalent to the usual one. Note also that it is clear that in a splitting the class given by $\{\mathbf{0}\}$ is contained in the set S_∞.

Fixed a chain ring R with quotient field $\mathbb{F}_q$, for a set S given as a union of cyclotomic classes we will denote the ideal on $R[Y_1, \ldots, Y_\delta]/I_A$ by I_S the ideal given by $I_S = \bigcap_{C \in \mathcal{C}_A, C \in S} I_C$. Note that in the case that R is a finite field, I_S denotes the polynomial ideal in $R[Y_1, \ldots, Y_\delta]/I_A$ whose elements vanish when evaluated in all the elements in S.

4.2 Polyadic Abelian codes over chain rings

Definition 11 (Polyadic Abelian codes over a chain ring) Let R be a chain ring and let A be a finite Abelian group. If $\mathcal{S} = (S_\infty, S_0, S_1, ..., S_{m-1})$ is a m-splitting of the cyclotomic classes $\mathcal{C}_A$ associated to I_A and $S'_\infty = S_\infty \setminus \{\mathbf{0}\}$. The ideals (codes)

$$K_i = I_{(S'_\infty \cup S_i)^c}, \widehat{K_i} = I_{S'_\infty \cup S_i}, L_i = I_{S_\infty \cup S_i}, \widehat{L_i} = I_{(S_\infty \cup S_i)^c} \tag{6}$$

in $R[Y_1, \ldots, Y_\delta]/I_A$ are called *polyadic codes*. K_i and $\widehat{K_i}$ are called even-like codes and L_i, $\widehat{L_i}$ are called odd-like codes.

The following result follows directly from the definition of polyadic codes. The reader can find in [12, Theorem 2.1] its counterpart for cyclic polyadic codes over finite fields.

Proposition 12 *For $i \neq j$, $i, j \in \{0, 1, \ldots, m-1\}$*

- *The following identities hold*
  1. $K_i \cap K_j = I_{S'^{\,c}_\infty}$ *and* $K_0 + K_1 + \ldots + K_{m-1} = I_{\{\mathbf{0}\}}$.
  2. $\widehat{K_i} + \widehat{K_j} = I_{S'_\infty}$ *and* $\widehat{K_0} \cap \widehat{K_1} \cap \ldots \cap \widehat{K_{m-1}} = I_{\{\mathbf{0}\}^c}$.
  3. $L_i + L_j = I_{S_\infty}$ *and* $L_0 \cap L_1 \cap \ldots \cap L_{m-1} = \{\mathbf{0}\}$.
  4. $\widehat{L_i} \cup \widehat{L_j} = I_{S_\infty{}^c}$ *and* $\widehat{L_0} + \widehat{L_1} + \ldots + \widehat{L_{m-1}} = R[Y_1, \ldots, Y_\delta]/I_A$.
- $K_i + \widehat{K_i} = R[Y_1, \ldots, Y_\delta]/I_A = L_i + \widehat{L_i}$.
- *For $0 \leq i \leq m-1$, all the codes K_i are equivalent codes. The same is true for the other families of codes $\widehat{K_i}$, L_i, and $\widehat{L_i}$.*

The last fact of the proposition is a straightforward consequence of $u_*(S_i) = S_{i+1}$. Since each K_i is uniquely determined by the m-splitting set S_i it can be regarded as a permutation thus each code K_i is equivalent to the other K_{i+1}.

Proposition 13 *For an m-adic code K over a finite chain ring R, let $\widetilde{K} = K \cdot I_{S_\infty}$, is an ideal in $R[Y_1, \ldots, Y_\delta]/I_A$. Then, for all $i \in \{0, 1, \ldots, m-1\}$ we have*

$$\widetilde{K_i} = \widetilde{\widetilde{L_i}} \quad \text{and} \quad \widetilde{\widetilde{\widehat{K_i}}} = \widetilde{L_i} = L_i.$$

The code $\widetilde{K} = I_{S_\infty}$ in the proposition above is called the *even-like* subcode of K when $S_\infty = \{0\}$ and the codewords in $K \setminus \widetilde{K}$ are called *odd-like* in that case.

For $0 \leq i \leq m-1$, let $\bar{e}_i$ and $\bar{e}'_i$ be the even-like idempotent generators of even-like codes K_i and $\widehat{K_i}$ respectively, $\bar{d}_i$ and $\bar{d}'_i$ be the odd-like idempotent generators of even-like codes L_i and $\widehat{L_i}$ respectively in $R[Y_1, \ldots, Y_\delta]/I_A$, given in Section 2.

If we take the element $\sigma = -1 \in A_\star$ it is clear that it induces a permutation of the cyclotomic cosets but it could be the case that it does not induce a permutation on the sets $S_0, \ldots, S_{m-1}$ of an m-splitting. The following result follows directly form the finite field case in [12, Proposition 2.2] and the characterization of the dual of an Abelian code over a chain ring in [13, Sections 4 & 5].

Proposition 14 *Suppose $\sigma_\star(S_\infty) = S_\infty$ and that $\sigma_\star$ is a permutation of $S_0, \ldots, S_{m-1}$ such that $\sigma_\star(X_i) = X_{\tilde{\sigma}(i)}$, for $i \in \{1, \ldots, m-1\}$. Then*

$$K_i^\perp = \widehat{K}_{\tilde{\sigma}(i)} \quad \text{and} \quad L_i^\perp = \widehat{L}_{\tilde{\sigma}(i)}.$$

Remark 15 Note that the existence of polyadic Abelian codes over the finite chain ring R relies on the existence of polyadic Abelian codes over $\mathbb{F}_q = R/\mathfrak{m}$ since we used the same cyclotomic classes. We refer to [12, Section 3] for that study.

4.3 Polyadic Consta-Abelian codes over chain rings

In this section, we will follow mainly the ideas in [11] for describing consta-Abelian serial codes over a chain ring R. As before, the Abelian group will be $A = \prod_{i=1}^{\delta} Z_i$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_\delta) \in (R^*)^\delta$. The ambient space for the codes is given by

$$R_{A,\boldsymbol{\lambda}} = R[Y_1, \ldots, Y_\delta]/I_{A,\boldsymbol{\lambda}} = R[Y_1, \ldots, Y_\delta]/\langle Y_1^{r_1} - \lambda_1, \ldots, Y_\delta^{r_\delta} - \lambda_\delta\rangle.$$

If we consider the set of roots of the polynomials $\overline{Y_1^{r_1} - \lambda_1}, \ldots, \overline{Y_\delta^{r_\delta} - \lambda_\delta}$ in an extension field of $R/\mathfrak{m}$ they are given by $(\beta_1\xi_1^{i_1}, \ldots, \beta_\delta\xi_\delta^{i_\delta})$ where β_j is a primitive r_j-th root of $\bar{\lambda}_j$, and ξ_j is a primitive r_j-th root of unity and $0 \leq i_j \leq n-1$ for $j = 1, \ldots, \delta$.

As before, we will consider the commutative group $A_\star = (A, \star)$. For each set $S \subset A$ one can define $\bar{S} = \{1 + r(s-1) \mid s \in S\}$. We say that $S \subset A$ defines an orbit with respect to r if $\bar{S}$ is a cyclotomic coset of A, in other words, once the root of unity is fixed, the set $\bar{S}$ defines the exponents of that root in a cyclotomic class of the associated Abelian code $R[A]$. We will denote $C_{\bar{S}}$ as the class related to $\bar{S}$.

Definition 16 Let Θ be a union of orbits in A w.r.t. r. For a positive integer $m \geq 2$ and a set $S_\infty \subset \Theta$, an $m-$ splitting of Θ w.r.t. r is a m-tuple $\mathcal{S} = (S_\infty, S_0, S_1, ..., S_{m-1})$ which satisfies the follow conditions:

1. Each set $S_\infty, S_0, S_1, ..., S_{m-1}$ is a union of orbits in A w.r.t. r,
2. The sets $S_\infty, S_0, S_1, ..., S_{m-1}$ are disjoint and form a partition of Θ,
3. There exists $u \in A_\star^*$ such that $u_*(C_{\bar{S}_\infty}) = C_{\bar{S}_\infty}$ and $u_*(C_{\bar{S}_i}) = C_{\bar{S}_{i+1}}$ where u_* is a multiplier.

We say that the splitting is non-trivial if $S_\infty \subsetneq \Theta$, that is $S_i \neq \emptyset$ for $i = 0, \ldots, n-1$.

Given a splitting of Θ w.r.t. r as stated above, the ideals (codes) given by its defining sets

$$K_i = I_{(\bar{S}_i)^c}, \widehat{K_i} = I_{\bar{S}_i}, L_i = I_{\bar{S}_\infty \cup \bar{S}_i}, \widehat{L_i} = I_{(\bar{S}_\infty \cup \bar{S}_i)^c} \tag{7}$$

defined in $R[Y_1, \dots, Y_\delta]/I_A$ are called *polyadic codes*. K_i and $\widehat{K_i}$ are called even-like codes and L_i, $\widehat{L_i}$ are called odd-like codes.

Note that in the case of A being a cyclic group and $\Theta = A$ we are in the case of splittings for constacyclic codes, moreover in that case if $S_\infty = \emptyset$ they are called Type I, otherwise they are called Type II (See for example [2]). Now we have a similar result to Proposition 12 that can be stated for chain rings. The reader can check [11, Theorem 7.2] for a proof in the finite field case.

Proposition 17 *For $i \neq j$, $i, j \in \{0, 1, \dots, m-1\}$*

- *The following identities hold*
  1. $L_i + L_j = I_{S_\infty}$ *and* $L_0 \cap L_1 \cap \dots \cap L_{m-1} = \{\mathbf{0}\}$.
  2. $\widehat{L_i} \cup \widehat{L_j} = I_{S_\infty{}^c}$ *and* $\widehat{L_0} + \widehat{L_1} + \dots + \widehat{L_{m-1}} = R[Y_1, \dots, Y_\delta]/I_{A,\boldsymbol{\lambda}}$.
- $R[Y_1, \dots, Y_\delta]/I_{A,\boldsymbol{\lambda}} = L_i + \widehat{L_i}$.
- *For $0 \leq i \leq m-1$, all the codes L_i are equivalent codes. The same statement is true for the family of codes $\widehat{L_i}$.*

5 Polyadic Abelian codes over serial rings

In this section, we describe polyadic codes of length n over the ring $\mathcal{R}$ expressed as in (1) by using the primitive idempotents e_C of $\mathcal{R}$ given in Section 1. Let us divide the classes of $\mathcal{C}$ from $\{1, \dots, |\mathcal{C}|\}$ into m disjoint sets denoted as A_i for $i = 1, \dots, m$. $\mathcal{C}$ can be written as follows:

$$\mathcal{C} = \{C_i \mid i = 1, \dots, |\mathcal{C}|\} = A_1 \cup \dots \cup A_m \tag{8}$$

Consider the condition that each A_i is non-empty set if $|\mathcal{C}| \geq m$. Moreover, in this case $|A_i| = t_i$, $1 \leq t_i \leq |\mathcal{C}| - m + 1$. Otherwise, $|\mathcal{C}|$ sets in the partition are non-empty having only one element, and the remaining $m - |\mathcal{C}|$ are the empty set. Thus, $|A_i| = t_i = 1$ if A_i is a non-empty set and $|A_i| = t_i = 0$ if A_i is the empty set. It can be easily seen that $|\mathcal{C}| = \sum_{i=1}^m t_i$.

We define $\theta_{A_i} = \sum_{C_j \in A_i} e_{C_j}$ for each $i = 1, \dots, m$. Assume that $\theta_{A_i} = 0$ when $A_i = \emptyset$. It is easily seen that $\sum_{i=1}^m \theta_{A_i} = \sum e_C = 1_{\mathcal{R}}$, $\theta_{A_i}^2 = \theta_{A_i}$ and $\theta_{A_i}.\theta_{A_j} = 0$ for all $i \neq j$. Also, note that these idempotents in $\mathcal{R}$ can be seen as generators corresponding to a disjoint decomposition of $\mathcal{R}$ as a sum of serial codes.

Let E_i, E_i', D_i, D_i' be the idempotent generators of polyadic codes over $R[A]$ defined as in the Section 4.1 for $i = 1, \dots, m$. E_i and E_i''s are even-like ones, while the others are odd-like ones.

From now on, we can define the idempotents to obtain polyadic codes over the serial ring $\mathcal{R}[A] = R[X_1, \dots, X_s, Y_1, \dots, Y_\delta]/\langle I, I_A\rangle$. These idempotents can be written as follows:

Note that the following index number k_{ij} we will use to enumerate all idempotents is the smallest positive integer which is equivalent to the number $i - j + 1$ (i.e $k_{ij} = i - j + 1$ *mod* (m)) and the structure of the positive integer k_{ij} forces the cyclicity of the mapping $u_\star$ over the new idempotents.

- Odd-like idempotent generators over the ring $\mathcal{R}[A]$ for each $j = 2, \dots, m$
  - $\mathcal{D}_1 = \sum_{i=1}^m \theta_{A_i} D_i$ where D_i is the idempotent generator for L_i.
  - $\mathcal{D}_j = u_*(\mathcal{D}_{j-1}) = \sum_{i=1}^m \theta_{A_i} D_{k_{ij}}$

- $\mathcal{D}'_1 = \sum_{i=1}^{m} \theta_{A_i} D'_i$ where D'_i is the idempotent generator for $\widehat{L_i}$.
- $\mathcal{D}'_j = u_*(\mathcal{D}'_{j-1}) = \sum_{i=1}^{m} \theta_{A_i} D'_{k_{ij}}$

- Even-like idempotent generators over the ring $\mathcal{R}[A]$ for each $j = 2, \ldots, m$
  - $\mathcal{E}_1 = \sum_{i=1}^{m} \theta_{A_i} E_i$ where E_i is the idempotent generator for K_i.
  - $\mathcal{E}_j = u_*(\mathcal{E}_{j-1}) = \sum_{i=1}^{m} \theta_{A_i} E_{k_{ij}}$
  - $\mathcal{E}'_1 = \sum_{i=1}^{m} \theta_{A_i} E'_i$ where E_i is the idempotent generator for $\widehat{K_i}$.
  - $\mathcal{E}'_j = u_*(\mathcal{E}'_{j-1}) = \sum_{i=1}^{m} \theta_{A_i} E'_{k_{ij}}$

Therefore, we can obtain odd-like (or even-like) polyadic codes over $\mathcal{R}$ by using the odd-like idempotents $\mathcal{D}_j$ and $\mathcal{D}'_j$ (or even-like idempotents $\mathcal{E}_j$ and $\mathcal{E}'_j$) for $j = 1, \ldots, m$. Let the polyadic codes associated with the idempotents $\mathcal{D}_j$, $\mathcal{D}'_j$, $\mathcal{E}_j$ and $\mathcal{E}'_j$ over $\mathcal{R}$ be called as $\mathcal{P}_j$, $\widehat{\mathcal{P}_j}$, $\mathcal{Q}_j$ and $\widehat{\mathcal{Q}_j}$, respectively. So, the desired polyadic codes are generated by the idempotents such that $\mathcal{P}_j = \langle \mathcal{D}_j \rangle$, $\widehat{\mathcal{P}_j} = \langle \mathcal{D}'_j \rangle$, $\mathcal{Q}_j = \langle \mathcal{E}_j \rangle$ and $\widehat{\mathcal{Q}_j} = \langle \mathcal{E}'_j \rangle$.

Example 18 (Continuation of Example 2) Consider $\mathcal{R} = \mathbb{Z}_4[x_1, x_2]/\langle x_1^2 + x_1 + 1, x_2^2 - x_2 \rangle$ as in Example 2, and consider the ambient space $\mathcal{R}[y]/\langle y^{17} - 1 \rangle$, i.e. we are considering the cyclic group of order 17. The 2-cyclotomic cosets modulo 17 are $\{0\}$, $\{1, 2, 4, 8, 9, 13, 15, 16\}$, and $\{3, 5, 6, 7, 10, 11, 12, 14\}$. It is well known that, over the field $\mathbb{F}_2 = \mathbb{Z}_4/\langle 2 \rangle$, the even-like idempotents are $e_0(y) = 0$, $e_1(y) = y + y^2 + y^4 + y^8 + y^9 + y^{13} + y^{15} + y^{16}$, $e_2(y) = y^3 + y^5 + y^6 + y^7 + y^{10} + y^{11} + y^{12} + y^{14}$, and $e_3(y) = y + y^2 + y^3 + y^4 + y^5 + y^6 + y^7 + y^8 + y^9 + y^{10} + y^{11} + y^{12} + y^{13} + y^{14} + y^{15} + y^{16}$. Note that $e_0(y)$ generates the space given by $\mathbf{0}$ and $e_3(y)$ generates the whole space. So the only possible generating idempotents for even-like duadic codes are $e_1(y)$ and $e_2(y)e_1(y) + e_2(y) = 1 - j(y)$, where $j(y)$ is the idempotent associate to the repetition code. Also, if we apply the multiplier $3_\star$ to $e_1(y)$ we get $e_2(y)$ and applied to $e_2(y)$ we get $e_1(y)$. Thus it is a pair of even-like duadic codes of length 17 over $\mathbb{F}_2$ with associated odd-like pair having generating idempotents $1 - e_1(y)$ and $1 - e_2(y)$. Now one can lift those idempotents to $GR(4, 2)$ and use the decomposition of $\mathcal{R}$ in Example 2 to multiply them by the idempotents in $\mathcal{R}$ and provide duadic codes in $\mathcal{R}[y]/\langle y^{17} - 1 \rangle$. From the previous Example 2, w get $\theta_{A_i} = e_{C_i}$, i.e $\theta_{A_1} = e_{C_1} = x_2$ and $\theta_{A_2} = e_{C_2} = 3x_2 + 1$. If we denote by E_1, E_2 the lifted of the idempotents $e_1(y), e_2(y)$ to $\mathbb{Z}_4$, the even-like idempotents are $\mathcal{E}_1 = \sum_{i=1}^{2} \theta_{A_i} E_i = \theta_{A_1} E_1 + \theta_{A_2} E_2$ and $\mathcal{E}_2 = 3_\star(\mathcal{E}_1)$ that generate the even like duadic codes over the serial ring. Odd-like idempotents and associated odd-like duadic codes can be obtained similarly.

Remark 19 If we fix the partition A_i $i = 1, \ldots, m$, we get idempotents that define a polyadic code (odd-like or even-like) and one can easily see that the polyadic codes obtained by taking the image of these idempotents under the multiplier u_* are all equivalent codes. But if we change our choice for the partition A_i we will get a new odd-like (or even-like) polyadic code that is inequivalent code to the other codes obtained from the previous arrangement. A counting procedure for the number of inequivalent codes can be found in the following theorem.

Theorem 20 (Number of polyadic codes) *The following statements hold:*

1. *If $|\mathcal{C}| \geq m$, then the number of inequivalent odd-like (or even-like) polyadic codes over the ring $\mathcal{R}$ is equal to*

$$\frac{2}{m} \sum_{t_{m-1}=1}^{|\mathcal{C}|-T_{m-2}-1} \cdots \sum_{t_2=1}^{|\mathcal{C}|-T_1-(m-2)} \sum_{t_1=1}^{|\mathcal{C}|-(m-1)} \binom{|\mathcal{C}|}{t_1} \binom{|\mathcal{C}| - T_1}{t_2} \cdots \binom{|\mathcal{C}| - T_{m-2}}{t_{m-1}},$$

where $T_i = \sum_{j=1}^{i} t_j$.

2. *If* $|\mathcal{C}| < m$, *then the number of inequivalent odd-like (or even-like) polyadic codes over the ring* $\mathcal{R}$ *is equal to*

$$\frac{2}{m}(|\mathcal{C}|)!\binom{m}{|\mathcal{C}|}.$$

Proof 1. First, we will tackle the case $|\mathcal{C}| \geq m$. To count the odd-like polyadic codes one should compute the number of idempotent generators given by $\mathcal{D}_j = u_*(\mathcal{D}_{j-1}) = \sum_{i=1}^{m} \theta_{A_i} D_{k_{ij}}$ and $\mathcal{D}_1 = \sum_{i=1}^{m} \theta_{A_i} D_i$ for $j = 2, \ldots, m$. The total number of odd-like idempotent generators is determined by the number of choices of θ_{A_i} for $i = 1, \ldots, m$. Recall that $|\theta_{A_i}| = t_i$ and the total number of idempotents of type e_C is $|\mathcal{C}|$. θ_{A_1} can be chosen in $\binom{|\mathcal{C}|}{t_1}$ different ways out of $|\mathcal{C}|$ idempotents. θ_{A_2} can be chosen in $\binom{|\mathcal{C}|-t_1}{t_2}$ different ways out of $|\mathcal{C}| - t_1$ remained idempotents. In the same fashion, $\theta_{A_{m-1}}$ can be chosen in $\binom{|\mathcal{C}|-(t_1+t_2+\ldots+t_{m-2})}{t_{m-1}}$ different ways and finally there is only one choice for θ_{A_m}. Thus, the total number of odd-like idempotent generators is equal to

$$\sum_{t_{m-1}=1}^{|\mathcal{C}|-T_{m-2}-1} \cdots \sum_{t_2=1}^{|\mathcal{C}|-T_1-(m-2)} \sum_{t_1=1}^{|\mathcal{C}|-(m-1)} \binom{|\mathcal{C}|}{t_1}\binom{|\mathcal{C}|-T_1}{t_2} \cdots \binom{|\mathcal{C}|-T_{m-2}}{t_{m-1}}.$$

Since $\mathcal{D}_1 = \sum_{i=1}^{m} \theta_{A_i} D_i$, $\mathcal{D}_j = u_*(\mathcal{D}_{j-1}) = \sum_{i=1}^{m} \theta_{A_i} D_{k_{ij}}$ for $j = 2, \ldots, m$ and the fact that u_* only permutes the idempotents D_i, the number of idempotent generators obtained by $\mathcal{D}_j$ is equal to each other for all $j = 1, \ldots, m$. Since there are m $\mathcal{D}_j$, only one $\mathcal{D}_j$ generates

$$\frac{1}{m}\sum_{t_{m-1}=1}^{|\mathcal{C}|-T_{m-2}-1} \cdots \sum_{t_2=1}^{|\mathcal{C}|-T_1-(m-2)} \sum_{t_1=1}^{|\mathcal{C}|-(m-1)} \binom{|\mathcal{C}|}{t_1}\binom{|\mathcal{C}|-T_1}{t_2} \cdots \binom{|\mathcal{C}|-T_{m-2}}{t_{m-1}}$$

odd-like idempotent generators (i.e. $1/m$ of the total number) and similarly the number of idempotent generators generated by $\mathcal{D}'_j$. Therefore we get the expression stated in the theorem.

The same quantity is valid for the number of even-like inequivalent polyadic codes (For that consider $\mathcal{E}_j$'s instead of $\mathcal{D}_j$). Hence, the number of idempotent generators is obtained as desired in the first statement of the theorem.

2. Now consider the case $|\mathcal{C}| < m$. Since each non-empty sets of the partition A_i has only one element, the total number of choices A_i is equal to $\binom{m}{|\mathcal{C}|}$ and the number of their different arrangements is $(|\mathcal{C}|)!$, the total number of odd-like idempotent generators is $(|\mathcal{C}|)!\binom{m}{|\mathcal{C}|}$. The number of inequivalent ones is $\frac{1}{m}(|\mathcal{C}|)!\binom{m}{|\mathcal{C}|}$ by using the same argument as the previous case. Considering the equality of quantities for each $\mathcal{D}_j$ and $\mathcal{D}'_j$, the total number of inequivalent odd-like polyadic codes is $\frac{2}{m}(|\mathcal{C}|)!\binom{m}{|\mathcal{C}|}$. The same counting procedure can be done to compute the number of inequivalent even-like polyadic codes. □

Definition 21 We will denote by $\mathrm{Rep}(n)$ *the repetition code of length* n, that is, the code generated by the polynomial $\sum_i Y^i \in \mathcal{R}[A]$ (i.e. the polynomial with all ones as coefficients), and, as usual, the *even weight code* is just $\mathrm{Rep}(n)^{\perp}$.

The following two theorems extend the results given in [7] to affine algebra rings and Abelian codes.

Theorem 22 *Let B be a subset of $\{1, 2, \ldots, m\}$ with at least two elements. The following propositions are satisfied for the polyadic codes $\mathcal{P}_i$, $\mathcal{Q}_i$ over $\mathcal{R}$ defined as above.*

1. $\bigcap_{i=1}^{m} \mathcal{P}_i = \mathrm{Rep}(n)$*, the repetition code over* $\mathcal{R}$.
2. $\sum_{i=1}^{m} \mathcal{P}_i = \sum_{j\in B} \mathcal{P}_j$.
3. $\bigcap_{i=1}^{m} \mathcal{Q}_i = \bigcap_{j\in B} \mathcal{Q}_j$
4. $\sum_{i=1}^{m} \mathcal{Q}_i = \mathrm{Rep}(n)^{\perp}$*, the even weight code over* $\mathcal{R}$.
5. $\mathcal{Q}_i \cap \mathrm{Rep}(n) = \{0\}$ *and* $\mathcal{P}_i \cap \mathrm{Rep}(n) = \mathrm{Rep}(n)$ *for* $1 \le i \le m$.
6. $\mathcal{P}_i + \mathcal{Q}_i = \mathcal{R}[A]$ *and* $\mathcal{P}_i \cap \mathcal{Q}_i = \{0\}$ *for* $1 \le i \le m$.

If we consider $\widehat{\mathcal{P}_i}$'s and $\widehat{\mathcal{Q}_i}$'s instead of $\mathcal{P}_i$'s and $\mathcal{Q}_i$'s respectively, the previous statements also hold.

Proof For proving the first statement of the theorem, recall that for any pair of Abelian codes C and D, the defining set of the Abelian code $C \cap D$ is the union of the defining set of C and D. So the union of all defining sets of $\mathcal{P}_i$ generates the repetition code. Recall also that for any two Abelian codes C and D whose idempotent generators of these codes are E_1 and E_2 respectively, the idempotent generators of $C \cap D$ and $C + D$ are E_1E_2 and $E_1 + E_2 - E_1E_2$ respectively. By generalizing these properties, we can obtain the second statement of the theorem. By using that the idempotent generators of $\mathcal{P}_i$ are $\mathcal{D}_i$ as above if we consider the basic properties of idempotents and the fact that $\mathcal{D}_j = u_*(\mathcal{D}_{j-1}) = \sum_{i=1}^{m} \theta_{A_i} D_{k_{ij}}$ $(j = 2, ..., m)$, we get the following expression for $j = 2$

$$\mathcal{D}_1\mathcal{D}_2 = \mathcal{D}_1 u_*(\mathcal{D}_1) = \sum_{i=1}^{m} \theta_{A_i} D_i \sum_{i=1}^{m} \theta_{A_i} D_{k_{i2}} = \sum_{i=1}^{m} \theta_{A_i} D_i D_{m-i+1}$$

$$= \sum_{i=1}^{m} \left(\theta_{A_i} \prod_{j=1}^{m} D_j \right) = \left(\sum_{i=1}^{m} \theta_{A_i} \right) \left(\prod_{j=1}^{m} D_j \right) = \prod_{j=1}^{m} D_j .$$

Applying a similar reasoning one can obtain the following equality $\prod_{j\in B} \mathcal{D}_j = \prod_{j=1}^{m} D_j$ and therefore 3) is proved. The expression in 4) can easily be obtained by considering 1).

Now, Let the idempotent generator of the $\mathrm{Rep}(n)$ be i_{Rep}. By using that the idempotent generators of $\mathcal{Q}_i$ are $\mathcal{E}_i$ as stated above and the fact that $\mathcal{E}_i \cdot i_{\mathrm{Rep}} = 0$, we get that $\mathcal{Q}_i \cap \mathrm{Rep}(n) = 0$. Now using the equality $\mathcal{D}_i \cdot i_{\mathrm{Rep}} = (1 - \mathcal{E}_i) \cdot i_{\mathrm{Rep}} = i_{\mathrm{Rep}} - \mathcal{E}_i \cdot i_{\mathrm{Rep}}$ and the previous result, the equality $\mathcal{D}_i \cdot i_{\mathrm{Rep}} = i_{\mathrm{Rep}}$ arises. Therefore, $\mathcal{P}_i \cap \mathrm{Rep}(n) = \mathrm{Rep}(n)$. Therefore we have proven statement 5) in the theorem.

For proving the last properties in the theorem, we will take $i = 1$ for convenience. Note that the idempotent generators of the code $\mathcal{P}_1 + \mathcal{Q}_1$ and $\mathcal{P}_1 \cap \mathcal{Q}_1$ can be written as $\mathcal{D}_1 + \mathcal{E}_1 - \mathcal{D}_1\mathcal{E}_1$ and $\mathcal{D}_1\mathcal{E}_1$, respectively. Since $\mathcal{D}_1\mathcal{E}_1 = \sum_{i=1}^{m} \theta_{A_i} D_i E_i = 0$ and $\mathcal{D}_1 + \mathcal{E}_1 = \sum_{i=1}^{m} \theta_{A_i}(D_i + E_i) = \sum_{i=1}^{m} \theta_{A_i} = 1$, we get that $\mathcal{P}_1 \cap \mathcal{Q}_1 = \langle \mathcal{D}_i\mathcal{E}_i \rangle = \{0\}$ and $\mathcal{P}_1 + \mathcal{Q}_1 = \langle \mathcal{D}_i + \mathcal{E}_i - \mathcal{D}_i\mathcal{E}_i \rangle = \langle 1 \rangle$. □

Theorem 23 *Let B be a subset of $\{1, 2, \ldots, m\}$ with at least two elements. The following statements are satisfied by the polyadic codes $\mathcal{P}_i$ and $\mathcal{Q}_i$ over $\mathcal{R}$ defined as above.*

1. $\mathcal{Q}_i + \mathrm{Rep}(n) = \widehat{\mathcal{P}_i}$ *and* $\widehat{\mathcal{Q}_i} + \mathrm{Rep}(n) = \mathcal{P}_i$
2. $\mathcal{Q}_i + \widehat{\mathcal{Q}_i} = \mathrm{Rep}(n)^{\perp}$ *and* $\mathcal{Q}_i \cap \widehat{\mathcal{Q}_i} = \{0\}$
3. $\mathcal{P}_i + \widehat{\mathcal{P}_i} = \mathcal{R}[A]$ *and* $\mathcal{P}_i \cap \widehat{\mathcal{P}_i} = \mathrm{Rep}(n)$

Proof Let $\mathcal{E}_i$ and i_{Rep} be the generator idempotents of $\mathcal{Q}_i$ and Rep(n) respectively. Then the idempotent generator of the code $\mathcal{Q}_i + \text{Rep}(n)$ can be written as

$$\begin{aligned}\mathcal{E}_i + i_{\text{Rep}} - \mathcal{E}_i \cdot i_{\text{Rep}} = \mathcal{E}_i + i_{\text{Rep}} &= \sum_{j=1}^{m} \theta_{A_j} E_j + i_{\text{Rep}} \\ &= \sum_{j=1}^{m} \theta_{A_j} E_j + i_{\text{Rep}} \cdot \sum_{j=1}^{m} \theta_{A_j} \\ &= \sum_{j=1}^{m} \theta_{A_j} (E_j + i_{\text{Rep}}) = \sum_{j=1}^{m} \theta_{A_j} D'_j = \mathcal{D}'_j.\end{aligned}$$

Hence $\mathcal{Q}_i + \text{Rep}(n) = \widehat{\mathcal{P}_i}$ is satisfied. The remaining parts of the result can be proven similarly. □

A *Linear Complementary Dual* (LCD) code is a code whose intersection with its dual codes is trivial. Consider $u = (-1, \ldots, -1) = -\mathbf{1}$, then $u_\star$ given by $a = (a_1, \ldots, a_\delta) \mapsto u_\star(a) = (-a_1, \ldots, -a_\delta)$ for all a in A and we will denote it by $-\mathbf{1}_\star$.

Theorem 24 (LCD codes) *Let $\mathcal{Q}_i$ and $\widehat{\mathcal{Q}_i}$ be a pair of even-like polyadic codes with the associated odd-like polyadic codes $\mathcal{P}_i$ and $\widehat{\mathcal{P}_i}$ over the ring $\mathcal{R}$ for $1 \leq i \leq m$. The following statements hold.*

1. $\mathcal{Q}_i^\perp = -\mathbf{1}_\star(\mathcal{P}_i)$ *and* $\widehat{\mathcal{Q}_i}^\perp = -\mathbf{1}_\star(\widehat{\mathcal{P}_i})$
2. *If* $-\mathbf{1}_\star(E_i) = E_i$ *then* $\mathcal{Q}_i^\perp = \mathcal{P}_i$, $\widehat{\mathcal{Q}_i}^\perp = \widehat{\mathcal{P}_i}$ *and* $\mathcal{Q}_i, \widehat{\mathcal{Q}_i}, \mathcal{P}_i, \widehat{\mathcal{P}_i}$ *are LCD codes over* $\mathcal{R}$, *for* $1 \leq i \leq m$.

Proof Without loss of generality we will assume that $i = 1$. The idempotent generator of $\mathcal{Q}_1^\perp$ is given by

$$\begin{aligned}1 - (-\mathbf{1}_\star)(\mathcal{E}_1) = \sum_{j=1}^{m} \theta_{A_j} - (-\mathbf{1}_\star)\left(\sum_{j=1}^{m} \theta_{A_j} E_j\right) &= \sum_{j=1}^{m} \left[\theta_{A_j}(1 - (-\mathbf{1}_\star)(E_j))\right] \\ = (-\mathbf{1}_\star)\left(\sum_{j=1}^{m} \theta_{A_j} D_j\right) &= (-\mathbf{1}_\star)(\mathcal{D}_1).\end{aligned}$$

Hence, $\mathcal{Q}_1^\perp = \langle 1 - (-\mathbf{1}_\star)(\mathcal{E}_1)\rangle = \langle(-\mathbf{1}_\star)(\mathcal{D}_1)\rangle = (-\mathbf{1}_\star)(\langle\mathcal{D}_1\rangle) = (-\mathbf{1}_\star)(\mathcal{P}_1)$. The equality $\widehat{\mathcal{Q}_i}^\perp = -\mathbf{1}_\star(\widehat{\mathcal{P}_i})$ can be proven in a similar way. For proving the second statement in the theorem we consider Theorem 22 6) and the equalities there for $\widehat{\mathcal{P}_i}$ and $\widehat{\mathcal{Q}_i}$. If $-\mathbf{1}_\star(E_i) = E_i$ then $\mathcal{Q}_i \cap \mathcal{Q}_i^\perp = \mathcal{Q}_i \cap \mathcal{P}_i = \{0\}$ and $\widehat{\mathcal{Q}_i} \cap \widehat{\mathcal{Q}_i}^\perp = \widehat{\mathcal{Q}_i} \cap \widehat{\mathcal{P}_i} = \{0\}$. So $\mathcal{Q}_i$ and $\widehat{\mathcal{Q}_i}$ are LCD codes over $\mathcal{R}$, for $1 \leq i \leq m$. Similarly, the case for $\mathcal{P}_i, \widehat{\mathcal{P}_i}$ also holds. □

6 Polyadic consta-Abelian codes over serial rings

In this section, we will take $\Theta = A$ and we will consider splitting as in Definition 16. Recall the set $\bar{S} = \{1 + r(s-1) | s \in S\}$ Section 4.3 and the associated cyclotomic class $C_{\bar{S}}$. As in the previous Section 5 we have the partition of the class $C_{\bar{S}}$ as

$$C_{\bar{S}} = \{C_i \mid i = 1, \ldots, |C_{\bar{S}}|\} = A_1 \cup \cdots \cup A_m \tag{9}$$

We have two different cases depending on whether the $|C_S|$ is greater than m or not. Let $|A_i| = t_i, i = 1, \dots, |C_{\bar{S}}|$.

- If $|C_{\bar{S}}| \geq m$ for $1 \leq t_i \leq |C_{\bar{S}}| - m + 1$, each A_i is non-empty set.
- Otherwise, $|C_{\bar{S}}|$ sets in the partition in (9) are non-empty and they have only one element and the remaining $m - |C_{\bar{S}}|$ sets are empty.

It can be easily checked that $|C_{\bar{S}}| = \sum_{i=1}^{m} t_i$. We will represent the idempotents of polyadic consta-Abelian codes defined on $R[A, \boldsymbol{\lambda}]$ with D_i, $\widehat{D_i}$, E_i and $\widehat{E_i}$. In the case of Type I, recall that there should be only one single idempotent and its pair, say D_i and $\widehat{D}_i$. Now we can define the polyadic consta-Abelian codes over $\mathcal{R}$ using polyadic consta-Abelian codes over a chain ring R Thus we get the following codes (in this case, also, we have Type I codes since S_∞ can be the empty set) over $\mathcal{R}[A, \boldsymbol{\lambda}] = R[X_1, \dots, X_s, Y_1, \dots, Y_\delta]/\langle I, I_{A,\boldsymbol{\lambda}}\rangle$.

Definition 25 Let $\Theta = A$ and $\mathcal{S} = (S_\infty, S_0, S_1, ..., S_{m-1})$ be an $m-$ splitting of Θ w.r.t. r. Let k_{ij} be integers such that $k_{ij} = i - j + 1 \ mod \ (m)$. Let L_i, $\widehat{L_i}$, K_i and $\widehat{K_i}$ be defined as in Section 4.3.

1. Type I codes
   The idempotent generators of the polyadic codes over the ring $\mathcal{R}[A, \boldsymbol{\lambda}]$ for each $j = 2, \dots, m$
   - $\mathcal{D}_1 = \sum_{i=1}^{m} \theta_{A_i} D_i$ where D_i is the idempotent generator for L_i.
   - $\mathcal{D}_j = u_*(\mathcal{D}_{j-1}) = \sum_{i=1}^{m} \theta_{A_i} D_{k_{ij}}$

   Let the polyadic consta-Abelian codes associated with the idempotents $\mathcal{D}_j$ over $\mathcal{R}[A, \boldsymbol{\lambda}]$ be called as $\mathcal{P}_j$. So, the desired polyadic consta-Abelian code is generated by the idempotent such that $\mathcal{P}_j = \langle \mathcal{D}_j \rangle$.
2. Type II codes
   - Odd-like idempotent generators over the ring $\mathcal{R}[A, \boldsymbol{\lambda}]$ for each $j = 2, \dots, m$
     - $\mathcal{D}_1 = \sum_{i=1}^{m} \theta_{A_i} D_i$ where D_i is the idempotent generator for L_i.
     - $\mathcal{D}_j = u_*(\mathcal{D}_{j-1}) = \sum_{i=1}^{m} \theta_{A_i} D_{k_{ij}}$
     - $\mathcal{D}'_1 = \sum_{i=1}^{m} \theta_{A_i} D'_i$ where D'_i is the idempotent generator for $\widehat{L}_i$.
     - $\mathcal{D}'_j = u_*(\mathcal{D}'_{j-1}) = \sum_{i=1}^{m} \theta_{A_i} D'_{k_{ij}}$
   - Even-like idempotent generators over the ring $\mathcal{R}[A, \boldsymbol{\lambda}]$ for each $j = 2, \dots, m$
     - $\mathcal{E}_1 = \sum_{i=1}^{m} \theta_{A_i} E_i$ where E_i is the idempotent generator for K_i.
     - $\mathcal{E}_j = u_*(\mathcal{E}_{j-1}) = \sum_{i=1}^{m} \theta_{A_i} E_{k_{ij}}$
     - $\mathcal{E}'_1 = \sum_{i=1}^{m} \theta_{A_i} E'_i$ where E_i is the idempotent generator for $\widehat{K_i}$.
     - $\mathcal{E}'_j = u_*(\mathcal{E}'_{j-1}) = \sum_{i=1}^{m} \theta_{A_i} E'_{k_{ij}}$

   Let the polyadic consta-Abelian codes associated with the idempotents $\mathcal{D}_j$, $\mathcal{D}'_j$, $\mathcal{E}_j$ and $\mathcal{E}'_j$ over $\mathcal{R}[A, \boldsymbol{\lambda}]$ be called as $\mathcal{P}_j$, $\widehat{\mathcal{P}_j}$, $\mathcal{Q}_j$ and $\widehat{\mathcal{Q}_j}$, respectively. So, the desired polyadic consta-Abelian codes are generated by the idempotents such that $\mathcal{P}_j = \langle \mathcal{D}_j \rangle$, $\widehat{\mathcal{P}_j} = \langle \mathcal{D}'_j \rangle$, $\mathcal{Q}_j = \langle \mathcal{E}_j \rangle$ and $\widehat{\mathcal{Q}_j} = \langle \mathcal{E}'_j \rangle$.

It is a straightforward exercise to check that the following results can be proven in the same fashion as the Abelian case in Section 5 above since they only rely on the decomposition of the idempotents in the polynomial rings $R[X_1, \dots, X_s]/\langle I \rangle$ and $R[Y_1, \dots, Y_\delta]/\langle I_{A,\boldsymbol{\lambda}} \rangle$. The first two theorems are related to Type I codes and the remaining ones to Type II codes.

Theorem 26 (Number of polyadic consta Abelian codes of Type I) *Let $\mathcal{R}[A, \boldsymbol{\lambda}]$ be the ring defined in Definition 25 and $|C_{\bar{S}}|$ be the cyclotomic classes related to a splitting of the abelian group A as in (9). The following statements hold.*

1. *If $|C_{\bar{S}}| \geq m$, then the number of inequivalent polyadic consta Abelian codes of Type I in $\mathcal{R}[A, \boldsymbol{\lambda}]$ is equal to*
$$\frac{1}{m} \sum_{t_{m-1}=1}^{|C_{\bar{S}}|-T_{m-2}-1} \cdots \sum_{t_2=1}^{|C_{\bar{S}}|-T_1-(m-2)} \sum_{t_1=1}^{|C_{\bar{S}}|-(m-1)} \binom{|C_{\bar{S}}|}{t_1} \binom{|C_{\bar{S}}| - t_1}{t_2} \cdots \binom{|C_{\bar{S}}| - (T_{m-2})}{t_{m-1}}$$
where $T_i = \sum_{j=1}^{i} t_j$.
2. *If $|C_{\bar{S}}| < m$, then the number of inequivalent polyadic consta-Abelian codes of Type I in $\mathcal{R}[A, \boldsymbol{\lambda}]$ is equal to*
$$\frac{1}{m}(|C_{\bar{S}}|)! \binom{m}{|C_{\bar{S}}|}$$

Proof Considering the same counting argument as in Theorem 20, it can be seen that the number of the polyadic consta-Abelian codes of Type I is equal to half of the number for old ones since there is only one single idempotent. □

Theorem 27 *Let B be a subset of $\{1, 2, \ldots, m\}$ with at least two elements. The following propositions are satisfied for polyadic consta-Abelian codes of Type I $\mathcal{P}_i$ over $\mathcal{R}[A, \boldsymbol{\lambda}]$ defined as above.*

1. $\bigcap_{i=1}^{m} \mathcal{P}_i = \bigcap_{j \in B} \mathcal{P}_j = \{0\}$
2. $\sum_{i=1}^{m} \mathcal{P}_i = \mathcal{R}[A, \boldsymbol{\lambda}]$
3. $\prod_{i=1}^{m} \mathcal{D}_i = \prod_{j \in B} \mathcal{D}_j = \{0\}$
4. $\sum_{i=1}^{m} \mathcal{D}_i = 1$

Theorem 28 (Number of polyadic consta-Abelian codes of Type II) *Let $\mathcal{R}[A, \boldsymbol{\lambda}]$ be the ring defined in Definition 25 and $|C_{\bar{S}}|$ be the cyclotomic classes related to a splitting of the abelian group A as in (9). The following statements hold.*

1. *If $|C_{\bar{S}}| \geq m$, then the number of inequivalent odd-like (or even-like) polyadic codes of Type II in $\mathcal{R}[A, \boldsymbol{\lambda}]$ is equal to*
$$\frac{2}{m} \sum_{t_{m-1}=1}^{|C_{\bar{S}}|-T_{m-2}-1} \cdots \sum_{t_2=1}^{|C_{\bar{S}}|-T_1-(m-2)} \sum_{t_1=1}^{|C_{\bar{S}}|-(m-1)} \binom{|C_{\bar{S}}|}{t_1} \binom{|C_{\bar{S}}| - T_1}{t_2} \cdots \binom{|C_{\bar{S}}| - T_{m-2}}{t_{m-1}},$$
where $T_i = \sum_{j=1}^{i} t_j$.
2. *If $|C_{\bar{S}}| < m$, then the number of inequivalent odd-like (or even-like) polyadic codes of Type II in $\mathcal{R}[A, \boldsymbol{\lambda}]$ is equal to*
$$\frac{2}{m}(|C_{\bar{S}}|)! \binom{m}{|C_{\bar{S}}|}.$$

Theorem 29 *Let B be a subset of $\{1, 2, \ldots, m\}$ with at least two elements. The following propositions are satisfied for polyadic consta-Abelian codes of Type II $\mathcal{P}_i$ and $\mathcal{Q}_i$ over $\mathcal{R}[A, \boldsymbol{\lambda}]$ given in Definition 25.*

1. $\bigcap_{i=1}^{m} \mathcal{P}_i = \mathrm{Rep}(n)$, *the repetition code over $\mathcal{R}[A, \boldsymbol{\lambda}]$*
2. $\sum_{i=1}^{m} \mathcal{P}_i = \sum_{j \in B} \mathcal{P}_j = \mathcal{R}[A, \boldsymbol{\lambda}]$.

3. $\bigcap_{i=1}^{m} \mathcal{Q}_i = \bigcap_{j\in B} \mathcal{Q}_j = \{0\}$.
4. $\sum_{i=1}^{m} \mathcal{Q}_i = \mathrm{Rep}(n)^{\perp}$.
5. $\mathcal{Q}_i \cap \mathrm{Rep}(n) = \{0\}$ *and* $\mathcal{P}_i \cap \mathrm{Rep}(n) = \mathrm{Rep}(n)$ *for* $1 \leq i \leq m$.
6. $\mathcal{P}_i + \mathcal{Q}_i = \mathcal{R}[A, \boldsymbol{\lambda}]$ *and* $\mathcal{P}_i \cap \mathcal{Q}_i = \{0\}$ *for* $1 \leq i \leq m$.

If we consider $\widehat{\mathcal{P}_i}$*'s and* $\widehat{\mathcal{Q}_i}$*'s instead of* $\mathcal{P}_i$*'s and* $\mathcal{Q}_i$*'s respectively, the previous statements also hold.*

Proof Just follow the proof of Theorem 22 using consta-Abelian polyadic code definition. □

Theorem 30 *Let* B *be a subset of* $\{1, 2, \ldots, m\}$ *with at least two elements. The following statements are satisfied for polyadic consta-Abelian codes of Type II* $\mathcal{P}_i$ *and* $\mathcal{Q}_i$ *over* $\mathcal{R}$ *given in Definition 25.*

1. $\mathcal{Q}_i + \mathrm{Rep}(n) = \widehat{\mathcal{P}_i}$ *and* $\widehat{\mathcal{Q}_i} + \mathrm{Rep}(n) = \mathcal{P}_i$
2. $\mathcal{Q}_i \cap \widehat{\mathcal{Q}_i} = \mathrm{Rep}(n)^{\perp}$ *and* $\mathcal{Q}_i \cap \widehat{\mathcal{Q}_i} = \{0\}$
3. $\mathcal{P}_i + \widehat{\mathcal{P}_i} = \mathcal{R}[A, \boldsymbol{\lambda}]$ *and* $\mathcal{P}_i \cap \widehat{\mathcal{P}_i} = \mathrm{Rep}(n)$

In the nega-Abelian case, that is if $\boldsymbol{\lambda} = (-1, \ldots, -1)$, an analogous result to Theorem 24 in the previous section can be proven.

Theorem 31 (Nega-Abelian LCD codes) *Let* $\mathcal{Q}_i$ *and* $\widehat{\mathcal{Q}_i}$ *be a pair of even-like polyadic negacyclic codes of Type II with the associated odd-like polyadic negacyclic codes of Type II* $\mathcal{P}_i$ *and* $\widehat{\mathcal{P}_i}$ *over the ring* $\mathcal{R}[A, \boldsymbol{\lambda}]$ *for* $1 \leq i \leq m$*. The following statements hold:*

1. $\mathcal{Q}_i^{\perp} = -\mathbf{1}_{\star}(\mathcal{P}_i)$ *and* $\widehat{\mathcal{Q}_i}^{\perp} = -\mathbf{1}_{\star}(\widehat{\mathcal{P}_i})$
2. *If* $-\mathbf{1}_{\star}(E_i) = E_i$ *then* $\mathcal{Q}_i^{\perp} = \mathcal{P}_i$, $\widehat{\mathcal{Q}_i}^{\perp} = \widehat{\mathcal{P}_i}$ *and* $\mathcal{Q}_i, \widehat{\mathcal{Q}_i}, \mathcal{P}_i, \widehat{\mathcal{P}_i}$ *are LCD codes over* $\mathcal{R}[A, \boldsymbol{\lambda}]$*, for* $1 \leq i \leq m$.

7 Conclusions

In this paper we have studied polyadic Abelian codes and consta-Abelian codes defined over some serial rings given by affine algebras of a certain type with a finite commutative chain coefficient ring. We have completely described them in terms of their generators associated with the concrete splitting of the Abelian group underlying the structure. As a follow-up applied work, it will be nice to check if the Gray mappings associated with the idempotent decomposition of these codes (see [6–8]) provide codes with good properties over the base chain ring.

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References

1. Brualdi, R.A., Pless, V.S.: Polyadic codes. vol. 25, pp. 3–17 (1989). https://doi.org/10.1016/0166-218X(89)90042-5. Combinatorics and complexity (Chicago, IL, 1987)
2. Chen, B., Dinh, H.Q., Fan, Y., Ling, S.: Polyadic constacyclic codes. IEEE Trans. Inform. Theory **61**(9), 4895–4904 (2015). https://doi.org/10.1109/TIT.2015.2451656. ISSN 0018-9448
3. Dinh, H.Q., López-Permouth, S.R.: Cyclic and negacyclic codes over finite chain rings. IEEE Trans. Inf. Theory **50**(8), 1728–1744 (2004). https://doi.org/10.1109/TIT.2004.831789. ISSN 0018-9448
4. Goyal, M., Raka, M.: Quadratic residue codes over the ring $\mathbb{F}_p[u]/\langle u^m - u\rangle$ and their Gray images. Cryptogr. Commun. **10**(2), 343–355 (2018). https://doi.org/10.1007/s12095-017-0223-z. ISSN 1936-2447,1936-2455
5. Goyal, M., Raka, M.: Duadic and triadic codes over a finite non-chain ring and their Gray images. Int. J. Inf. Coding Theory **5**(1), 36–54 (2018). https://doi.org/10.1504/ijicot.2018.091834. ISSN 1753-7703,1753-7711
6. Goyal, M., Raka, M.: Polyadic constacyclic codes over a non-chain ring $\mathbb{F}_q[u, v]/\langle f(u), g(v), uv - vu\rangle$. J. Appl. Math. Comput. **62**(1–2), 425–447 (2020). https://doi.org/10.1007/s12190-019-01290-x. ISSN 1598-5865
7. Goyal, M., Raka, M.: Polyadic cyclic codes over a non-chain ring. J. Comput. Commun. (2021). https://doi.org/10.4236/jcc.2021.95004
8. Islam, H., Martínez-Moro, E., Prakash, O.: Cyclic codes over a non-chain ring $R_{e,q}$ and their application to LCD codes. Discrete Math. **344**(10), 11 (2021). https://doi.org/10.1016/j.disc.2021.112545. Paper No. 112545, ISSN 0012-365X
9. Kürüz, F., Sari, M., Köroglu, M.E.: m-adic residue codes over the ring $\mathbb{F}_q[v]/(v^s - v)$ and their applications to quantum codes. Quantum Inf. Comput. **22**(5–6), 427–439 (2022). https://doi.org/10.26421/qic22.5-6-4. ISSN 1533-7146
10. Lim, C.J.: Consta-abelian polyadic codes. IEEE Trans. Inform. Theory **51**(6), 2198–2206 (2005). https://doi.org/10.1109/TIT.2005.847734. ISSN 0018-9448
11. Lim, C.J.: Consta-abelian polyadic codes. IEEE Trans. Inform. Theory **51**(6), 2198–2206 (2005). https://doi.org/10.1109/TIT.2005.847734. ISSN 0018-9448
12. Ling, S., Xing, C.: Polyadic codes revisited. IEEE Trans. Inform. Theory, 50 (1): 200–207, 2004. https://doi.org/10.1109/TIT.2003.821986. ISSN 0018-9448
13. Martínez-Moro, E., Rúa, I.F.: Multivariable codes over finite chain rings: serial codes. SIAM J. Discrete Math. **20**(4), 947–959 (2006). https://doi.org/10.1137/050632208. ISSN 0895-4801
14. Martínez-Moro, E., Rúa, I.F.: On repeated-root multivariable codes over a finite chain ring. Des. Codes Cryptogr. **45**(2), 219–227 (2007). https://doi.org/10.1007/s10623-007-9114-1. ISSN 0925-1022
15. Martínez-Moro, E., Piñera Nicolás, A., Rúa, I.F.: Multivariable codes in principal ideal polynomial quotient rings with applications to additive modular bivariate codes over $\mathbb{F}_4$. J. Pure Appl. Algebra **222**(2), 359–367 (2018). https://doi.org/10.1016/j.jpaa.2017.04.007. ISSN 0022-4049
16. Martínez-Moro, E., Piñera Nicolás, A., Rúa, I.F.: Codes over affine algebras with a finite commutative chain coefficient ring. Finite Fields Appl. **49**, 94–107 (2018). https://doi.org/10.1016/j.ffa.2017.09.008. ISSN 1071-5797
17. McDonald, B.R.: Finite rings with identity. Pure and Applied Mathematics, vol. 28. Marcel Dekker, Inc., New York (1974)
18. Norton, G.H., Sălăgean, A.: On the structure of linear and cyclic codes over a finite chain ring. Appl. Algebra Engrg. Comm. Comput. **10**(6), 489–506 (2000). https://doi.org/10.1007/PL00012382. ISSN 0938-1279

19. Puninski, G.: Serial rings. Kluwer Academic Publishers, Dordrecht (2001). https://doi.org/10.1007/978-94-010-0652-1. isbn:0-7923-7187-9

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