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RESEARCH

Low-hit-zone frequency hopping sequence sets under aperiodic Hamming correlation

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Abstract
The study of aperiodic Hamming correlation (APC) of frequency hopping sequences (FHSs) is a hard problem that has not been paid enough attention in the literature. For low-hit-zone (LHZ) FHSs, the study of APC becomes more difficult. We call them LHZ FHSs under APC (LHZ-APC FHSs). LHZ-APC FHSs are studied for the first time in this paper. First, we establish a bound on the family sizes of LHZ-APC FHS sets. Then we present a method for constructions of LHZ-APC FHS sets based on conventional FHS sets under periodic Hamming correlation (conventional PC FHS sets). By choosing different conventional PC FHS sets, we obtain three classes of LHZ-APC FHS sets whose family sizes are optimal or near optimal according to this new bound. Further, we modify the construction method and get more new LHZ-APC FHS sets with optimal family sizes.

Keywords Frequency hopping sequences · Aperiodic Hamming correlation · Low hit zone · Theoretical bound · Quasi-synchronous frequency hopping systems

Mathematics Subject Classification (2010) 94A55 · 94B05

1 Preliminaries

Aperiodic Hamming correlation (APC) of frequency hopping sequences (FHSs) provides a better evaluation for frequency hopping (FH) spread-spectrum (SS) systems than periodic Hamming correlation (PC). However, it is hard to find mathematical tools to design FHSs with good APC properties. APC of FHSs has not been paid enough attention in the literature compared with PC. There are a lot of constructions for FHSs with good PC [1–3, 6–9, 12, 24, 27, 29]. Recently, several results on APC of conventional FHSs have emerged [13, 14, 17, 31]. We call them conventional FHSs under APC (conventional APC FHSs). For quasi-synchronous FH SS systems, low-hit-zone (LHZ) FHSs are necessary as the time delay can be controlled within the LHZ. The interference is reduced as the time delay is kept in the LHZ [30]. There are some results on PC of LHZ FHSs [4, 16, 19, 20, 22, 23, 25]. For APC

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of LHZ FHSs, however, there are still not good results [21]. The study of APC of LHZ FHSs is very difficult.

This paper focuses on the APC of LHZ FHSs. We call them LHZ FHSs under APC (LHZ-APC FHSs). We first derive a bound on the family sizes of LHZ-APC FHS sets. Then we present a method for constructions of LHZ-APC FHS sets based on conventional FHS sets under PC (conventional PC FHS sets) and get three classes of LHZ-APC FHS sets. Further, we modify the construction method and get another class of LHZ-APC FHS sets. All of the LHZ-APC FHS sets in this paper have optimal or near optimal family sizes according to this new bound.

Next we give some notations and definitions.

Let $Q = \{q_0, q_1, \cdots, q_{\lambda-1}\}$ be a set of frequency slots. Let $G = \{G^0, G^1, \cdots, G^{M-1}\}$ be an FHS set over Q, where $G^k = (g_0^k, g_1^k, \cdots, g_{N-1}^k)$ for $k = 0, 1, \cdots, M-1$. For any integers x, y, let $\langle y \rangle_x$ satisfy that $y \equiv \langle y \rangle_x \bmod x$ and $0 \le \langle y \rangle_x \le x-1$. For $G^{k_1}, G^{k_2} \in G$, $0 \le k_1, k_2 \le M-1$, their PC function is

$$P_{G^{k_1}G^{k_2}}(\varsigma) = \sum_{t=0}^{N-1} c(g_t^{k_1}, g_{\langle t+\varsigma \rangle_N}^{k_2}), \tag{1}$$

where $c(u, v) = 0$ for $u \ne v$ and $c(u, v) = 1$ for $u = v$, and ς is the time delay. Their APC function at time delay ς $(0 \le \varsigma \le N-1)$ is

$$C_{G^{k_1}G^{k_2}}(\varsigma) = \sum_{t=0}^{N-\varsigma-1} c(g_t^{k_1}, g_{t+\varsigma}^{k_2}). \tag{2}$$

Let P_a, P_c, P_m be the maximum periodic Hamming auto-correlation (PAC), maximum periodic Hamming cross-correlation (PCC), and maximum PC of G respectively, which are given as follows:

$$\begin{aligned}
P_a &= \max\{P_{G^{k_1}G^{k_1}}(\varsigma) : 0 \le k_1 \le M-1,\ \varsigma = 1, 2, \cdots, N-1\},\\
P_c &= \max\{P_{G^{k_1}G^{k_2}}(\varsigma) : 0 \le k_1, k_2 \le M-1,\ k_1 \ne k_2,\ \varsigma = 0, 1, \cdots, N-1\},\\
P_m &= \max\{P_a, P_c\}.
\end{aligned}$$

By considering the PC in above definitions, we call G a conventional PC FHS set.

Let G be an LHZ FHS set whose LHZ is Z. Let C_a, C_c, C_m be the maximum aperiodic Hamming auto-correlation (APAC), maximum aperiodic Hamming cross-correlation (APCC), and maximum APC of G respectively, which are given as follows:

$$\begin{aligned}
C_a &= \max\{C_{G^{k_1}G^{k_1}}(\varsigma) : 0 \le k_1 \le M-1,\ \varsigma = 1, 2, \cdots, Z\},\\
C_c &= \max\{C_{G^{k_1}G^{k_2}}(\varsigma) : 0 \le k_1, k_2 \le M-1,\ k_1 \ne k_2,\ \varsigma = 0, 1, \cdots, Z\},\\
C_m &= \max\{C_a, C_c\}.
\end{aligned}$$

By considering the APC in above definitions, G is an LHZ-APC FHS set. When $Z = N-1$, G becomes a conventional APC FHS set.

In the remainder, we denote an LHZ-APC FHS set with family size M, sequence length N, frequency slot number λ, and LHZ Z, by $[N, M, \lambda, Z]$, a conventional APC FHS set with family size M, sequence length N, frequency slot number λ, by $[N, M, \lambda]$, and a conventional PC FHS set with family size M, sequence length N, frequency slot number λ, by (N, M, λ).

By utilizing the Singleton bound, Ding et al. [5] and Yang et al. [28] obtained a bound on conventional PC FHS sets.

Lemma 1 (Singleton bound on conventional PC FHS sets) For an (N, M, λ) FHS set with maximum PC P_m, we have

$$M \leq \frac{\lambda^{P_m+1}}{N}. \tag{3}$$

Liu et al. [15, 18] then improved this bound.

Lemma 2 For an (N, M, λ) FHS set with maximum PC P_m, we have

$$\sum_{\kappa|\gcd(P_m+1,N)} \varphi(\kappa)\lambda^{\frac{P_m+1}{\kappa}} \geq MN \tag{4}$$

where $\varphi(\kappa)$ is the Möbius function, i.e.

$$\varphi(\kappa) = \begin{cases} 1, & \kappa = 1 \\ (-1)^i, & \kappa \text{ is equal to the product of } i \text{ different primes} \\ 0, & \text{otherwise.} \end{cases}$$

Definition 1 Let G' be an (N', M', λ') FHS set with maximum PC P'_m. If $M = M'$ is the largest integer satisfying (3) or (4), then the family size of G' is optimal.

2 A bound on LHZ-APC FHS sets

We establish a bound on the family sizes of LHZ-APC FHS sets in this section.

Theorem 1 For an $[N, M, \lambda, Z]$ LHZ-APC FHS set with maximum APC C_m, we have

$$M \leq \frac{\lambda^{C_m+1}}{\min\{N - C_m, Z + 1\}}. \tag{5}$$

Proof Let $G = \{G^0, G^1, \cdots, G^{M-1}\}$ be an $[N, M, \lambda, Z]$ LHZ-APC FHS set with maximum APC C_m, where $G^k = (g_0^k, g_1^k, \cdots, g_{N-1}^k)$ for $k = 0, 1, \cdots, M-1$. For $0 \leq k_1, k_2 \leq M-1$ and $0 \leq \varsigma \leq Z$, the APC between G^{k_1} and G^{k_2} at time delay ς is

$$C_{G^{k_1}G^{k_2}}(\varsigma) = \sum_{t=0}^{N-\varsigma-1} c(g_t^{k_1}, g_{t+\varsigma}^{k_2}). \tag{6}$$

Since the maximum APC of G is C_m, we have

$$\sum_{t=0}^{N-\varsigma-1} c(g_t^{k_1}, g_{t+\varsigma}^{k_2}) \leq C_m \tag{7}$$

for $(k_1 - k_2)^2 + \varsigma^2 \neq 0$.

Case 1: $N - C_m \leq Z + 1$. For two vectors $(g_{t_1}^{k_1}, g_{t_1+1}^{k_1}, \cdots, g_{t_1+C_m}^{k_1})$ and $(g_{t_2}^{k_2}, g_{t_2+1}^{k_2}, \cdots, g_{t_2+C_m}^{k_2})$ where $0 \leq t_1 \leq t_2 \leq N - C_m - 1$, $(t_1 - t_2)^2 + (k_1 - k_2)^2 \neq 0$, we have

$$\begin{aligned} \sum_{t=0}^{C_m} c(g_{t_1+t}^{k_1}, g_{t_2+t}^{k_2}) &\leq \sum_{t=-t_1}^{N-t_2-1} c(g_{t_1+t}^{k_1}, g_{t_2+t}^{k_2}) \\ &= \sum_{t=0}^{N-(t_2-t_1)-1} c(g_t^{k_1}, g_{t+t_2-t_1}^{k_2}). \end{aligned} \tag{8}$$

Since $0 \le t_2 - t_1 \le N - C_m - 1 \le Z$ and $(k_1 - k_2)^2 + (t_2 - t_1)^2 \neq 0$, by (7) and (8) we have

$$\sum_{t=0}^{C_m} c(g_{t_1+t}^{k_1}, g_{t_2+t}^{k_2}) \le C_m. \tag{9}$$

This means that in the vector set

$$\{(g_t^k, g_{t+1}^k, \cdots, g_{t+C_m}^k) : t = 0, 1, \cdots, N - C_m - 1,\ k = 0, 1, \cdots, M - 1\}$$

the elements are distinct. Since it is up to λ^{C_m+1} different sequences with length $C_m + 1$, we have $(N - C_m)M \le \lambda^{C_m+1}$ which leads to

$$M \le \frac{\lambda^{C_m+1}}{N - C_m}. \tag{10}$$

Case 2: $N - C_m > Z + 1$. Let $(g_{t_1}^{k_1}, g_{t_1+1}^{k_1}, \cdots, g_{t_1+C_m}^{k_1})$ and $(g_{t_2}^{k_2}, g_{t_2+1}^{k_2}, \cdots, g_{t_2+C_m}^{k_2})$ be two vectors, where $0 \le t_1 \le t_2 \le Z$, $(t_1 - t_2)^2 + (k_1 - k_2)^2 \neq 0$. Note that $C_m < N - Z - 1 \le N - t_2 - 1$. Similar to Case 1, we have

$$\sum_{t=0}^{C_m} c(g_{t_1+t}^{k_1}, g_{t_2+t}^{k_2}) \le C_m \tag{11}$$

which means that in the vector set

$$\{(g_t^k, g_{t+1}^k, \cdots, g_{t+C_m}^k) : t = 0, 1, \cdots, Z,\ k = 0, 1, \cdots, M - 1\}$$

the elements are distinct. Thus, we have $(Z + 1)M \le \lambda^{C_m+1}$. It follows that

$$M \le \frac{\lambda^{C_m+1}}{Z + 1}. \tag{12}$$

Case 1 and Case 2 imply that $M \le \frac{\lambda^{C_m+1}}{\min\{N - C_m, Z+1\}}$. □

Let $Z = N - 1$ in (5). It degenerates into the bound on conventional APC FHS sets. It was first derived by Liu and Peng [13] in 2014.

Corollary 1 For an $[N, M, \lambda]$ conventional APC FHS set with maximum APC C_m, we have

$$M \le \frac{\lambda^{C_m+1}}{N - C_m}. \tag{13}$$

Definition 2 Let G' be an $[N', M', \lambda', Z']$ LHZ-APC FHS set with maximum APC C'_m. If $M = M'$ is the largest integer satisfying (5), then the family size of G' is optimal. If $M = M' + 1$ is the largest integer satisfying (5), then the family size of G' is near optimal.

Definition 3 Let G' be an $[N', M', \lambda']$ FHS set with maximum APC C'_m. If $M = M'$ is the largest integer satisfying (13), then the family size of G' is optimal. If $M = M' + 1$ is the largest integer satisfying (13), then the family size of G' is near optimal.

Example 1 For a frequency slot set $Q = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$, a [10, 20, 9, 3] LHZ-APC FHS set G is

$$
\begin{aligned}
G = \{ &(4, 3, 8, 5, 7, 0, 8, 3, 5, 3), && (7, 0, 8, 3, 5, 3, 3, 1, 6, 8),\\
&(5, 3, 3, 1, 6, 8, 4, 0, 6, 1), && (6, 8, 4, 0, 6, 1, 8, 1, 1, 5),\\
&(6, 1, 8, 1, 1, 5, 2, 6, 3, 0), && (1, 5, 2, 6, 3, 0, 2, 5, 6, 5),\\
&(3, 0, 2, 5, 6, 5, 5, 8, 7, 2), && (6, 5, 5, 8, 7, 2, 1, 0, 7, 8),\\
&(7, 2, 1, 0, 7, 8, 2, 8, 8, 6), && (7, 8, 2, 8, 8, 6, 4, 7, 5, 0),\\
&(8, 6, 4, 7, 5, 0, 4, 6, 7, 6), && (5, 0, 4, 6, 7, 6, 6, 2, 3, 4),\\
&(7, 6, 6, 2, 3, 4, 8, 0, 3, 2), && (3, 4, 8, 0, 3, 2, 4, 2, 2, 7),\\
&(3, 2, 4, 2, 2, 7, 1, 3, 6, 0), && (2, 7, 1, 3, 6, 0, 1, 7, 3, 7),\\
&(6, 0, 1, 7, 3, 7, 7, 4, 5, 1), && (3, 7, 7, 4, 5, 1, 2, 0, 5, 4),\\
&(5, 1, 2, 0, 5, 4, 1, 4, 4, 3), && (5, 4, 1, 4, 4, 3, 8, 5, 7, 0)\}.
\end{aligned}
$$

It can be verified that the maximum APC of the LHZ-APC FHS set G is 1. By (5)

$$M \leq \left\lfloor \frac{81}{4} \right\rfloor = 20,$$

which implies that the family size of LHZ-APC FHS set G is optimal.

3 A new method for constructions of LHZ-APC FHS sets

Now a new method which is suitable for constructions of LHZ-APC FHS sets is given. Then we construct LHZ-APC FHS sets, whose family sizes are optimal/near optimal with respect to the new bound.

A construction method:

Step 1: Choose an (N, M, λ) conventional PC FHS set $T = \{T^0, T^1, \cdots, T^{M-1}\}$, where $T^k = (t_0^k, t_1^k, \cdots, t_{N-1}^k)$ for $k = 0, 1, \cdots, M-1$. P_m is the maximum PC of T.

Step 2: Let Z, l be two integers such that $Z + 1 \mid N$ and $P_m \leq l \leq N - Z - 1$. An LHZ-APC FHS set S is given by

$$S = \Big\{(t^k_{i(Z+1)}, t^k_{i(Z+1)+1}, \cdots, t^k_{i(Z+1)+Z+l}) : \quad i = 0, 1, \cdots, \frac{N}{Z+1} - 1,\ k = 0, 1, \cdots, M-1\Big\} \tag{14}$$

where the subscripts are performed modulo N.

We illustrate this construction method by Fig. 1. We choose $Z = 2, l = 5$ by giving a conventional PC FHS set T with sequence length $N = 12$. For a sequence $(t_0^k, t_1^k, t_2^k, t_3^k, t_4^k, t_5^k, t_6^k, t_7^k, t_8^k, t_9^k, t_{10}^k, t_{11}^k) \in T$, select red square parts in Fig. 1 as sequences S^0, S^1, S^2, S^3 respectively. Note that the red square parts of S^0, S^1, S^2, S^3 overlap each other partially. The size of each overlapping area is l. We can get other sequences in this way from each sequence of T. Then a sequence set $S = \{S^0, S^1, S^2, S^3, \cdots\}$ can be obtained. Next we will discuss the fact that S is an LHZ-APC FHS set whose family size is optimal under some conditions.

For the LHZ-APC FHS set S obtained by the construction method, the following result is obtained.

Theorem 2 S is an $[l + Z + 1, \frac{MN}{Z+1}, \lambda, Z]$ LHZ-APC FHS set with maximum APC P_m.

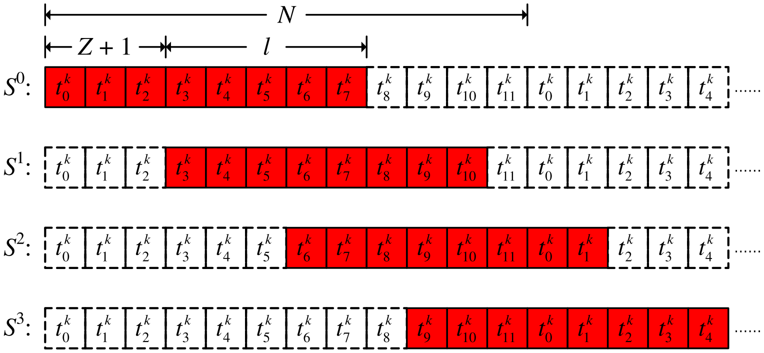

Fig. 1 Construction method for LHZ-APC FHS sets

Proof Obviously, the sequence length and family size of S are $l + Z + 1$ and $\frac{MN}{Z+1}$ respectively. The frequency slot number is also λ. For $S^{j_1} = (t^{k_1}_{i_1(Z+1)}, t^{k_1}_{i_1(Z+1)+1}, \cdots, t^{k_1}_{i_1(Z+1)+Z+l}), S^{j_2} = (t^{k_2}_{i_2(Z+1)}, t^{k_2}_{i_2(Z+1)+1}, \cdots, t^{k_2}_{i_2(Z+1)+Z+l}) \in S$, the APC function between them at time delay ς is

$$\begin{aligned} C_{S^{j_1}S^{j_2}}(\varsigma) &= \sum_{m=0}^{l+Z-\varsigma} c(t^{k_1}_{i_1(Z+1)+m}, t^{k_2}_{i_2(Z+1)+m+\varsigma}) \\ &\le \sum_{m=0}^{N-1} c(t^{k_1}_{i_1(Z+1)+m}, t^{k_2}_{i_2(Z+1)+m+\varsigma}) \\ &= \sum_{m=0}^{N-1} c(t^{k_1}_{m}, t^{k_2}_{m+i_2(Z+1)-i_1(Z+1)+\varsigma}). \end{aligned} \tag{15}$$

Case 1: $i_1 = i_2$, $k_1 = k_2$, $1 \le \varsigma \le Z$. In this case, (15) becomes $C_{S^{j_1}S^{j_2}}(\varsigma) \le \sum_{m=0}^{N-1} c(t^{k_1}_m, t^{k_1}_{m+\varsigma}) \le P_m$.

Case 2: $k_1 \ne k_2$, $0 \le \varsigma \le Z$. Similarly, we can get $C_{S^{j_1}S^{j_2}}(\varsigma) \le P_m$.

Case 3: $i_1 \ne i_2$, $0 \le \varsigma \le Z$. Since $\langle i_2(Z+1) - i_1(Z+1) + \varsigma\rangle_N \ne 0$, we have $C_{S^{j_1}S^{j_2}}(\varsigma) \le P_m$.

Thus, within LHZ Z, the maximum APC of S is P_m. □

Then we discuss the optimality of S.

Theorem 3 The LHZ-APC FHS set S has optimal family size by (5) if T has optimal family size by bound (3) and $\left\lfloor \frac{\lambda^{P_m+1}}{Z+1} \right\rfloor = \frac{N}{Z+1} \cdot \left\lfloor \frac{\lambda^{P_m+1}}{N} \right\rfloor$.

Proof If the conventional PC FHS set T has optimal family size by (3), then

$$M = \left\lfloor \frac{\lambda^{P_m+1}}{N} \right\rfloor. \tag{16}$$

Since $\left\lfloor \frac{\lambda^{P_m+1}}{Z+1} \right\rfloor = \frac{N}{Z+1} \cdot \left\lfloor \frac{\lambda^{P_m+1}}{N} \right\rfloor$, we have

$$\frac{MN}{Z+1} = \left\lfloor \frac{\lambda^{P_m+1}}{Z+1} \right\rfloor. \tag{17}$$

$\frac{MN}{Z+1} = \left\lfloor \frac{\lambda^{P_m+1}}{\min\{l+Z+1-P_m, Z+1\}} \right\rfloor$ holds for $l \geq P_m$. □

In order to illustrate the construction method by a concrete example, we choose a conventional PC FHS (1, 0, 1, 1, 2, 0, 2, 2) whose maximum PC is 2. By choosing $Z = 3$ and $l = 3$, an LHZ-APC FHS set $\{(1, 0, 1, 1, 2, 0, 2), (2, 0, 2, 2, 1, 0, 1)\}$ can be obtained. It is easy to verify that the set is a [7, 2, 3, 3] LHZ-APC FHS set with maximum APC 2. Note that this example only illustrates the construction method without considering its optimality.

Next we give three classes of conventional PC FHS sets which will be used to generate LHZ-APC FHS sets which have optimal or near optimal family sizes. A finite field with prime power q is defined by F_q.

A. First Class

For F_q, let α be its generator and β be a primitive n'th root of unity in F_q where $n' \mid q-1$. Let

$$\mathbb{U} = \left\{ \sum_{j=1}^{e} a_j x^j : a_j \in \mathrm{F}_q, j = 1, 2, \cdots, e \right\} \tag{18}$$

and

$$\mathbb{V} = \left\{ (u(\alpha^i), u(\beta\alpha^i), \cdots, u(\beta^{n'-1}\alpha^i)) : u(x) \in \mathbb{U}, i = 0, 1, \cdots, \frac{q-1}{n'} - 1 \right\}, \tag{19}$$

where $1 \leq e < \min\{x : x \mid n',\ x > 1\}$.

Definition 4 For a sequence $g = (g_0, g_1, \cdots, g_{N-1})$, define $\varrho^i(g) = (g_i, g_{\langle i+1\rangle_N}, \cdots, g_{\langle i+N-1\rangle_N})$ for $i = 0, 1, \cdots, N-1$.

Definition 5 For two different sequences x, y, if there exists ς making $x = \varrho^{\varsigma}(y)$, then x and y are cyclic equivalent.

Partition $\mathbb{V}$ into cyclic equivalence classes $\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3, \cdots$ [26], that is, all the sequences that are cyclic equivalent constitute some $\mathbb{V}_i$, $i = 1, 2, 3, \cdots$. Note that the cycle lengths of each cyclic equivalence class can be different. We only consider the cyclic equivalence classes whose cycle lengths are n'. Select one element from each cyclic equivalence class with cycle length n', and put them together as set $\mathbb{V}^*$.

Take $\mathbb{V}^*$ as a conventional PC FHS set in Step 1 of the construction method. Let Z, l be two integers such that $Z + 1 \mid n'$ and $e - 1 \leq l \leq n' - Z - 1$. Then an LHZ-APC FHS set S_1 can be obtained by Step 2 of the construction method.

Theorem 4 S_1 is an $[l + Z + 1, \frac{q^e-1}{Z+1}, q, Z]$ LHZ-APC FHS set with maximum APC $e - 1$ and has optimal family size by (5).

Proof $\mathbb{V}^*$ is an $(n', \frac{q^e-1}{n'}, q)$ conventional PC FHS set with maximum PC $e-1$ by Lemma 20 in [28], and has optimal family size by bound (3). By Theorem 2, S_1 is an $[l+Z+1, \frac{q^e-1}{Z+1}, q, Z]$ LHZ-APC FHS set with maximum APC $e - 1$. Since $Z + 1 \mid n'$ and $n' \mid q - 1$, we have

$Z+1 \mid q^e - 1$. Then

$$\left\lfloor \frac{q^{(e-1)+1}}{Z+1} \right\rfloor = \left\lfloor \frac{q^e - 1 + 1}{Z+1} \right\rfloor$$
$$= \frac{q^e - 1}{Z+1} + \left\lfloor \frac{1}{Z+1} \right\rfloor$$
$$= \frac{n'}{Z+1} \cdot \frac{q^e - 1}{n'}$$
$$= \frac{n'}{Z+1} \cdot \left(\frac{q^e - 1}{n'} + \left\lfloor \frac{1}{n'} \right\rfloor \right)$$
$$= \frac{n'}{Z+1} \cdot \left\lfloor \frac{q^{(e-1)+1}}{n'} \right\rfloor .$$

By Theorem 3, S_1 has optimal family size by (5). □

Example 2 For $q = 512, n' = 511, e = 3, Z = 72, l = 427$, we can obtain a [500, 1838599, 512, 72] LHZ-APC FHS set with maximum APC 2. By (5)

$$M \le \left\lfloor \frac{512^3}{73} \right\rfloor = 1838599.$$

Hence, it has optimal family size.

B. Second Class

For F_{q^2} with prime power q, let δ be a primitive n'th root of unity in F_{q^2} where n' is odd and $n' \mid q+1$. Define

$$E_0 = \{0\},\ \ E_j = \{j, n'-j\},\ j = 1, 2, \cdots, \frac{n'-1}{2}. \tag{20}$$

For an integer e', $1 \le e' < \frac{\min\{x : x|n',\ x>1\}+1}{2}$, let

$$A(x) = A_{\frac{n'-1}{2}-e'+1}(x) A_{\frac{n'-1}{2}-e'+2}(x) \cdots A_{\frac{n'-1}{2}}(x) \tag{21}$$

where

$$A_i(x) = \prod_{k \in E_i} (x - \delta^k),\ \ i = \frac{n'-1}{2} - e' + 1, \frac{n'-1}{2} - e' + 2, \cdots, \frac{n'-1}{2}.$$

Let $A(x)$ be the parity-check polynomial of $\mathbb{D}$ whose length is n'. Partition $\mathbb{D}$ into cyclic equivalence classes $\mathbb{D}_1, \mathbb{D}_2, \mathbb{D}_3, \cdots$, that is, all the codewords that are cyclic equivalent constitute some $\mathbb{D}_i$, $i = 1, 2, 3, \cdots$. We only consider the cyclic equivalence classes whose cycle lengths are n'. Select one codeword from each cyclic equivalence class with cycle length n', and put them together as set $\mathbb{D}^*$ which is a subset of $\mathbb{D}$.

Take $\mathbb{D}^*$ as a conventional PC FHS set in Step 1 of the construction method. Let Z, l be two integers such that $Z+1 \mid n'$ and $2e'-1 \le l \le n'-Z-1$. By Step 2 of the construction method, an LHZ-APC FHS set S_2 can be generated.

Theorem 5 S_2 is an $[l+Z+1, \frac{q^{2e'}-1}{Z+1}, q, Z]$ LHZ-APC FHS set with maximum APC $2e'-1$ and has optimal family size by (5).

Proof In [17], we can deduce that $\mathbb{D}^*$ is an $(n', \frac{q^{2e'}-1}{n'}, q)$ conventional PC FHS set with maximum PC $2e'-1$. It is clear that

$$\frac{q^{2e'}-1}{n'} = \frac{q^{2e'}-1}{n'} + \left\lfloor \frac{1}{n'} \right\rfloor = \left\lfloor \frac{q^{(2e'-1)+1}}{n'} \right\rfloor.$$

Thus, $\mathbb{D}^*$ has optimal family size by (3). By Theorem 2, S_2 is an $[l+Z+1, \frac{q^{2e'}-1}{Z+1}, q, Z]$ LHZ-APC FHS set with maximum APC $2e'-1$. Since $Z+1 \mid n'$ and $n' \mid q+1$, we have $Z+1 \mid q^{2e'}-1$. Then

$$\begin{aligned}\left\lfloor \frac{q^{(2e'-1)+1}}{Z+1} \right\rfloor &= \left\lfloor \frac{q^{2e'}-1+1}{Z+1} \right\rfloor \\ &= \frac{q^{2e'}-1}{Z+1} + \left\lfloor \frac{1}{Z+1} \right\rfloor \\ &= \frac{n'}{Z+1} \cdot \frac{q^{2e'}-1}{n'} \\ &= \frac{n'}{Z+1} \cdot \left(\frac{q^{2e'}-1}{n'} + \left\lfloor \frac{1}{n'} \right\rfloor \right) \\ &= \frac{n'}{Z+1} \cdot \left\lfloor \frac{q^{(2e'-1)+1}}{n'} \right\rfloor.\end{aligned}$$

By Theorem 3, S_2 has optimal family size by (5). □

Example 3 For $q = 64, n' = 65, e' = 2, Z = 12, l = 47$, a [60, 1290555, 64, 12] LHZ-APC FHS set with maximum APC 3 can be obtained. By bound (5), we have

$$M \le \left\lfloor \frac{64^4}{13} \right\rfloor = 1290555.$$

Thus, it has optimal family size.

C. Third Class

For F_{q^2} with prime power q, let α be its generator and $\beta = \alpha^{q-1}$. Let $\mathrm{tr}(x)$ be a trace function from F_{q^2} to F_q. Define a one-to-one mapping from $\{0, 1, \cdots, q-1\}$ to F_q, by $\pi(x)$. Let $B = \{(b_0^j, b_1^j, \cdots, b_q^j) : 0 \le j \le q(q-1)-1\}$ where

$$b_i^{x(q-1)+y} = \mathrm{tr}(\beta^i \alpha^y) + \pi(x)$$

for $0 \le x \le q-1, 0 \le y \le q-2, 0 \le i \le q$.

Take B as a conventional PC FHS set in Step 1 of the construction method. Let Z, l satisfy that $Z+1 \mid q+1$, $Z > \frac{q}{2}-1$, and $2 \le l \le q-Z$. By Step 2 of the construction method, an LHZ-APC FHS set S_3 can be generated.

Theorem 6 S_3 is an $[l+Z+1, \frac{q(q^2-1)}{Z+1}, q, Z]$ LHZ-APC FHS set with maximum APC 2 and has near optimal family size by (5).

Proof In [17], it is known that B is a $(q+1, q(q-1), q)$ conventional PC FHS set with maximum PC 2. By Theorem 2, S_3 is an $[l+Z+1, \frac{q(q^2-1)}{Z+1}, q, Z]$ LHZ-APC FHS set with maximum APC 2. Since $Z > \frac{q}{2}-1$ and $2 \le l \le q-Z$, we have $\left\lfloor \frac{q}{Z+1} \right\rfloor = 1$. For the LHZ-APC FHS set S_3, by (5)

$$\begin{aligned} M &\le \left\lfloor \frac{q^{2+1}}{\min\{l+Z+1-2, Z+1\}} \right\rfloor \\ &= \left\lfloor \frac{q^3-q+q}{Z+1} \right\rfloor \\ &= \frac{q^3-q}{Z+1} + \left\lfloor \frac{q}{Z+1} \right\rfloor \\ &= \frac{q^3-q}{Z+1} + 1. \end{aligned}$$

This implies that S_3 has near optimal family size. □

Example 4 For $q = 11$, $Z = 5$, $l = 5$, an $[11, 220, 11, 5]$ LHZ-APC FHS set with maximum APC 2 is obtained. By (5)

$$M \le \left\lfloor \frac{11^3}{6} \right\rfloor = 221.$$

Thus, it has near optimal family size.

4 A modified method for constructions of LHZ-APC FHS sets

In this section, we modify the construction method for LHZ-APC FHS sets which was presented in Section 3.

A modified construction method:

Step 1: Let n, N satisfy that $n \mid N$. The number of frequency slots is λ. Choose a sequence set $T = \{U^0, U^1, \cdots, U^{M_1-1}, V^0, V^1, \cdots, V^{M_2-1}\}$ with sequence length N, where $U^{k_1} = (u_0^{k_1}, u_1^{k_1}, \cdots, u_{N-1}^{k_1})$ for $k_1 = 0, 1, \cdots, M_1-1$ and $V^{k_2} = (v_0^{k_2}, v_1^{k_2}, \cdots, v_{n-1}^{k_2}, v_0^{k_2}, v_1^{k_2}, \cdots, v_{n-1}^{k_2}, \cdots, v_0^{k_2}, v_1^{k_2}, \cdots, v_{n-1}^{k_2})$ for $k_2 = 0, 1, \cdots, M_2-1$. It is clear that V^{k_2} has cycle length n. Here, $P_{U^{k_1}U^{k_1}}(\varsigma) \le P_m$ for $1 \le \varsigma \le N-1, k_1 = 0, 1, \cdots, M_1-1$, and $P_{V^{k_2}V^{k_2}}(\varsigma) \le P_m$ for $1 \le \varsigma \le n-1, k_2 = 0, 1, \cdots, M_2-1$. The maximum PCC of T is P_m.

Step 2: Let Z, l be two integers such that $Z+1 \mid n$ and $P_m \le l \le N-Z-1$. An LHZ-APC FHS set S is given as

$$\begin{aligned} S = \\ &\left\{(u_{i(Z+1)}^{k_1}, u_{i(Z+1)+1}^{k_1}, \cdots, u_{i(Z+1)+Z+l}^{k_1}) : i = 0, 1, \cdots, \frac{N}{Z+1}-1, k_1 = 0, 1, \cdots, M_1-1\right\} \bigcup \\ &\left\{(v_{j(Z+1)}^{k_2}, v_{j(Z+1)+1}^{k_2}, \cdots, v_{j(Z+1)+Z+l}^{k_2}) : j = 0, 1, \cdots, \frac{n}{Z+1}-1, k_2 = 0, 1, \cdots, M_2-1\right\}, \end{aligned} \quad (22)$$

where the subscripts of $u_{i(Z+1)+i'}^{k_1}$ and $v_{j(Z+1)+i'}^{k_2}$, $i' = 0, 1, \cdots, Z+l$, are performed modulo N and n, respectively.

We illustrate the LHZ-APC FHSs generated by a sequence with cycle length n as follows.

In Fig. 2, the sequence set T has sequence length $N = 12$. For $V^k = (v_0^k, v_1^k, v_2^k, v_3^k, v_4^k, v_5^k, v_0^k, v_1^k, v_2^k, v_3^k, v_4^k, v_5^k) \in T$, $k = 0, 1, \cdots, M_2 - 1$, we choose $Z = 2, l = 5$. It is clear that V^k has cycle length $n = 6$. Select red square parts in Fig. 2 as sequences S^0, S^1 respectively. It is easy to see that S^0 and S^1 overlap each other partially within the cycle length $n = 6$. The size of each overlapping area is l. Note that $l > n - Z - 1$ in Fig. 2. The other sequences generated by $U^0, U^1, \cdots, U^{M_1-1}$ are performed in the same way as that in Section 3. Then $S = \{S^0, S^1, \cdots\}$ can be generated.

Theorem 7 S is an $[l+Z+1, \frac{M_1N+M_2n}{Z+1}, \lambda, Z]$ LHZ-APC FHS set with maximum APC P_m.

Proof Obviously, the sequence length and family size of S are $l + Z + 1$ and $\frac{M_1N+M_2n}{Z+1}$ respectively. The number of frequency slots is also λ. For $S^{m_1} = (v^{k_{21}}_{j_1(Z+1)}, v^{k_{21}}_{j_1(Z+1)+1}, \cdots, v^{k_{21}}_{j_1(Z+1)+Z+l})$, $S^{m_2} = (v^{k_{22}}_{j_2(Z+1)}, v^{k_{22}}_{j_2(Z+1)+1}, \cdots, v^{k_{22}}_{j_2(Z+1)+Z+l}) \in S$, their APC function at time delay ς is

$$\begin{aligned}
C_{S^{m_1}S^{m_2}}(\varsigma) &= \sum_{m=0}^{l+Z-\varsigma} c(v^{k_{21}}_{j_1(Z+1)+m}, v^{k_{22}}_{j_2(Z+1)+m+\varsigma}) \\
&\le \sum_{m=0}^{N-1} c(v^{k_{21}}_{j_1(Z+1)+m}, v^{k_{22}}_{j_2(Z+1)+m+\varsigma}) \\
&= \sum_{m=0}^{N-1} c(v^{k_{21}}_{m}, v^{k_{22}}_{m+j_2(Z+1)-j_1(Z+1)+\varsigma}) \\
&\le P_{V^{k_{21}}V^{k_{22}}}(j_2(Z+1) - j_1(Z+1) + \varsigma). \qquad (23)
\end{aligned}$$

The subscripts are performed modulo n.

Case 1: $j_1 = j_2$, $k_{21} = k_{22}$, $1 \le \varsigma \le Z$. In this case, (23) becomes $C_{S^{m_1}S^{m_2}}(\varsigma) \le P_{V^{k_{21}}V^{k_{21}}}(\varsigma) \le P_m$.

Case 2: $k_{21} \ne k_{22}$, $0 \le \varsigma \le Z$. Similarly, we can get $C_{S^{m_1}S^{m_2}}(\varsigma) \le P_m$.

Case 3: $j_1 \ne j_2$, $0 \le \varsigma \le Z$. Note that $j_1, j_2 = 0, 1, \cdots, \frac{n}{Z+1} - 1$. If $j_1 < j_2$, then $Z+1 \le j_2(Z+1) - j_1(Z+1) + \varsigma \le n-1$ which leads to $C_{S^{m_1}S^{m_2}}(\varsigma) \le P_m$. If $j_1 > j_2$, then $Z+1-n \le j_2(Z+1) - j_1(Z+1) + \varsigma \le -1$. Since $P_{V^{k_{21}}V^{k_{22}}}(j_2(Z+1) - j_1(Z+1) + \varsigma) = P_{V^{k_{22}}V^{k_{21}}}(-(j_2(Z+1) - j_1(Z+1) + \varsigma))$, we have $C_{S^{m_1}S^{m_2}}(\varsigma) \le P_{V^{k_{22}}V^{k_{21}}}(-(j_2(Z+1) - j_1(Z+1) + \varsigma)) \le P_m$.

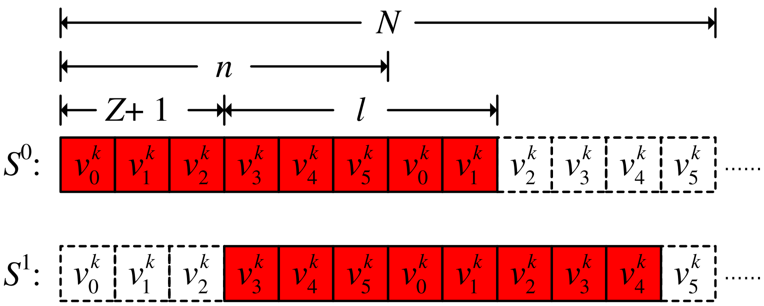

Fig. 2 LHZ-APC FHSs generated by a sequence with cycle length n

For $S^{m_1} = (u^{k_1}_{i(Z+1)}, u^{k_1}_{i(Z+1)+1}, \cdots, u^{k_1}_{i(Z+1)+Z+l}), S^{m_2} = (v^{k_2}_{j(Z+1)}, v^{k_2}_{j(Z+1)+1}, \cdots, v^{k_2}_{j(Z+1)+Z+l}) \in S$, the APC function between them at time delay ς is

$$
\begin{aligned}
C_{S^{m_1}S^{m_2}}(\varsigma) &= \sum_{m=0}^{l+Z-\varsigma} c(u^{k_1}_{i(Z+1)+m}, v^{k_2}_{j(Z+1)+m+\varsigma}) \\
&\leq \sum_{m=0}^{N-1} c(u^{k_1}_{i(Z+1)+m}, v^{k_2}_{j(Z+1)+m+\varsigma}) \\
&= \sum_{m=0}^{N-1} c(u^{k_1}_{m}, v^{k_2}_{m+j(Z+1)-i(Z+1)+\varsigma}) \\
&\leq P_{U^{k_1}V^{k_2}}(j(Z+1)-i(Z+1)+\varsigma) \\
&\leq P_m.
\end{aligned} \tag{24}
$$

The subscripts of $u^{k_1}_{i(Z+1)+m}$ and $v^{k_2}_r$, $r = j(Z+1)+m+\varsigma, m+j(Z+1)-i(Z+1)+\varsigma$, are performed modulo N and n, respectively.

For $S^{m_1} = (u^{k_{11}}_{i_1(Z+1)}, u^{k_{11}}_{i_1(Z+1)+1}, \cdots, u^{k_{11}}_{i_1(Z+1)+Z+l}), S^{m_2} = (u^{k_{12}}_{i_2(Z+1)}, u^{k_{12}}_{i_2(Z+1)+1}, \cdots, u^{k_{12}}_{i_2(Z+1)+Z+l}) \in S$, we can get the same results in a similar way.

Thus, the maximum APC of S is P_m within LHZ Z. □

Next we give a class of sequence sets which was described by Step 1 of the modified construction method. Then a class of LHZ-APC FHS sets can be generated.

In (18), we set $\min\{x : x \mid n', x > 1\} \leq e < \min\{y : y \mid n', y > \min\{x : x \mid n', x > 1\}\}$. Let $n' = q-1$ in (19). Select an element from each equivalence class of $\mathbb{V}$, and $\mathbb{H}'$ is obtained. Define a set consisting of those elements $x \in \mathbb{H}'$ which have cycle length $q-1$, by $\mathbb{U}'$. Let $\mathbb{V}' = \{V : V \in \mathbb{H}' \backslash \mathbb{U}', V \neq (0, 0, \cdots, 0)\}$. By Theorem 3.3 in [26], the elements in $\mathbb{V}'$ have cycle length $\frac{q-1}{\min\{x:x|q-1, x>1\}}$. Next we will determine the sizes of $\mathbb{U}'$ and $\mathbb{V}'$.

For simplicity and convenience, let $\theta = \min\{x : x \mid q-1, x > 1\}$.

Lemma 3 The size of $\mathbb{H}'$ is

$$
\mid \mathbb{H}' \mid = \frac{q^e + (\theta-1)q^{\lfloor \frac{e}{\theta} \rfloor} - \theta + q - 1}{q-1}. \tag{25}
$$

Proof Let ϱ be defined in Definition 4. By the Polya's counting formula [10], we have

$$
\mid \mathbb{H}' \mid = \frac{1}{q-1} \sum_{l|q-1} I(\varrho^l)\phi(\frac{q-1}{l}) \tag{26}
$$

under the cyclic group $\{\varrho, \varrho^2, \cdots, \varrho^{q-1}\}$, where $\phi(\cdot)$ is the Euler ϕ-function and $I(\varrho^l)$ the number of elements in $\mathbb{V}$ fixed by ϱ^l. Let J_m be the number of equivalence classes whose cycle lengths are m. Thus, $I(\varrho^l) = \sum_{m|l} mJ_m$. For a codeword with cycle length $m \mid l$, we have

$$
u(\beta^j) = u(\beta^{j+l}), \ j = 0, 1, \cdots, q-2,
$$

which leads to

$$
\sum_{i=1}^{e} a_i(\beta^{il} - 1)\beta^{ij} = 0, \ j = 0, 1, \cdots, q-2.
$$

Then $\sum_{i=1}^{e} a_i(\beta^{il}-1)x^i = 0$ has $q-1$ different roots. Since $e < q-1$, we have

$$a_i(\beta^{il}-1)=0,\ i=1,2,\cdots,e.$$

For $il \not\equiv 0 \bmod q-1$, since $\beta^{il}-1 \neq 0$, we have $a_i = 0$. Thus any codeword with cycle length $m \mid l$ can be given as follows

$$u(x) = \sum_{q-1|il, 1\le i\le e} a_i x^i. \tag{27}$$

Note that the size of $\{i : q-1 \mid il,\ 1 \le i \le e\}$ is $\left\lfloor \frac{el}{q-1} \right\rfloor$. For $l \mid q-1$ we have

$$I(\varrho^l) = \sum_{m|l} mJ_m = q^{\left\lfloor \frac{el}{q-1} \right\rfloor}.$$

Thus

$$\mid \mathbb{H}' \mid = \frac{1}{q-1}\sum_{l|q-1} q^{\left\lfloor \frac{el}{q-1} \right\rfloor}\phi\left(\frac{q-1}{l}\right) = \frac{1}{q-1}\sum_{l|q-1} q^{\lfloor \frac{e}{l} \rfloor}\phi(l). \tag{28}$$

By the properties of the Euler's function [10, 11], $\phi(p) = p-1$ for a prime p, $\phi(1) = 1$, and $\sum_{l|q-1}\phi(l) = q-1$. Since $\theta \le e < \min\{y : y \mid q-1,\ y > \theta\}$, we have

$$\begin{aligned}
\mid \mathbb{H}' \mid &= \frac{1}{q-1}\left(q^e + \phi(\theta)q^{\lfloor \frac{e}{\theta} \rfloor} + \sum_{l|q-1, l\neq 1,\theta} \phi(l)q^{\lfloor \frac{e}{l} \rfloor}\right)\\
&= \frac{1}{q-1}\left(q^e + (\theta-1)q^{\lfloor \frac{e}{\theta} \rfloor} - \phi(1) - \phi(\theta) + \sum_{l|q-1}\phi(l)\right)\\
&= \frac{1}{q-1}\left(q^e + (\theta-1)q^{\lfloor \frac{e}{\theta} \rfloor} - \theta + q - 1\right).
\end{aligned}$$

□

Theorem 8 The size of $\mathbb{U}'$ is

$$\mid \mathbb{U}' \mid = \frac{q^e - q^{\lfloor \frac{e}{\theta} \rfloor}}{q-1}. \tag{29}$$

Proof We have

$$\mid \mathbb{U}' \mid = \frac{1}{q-1}\sum_{l|q-1} \varphi(l)q^{\lfloor \frac{e}{l} \rfloor}$$

by Lemma 19 in [28], where $\varphi(l)$ is the Möbius function. By the properties of the Möbius function [10, 11], $\varphi(1) = 1$ and $\sum_{l|q-1}\varphi(l) = 0$. Since $\theta \le e < \min\{y : y \mid q-1,\ y > \theta\}$, we have

$$
\begin{aligned}
|\ \mathbb{U}'\ | &= \frac{1}{q-1}\left(q^e - q^{\lfloor \frac{e}{\theta} \rfloor} + \sum_{l|q-1, l\neq 1,\theta} \varphi(l) q^{\lfloor \frac{e}{l} \rfloor}\right) \\
&= \frac{1}{q-1}\left(q^e - q^{\lfloor \frac{e}{\theta} \rfloor} - \varphi(1) - \varphi(\theta) + \sum_{l|q-1} \varphi(l)\right) \\
&= \frac{1}{q-1}\left(q^e - q^{\lfloor \frac{e}{\theta} \rfloor}\right).
\end{aligned}
$$

□

Theorem 9 The size of $\mathbb{V}'$ is

$$
|\ \mathbb{V}'\ |= \frac{\theta q^{\lfloor \frac{e}{\theta} \rfloor} - \theta}{q-1}. \tag{30}
$$

Proof We have

$$
\begin{aligned}
|\ \mathbb{V}'\ | &= |\ \mathbb{H}'\ | - |\ \mathbb{U}'\ | - 1 \\
&= \frac{q^e + (\theta-1) q^{\lfloor \frac{e}{\theta} \rfloor} - \theta + q - 1}{q-1} - \frac{q^e - q^{\lfloor \frac{e}{\theta} \rfloor}}{q-1} - 1 \\
&= \frac{\theta q^{\lfloor \frac{e}{\theta} \rfloor} - \theta}{q-1}.
\end{aligned}
$$

□

Take $\mathbb{U}' \bigcup \mathbb{V}'$ as a sequence set in Step 1 of the modified construction method. It is easily checked that the sequence set meets the conditions in Step 1 with $P_m = e-1$. Let Z, l be two integers such that $Z+1 \mid \frac{q-1}{\theta}$ and $e-1 \leq l \leq q-Z-2$. Then an LHZ-APC FHS set S_4 is obtained by Step 2 of the modified construction method.

Theorem 10 S_4 is an $[l+Z+1, \frac{q^e-1}{Z+1}, q, Z]$ LHZ-APC FHS set with maximum APC $e-1$ and has optimal family size by (5).

Proof By Theorems 7, 8, and 9, S_4 is an $[l+Z+1, \frac{q^e-1}{Z+1}, q, Z]$ LHZ-APC FHS set with maximum APC $e-1$. Since $Z+1 \mid \frac{q-1}{\theta}$, we have $Z+1 \mid q^e-1$. By bound (5), S_4 has optimal family size. □

Remark 1 The parameters of S_4 are different from those of S_1 in Theorem 4, since the restrictions for S_4 include $\theta \leq e < \min\{y : y \mid q-1,\ y > \theta\}$ and $e-1 \leq l \leq q-Z-2$ which are not same as S_1.

Example 5 For $q = 11, e = 4, Z = 4, l = 4$, one can obtain a [9, 2928, 11, 4] LHZ-APC FHS set with maximum APC 3. By bound (5), we have

$$
M \leq \left\lfloor \frac{11^4}{5} \right\rfloor = 2928.
$$

Then it has optimal family size.

Note that by setting $q = 11, n' = 5, e = 4, Z = 4$, we can still not get a [9, 2928, 11, 4] LHZ-APC FHS set with maximum APC 3 by First Class in Section 3. This is because l is restricted by $l \leq n' - Z - 1$ in Section 3. But by utilizing the modified construction method in this section, an LHZ-APC FHS set with these parameters can be obtained.

Table 1 The LHZ-APC FHS sets in this paper

Parameters $[N, M, \lambda, Z]$, C_m	Restriction	According to the new bound (5)	Remark
$[l+Z+1, \frac{q^e-1}{Z+1}, q, Z], e-1$	$n' \mid q-1, 1 \le e < \theta', Z+1 \mid n', e-1 \le l \le n'-Z-1$	Optimal family size	Theorem 4
$[l+Z+1, \frac{q^{2e'}-1}{Z+1}, q, Z], 2e'-1$	$n' \mid q+1, 1 \le e' < \frac{\theta'+1}{2}, Z+1 \mid n', 2e'-1 \le l \le n'-Z-1$	Optimal family size	Theorem 5
$[l+Z+1, \frac{q(q^2-1)}{Z+1}, q, Z], 2$	$Z+1 \mid q+1, Z > \frac{q}{2}-1, 2 \le l \le q-Z$	Near optimal family size	Theorem 6
$[l+Z+1, \frac{q^e-1}{Z+1}, q, Z], e-1$	$\theta \le e < \min\{y : y \mid q-1,\ y > \theta\}, Z+1 \mid \frac{q-1}{\theta}, e-1 \le l \le q-Z-2$	Optimal family size	Theorem 10

- q is a prime power.
- $\theta' = \min\{x : x \mid n',\ x > 1\}$.
- $\theta = \min\{x : x \mid q-1,\ x > 1\}$.

5 Conclusion

In this paper, a bound on the family sizes of LHZ-APC FHS sets was established. Then a method for constructions of LHZ-APC FHS sets was proposed, which is based on conventional PC FHS sets. By choosing different conventional PC FHS sets, three classes of LHZ-APC FHS sets were obtained. Moreover, the construction method was modified, which generated another class of LHZ-APC FHS sets. It should be noted that all of the LHZ-APC FHS sets in this paper have optimal or near optimal family sizes with respect to the new bound.

Table 1 lists the parameters of the LHZ-APC FHS sets in this paper.

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Declarations

Competing interests The authors declare no competing interests.

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