Mathematical Musings on the External Anatomy of the Novel Corona Virus*

Part 3: Spherical Triangles

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What is the shape of the novel coronavirus (n-CoV) which has turned our world upside down? Even though under a microscope, it looks dull, unattractive, and even disgusting, creative artists have attributed to it bright colors, made it look pretty, and depicted it as a thing of beauty. What can a mathematician contribute to this effort? We take a purist's point of view by imposing on it a quasi-symmetry and then deriving some consequences. In an idealistic world, far removed from reality but still constrained by the rules of mathematics, anyone can enjoy this ethereal beauty of the mind's creation, beckoning others to join in the pleasure.

Our musings are split into four parts. We fondly hope while readers wait for the future parts to appear, they will indulge in their own musings, tell others about them, and propagate the good virus of mathematical thinking.

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Gist of Parts 1 and 2[1]

In Part 1, we described the external shape of the n-Cov as a sphere with three kinds of proteins protruding out of it and studied many properties of a sphere. In Part 2, by chasing after quasi-symmetry, we modeled some locations of the S-Proteins as the vertices of icosahedron and dodecahedron inscribed within a sphere.

Here, in Part 3, we study the properties of spherical triangles, which will aid us in counting and locating M- and E-proteins. Equipped with our model, we can answer the challenge posed in

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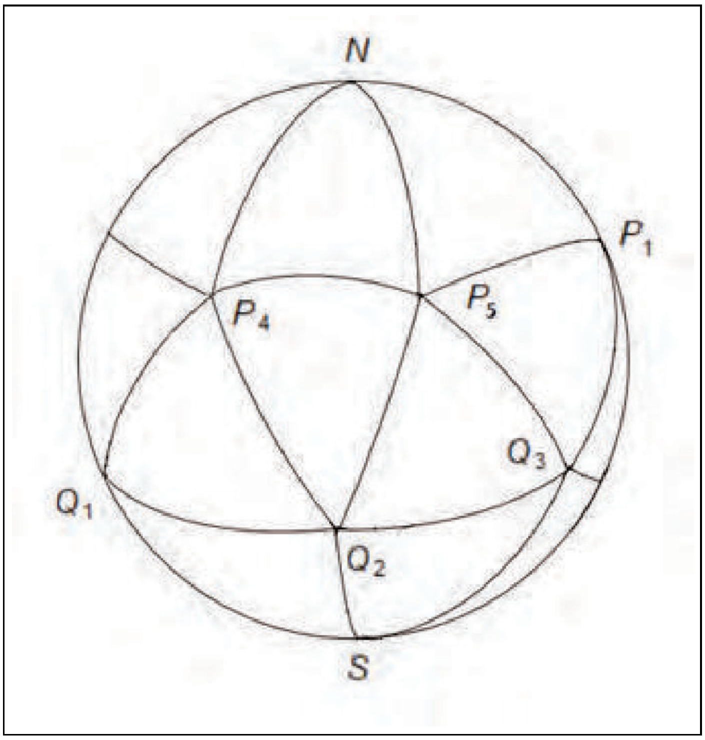

Figure 1. Radial projection of a unit icosahedron on its circumscribing sphere.

[1] See Vol.27, Nos.4&5, pp.609–622; pp.855–866, 2022.

Part 1 as to which depictions in *Figure* 1 (in Part 1) are closer to truth and which are far-fetched.

6.1 Radial Projection of an Unit Icosahedron on Its Circumscribing Sphere

Keywords
Great circle, platonic solids, duality, spherical geometry, Girard's theorem, Hamiltonian cycle, intermediate value theorem.

Consider a wire net of a unit icosahedron that is circumscribed by a sphere of radius $R_0 = \sqrt{(5+\sqrt{5})/8}$. (see Part 2) Next, imagine that a point source of light is placed at the center of the unit icosahedron (or of the circumscribing sphere). Then the shadows of the edges of the icosahedron cast on the sphere are shown in *Figure* 1. Such a casting of shadows of an object on the sphere, by a point source of light placed at the center, is called a *radial projection*.

The most prolific tool in the mathematician's toolbox is imagination.

The shadows in *Figure* 1 collectively form a spherical icosahedron. In particular, the radial projection of each (equilateral) triangular face of the unit icosahedron on the surface of the circumscribing sphere is a 'spherical equilateral triangle'. There are infinitely many spherical equilateral triangles on a sphere of various sizes and shapes depending on how far from the center the plane passing through the vertices is. Make no mistake, not only the sizes but also the shapes are different! Whenever you pick three points on a sphere that are pairwise equidistant, you can generate, via the radial projection of the three line segments joining these points pairwise, a spherical equilateral triangle. To avoid confusion with other similar objects, let us formally define our special spherical equilateral triangle obtained by radial projection of a face of an icosahedron on its circumscribing sphere.

Not only the sizes but also the shapes are different!

Definition 1. The image of a face of the unit icosahedron under radial projection on the circumscribing sphere is called a *spherical equilateral triangle* having area one-twentieth that of the entire surface of the sphere, or in short SET/20.

SET/20 defined.

Two other Platonic solids have triangular faces—tetrahedron and octahedron. Hence, we can define a spherical equilateral triangle with an area one-fourth of a sphere or SET/4 by radially projecting a unit tetrahedron on its circumscribing sphere and a spherical equilateral triangle with an area one-eighth of a sphere or SET/8 by radially projecting a unit octahedron on its circumscribing sphere. We encourage readers to study those SETs on their own.

Readers should study SET/4 and SET/8.

6.2 Properties of Spherical Equilateral Triangle SET/20

Much is known about a planar equilateral triangle of side length a: Its circumcenter, in-center, orthocenter, and centroid all coincide. Hence, this point is called 'the center.' Furthermore, each of the three perpendicular bisectors of sides (or angle bisectors or altitudes or medians), of length $\sqrt{3}\frac{a}{2}$, is split in the ratio 2:1 at the center; that is, the center is twice as far from each vertex as from each side. The area of an equilateral triangle is $\sqrt{3}a^2/4$.

Recall what you know about a planar equilateral triangle.

If the three mid-points of sides are pairwise connected by line segments, the equilateral triangle is split into four equilateral triangles of side length $a/2$ each.

It is desirable to translate as many of these features as possible to the spherical equilateral triangles. To do so, let us continue to use the same terminologies as used for a planar equilateral triangle to talk about the corresponding quantities of a SET/20 obtained by radial projection. For example, by the side of a SET/20, we mean the part of the great circle that forms one of the three boundaries of the SET/20; by an angle-bisector, we mean the arc of another great circle that forms the same angle (rather angles between the tangents along the great circles) with two adjacent sides; etc.

Use the same terminologies as are used for a planar equilateral triangle.

Let us first mention one property of SET/20 that starkly departs from the corresponding property of a planar equilateral triangle. Whereas each angle of a planar equilateral triangle measures $\pi/3$ or $60°$, each angle between the sides of SET/20 (rather between the tangents to the sides along their great circles) measures $2\pi/5$ or $72°$. Consequently, the sum of the three angles of SET/20 equals $6\pi/5$, which exceeds π, the sum of angles of any triangle!

Each angle of SET/20 measures $72°$.

Using Lemma 2 (see Part 1), the spherical distance between any two neighboring vertices of the inscribed unit icosahedron is $c = g(1) = 2R_0 \sin^{-1}(1/2R_0) = 1.052961$. Let us define the straight-edge distance between a vertex and the mid-point of an adjacent edge as b, between an edge-center and the face-center as u, and between the face-center and a vertex as v. Moreover, by symmetry, the three perpendicular bisectors of sides (or angle bisectors or altitudes or medians) are concurrent at the center (the shadow of the center of a face of the icosahedron), and each has length $u + v$. Using the same Lemma 2, and aided by *Figure* 2, we shall evaluate these lengths; and we shall determine in what ratio each of the three medians AX, BY, CZ is split at the center T, or the ratio $g(v) : g(u)$.

Straightedge distances are measured piercing through the sphere.

Indeed, from *Figure* 2 (c), we have
$b^2 = \frac{1}{4} + \left(R_0 - \sqrt{R_0^2 - \frac{1}{4}}\right)^2 = 2R_0^2\left(1 - \sqrt{1 - \frac{R_0^{-2}}{4}}\right).$

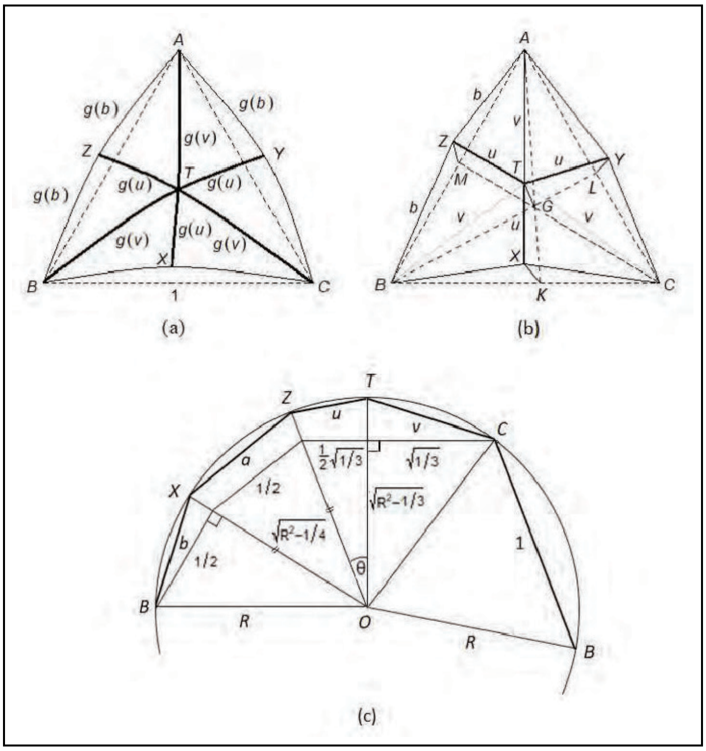

Figure 2. **(a)** A radial projection of an SET/20 and its medians on the circumscribing sphere; **(b)** Nine auxiliary planar faces atop an SET/20 whose redial projections match those of the SET/20 and its three altitudes; **(c)** Several triangles, though they exist in different planes, are reassembled to assist computations.

$$v^2 = \frac{1}{3} + \left(R_0 - \sqrt{R_0^2 - \frac{1}{3}}\right)^2 = 2R_0^2\left(1 - \sqrt{1 - \frac{R_0^{-2}}{3}}\right).$$

$$u^2 = 2R_0^2\,(1 - \cos\theta) = 2R_0^2\left(1 - \frac{\sqrt{1-\frac{R_0^{-2}}{3}}}{\sqrt{1-\frac{R_0^{-2}}{4}}}\right).$$

In the last equation, we have used the law of cosines, which is an extension of the Pythagorean theorem to non-right-triangles. Next, from Theorem 2.2 (see Part 2), we have $R_0^2 = \frac{5+\sqrt{5}}{8}$, and so $R_0^{-2} = 2\left(\frac{5-\sqrt{5}}{5}\right)$. Leaving aside the exact radical expressions for the reader to discover, we simply report the decimal values of straightedge distances b, v, u. Thereafter, using Lemma 1, we evaluate the corresponding spherical distances $g(b)$, $g(v)$, $g(u)$ as follow:

$b = 0.5198, g(b) = 0.5265; v = 0.6095, g(v) = 0.6204; u = 0.3451, g(u) = 0.3470$.

Furthermore, the solid in *Figure* 2(a) has curved surface area $4\pi\frac{R_0^2}{20}$ and volume $\frac{4}{3}\frac{\pi R_0^3}{20} - \frac{1}{3}\frac{\sqrt{3}}{4}\sqrt{R_0^2 - \frac{1}{3}} = \frac{\pi R_0^3}{15} - \frac{1}{4}\sqrt{\frac{R_0^2}{3} - \frac{1}{9}}$ with $R_0 = 0.9510565$.

Since each side of an SET/20 is bisected, we must have $g(b) = g(1)/2$. Each median (going from a vertex towards the mid-point of the opposite edge along a great circle) is split at the center T in the ratio $g(v)/g(u) = 1.78795$, unlike a planar triangle where the ratio is 2. Further, as shown in *Figure* 3, starting from the north pole N if a bug travels along a side of SET/20 and continues along that same great circle (this bug is really smart) following the dotted curve, which traces the medians of two copies of SET/20, it reaches the south pole, traversing a distance of half the perimeter of the great circle. Hence, $2[g(b) + g(u) + g(v)] = \pi R$. The area of SET/20, shown by the shaded area in *Figure* 3 (a), is one-twentieth of the total surface area of the circumscribing sphere of radius $R_0 = 0.9510565$, or $\pi R_0^2/5 = 0.5683194$ unit^2. Hence, the total area of five spherical equilateral triangles, the shadow cast by the five faces of the icosahedron incident at any one vertex, is exactly πR^2, which is the area of a great circle. Compared to the area $2\left(1 - \sqrt{\frac{1}{5}}\right)\pi R^2$ of the curved surface of the cap when the sphere is sliced through five vertices adjacent to the North Pole, this is an amazingly beautiful result!

This bug is really smart!

This is an amazingly beautiful result!

One may be curious about the angles subtended by various pairs of points on the SET/20 at the center of the sphere. These angles can be seen also in *Figure* 2(c): Any two vertices subtend angle $BOC = g(1)/R_0 = 63.435°$; a vertex and the face-center subtend angle $COT = g(v)/R_0 = 37.37°$; the face-center and a mid-point of a side subtend angle $TOZ = g(u)/R_0 = 20.91°$; a mid-point of a side and either of the two nearest vertices subtend angle $XOB = g(b)/R_0 = g(1)/2R_0) = 31.72°$; and a mid-point of a side and the farthest vertex along a median subtend angle $COZ = [g(u) + g(v)]/R_0 = 58.28°$. Most surprisingly, the mid-points of any two sides of SET/20 subtend angle $XOZ = \pi/5 =$

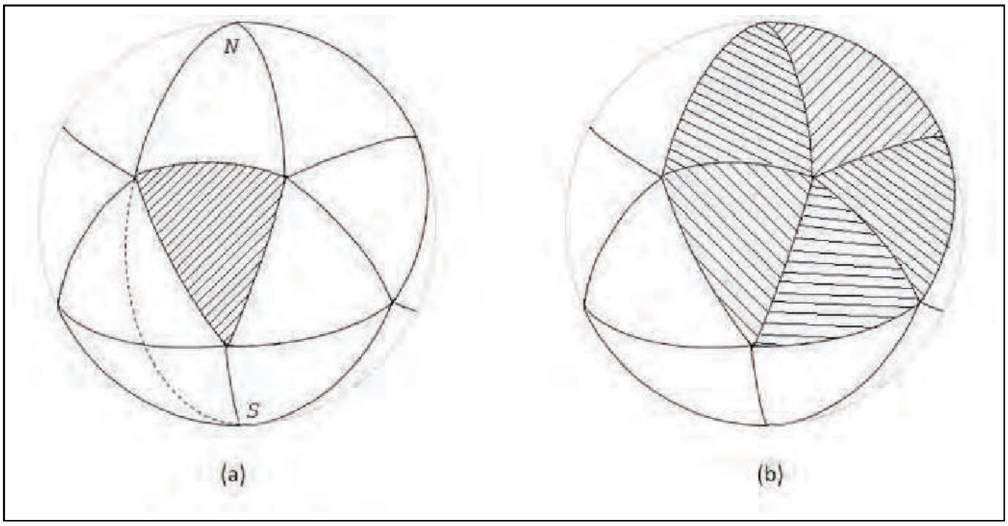

Figure 3. **(a)** Half the perimeter of a great circle is $\pi R = 2[g(b) + g(u) + g(v)]$. **(b)** The total curved surface area of five faces of SET/20 equals that of a great circle.

36° at the center. Why is this answer a perfect integer when measured in degrees?

Why a perfect integer?

We could leave this discovery to the interested reader. But the joy of mathematics overwhelms us. So please allow us to disclose the secret. Indeed, $a = XZ$ is a magnification of KM, of length $1/2$, under the radial projection. As such

$$a = XZ = \frac{1}{2}\frac{R_0}{\sqrt{R_0^2 - \frac{1}{4}}} = R_0\frac{\sqrt{5}-1}{2} = \sqrt{\frac{5-\sqrt{5}}{8}} = 0.5877853$$

Thus, $R_0/XZ = 2/(\sqrt{5}-1)$ is, in fact, the golden ratio $\varphi = (\sqrt{5}+1)/2$, the positive solution to the quadratic equation $x^2 - x - 1 = 0$. Yet another beautiful mathematical gem just popped in to greet us! Was there a prior hint of it anywhere? You can learn more about the golden ratio from [1]. In case you regret that we have robbed you of the opportunity to discover this fascinating truth on your own, we invite the astute reader to think of a simpler justification, which we will reveal in Section 8 in Part 4.

Yet another beautiful mathematical gem just popped in to greet us!

6.3 Splitting SET/20

Let us now comment on the various parts into which SET/20 is split by the spherical equilateral triangle XYZ. Whereas the planar XYZ is an equilateral triangle, planar triangles AZY, BXZ, CYX are only isosceles with two sides equal to b and the third side equal to R_0/φ. Therefore, their spherical counterparts exhibit quasi-symmetry. The corresponding spherical isosceles triangles have two sides measuring $g(b)$ and the third measuring $g(a) = g(R_0/\varphi) = R_0\pi/5$; and they have two angles measuring about 58.3° and one measuring exactly 72°. The angles can be found using the spherical law of sines [2]. The central spherical triangle XYZ is equilateral with side length $g(a) = R_0\pi/5$; but it is not like SET/20 ABC! The former has all three angles measuring about 63.435°, whereas the latter has all three angles measuring $2\pi/5 = 72°$.

SET XYZ is not similar to SET ABC!

According to Girard's theorem [see [2] again], the surface area of any spherical triangle (no need for it to be isosceles, equilateral, or right) equals the squared radius times the excess over π attained by the sum of the three angles. For instance, SET/20 has excess angle $\pi/5$, so its surface area is one-fifth that of the great circle. The spherical equilateral triangle XYZ has an excess angle of 10.305°; so, it has an area $10.305/180 = 0.0572$ times that of the great circle. Each of the three peripheral spherical isosceles triangles, AZY, BXZ, CYX, has an excess angle 8.565°; so each has surface area $8.565/180 = 0.0476$ times that of the great circle.

The area of a spherical triangle equals squared radius times (sum of three angles minus π).

Next, let us comment on the various parts into which SET/20 is split by the three medians. Indeed, the three medians split SET/20 into six congruent (in both shape and size) right spherical triangles of side lengths $g(b)$, $g(v)$, $g(u)$, and angles $36°, 90°, 60°$ at the vertex, edge-center and face-center, respectively. Each right triangle has an excess of 6° over their planar counterparts; hence, each has area $\frac{6}{180}$ times that of the great circle, or $\pi R^2/30$, using Girard's theorem. In fact, using symmetry, we already know that the area of each scalene triangle, being one-sixth that of an SET/20, must be $\left(4\pi R^2/20\right)/6 = \pi R^2/30$. We will say more about

Symmetry is the spice of mathematics.

these right triangles in subsection 7.2 in Part 4.

We can now reveal the length of line segments in *Figure* 2 in Part 2: Solid g(b), dotted $g(v)$, gray $g(u)$, and dashed $g(a) = R_0\pi/5$. Also, in *Figure* 2(c) of Part 2, the angle between the solid and dashed segments is 58.28253°; and between dashed and gray is 31.71747°.

6.4 Inscribed Dodecahedron Within the Sphere Circumscribing the Unit Icosahedron

In this subsection, we calculate the side length of the inscribed dodecahedron formed by taking the convex hull of the radial projections of the face-centers of the icosahedron onto its circumscribing sphere. We leave it to the reader to check that the ratio of the in-radius to the circumradius of a regular pentagon is $\sin 54° = 0.809017$. Hence, for any two adjacent faces of the unit icosahedron, their face-centers have a straight-edge distance of $(2/3)\sin 54° = 0.5393447$. When these points are radially projected onto the circumscribing sphere, the distance is magnified by a factor $R_0 / \sqrt{R_0^2 - 1/3}$. Therefore, each side of a dodecahedron inscribed within the circumscribing sphere of radius $R_0 = 0.9510565$ is of length

We leave to the reader some more work.

$$D = \frac{2}{3}\sin 54° \frac{R_0}{\sqrt{R_0^2 - 1/3}} = 0.6787159.$$

Hence, using Lemma 2 (see Part 1), the adjacent vertices of the inscribed dodecahedron have a spherical distance of $g(D) = 0.6940122$.

Thus, within any sphere, the inscribed dodecahedron has a side length 0.6787159 times that of the inscribed icosahedron; and the spherical counterpart of the former is $g(D)/g(1) = 0.6591054$ part of the spherical counterpart of the latter. Furthermore, the edges of the icosahedron and the dodecahedron do not intersect, even though their mid-points have radial projections at the same point on the sphere! In fact, their mid-points differ by a distance of

The edges of the icosahedron and the dodecahedron do not intersect.

$$\sqrt{R_0^2 - D^2/4} - \sqrt{R_0^2 - 1/4} = 0.07943371,$$

or 8.35% of the radius. Consequently, as does the icosahedron, the centers of any two adjacent edges of the dodecahedron also subtend an angle $\pi/5$ at the center of the circumscribing sphere. While two adjacent vertices of an icosahedron subtend an angle of $2\sin^{-1}(12R_0) = 63.435°$ at the center of the circumscribing sphere, two adjacent vertices of an dodecahedron subtend an angle of $2\sin^{-1}(0.6787159/(2R_0)) = 41.81031°$ at the center.

What to Expect in Part 4

Having built our model for the n-Cov as the superposition of an inscribed icosahedron and an inscribed dodecahedron within a sphere, we are ready to propose the number and locations of the S-Proteins, and thereafter, settle the challenge of which diagrams within *Figure* 1 of Part 1 are closer to truth and which are far-fetched. Moreover, we are also equipped to conjecture the numbers and locations of the M- and E-Proteins. Stay tuned.

We are ready to propose the number and locations of the S-Proteins.

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Suggested Reading

[1] Wikipedia (c), Golden Ratio, Retrieved from *Wikipedia*, The Free Encyclopedia on 10 December 2020. https://en.wikipedia.org/wiki/Golden_ratio
[2] Wikipedia (d), Spherical Trigonometry, Retrieved from *Wikipedia*, The Free Encyclopedia on 15 December 2020. https://en.wikipedia.org/wiki/Spherical_trigonometry