GENERAL ARTICLE

# Mathematical Musings on the External Anatomy of the Novel Corona Virus*

## Part 1: The Overall Shape of the n-CoV

*Jyotirmoy Sarkar and Mamunur Rashid*

**What is the shape of the novel coronavirus which has turned our world upside down? Even though under a microscope it looks dull, unattractive, and even disgusting, creative artists have attributed to it bright colors, made it look pretty, and depicted it as a thing of beauty. What can a mathematician contribute to this effort? We take a purist's point of view by imposing on it a quasi-symmetry and then deriving some consequences. In an idealistic world, far removed from reality but still obeying the rules of mathematics, anyone can enjoy this ethereal beauty of the mind's creation, beckoning others to join in the pleasure.**

**Our musings are split into four parts. We fondly hope that while readers await the future parts to appear, they will indulge in their own musings, tell others about them, and propagate the good virus of mathematical thinking.**

**Jyotirmoy Sarkar is a Professor at IUPUI, Indianapolis, Indiana. His research includes mathematics, probability and statistics. He enjoys reading, 'riting, 'rithmetic and R-coding. He devotes his spare times to puzzle making.**

**Mamunur Rashid is an Associate Professor at DePauw University, Greencastle, Indiana. His research interests include statistical computing, data visualization, data analysis, and applications of statistics to public health/population studies.**

### Preamble

Neither of us has seen the novel coronavirus (n-CoV) under a microscope. In fact, neither of us has access to a laboratory with the privilege to see the virus. Most assuredly, neither of us has the least interest in adopting the virus as a pet. We suspect most of our readers are in the same predicament. Formally known as the severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2), the n-CoV has caused a global pandemic of coronavirus disease 2019 (COVID-19), which originated in Wuhan,

*Vol.27, No.4, DOI: https://doi.org/10.1007/s12045-022-1352-z

**Keywords**
Great circle, platonic solids, duality, spherical geometry, Girard's theorem, Hamiltonian cycle, intermediate value theorem.

Hubei, China in December 2019. With its unprecedented ravages, this tiny virus has turned our world upside down. In one fell swoop, the pathogen has profoundly changed our lives and lifestyles. So many lives have been lost; so many businesses have gone bankrupt; so many people have become jobless; so much hunger, insecurity, and devastation have been inflicted globally by this invisible enemy! Both of us have shed many a tear over this calamity.

This tiny n-CoV virus has turned our world upside down.

During the stay-at-home period that followed, intending to curtail the rate of spread of COVID-19, creative artists in all walks of life have found novel ways to practice, promote and disseminate their arts. The mathematician is in a relatively fortunate position, for our trade only requires a brain, a computer, and access to some books and the internet. Therefore, we decided to make good of our time of seclusion towards promoting some mathematics education. If this article is received as a small gift to humanity by a few sympathizers, we will be glad our intense labor of love has not been in vain.

Creative artists have found novel ways to practice, promote and disseminate their arts.

## 1. An Invitation to the Reader

An important step in combating one's enemy is to know its nature. So, we looked around at media resources and websites of reputed organizations to find out what this ominous SARS-CoV-2 looks like. We found a plethora of pictures of the virus: Can all of them be true depictions when they are at variance with one another? A sample of diagrams is shown in *Figure* 1. Amid this diversity, what unifying features can we see as mathematicians?

What unifying features can we see as mathematicians?

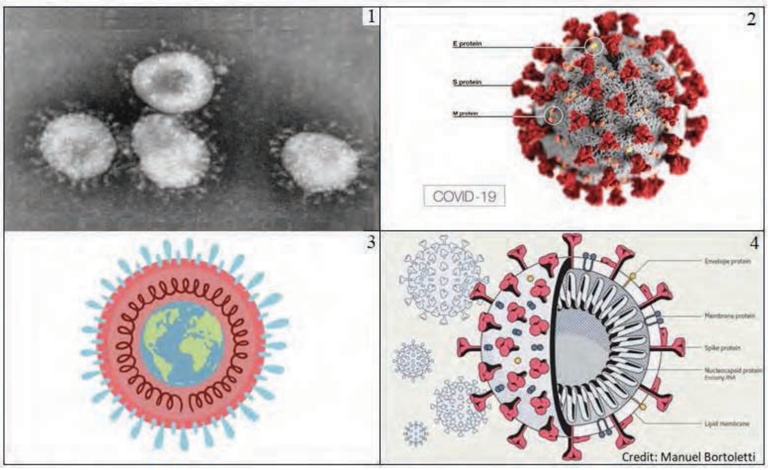


**Figure 1.** A few diagrams of the novel coronavirus gleaned from the internet. Credits: (1) and (2) Center for Disease Control and Prevention, (3) Innovative Genomics Institute, (4) Manuel Bartoletti.

**Box 1. For additional figures see:**

5. A 2D model with 16 S-Proteins at
`https://www.legacyias.com/explained-why-is-covid-19-not\break-a-pandemic-yet/`
6. A 2D model with 18 S-Proteins at
`https://www.dailynews.lk/2020/01/31/features/210052/australian-\\scientists-replicate-virus`
7. A 3D model with about 52 S-Proteins and a 2D model with 20 S-Proteins `https://commons.wikimedia.org/wiki/File:3D_medical_animation_corona_virus.jpg`

We tried to congeal some common themes out of other peoples' renditions of the virus and then formulated our model from the perspective of mathematics. In this article, we document some musings that we found both educational and delightful; and they will reveal to you, dear readers, some secrets of the mathematician's art. Therefore, we invite you to partake in our intellectual pleasure and spread the word to others.

As do most others, we too deviate from the absolute truth. Nonetheless, we offer you some mathematical nuggets, including some precious gems, which we hope will both educate and delight you, and may inspire you to indulge in your own musings.

Mathematical nuggets and precious gems will educate and delight you.

In the remainder of Part 1, we justify why the overall shape of the

n-CoV is spherical. Then recognizing that there are three kinds of proteins protruding from the sphere, but not knowing their exact numbers, we venture the possible number of S-proteins, and patterns formed by them, speaking mostly qualitatively. In Part 2, we describe our model in full and show many quantitative calculations that explain the nature of quasi-symmetry in our model. Part 3 identifies some locations of the S-proteins by utilizing this quasi-symmetry property and studies the mathematical properties of spherical triangles in preparation for locating M- and E-proteins. In Part 4, we choose the numbers and the locations of M- and E-proteins. Moreover, we discuss some alternative models that also achieve other forms of quasi-symmetry. Several suggestions are given to astute readers to formulate their own models.

Quantitative calculations explain the nature of quasi-symmetry.

## 2. The Shape We Propose and Why

The prevalent features noticeable in *Figure* 1, where the n-CoV is depicted in the left column in three dimensions (3D) and the right column in two dimensions (2D), indicate that the external shape of the n-CoV is typically assumed to be spherical, with three kinds of proteins protruding out of the sphere—Spike (S) protein, membrane (M) protein, and small envelope (E) membrane protein. These proteins have differing lengths, and their endpoints are respectively three-headed, two-headed, and single-headed. When these proteins are given distinctive shapes and colors, who can deny the beauty of this tiny devil?

Who can deny the beauty of this tiny devil?

Beyond these general agreements, the diagrams vary substantially, raising two questions of special interest to the mathematician: (1) How many of each kind of protein are there? (2) Where on the sphere are they located? These are profound questions: If the answer to the first question is an arbitrary number, then the answer to the second question is extremely hard, in general. See Problem 7 in [1]. We make no attempt to solve that problem. We are fortunate that we can choose our favorite answers to the first question so that the answers to the second question become easier by invoking symmetry or quasi-symmetry as we shall see.

Answers become easier by invoking symmetry or quasi-symmetry.

This article (split in four parts) is devoted to theorizing idealistic answers to these two questions. Admittedly, any ideal shape is too good to be true, including the one we propose. Nonetheless, our proposed model serves as a baseline against which any manifestation (of which there is a multitude) of the truth can be measured. More importantly, our proposal offers yet another type of beauty—mathematical.

## 3. External Shape: Why a Sphere?

Agreeing with the prevalent feature, we subscribe to the ideal shape of the n-CoV as a sphere with three types of projections. However, our subscription to perfect sphericity is also based on a mathematical truth: An object having a fixed total amount of material (measured by its volume) occupies a space with the least amount of exposed external surface area when it is spherical. Although some deviations from the ideal shape are usually noticeable because of gravity, contaminants, and interaction with other objects, the overall shape of raindrops, hails, soap bubbles, planets, and stars is spherical. Other than saying that the mathematical truth follows from the isoperimetric inequality, we do not have room here to prove it in 3D. We restate its 2D version and the associated dual problem and leave it to the interested reader to pursue its proof in [2].

The overall shape of raindrops, hails, soap bubbles, planets, and stars is spherical.

**Lemma 1.** The following primal and dual claims hold true.

(a) Minimum Boundary Problem: A planar region of a fixed area that has the shortest boundary is a circle.

(b) Maximum Area Problem: A planar region with a constant perimeter has the largest area when it is a circle.

### 3.1 *Properties of a Sphere*

Having idealized the shape of the n-CoV as a sphere and having justified this choice, we will do wisely to recall a few well-known properties of the sphere. Let us begin with its definition. In 3D,

Recall a few well-known properties of the sphere.

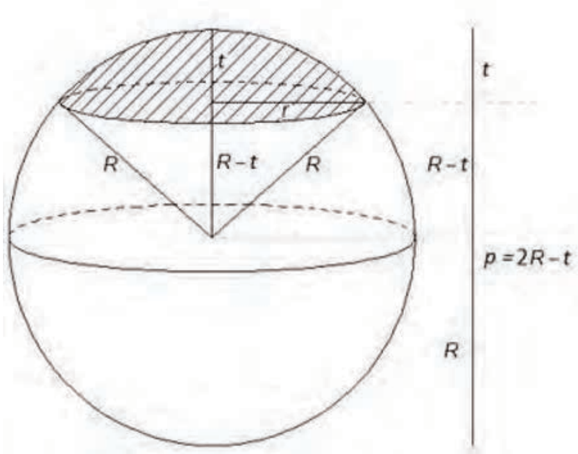


**Figure 2.** A plane cut splits a sphere into a cap (smaller) and a pedestal (bigger).

if we fix a point $O$ and consider all points at a constant distance $R$ from $O$, then we have a 2-sphere, the boundary surface of a 3D ball, which is the set of all points at a distance no more than $R$ from $O$. Thus, a ball is a genuine 3D object that includes the sphere as a boundary; but a sphere (or the boundary of the ball) is a 2D object that lives in 3D. The fixed point $O$ is called the center, and the constant distance $R$ is called the radius. The shape is unique, and the size of a ball (and the associated sphere) is fully determined by its radius $R$. Any line segment passing through the center and terminated by the sphere is called a diameter, and it is of length $2R$. The ball is convex in the sense that a line segment joining any two points in the ball lies entirely in the ball.

### 3.2 *When a plane cuts a sphere*

A plane cuts the ball into a cap and a pedestal.

A plane cut of the ball of radius $R$ slices it into a cap (the smaller part) and a pedestal (the larger part), as shown in *Figure* 2. These two parts overlap in the shape of a disc of radius $r$ which depends on $t$, the depth of the cap, or equivalently, on $h = R-t$, the distance of the cutting plane from the center of the sphere. The height of the pedestal is $p = 2R - t$.

If the cap is available but the pedestal is lost or unavailable (as might happen during an archaeological dig), then the radius of

the ball can be found as follows: Measure the radius $r$ of the cross-sectional disc of the cap. Then measure the depth $t$ of the cap as the distance between the disc and another parallel plane sandwiching the cap. Then the radius of the ball is

The radius of the ball can be found from two linear measures on the cap/pedestal.

$$R = \frac{1}{2}\left(\frac{r^2}{t} + t\right),$$

by the Pythagorean theorem. Also, the distance of the cutting plane from the center is

$$h = R - t = \frac{1}{2}\left(\frac{r^2}{t} - t\right),$$

and the depth of the pedestal is

$$p = R + h = 2R - t = \frac{r^2}{t}.$$

Conversely, when only the pedestal is available and the cap is lost, the reader can measure the cross-sectional radius $r$ of the disc and the height $p$ of the pedestal, and recover $R, h, t$ as

$$R = \frac{1}{2}\left(\frac{r^2}{p} + p\right); \quad h = p - R = \frac{1}{2}\left(p - \frac{r^2}{p}\right); \quad t = 2R - p = \frac{r^2}{t}.$$

Specifically, a plane cut passing through the center renders $t = p = r = R$, and makes the cap and the pedestal equal in size, each of which is called a hemisphere. Among all spherical caps, the hemisphere attains the largest possible cross-sectional disc with radius $R$, whose boundary is called a great circle. Among all possible circles traced by plane cuts, a great circle has the largest perimeter, $2\pi R$, and it encloses the largest area, $\pi R^2$. Given any great circle, by spinning it around any diameter (of the great circle) as an axis, one can regenerate the sphere. We shall report the curved surface area and the volume of the spherical cap towards the end of subsection 3.4. First, let us talk about the geodesic (or the shortest distance) between any two points on the sphere when the path joining them lies entirely on the sphere.

Wait to see the greatness of the great circle.

### 3.3 *Geodesics on a Sphere*

Many terminologies for a sphere are borrowed from geographic terms applied to a globe.

Two points on the sphere are called diametrically opposite (or antipodal) if they are co-linear with the center. Given two antipodal points (call them the North Pole and the South Pole, respectively), there are uncountable-many great circles (meridians) passing through them. Half-perimeter of any one of them is the 'spherical distance' between the antipodal points. A plane orthogonal to the diameter joining the North Pole and the South Pole cuts the sphere to generate a circle whose points have a constant latitude. Among the circles so generated, there is only one great circle, which is called the equator (having latitude 0°).

The honey-hungry bug will trace a geodesic.

For any two non-antipodal points, there is a unique shortest path (or geodesic) on the sphere between these two points—it is the shorter arc of the great circle traced by a plane passing through these two points and the center. If on a globe you put a drop of honey at one point and release a bug with an extraordinary sense of smell at another point, then to quickly reach the honey, the bug will walk along the geodesic. The spherical distance between these two points is the length of this shortest path (geodesic) on the sphere, and it is a function of the straightedge distance between the two points (along a line segment piercing through the sphere) and the radius $R$ of the sphere, given by the following result.

**Lemma 2.** If two points on a sphere of radius $R$ are at a straightedge distance $d$ (piercing through the sphere) from each other, then they are at a great-circle distance $g(d)$ given by

$$g(d) = 2R\sin^{-1}\left(\frac{d}{2R}\right). \tag{1}$$

*Proof:* Refer to *Figure* 3, which shows the great circle on the plane passing through the given points $A, B$ and the center $O$. Note that $\sin(g(d)/(2R)) = (d/2)/R$. Hence, proved. □

An application of this great circle principle is how planes fly between two cities. For example, Chicago, USA, and Rome, Italy, have the same latitude 42°N. If a plane flies parallel to the

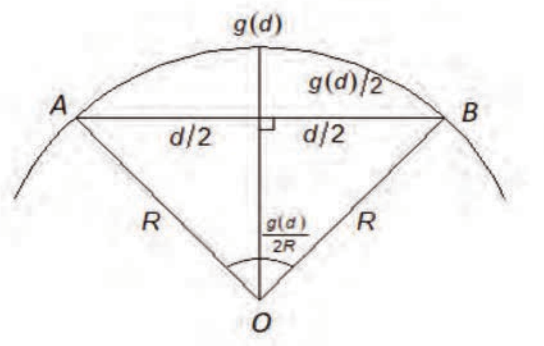


**Figure 3.** Relation between the lengths of a chord and the minor arc.

equator (or stays at 42°N latitude and hence equally far from the North Pole) throughout its flight, the distance between these cities would be 8,312 km, calculated based on the longitudinal difference between these cities being $12.5° - (-87.9°) = 100.4°$, the radius of the Earth being $R = 6,371$ km and assuming that the plane flies 12 km above the Earth's surface, using the formula for constant latitudinal distance (CLD)

$$CLD = \frac{100.4}{180}\pi\,(R + 12)\ \cos\left(\pi\frac{42}{180}\right). \tag{2}$$

However, when the plane flies along the great circle (initially making an angle of 48°E with the meridian at Chicago, increasing this angle continuously and moving closer to the North Pole until half the (longitudinal) distance is covered) then the shortest path between those same two cities is 7,757.4 km, calculated by applying Lemma 2 after finding the straightedge distance (piercing through the Earth) between the two cities to be 7,288.7 km using the formula

Traveling on the geodesic, as the honey-hungry bug did, is time- and cost-efficient.

$$d = 2(R + 12)\sin\left(\pi\frac{100.4}{360}\right)\cos\left(\pi\frac{42}{180}\right). \tag{3}$$

Compared to the CLD, the geodesic route yields a 6.67% savings in time and fuel cost!

### 3.4 *Surface Area and Volume of a Sphere*

Having studied the 1D measure of distance on a sphere, we now move on to the 2D measure of area and the 3D measure of volume. We trust most of our readers know the surface area formula for a sphere of radius $R$. We believe many can prove this result using calculus: First, using calculus, find the volume of the ball as $4\pi R^3/3$; then decompose the ball into a multitude of pyramids each having an apex at the center $O$, and base on the sphere. Thus, all such pyramids have height $R$, and their bases together span the entire surface of the sphere. Recalling that the volume of a pyramid is one-third of the height times area of the base, the volume of the ball equals the surface area of the sphere times $R/3$, whence the surface area is $4\pi R^2$.

Don't do it! Just imagine decomposing the ball.

Nonetheless, we like to share a simple geometric proof, attributed to Archimedes (287–212/211 BCE), that does not rely on the volume formula. Next, reversing the logic in the above paragraph, the volume formula can be proved from the surface area formula! Moreover, this proof makes easy the task of finding the surface area and the volume of any spherical cap!

This proof makes trivial some other tasks.

**Lemma 3.** The total surface area of a sphere of radius $R$ is $4\pi R^2$.

*Proof.* Refer to *Figure* 4. Circumscribe the sphere with a right circular cylinder of radius $R$ and height $2R$. Slice up both the cylinder and the sphere by making plane cuts parallel to the plane faces of the cylinder at successive regular distance $\Delta$. The portion of the curved surface of the cylinder sandwiched between two successive cuts has area $2\pi R\Delta$, since the surface of the cylinder can be unrolled into a rectangle of sides $2\pi R$ and $\Delta$.

The next time you slice a cucumber or a tomato, remember Archimedes.

The portion of the surface of the sphere sandwiched between two successive cuts cannot be unrolled into a planar region. However, when $\Delta$ is close to 0, that same portion can be approximated by the curved surface of a frustum of a cone sandwiched between two right circular cuts of nearly comparable radius $x$. If the frustum has slant height $l$, then the curved surface of the frustum has an area $2\pi xl$. Next, by the similarity of triangles, which we leave

A frustum of a cone sandwich...yummy!

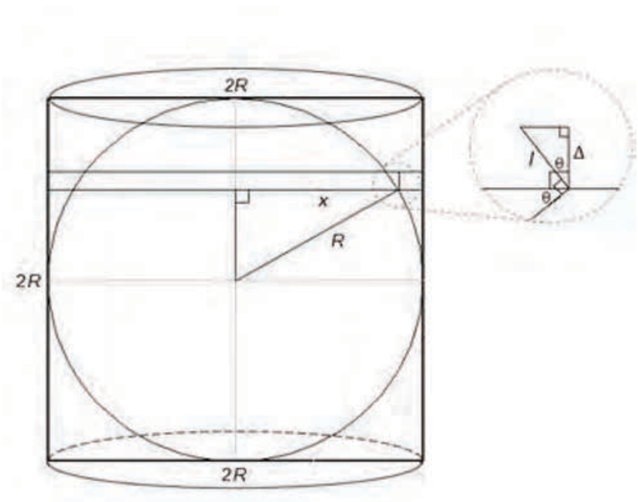


**Figure 4.** Between successive plane cuts, the curved surface of the sphere is approximated by that of a frustum of a cone obtained by right circular cuts. The marked right triangles with leg lengths $(R, x)$ and $(l, \Delta)$ are similar.

to the reader to justify using the magnified portion shown in *Figure* 4 as an aid, we have $xl = R\Delta$. Consequently, the frustum (and hence the sphere) has the same curved area as that of the thin cylinder, sandwiched between those same two planar cuts.

Thereafter, by aggregating the surface areas between successive slices, we establish that the total surface area of the sphere equals the total area of the curved surface of the cylinder. But the later surface can be unfolded into a rectangle whose two sides equal the perimeter and the height of the cylinder respectively, and as such, it has an area of $2\pi R \cdot 2R = 4\pi R^2$. This completes the proof. $\square$

The circumscribing right cylinder, of course, has two planar circular surfaces as well, each with area $\pi R^2$, making its total area $6\pi R^2$. Archimedes was so fascinated by the ratio of surface area (as well as volume) of the sphere to its circumscribing right cylinder being 2:3 that at his request a 3D sculpture of a sphere and its circumscribing right cylinder was placed on his tomb, as reported in *Wikipedia* [3]. While it is fine to logically equate the area of the surface of a sphere to that of the curved surface of the circumscribing right cylinder, the former cannot be flattened into the latter. Attempts to do so has caused the cartographer's dilemma as they tried to draw the world map on a flat paper, so reports

The areas of a sphere and its circumscribing cylinder have ratio 2:3, the same ratio as their volumes.

*NatGeoEd* [4].

Not haphazardly, but uniformly.

A further application of Lemma 3 is how to generate a random point on a sphere: Generate two independent, uniform(0, 1) random variables $U$ and $V$; define $\theta = 2\pi U, z = 2V-1, w = \sqrt{1-z^2}$; choose the point $R(w\cos(\theta), w\sin(\theta), z)$ on the sphere.

Various measures of a spherical cap.

Returning to the spherical cap, obtained by a plane cut $h$ units away from the center of a ball of radius $R$, we declare that the cap has a base radius of $r = \sqrt{R^2 - h^2}$, a planar surface area of $\pi(R + h)(R - h)$; and using the proof of Lemma 3, the cap has a curved surface area of $2\pi R(R-h)$ and a volume (after subtracting the planar cone from the spherical cone) of

$$\frac{1}{3}\pi(2R + h)(R - h)^2. \tag{4}$$

Having studied the properties of a sphere, let us return to modeling the external anatomy of the n-CoV. Later, in Part 3, we will review the formulas for the three angles and the area of a spherical triangle formed by geodesics joining pairwise three points on the sphere.

## 4. Counting S-proteins

It is difficult to know for sure how many of each type of protein protrudes from the spherical surface of n-CoV, based on the 2D prints on paper or displays on the computer screen of a truly 3D object. Therefore, we attentively watched several 3D videos and TV animations. Still, we failed to count the proteins with certainty. Whoever created these videos have relied on some imaginations of their own; so must we rely on imagination.

Rely on imagination.

Let us first count the S-proteins, the dominant and longer protrusions, which we might affectionately (why not?) call the three-headed mini-monsters. Depending on how you count, the 3D pictures in *Figure* 1 indicate between 58 and 72 S-proteins. Perhaps a forensic investigator would count better than we can!

We were hoping that the 2D diagrams, being sketches of thin, planar, parallel cuts passing close to the center, would be more

helpful. No luck there either: The four 2D diagrams shown in *Figure* 1 depict respectively 16, 24, 18, 20, S-proteins. One of the 3D pictures indicates the number is only 12. If we subscribe to the notion of crowdsourcing, we might go with the mean value of 20. The TV contestant in us would find some solace in this approach, but the statistician in us knows too well the selection bias inherent in our sampling of the diagrams, and the mathematician within us finds the method simply abhorrent. Dear readers, why don't you make your guesses?

Guessing is an indispensable tool in the mathematician's toolbox.

## What to Expect in Future Parts?

Let us leave the mathematical musings on counting and locating S-proteins for Part 2. Next, in Part 3, we shall build our model for the n-Cov and study the properties of spherical triangles. In Part 4, we will declare which picture is likely right (or close to the truth) and which is possibly wrong (or far from the truth). Stay tuned!

## Acknowledgement

We are thankful for the uninterrupted time during the COVID-19 quarantine when we could engage in mathematical musings. We thank Dr. Roeder for some comments, especially for drawing our attention to [1]. Thanks are also due to the editor and an anonymous referee for their encouragement.

## Suggested Reading

[1] S Smale, Mathematical problems for the next century, In: *The Mathematical Intelligencer*, Vol.20, No.2, pp.7–15, Springer-Verlag, New York, 1998.

[2] R Courant, H Robbins and I Stewart, *What is Mathematics?*, 2nd ed. New York: Oxford University Press, 1996.

[3] Archimedes, *Wikipedia, The Free Encyclopedia*, `https://en.wikipedia.org/wiki/Archimedes`.

[4] The Cartographer's Dilemma, *National Geographic Education Resource Library* `www.nationalgeographic.org/video/the-cartographers-dilemma/`.

[5] **Cavalieri's principle,** ***Wikipedia, The Free Encyclopedia*****,**
**https://en.wikipedia.org/wiki/Cavalieri\%27s_principle.**

[6] **Spherical cap,** ***Wikipedia, The Free Encyclopedia***
**https://en.wikipedia.org/wiki/Spherical_cap.**

*Address for Correspondence*
Jyotirmoy Sarkar
Mathematical Sciences
Indiana University-Purdue
University Indianapolis
402 North Blackford Street
Indianapolis, IN 46202, USA.
Email: jsarkar@iupui.edu

Mamunur Rashid
Mathematical Sciences
DePauw University
2 East Hanna Street
Greencastle, IN 46135, USA.
Email: mrashid@depauw.edu