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A more direct way to the Cauchy problem for effectively hyperbolic operators

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Abstract

This paper is devoted to a simpler derivation of energy estimates and a proof of the well-posedness, compared to previously existing ones, for effectively hyperbolic Cauchy problem. One difference is that instead of using the general Fourier integral operator, we only use a change of local coordinates x (of the configuration space) leaving the time variable invariant. Another difference is an efficient application of the Weyl-Hörmander calculus of pseudodifferential operators associated with several different metrics.

Keywords Effective hyperbolicity · Coordinates changes · Weyl-Hörmander calculus · Geometric characterization · Localized symbols

Mathematics Subject Classification 35L15 · 35L80 · 35S05 · 35S10

1 Introduction

Consider

$$P = -D_t^2 + A_2(t, x, D) + A_0(t, x, D)D_t + A_1(t, x, D) \tag{1.1}$$

where $A_j(t, x, D)$ are differential operators of order j depending smoothly on t, having the principal symbol

$$p(t, x, \tau, \xi) = -\tau^2 + a(t, x, \xi)$$

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where $a(t, x, \xi)$ is positively homogeneous of degree 2 in ξ and nonnegative for any $(t, x, \xi) \in U \times \mathbb{R}^d$ with some neighborhood U of $(0, 0) \in \mathbb{R}^{d+1}$.

In [6], Ivrii and Petkov proved that if the Cauchy problem for P is C^∞ well posed for any lower order term then every critical point of $p = 0$ is *effectively hyperbolic*, namely the Hamilton map has a pair of non-zero real eigenvalues there. In [7], Ivrii has proved that if every critical point is effectively hyperbolic and p admits a decomposition $p = q_1 q_2$ nearby with real smooth symbols q_i vanishing at the reference point, then the Cauchy problem is C^∞ well-posed for every lower order term, transforming the original P by operator powers of operator to one with a suitable lower order term for which a standard energy method can be applied, and has conjectured that this is true without any restriction.

If a critical point (t, x, τ, ξ) is effectively hyperbolic then τ is a characteristic root of multiplicity at most 3 ([6, Lemma 8.1]) and if every multiple characteristic root is at most double, the conjecture has been proved in [9–11, 16, 17]. In [9, 10] the proof is based on the reduction of the original P to an operator for which an improved version of the method of [7] can be applied, where the reduction is made applying the Nash-Moser implicit function theorem. On the other hand, in [16] (see also [19]) the proof is based on energy estimates with pseudodifferential weights of which symbol comes from a geometric characterization of effectively hyperbolic characteristic points, after some preliminary transformations by Fourier integral operators, while in [20] another way to obtain microlocal energy estimates without the use of Fourier integral operators was given, where the original P is transformed by Gevrey pseudodifferential operators on the (t, x)-space to one with symbol extended in the complex directions, to which one can apply the classical separating operator method.

In this paper, we propose a simpler derivation of energy estimates and proof of the well-posedness of the Cauchy problem for effectively hyperbolic operators. Although we follow [19] mainly, one difference is that instead of using the general Fourier integral operator when transforming the operator, we only use a change of local coordinates x (of the configuration space which extends as a linear transformation outside a compact set) leaving the time variable invariant. This allows us to simplify the analysis of deducing the result for the original operator from that obtained for the transformed operator. Another difference is the application of Weyl-Hörmander calculus of pseudodifferential operators associated with several different metrics. The method has been used in a naive way in [19], but here we aim to organize the approach thoroughly. As a result, the argument to derive energy estimates for localized operators is made simpler and clearer and so is the proof of the local existence and uniqueness of the solution to the original Cauchy problem.

For the Cauchy problem for operators with triple effectively hyperbolic characteristics, where p cannot be smoothly factorized, see [22] and the references given there.

2 Geometric characterization of effectively hyperbolic characteristics

In this section, we prove the following proposition, which provides a geometric characterization of effectively hyperbolic characteristics ([18, Lemmas 3.1, 3.2], [19, Section 2.1]).

Proposition 2.1 *Assume that* $(0, 0, 0, \bar{\xi})$ *is effectively hyperbolic. One can choose a local coordinates* x *around* $x = 0$ *such that* $\bar{\xi} = e_d = (0, \dots, 0, 1)$ *and smooth function* $\psi(x, \xi)$*, positively homogeneous of degree* 0 *vanishing at* $(0, e_d)$*, such that either* $d\psi = d\xi_1$ *or* $d\psi = \varepsilon dx_1 + c dx_d$ *at* $(0, e_d)$ *where* $c \in \mathbb{R}$ *and* $\varepsilon = 0$ *or* 1*, and smooth* $\ell(t, x, \xi)$*,* $q(t, x, \xi) \geq 0$ *vanishing at* $(0, 0, e_d)$*, positively homogeneous of degree* $1, 2$ *respectively such that*

$$p(t, x, \tau, \xi) = -\tau^2 + \ell^2(t, x, \xi) + q(t, x, \xi), \quad q(t, x, \xi) \geq \bar{c}(t - \psi)^2 |\xi|^2 \tag{2.1}$$

with some $\bar{c} > 0$ *on a conic neighborhood of* $(0, 0, e_d)$ *where*

$$|\{\ell, \psi\}(0, 0, e_d)| < 1, \quad \{\psi, \{\psi, q\}\}(0, 0, e_d) = 0. \tag{2.2}$$

The change of coordinates $x \mapsto \chi(x)$ *can be extended to a diffeomorphism on* $\mathbb{R}^d$ *such that* $\chi(x)$ *is a linear transformation outside a neighborhood of* $x = 0$*.*

The coordinates change is called (a) or (b) according to the resulting form $d\psi = d\xi_1$ or $d\psi = \varepsilon x_1 + c x_d$, in each case one can write

$$\psi(x, \xi) = \xi_1/|\xi| + r(x, \xi), \quad \psi(x, \xi) = \varepsilon x_1 + c x_d + r(x, \xi) \tag{2.3}$$

where $dr(0, 0, e_d) = 0$. Note that $\{\psi, \{\psi, q\}\}(0, 0, e_d) = 0$ implies that

$$\partial_{x_1}^2 q(0, 0, e_d) = 0, \qquad \varepsilon \partial_{\xi_1}^2 q(0, 0, e_d) = 0 \tag{2.4}$$

according to the case (a) or (b) since $\partial_{\xi_j \xi_d}^2 q(0, 0, d_d) = 0$ by the Euler's identity for homogeneous functions.

2.1 A key lemma

In this subsection, for typographical reason, we write x_0 for t and ξ_0 for τ and denote $x = (x_0, x') = (x_0, x_1, \dots, x_d)$ and $\xi = (\xi_0, \xi') = (\xi_0, \xi_1, \dots, \xi_d)$ so that $p(x, \xi) = -\xi_0^2 + a(x, \xi')$. We also write $z = (x, \xi)$, $v = (y, \eta) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} = V$. Let $\rho = (0, \bar{\xi})$ be a critical point of $p = 0$ and hence $\bar{\xi}_0 = 0$ and $p(\rho) = \nabla p(\rho) = (\partial p(\rho)/\partial x, \partial p(\rho)/\partial \xi) = 0$. Consider the Hamilton equation

$$\frac{d}{ds}\begin{pmatrix} x \\ \xi \end{pmatrix} = H_p(x, \xi) = \begin{pmatrix} \partial p/\partial \xi \\ -\partial p/\partial x \end{pmatrix}$$

then it is clear that the linearized equation at ρ is given by

$$\frac{d}{ds}\begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} \partial^2 p(\rho)/\partial x\partial\xi & \partial^2 p(\rho)/\partial\xi\partial\xi \\ -\partial^2 p(\rho)/\partial x\partial x & -\partial^2 p(\rho)/\partial\xi\partial x \end{pmatrix}\begin{pmatrix} x \\ \xi \end{pmatrix}$$

where the half of the coefficient matrix is denoted by $F_p(\rho)$ and called the Hamilton map (matrix) of p at ρ. Denoting the quadratic (polarized) form associated with the Hesse matrix of p at ρ by $Q(z, v)$ it is clear that

$$Q(z, v) = \sigma(z, F_p(\rho)v)$$

where $\sigma(z, v) = \langle\xi, y\rangle - \langle x, \eta\rangle$, $z = (x, \xi)$, $v = (y, \eta)$ is the symplectic two form on V. From the definition we see $p(\rho + \epsilon z) = \epsilon^2 Q(z)/2 + O(\epsilon^3)$ as $\epsilon \to 0$ and Q has the signature $(r, 1)$ with some $r \in \mathbb{N}$ since $a(x, \xi')$ is nonnegative near $\rho' = (0, \bar{\xi}') \in \mathbb{R}^{d+1} \times \mathbb{R}^d$. Moreover, it follows from the Morse lemma (see, e.g. [4, Lemma C.6.2]) that one can find $\phi_1, \ldots, \phi_r$ and g vanishing at ρ', homogeneous of degree 1, 2 in ξ' respectively, C^∞ in a conic neighborhood of ρ' such that $\nabla\phi_1, \ldots, \nabla\phi_r$ are linearly independent at ρ' and $g \geq 0$, $\nabla^2 g(\rho') = O$ and

$$a(x, \xi') = \sum_{j=1}^{r} \phi_j^2(x, \xi') + g(x, \xi'). \tag{2.5}$$

With $\phi_0 = \xi_0$ it is clear $Q(z, v) = -\langle\nabla\phi_0, z\rangle\langle\nabla\phi_0, v\rangle + \sum_{j=1}^{r}\langle\nabla\phi_j, z\rangle\langle\nabla\phi_j, v\rangle$. Then noticing $\langle\nabla\phi_j, z\rangle = \sigma(z, H_{\phi_j})$ we see that

$$Q(z, v) = \sigma(z, F_p v) = \sigma\Big(z, -\sigma(v, H_{\phi_0})H_{\phi_0} + \sum_{j=1}^{r}\sigma(v, H_{\phi_j})H_{\phi_j}\Big)$$

and hence $F_p v = -\sigma(v, H_{\phi_0})H_{\phi_0} + \sum_{j=1}^{r}\sigma(v, H_{\phi_j})H_{\phi_j}$. Therefore the kernel and the image of F_p are given by

$$\begin{aligned} \mathrm{Im} F_p &= \{z \in V \mid z = \sum_{j=0}^{r}\alpha_j H_{\phi_j}, \alpha_j \in \mathbb{R}\}, \\ \mathrm{Ker} F_p &= \{z \in V \mid \sigma(z, H_{\phi_j}) = 0, j = 0, \ldots, r\}. \end{aligned} \tag{2.6}$$

Consider the following open convex cone in V

$$\Gamma = \{z \in V \mid Q(z) = Q(z, z) = -\xi_0^2 + \sum_{j=1}^{r}\langle\nabla\phi_j, z\rangle^2 < 0, \xi_0 > 0\} \tag{2.7}$$

which is the connected component of $\{z \in V \mid Q(z) \neq 0\}$ containing the positive ξ_0 axis. Recall [3, Corollary 1.4.7] for which we give a more direct proof here.

Lemma 2.1 *If $F_p(\rho)$ has a nonzero real eigenvalue then $\Gamma \cap \mathrm{Im} F_p \neq \{0\}$.*

Proof Let $\lambda \neq 0$ be a real eigenvalue and $F_p z = \lambda z$ with $0 \neq z \in V$. Then from $0 = \sigma((F_p - \lambda)z, v) = \sigma(z, (-F_p - \lambda)v)$ for all $v \in V$ we see that $F_p + \lambda$ is not surjective which proves that $-\lambda$ is also an eigenvalue. Let $F_p z_\pm = \pm\lambda z_\pm$, $z_\pm \neq 0$ then $z_\pm \in \mathrm{Im} F_p$ for $\lambda \neq 0$. Note that the signature of Q is $(r, 1)$ with $r \geq 1$ otherwise $Q(z)$ would be $-\xi_0^2$ and hence F_p has no nonzero eigenvalues. The quadratic form Q induces a quadratic form $\bar{Q}$ in $V_0 = V/\mathrm{Ker} F_p$ which is non-degenerate and of Lorenz signature. If $\sigma(z_+, z_-) = 0$ then $\bar{Q}$ would vanish on the 2 dimensional linear subspace of V_0 spanned by $[z_+], [z_-]$ which is a contradiction. Thus with $z = \alpha z_+ + \beta z_- \in \mathrm{Im} F_p$ we have

$$Q(z) = \sigma(\alpha z_+ + \beta z_-, \lambda\alpha z_+ - \lambda\beta z_-) = -2\alpha\beta\lambda\sigma(z_+, z_-).$$

Choosing α, β such that $\alpha\beta\lambda\sigma(z_+, z_-) > 0$ we get $Q(z) < 0$ hence either $z \in \Gamma$ or $-z \in \Gamma$. □

For a linear subspace $S \subset V$ we denote $S^\sigma = \{z \in V \mid \sigma(z, S) = 0\}$ hence $(S^\sigma)^\sigma = S$ and for $0 \neq z \in V$, $\langle z \rangle$ stands for the line $\mathbb{R}z$. Introduce the dual cone of Γ with respect to σ defined by

$$C = \{z \in V; \sigma(z, w) \leq 0, \forall w \in \Gamma\}.$$

The next lemma [19, Lemma 1.1.3] is the key to the geometric characterization of effectively hyperbolic characteristics.

Lemma 2.2 *Let θ be the unit vector directed to positive ξ_0 axis. The following three conditions are equivalent;*

(i) $\Gamma \cap \mathrm{Im} F_p \neq \{0\}$,
(ii) *there is a linear subspace $H \subset V$ of codimension 1 such that $H \cap C = \{0\}$ and $\mathrm{Ker} F_p + \langle\theta\rangle \subset H$,*
(iii) $\Gamma \cap \mathrm{Im} F_p \cap \langle\theta\rangle^\sigma \neq \{0\}$.

Proof First note that

$$z \in \Gamma \Longrightarrow \langle z \rangle^\sigma \cap C = \{0\}. \tag{2.8}$$

In fact if there were $0 \neq v \in \langle z \rangle^\sigma \cap C$ we would have $\sigma(v, z + w) = \sigma(v, w) \leq 0$ for any small w since Γ is open leads to a contradiction.
(i) $\Longrightarrow$ (ii). We first assume $\theta \in \mathrm{Ker} F_p + \mathrm{Im} F_p$ so that $\theta = z_1 + z_2$ with $z_1 \in \mathrm{Ker} F_p$ and $z_2 \in \mathrm{Im} F_p$. Then $0 \neq z_2 \in \Gamma$ since $\theta \in \Gamma$ and $\Gamma + \mathrm{Ker} F_p \subset \Gamma$ and $\Gamma \cap \mathrm{Ker} F_p = \emptyset$. It is clear that $\theta \in \langle z_2 \rangle^\sigma$ because $\mathrm{Ker} F_p \subset \langle z_2 \rangle^\sigma$ and $z_2 \in \langle z_2 \rangle^\sigma$ therefore $H = \langle z_2 \rangle^\sigma$ is a desired subspace by (2.8).

Next consider the case $\theta \notin \mathrm{Ker} F_p + \mathrm{Im} F_p$ and hence $(\mathrm{Ker} F_p + \mathrm{Im} F_p) \cap \langle\theta\rangle = \{0\}$. Take $0 \neq w \in \Gamma \cap \mathrm{Im} F_p$ then $\mathrm{Ker} F_p = (\mathrm{Im} F_p)^\sigma \subset \langle w \rangle^\sigma$ and $\langle w \rangle^\sigma \cap C = \{0\}$ by (2.8), while $C \subset \mathrm{Im} F_p$ for $\Gamma + \mathrm{Ker} F_p \subset \Gamma$ one concludes $\mathrm{Ker} F_p + \mathrm{Im} F_p \not\subset \langle w \rangle^\sigma$. Therefore we have $\langle w \rangle^\sigma + (\mathrm{Ker} F_p + \mathrm{Im} F_p) = V$ and hence $\langle w \rangle^\sigma \cap (\mathrm{Ker} F_p + \mathrm{Im} F_p)$

is of codimension 1 in $\mathrm{Ker}F_p + \mathrm{Im}F_p$. Now writing $V = (\mathrm{Ker}F_p + \mathrm{Im}F_p) \oplus \langle\theta\rangle \oplus W$ (direct sum) it is clear that $H = (\langle w\rangle^\sigma \cap (\mathrm{Ker}F_p + \mathrm{Im}F_p)) \oplus \langle\theta\rangle \oplus W$ is a desired subspace.
(ii) $\Longrightarrow$ (iii). Choose $0 \neq v \in V$ such that $\langle v\rangle = H^\sigma$. It is clear that $\langle v\rangle \subset \mathrm{Im}F_p \cap \langle\theta\rangle^\sigma$ for $\mathrm{Ker}F_p + \langle\theta\rangle \subset H$. Show that v or $-v$ belongs to Γ. If not we would have $\langle v\rangle \cap \Gamma = \emptyset$ and by the Hahn-Banach theorem there were $0 \neq w \in V$ such that $\sigma(w, z) \leq 0, \forall w \in C$ and $w \in \langle v\rangle^\sigma = H$ which contradicts with (ii).
(iii) $\Longrightarrow$ (i) is trivial. □

2.2 Proof of Proposition 2.1

In this subsection we return to the original notation and write t for x_0 and τ for ξ_0 and denote $x = (x_1, \dots, x_d)$, $\xi = (\xi_1, \dots, \xi_d)$. After a suitable linear change of local coordinates x we may assume that $\bar{\xi} = (0, \dots, 0, 1) = e_d$. We write $\rho' = (0, 0, e_d) \in \mathbb{R}^{d+1} \times \mathbb{R}^d$ and $\rho'' = (0, e_d) \in \mathbb{R}^d \times \mathbb{R}^d$. Thanks to Lemma 2.2 one can take $0 \neq z \in \Gamma \cap \mathrm{Im}F_p \cap \langle\theta\rangle^\sigma$ where $z = \sum_{j=1}^r \alpha_j H_{\phi_j}(\rho) + \alpha_0 H_{\phi_0}(\rho)$ in view of (2.6), where we see $\alpha_0 = -\sigma(z, \theta) = 0$ for $z \in \langle\theta\rangle^\sigma$. Let

$$f(t, x, \xi) = \sum_{j=1}^{r} \alpha_j \phi_j(t, x, \xi)/|\xi|.$$

Since $H_f(\rho') = z \in \Gamma$ it is clear that $\partial f/\partial t < 0$ at ρ' in view of (2.7) then one can write $f(t, x, \xi) = e(t, x, \xi)(t - \psi(x, \xi))$ where $e(\rho') < 0$. It is clear from (2.5)

$$a(t, x, \xi) \geq c_1(t - \psi(x, \xi))^2|\xi|^2 \tag{2.9}$$

with some $c_1 > 0$. Since $-H_{x_0 - \psi}(\rho') \in \Gamma$ we see from (2.7) that

$$1 > \sum_{j=1}^{r} \langle\nabla\phi_j(\rho'), H_{t-\psi}(\rho)\rangle^2 = \sum_{j=1}^{r} \{\phi_j, \psi\}^2(\rho')$$

from which, taking (2.5) and $\nabla^2 g(\rho') = O$ into account, we conclude that

$$|\{\psi, \{\psi, a\}\}(\rho')| = 2\Big|\sum_{j=1}^{r} \{\psi, \phi_j\}^2(\rho')\Big| < 2. \tag{2.10}$$

The next lemma is well known.

Lemma 2.3 *Assume $d\psi \neq 0$ and not proportional to dx_d at ρ''. Then one can find a system of local coordinates $x = (x_1, \dots, x_d)$ such that either $d\psi = d\xi_1$ or $d\psi = dx_1 + c\,dx_d$ with some $c \in \mathbb{R}$ at ρ''.*

Proof Since $\partial_{\xi_d}\psi(\rho'') = 0$ by the Euler's identity one can write $\psi(x, \xi) = \langle\tilde{a}, \tilde{\xi}\rangle + \langle\tilde{b}, \tilde{x}\rangle + b_d x_d + r(x, \xi)$ where $\tilde{\xi} = (\xi_1, \dots, \xi_{d-1})$, $\tilde{x} = (x_1, \dots, x_{d-1})$ and r vanishes at ρ'' of order 2. If $\tilde{a} = 0$ hence $\tilde{b} \neq 0$ a linear change of coordinates $\tilde{x}$ gives a desired form. If $\tilde{a} \neq 0$ one can assume $\langle \tilde{a}, \tilde{\xi} \rangle = \xi_1 + \cdots + \xi_k$ renumbering and dilating x_j, $1 \leq j \leq d-1$. Changing the coordinate x_d to $x_d - \sum_{j=1}^{k} b_j x_j^2/2$ yields $\langle \tilde{b}, \tilde{x} \rangle + b_d x_d = \sum_{j=k+1}^{d} b_j x_j$. Changing again the coordinate x_d to $x_d - x_1 \sum_{j=k+1}^{d} b_j x_j$ yields $b_{k+1} = \cdots = b_d = 0$ hence after a linear change of coordinates $(x_1, \ldots, x_k)$ one has $d\psi = d\xi_1$ at ρ''. □

Proof of Proposition 2.1 Let ψ be the one given in (2.9). If $d\psi = 0$ or proportional to dx_d at ρ'' it suffices to take $\ell = 0$ and $q = a$ because $\partial_{\xi_d}^2 a(\rho') = 0$ by the Euler's identity. Assume $d\psi(\rho'') \neq 0$ and not proportional to dx_d. From Lemma 2.3 we may assume $d\psi = d\xi_1$ or $d\psi = dx_1 + c dx_d$. Assume $d\psi = d\xi_1$ at ρ''. If $\partial_{x_1}^2 a(\rho') = 0$ it suffices to take $\ell = 0$ and $q = a$. Otherwise, thanks to the Malgrange preparation theorem (e.g. [5, Theorem 7.5.5]) one can write

$$a(t, x, \xi) = e(t, x, \xi)((x_1 - h(t, x', \xi))^2 + g(t, x', \xi)), \quad x' = (x_2, \ldots, x_d)$$

where $e(\rho') > 0$ and h, g, vanishing at ρ', are of homogeneous of degree 0. Choose

$$\ell(t, x, \xi) = e^{1/2}(t, x, \xi)(x_1 - h(t, x', \xi)), \quad q(t, x, \xi) = e(t, x, \xi) g(t, x', \xi)$$

and set $\psi_1(t, x', \xi) = \psi(h(t, x', \xi), x', \xi)$ then $d\psi_1 = d\psi$ at ρ'. From (2.9) there is $c_2 > 0$ such that

$$q(t, x, \xi) \geq c_2 (t - \psi_1(t, x', \xi))^2 |\xi|^2.$$

Since $\partial \psi_1 / \partial t = 0$ at ρ' one can write $t - \psi_1(t, x', \xi) = e'(t, x', \xi)(t - \psi_2(x', \xi))$. Since $d\psi_2 = d\psi_1 = d\xi_1$ at ρ' then $\{\psi_2, \{\psi_2, q\}\}(\rho') = 0$ hence it follows from (2.10) that $\{\ell, \psi_2\}^2(\rho') < 1$. Thus ψ_2 is a desired one. When $d\psi = dx_1 + c dx_d$ the proof is similar. In Lemma 2.3 we used coordinates changes such that $y = Ax + q(x)$ where A is a non-singular matrix and $q(x)$ is a quadratic form in x, thus cutting $q(x)$ off outside a neighborhood of $x = 0$ it is clear that the resulting change of coordinates satisfies the requirements in Proposition 2.1. □

3 Quantitative expression of Proposition 2.1 by localized symbols

In this section, we localize the symbols obtained in Proposition 2.1 around $(0, e_d)$ with a positive parameter M and we will use this M to quantitatively express the condition (2.2). We first localize coordinates functions. Let $\chi(s) \in C^\infty(\mathbb{R})$ be such that $\chi(s) = s$ on $|s| \leq 1$, $|\chi(s)| = 2$ on $|s| \geq 2$ and $0 \leq d\chi(s)/ds = \chi^{(1)}(s) \leq 1$ everywhere. Define $y(x) = (y_1(x), \ldots, y_d(x))$ and $\eta(\xi) = (\eta_1(\xi), \ldots, \eta_d(\xi))$ by

$$y_j(x) = M^{-1} \chi(M x_j), \quad \eta_j(\xi) = M^{-1} \chi(M(\xi_j \langle \xi \rangle_\gamma^{-1} - \delta_{jd})), \quad 1 \leq j \leq d$$

where $\langle\xi\rangle_\gamma = (\gamma^2 + |\xi|^2)^{1/2}$ and δ_{ij} is the Kronecker's delta. Here M and γ are positive parameters constrained by

$$\gamma \ge M^4 \ge 1. \tag{3.1}$$

Clearly there is $C > 0$ such that

$$|y(x)| \le CM^{-1}, \quad |\eta(\xi)| \le CM^{-1}, \quad (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d \tag{3.2}$$

so that $(y(x), \eta(\xi) + e_d)$ is contained in a neighborhood of $(0, e_d)$ which shrinks with M. Note that $(y(x), (\eta + e_d)\langle\xi\rangle_\gamma) = (x, \xi)$ in a "conic like" neighborhood $W_{M,\gamma}$ of $(0, e_d)$ given by

$$W_{M,\gamma} = \{(x,\xi) \mid |x| \le M^{-1},\ |\xi/|\xi| - e_d| \le M^{-1}/2,\ |\xi| \ge \gamma M^{1/2}\} \tag{3.3}$$

because if $(x,\xi) \in W_{M,\gamma}$ then

$$\begin{aligned}|\xi/\langle\xi\rangle_\gamma - e_d| &\le |\xi/\langle\xi\rangle_\gamma - \xi/|\xi|| + |\xi/|\xi| - e_d| \le M^{-1}/2\\ &+ |\langle\xi\rangle_\gamma - |\xi||/\langle\xi\rangle_\gamma \le M^{-1}/2 + \gamma^2\langle\xi\rangle_\gamma^{-1}(\langle\xi\rangle_\gamma + |\xi|)^{-1} \le M^{-1}.\end{aligned}$$

From now on, fixing a $T_0 > 0$, we assume that the range of t is also constrained by

$$|t| < T_0 M^{-1}. \tag{3.4}$$

Definition 3.1 For a smooth function $f(t,x,\xi)$ near $(0,0,e_d)$ the localization f_M is defined to be $f(t, y(x), \eta(\xi) + e_d)$. When f is defined in a conic neighborhood of $(0,0,e_d)$ and of homogeneous of degree m in ξ we define $f_M = f(t, y(x), \eta(\xi) + e_d)\langle\xi\rangle_\gamma^m = f(t, y(x), (\eta(\xi) + e_d)\langle\xi\rangle_\gamma)$.

Throughout the paper, $A \lesssim B$ means $A \le CB$ with some constant C independent of *all involved parameters* (M, γ here) if otherwise stated. We denote $A_1 \approx A_2$ if $A_1 \lesssim A_2$ and $A_2 \lesssim A_1$. To express (2.2) quantitatively introduce a preliminary metric

$$G_z(w) = M^2(|y|^2 + \langle\xi\rangle_\gamma^{-2}|\eta|^2), \quad z = (x,\xi),\ w = (y,\eta) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{3.5}$$

It is clear that $y_j \in S(M^{-1}, G)$ and

$$\begin{aligned}\left|\partial_\xi^\alpha \eta_j(\xi)\right| &\lesssim \sum_{\alpha=\alpha^1+\cdots+\alpha^s, |\alpha^i|\ge 1} M^{-1}|\chi^{(s)}(M(\xi_j\langle\xi\rangle_\gamma^{-1} - \delta_{jd}))|\\ &\quad\times|\partial_\xi^{\alpha^1}(M(\xi_j\langle\xi\rangle_\gamma^{-1} - \delta_{jd}))|\cdots|\partial_\xi^{\alpha^s}(M(\xi_j\langle\xi\rangle_\gamma^{-1} - \delta_{jd}))|\\ &\lesssim \sum_{s\le|\alpha|} M^{-1}M^s\langle\xi\rangle_\gamma^{-|\alpha|} \lesssim M^{-1+|\alpha|}\langle\xi\rangle_\gamma^{-|\alpha|}, \quad |\alpha| \ge 1\end{aligned}$$

shows $\eta_j \in S(M^{-1}, G)$.

Lemma 3.1 *Let $f(t,x,\xi)$ be a smooth function in a neighborhood of $(0,0,e_d)$ such that $\partial_t^k\partial_x^\alpha\partial_\xi^\beta f(0,0,e_d)=0$ for $k+|\alpha+\beta|<r$. Then $f_M\in S(M^{-r},G)$ and*

$$f_M(t,x,\xi)-\sum_{k+|\alpha+\beta|=r}\frac{1}{k!\alpha!\beta!}\partial_t^k\partial_x^\alpha\partial_\xi^\beta f(0,0,e_d)t^k y^\alpha\eta^\beta\in S(M^{-r-1},G)$$

and $\partial_t f_M\in S(M^{-r+1},G)$. If the term $\sum_{k+|\alpha+\beta|=r}\cdots$ contains no y_l then $\partial_{x_l}f_M\in S(M^{-r},G)$ and contains no η_d then $\partial_{\xi_d}f_M\in S(M^{-r}\langle\xi\rangle_\gamma^{-1},G)$. Moreover if the term contains neither η_d nor η_l $(1\le l\le d-1)$ then we have $\partial_{\xi_l}f_M\in S(M^{-r}\langle\xi\rangle_\gamma^{-1},G)$.

Proof Noting

$$\partial\eta_j/\partial\xi_k-\delta_{jk}\chi^{(1)}(M\xi_j\langle\xi\rangle_\gamma^{-1})\langle\xi\rangle_\gamma^{-1}\in S(M^{-1}\langle\xi\rangle_\gamma^{-1},G)$$

for $1\le j\le d-1$, $1\le k\le d$ the proof follows from the Taylor formula

$$\begin{aligned}f(t,y,\eta+e_d)=&\sum_{k+|\alpha+\beta|=r}\frac{1}{k!\alpha!\beta!}\partial_t^k\partial_x^\alpha\partial_\xi^\beta f(0,0,e_d)t^k y^\alpha\eta^\beta\\&+\sum_{k+|\alpha+\beta|=r+1}\frac{r+1}{k!\alpha!\beta!}t^k y^\alpha\eta^\beta\int_0^1(1-\theta)^r\partial_t^k\partial_x^\alpha\partial_\xi^\beta f(\theta t,\theta y,\theta\eta+e_d)d\theta\end{aligned}\tag{3.6}$$

where the integral belongs to $S(1,G)$ since $|(t,y,\eta)|\le CM^{-1}$. □

Let $x\mapsto\chi(x)$ be the diffeomorphism on $\mathbb{R}^d$ obtained in Proposition 2.1 and denoting $(Tu)(t,x)=u(t,\kappa(x))$ the localized symbol of $T^{-1}PT$ is given by

$$\hat P(t,x,\tau,\xi)=-\tau^2+\ell_M^2(t,x,\xi)+q_M(t,x,\xi)+a_1(t,x,\xi)+a_0(t,x,\xi)\tau$$

where $\ell_M\in S(M^{-1}\langle\xi\rangle_\gamma,G)$, $q_M\in S(M^{-2}\langle\xi\rangle_\gamma^2,G)$ and $a_j\in S(\langle\xi\rangle_\gamma^j,G)$. Noting $|\eta(\xi)+e_d|\ge(1-CM^{-1})$ from (2.1) one finds $M_1>0$, $\underline{c}>0$ such that

$$q_M(t,x,\xi)\ge\underline{c}\,(t-\psi_M(x,\xi))^2\langle\xi\rangle_\gamma^2.\tag{3.7}$$

The following two propositions are quantitative expressions of (2.2).

Proposition 3.1 *We have $\{\psi_M,q_M\}\in S(M^{-2}\langle\xi\rangle_\gamma,G)$ and that $|\{\psi_M,q_M\}|\le CM^{-1/2}\sqrt{q_M}$.*

Proof Choose $f=q$ and $r=2$ in (3.6) then the quadratic form in (t,y,η) is nonnegative definite since $q(t,y,\eta+e_d)$ is nonnegative. In the case (a) this quadratic form contains no y_1 because of (2.4) hence $\partial_{x_1}^2 q_M(t,x,\xi)\in S(M^{-1}\langle\xi\rangle_\gamma^2,G)$ and $\partial_{x_j}^2 q_M(t,x,\xi)\in S(\langle\xi\rangle_\gamma^2,G)$ by Lemma 3.1 then from the Glaeser inequality one obtains

$$\left|\partial_{x_j}q_M\right|\le CM^{-\delta_{1j}/2}\sqrt{q_M}\,\langle\xi\rangle_\gamma,\quad\forall j.\tag{3.8}$$

In the case (b), thanks to Euler's identity and (2.4) we have $\partial_{\xi_d}^2 q(0,0,e_d)=0$ and $\varepsilon\partial_{\xi_1}^2 q(0,0,e_d)=0$ hence repeating the same arguments as above one obtains

$$|\partial_{\xi_d} q_M| \le CM^{-1/2}\sqrt{q_M}, \quad |\partial_{\xi_j} q_M| \le CM^{-\varepsilon\delta_{1j}/2}\sqrt{q_M}, \quad j \neq d. \tag{3.9}$$

Next study ψ_M. In the case (a) since $|\eta(\xi)+e_d|^2=\sum_{j=1}^{d-1}\eta_j^2+(\eta_d+1)^2=1+k$ with $k\in S(M^{-1},G)$ hence $1/|\eta(\xi)+e_d|=1+\tilde{k}$ with $\tilde{k}\in S(M^{-1},G)$ one sees $\eta_1(\xi)/|\eta(\xi)+e_d|-\eta_1(\xi)\in S(M^{-2},G)$. Then noting (2.3) it follows from Lemma 3.1 that

$$\begin{aligned}&\psi_M(x,\xi)-\eta_1(\xi)\in S(M^{-2},G), \quad \partial_{x_j}\psi_M(x,\xi)\in S(M^{-1},G), \quad \forall j,\\ &\partial_{\xi_j}\psi_M(x,\xi)-\delta_{1j}\chi^{(1)}(M\xi_1\langle\xi\rangle_\gamma^{-1})\langle\xi\rangle_\gamma^{-1}\in S(M^{-1}\langle\xi\rangle_\gamma^{-1},G), \quad \forall j.\end{aligned} \tag{3.10}$$

In the case (b) we have similarly that

$$\begin{aligned}&\psi_M(x,\xi)-\varepsilon y_1(x)-cy_d(x)\in S(M^{-2},G),\\ &\partial_{\xi_j}\psi_M(x,\xi)\in S(M^{-1}\langle\xi\rangle_\gamma^{-1},G), \quad \forall j\\ &\partial_{x_j}\psi_M-\varepsilon\delta_{1j}\chi^{(1)}(Mx_1)-c\delta_{dj}\chi^{(1)}(Mx_d)\in S(M^{-1},G), \quad \forall j.\end{aligned} \tag{3.11}$$

Now proceed to the proof of the proposition. In the case (a), noting $\partial_{x_1}q_M\in S(M^{-2}\langle\xi\rangle_\gamma^2,G)$, the first assertion follows from (3.10) and Lemma 3.1. The second assertion follows from (3.8) and (3.10). The proof for the case (b) is similar. □

Proposition 3.2 *We have* $\{\ell_M,\psi_M\}\in S(1,G)$ *and* $\sup|\{\ell_M,\psi_M\}|\le|\kappa|+CM^{-1}$ *where* $|\kappa|<1$.

Proof Note that $\partial_{\xi_d}\ell_M\in S(M^{-1},G)$ for $\partial_{\xi_d}\ell(0,0,e_d)=0$ by Euler's identity. According to the case (a) or (b) we have $\partial_x^\alpha\psi_M\in S(M^{-1},G)$ or $\partial_\xi^\alpha\psi_M(M^{-1}\langle\xi\rangle_\gamma^{-1},G)$ for $|\alpha|=1$ in view of (2.3) then it follows from (3.10) and (3.11) that

$$\{\ell_M,\psi_M\}+\kappa\chi^{(1)}(Mx_1)\chi^{(1)}(M\xi_1\langle\xi\rangle_\gamma^{-1})\in S(M^{-1},G)$$

where $\kappa=\partial_{x_1}\ell(0,e_d)$ or $\kappa=-\varepsilon\partial_{\xi_1}\ell(0,e_d)$ and $|\kappa|<1$ by (2.2). Noting that $\chi^{(1)}(Mx_1)\chi^{(1)}(M\xi_1\langle\xi\rangle_\gamma^{-1})\in S(1,G)$ and whose modulus is at most 1 the proof is complete. □

From now on, for notational simplicity we simply write ψ, ℓ and q instead of ψ_M, ℓ_M and q_M.

4 Energy estimates for localized operators

In this section, we utilize $t-\psi(x,\xi)$ obtained from the geometric characterization of effectively hyperbolic characteristic points to derive the weighted energy estimate for the localized operator $\hat{P}=\mathrm{op}(\hat{P}(t,x,\tau,\xi))$.

4.1 Metrics and weights related to energy estimates

In this paper the following simple metrics are used;

$$\begin{aligned}\bar{g} &= \langle\xi\rangle_\gamma |dx|^2 + \langle\xi\rangle_\gamma^{-1}|d\xi|^2, \quad \underline{g} = |dx|^2 + \langle\xi\rangle_\gamma^{-2}|d\xi|^2, \quad \gamma \geq 1,\\ g_\epsilon &= M^{-2\delta_{\epsilon a}}\langle\xi\rangle_\gamma |dx|^2 + M^{-2\delta_{\epsilon b}}\langle\xi\rangle_\gamma^{-1}|d\xi|^2, \quad \gamma \geq M^4 \geq 1\end{aligned} \tag{4.1}$$

where g_ϵ is related to the coordinates change (a) or (b), namely ϵ is either a or b and $\delta_{\epsilon\epsilon'} = 1$ if $\epsilon = \epsilon'$ and 0 otherwise. The properties of pseudodifferential operators associated with metrics (4.1) are summarized in the Appendix. It is clear that

$$g_\epsilon / g_\epsilon^\sigma \leq M^{-2}, \quad M^{-2}\bar{g} \leq g_\epsilon \leq \bar{g}$$

such that g_ϵ satisfies (6.31). Noting that $a \in S(m, g_\epsilon)$ if and only if

$$|\partial_x^\alpha \partial_\xi^\beta a| \lesssim m M^{-\epsilon(\alpha,\beta)}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}, \quad \epsilon(\alpha,\beta) = |\alpha|\delta_{\epsilon a} + |\beta|\delta_{\epsilon b}, \quad \alpha, \beta \in \mathbb{N}^d$$

and $M^{|\alpha+\beta|}\langle\xi\rangle_\gamma^{-|\beta|} \leq (M^4\langle\xi\rangle_\gamma^{-1})^{|\alpha+\beta|/2} M^{-\epsilon(\alpha,\beta)}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}$ it is clear that $S(m, G) \subset S(m, g_\epsilon)$. Following Sect. 6.2 we set

$$b = (q(t,x,\xi) + \lambda\langle\xi\rangle_\gamma)^{1/2}$$

then there exists $\bar{\lambda}$ such that for $\lambda \geq \bar{\lambda}$ both Proposition 6.1 and Lemma 6.8 hold. From now on we fix such a $\lambda = \bar{\lambda}$, while M and γ remain to be free under the constraints (3.1) and (6.21). Introducing

$$\omega(t,x,\xi) = ((t - \psi(x,\xi))^2 + \langle\xi\rangle_\gamma^{-1})^{1/2} \tag{4.2}$$

and taking (3.7) and $\langle\xi\rangle_\gamma^{-1/2} \leq \omega$ into account one sees that b satisfies ($\bar{\lambda} \geq \underline{c}$ can be assumed)

$$\begin{aligned} b &= (q + \bar{\lambda}\langle\xi\rangle_\gamma)^{1/2} \geq \big(\underline{c}\,(t-\psi)^2\langle\xi\rangle_\gamma^2 + \bar{\lambda}\langle\xi\rangle_\gamma\big)^{1/2}\\ &\geq \sqrt{\underline{c}}\;\omega^{-1}\langle\xi\rangle_\gamma\big((t-\psi)^2\omega^2 + \omega^2\langle\xi\rangle_\gamma^{-1}\big)^{1/2}\\ &\geq \sqrt{\underline{c}}\;\omega^{-1}\langle\xi\rangle_\gamma\big(|t-\psi|^4 + \langle\xi\rangle_\gamma^{-2}\big)^{1/2} \geq \sqrt{\underline{c}/2}\;\omega\langle\xi\rangle_\gamma. \end{aligned} \tag{4.3}$$

Lemma 4.1 *We have* $\partial_x^\alpha \partial_\xi^\beta q \in S(\langle\xi\rangle_\gamma^{1-|\beta|} b, \bar{g})$ *for* $|\alpha + \beta| = 1$, $\partial_t q \in S(\langle\xi\rangle_\gamma b, \bar{g})$ *and* $\{q, \psi\} \in S(M^{-1/2} b, \bar{g})$.

Proof The first two assertions are immediate consequences of Lemma 6.7. The third assertion follows from Proposition 3.1 and (6.30). □

The following weight is a key to energy estimates

$$\phi(t,x,\xi) = \omega(t,x,\xi) + t - \psi(x,\xi)$$

where it is clear that ϕ verifies

$$M\langle\xi\rangle_\gamma^{-1}/C \le \langle\xi\rangle_\gamma^{-1}/(2\omega) \le \phi \le CM^{-1}, \tag{4.4}$$

$$\partial_t\phi = \omega^{-1}\phi, \quad \partial_x^\alpha\partial_\xi^\beta\phi = \frac{-\partial_x^\alpha\partial_\xi^\beta\psi}{\omega}\phi + \frac{\partial_x^\alpha\partial_\xi^\beta\langle\xi\rangle_\gamma^{-1}}{2\omega}, \quad |\alpha+\beta| = 1. \tag{4.5}$$

Lemma 4.2 *We have* $\partial_x^\alpha\partial_\xi^\beta\psi \in S(\langle\xi\rangle_\gamma^{-1/2}M^{-\epsilon(\alpha,\beta)}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}, g_\epsilon)$ *for* $|\alpha+\beta| \ge 1$.

Proof Recall that $\psi = \eta_1(\xi) + r$ or $\psi = \varepsilon y_1(x) + cy_d(x) + r$ with $r \in S(M^{-2}, G)$ according to the coordinates change (a) or (b). For $\nu = \beta' + \beta$, $|\beta| \ge 1$ we have

$$|\partial_\xi^\nu\psi| \lesssim M^{-1-\delta_{\epsilon b}+|\nu|}\langle\xi\rangle_\gamma^{-|\nu|} \lesssim \langle\xi\rangle_\gamma^{-1/2}M^{-\epsilon(0,\nu)}\langle\xi\rangle_\gamma^{-|\nu|/2}(M^{2+2\delta_{\epsilon b}}\langle\xi\rangle_\gamma^{-1})^{(|\nu|-1)/2}.$$

For $\mu = \alpha' + \alpha$, $|\alpha| \ge 1$ one has

$$|\partial_x^\mu\psi| \lesssim M^{-1-\delta_{\epsilon a}+|\mu|} \lesssim \langle\xi\rangle_\gamma^{-1/2}M^{-\epsilon(\mu,0)}\langle\xi\rangle_\gamma^{|\mu|/2}(M^{2+2\delta_{\epsilon a}}\langle\xi\rangle_\gamma^{-1})^{(|\mu|-1)/2}.$$

Let $\mu = \alpha' + \alpha$, $|\alpha| \ge 1$ and $\nu = \beta' + \beta$, $|\beta| \ge 1$ then noting $|\mu + \nu| \le 2|\mu+\nu| - \epsilon(\mu,\nu)$ one has

$$\begin{aligned}|\partial_x^\mu\partial_\xi^\nu\psi| &\lesssim M^{-2+|\mu+\nu|}\langle\xi\rangle_\gamma^{-|\nu|} \\ &\lesssim \langle\xi\rangle_\gamma^{-1/2}M^{-\epsilon(\mu,\nu)}\langle\xi\rangle_\gamma^{(|\mu|-|\nu|)/2}(M^4\langle\xi\rangle_\gamma^{-1})^{(|\mu+\nu|-1)/2}.\end{aligned}$$

Since $M^{2+2\delta_{\epsilon\epsilon'}}\langle\xi\rangle_\gamma^{-1} \le M^4\langle\xi\rangle_\gamma^{-1} \le 1$ by (3.1) the assertion is proved. □

Lemma 4.3 *We have* $\partial_x^\alpha\partial_\xi^\beta\omega^s \in S(\omega^{s-1}\langle\xi\rangle_\gamma^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}, g_\epsilon)$ *for* $|\alpha+\beta| \ge 1$ *and* $s \in \mathbb{R}$. *In particular* $\omega^s \in S(\omega^s, g_\epsilon)$.

Proof We first show the assertion for $s = 2$. Since $\omega^2 = (t-\psi)^2 + \langle\xi\rangle_\gamma^{-1}$ noting $\omega\langle\xi\rangle_\gamma^{1/2} \ge 1$ and $|t-\psi| \le \omega$ one sees for $\nu = \beta' + \beta$, $|\beta| \ge 1$ that

$$\begin{aligned}|\partial_\xi^\nu\omega^2| &\lesssim \omega M^{-1-\delta_{\epsilon b}+|\nu|}\langle\xi\rangle_\gamma^{-|\nu|} + M^{-2-2\delta_{\epsilon b}+|\nu|}\langle\xi\rangle_\gamma^{-|\nu|} + \langle\xi\rangle_\gamma^{-1-|\nu|} \\ &\lesssim \omega\langle\xi\rangle_\gamma^{-1/2}M^{-\epsilon(0,\nu)}\langle\xi\rangle_\gamma^{-|\nu|/2}(M^{2+2\delta_{\epsilon b}}\langle\xi\rangle_\gamma^{-1})^{(|\nu|-1)/2} \\ &\quad+\omega\langle\xi\rangle_\gamma^{-1/2}M^{-\epsilon(0,\nu)}\langle\xi\rangle_\gamma^{-|\nu|/2}(M^{2+2\delta_{\epsilon b}}\langle\xi\rangle_\gamma^{-1})^{(|\nu|-2)/2} + \omega\langle\xi\rangle_\gamma^{-1/2}\langle\xi\rangle_\gamma^{-|\nu|}\end{aligned}$$

where it should be understood that the second term on the right-hand side is absent when $|\nu| = 1$. To estimate the last term it suffices to note $\langle\xi\rangle_\gamma^{-|\nu|} \le (M^2\langle\xi\rangle_\gamma^{-1})^{-|\nu|/2}M^{-\epsilon(0,\nu)}\langle\xi\rangle_\gamma^{-|\nu|/2}$. When $\mu = \alpha' + \alpha$, $|\alpha| \ge 1$ we see

$$\begin{aligned}|\partial_x^\mu\omega^2| &\lesssim \omega M^{-1-\delta_{\epsilon a}+|\mu|} + M^{-2-2\delta_{\epsilon a}+|\mu|} \\ &\lesssim \omega\langle\xi\rangle_\gamma^{-1/2}M^{-\epsilon(\mu,0)}\langle\xi\rangle_\gamma^{|\mu|/2}(M^{2+2\delta_{\epsilon a}}\langle\xi\rangle_\gamma^{-1})^{(|\mu|-1)/2} \\ &\quad+\omega\langle\xi\rangle_\gamma^{-1/2}M^{-\epsilon(\mu,0)}\langle\xi\rangle_\gamma^{|\mu|/2}(M^{2+2\delta_{\epsilon a}}\langle\xi\rangle_\gamma^{-1})^{(|\mu|-2)/2}\end{aligned}$$

where if $|\alpha| = 1$ the second term on the right-hand side is absent as above. When $\mu = \alpha' + \alpha$, $\nu = \beta' + \beta$, $|\alpha + \beta| \geq 1$ and $|\mu| \geq 1$, $|\nu| \geq 1$ noting that $\partial_x^\mu \partial_\xi^\nu \psi = \partial_x^\mu \partial_\xi^\nu r$ and $\partial_x^\mu \partial_\xi^\nu \psi \in S(M^{-3+|\mu+\nu|} \langle\xi\rangle_\gamma^{-|\nu|}, G)$ we have

$$\begin{aligned}\left|\partial_x^\mu \partial_\xi^\nu \omega^2\right| &\lesssim |\omega \partial_x^\mu \partial_\xi^\nu r| + M^{-3+|\mu+\nu|} \langle\xi\rangle_\gamma^{-|\nu|} \\ &\lesssim \omega M^{-2+|\mu+\nu|} \langle\xi\rangle_\gamma^{-|\nu|} + M^{1-|\mu+\nu|} \langle\xi\rangle_\gamma^{-1} (M^4 \langle\xi\rangle_\gamma^{-1})^{(|\mu+\nu|-2)/2} \langle\xi\rangle_\gamma^{-(|\mu|-|\nu|)/2} \\ &\lesssim \omega \langle\xi\rangle_\gamma^{-1/2} M^{-\epsilon(\mu,\nu)} \langle\xi\rangle_\gamma^{(|\mu|-|\nu|)/2} (M^4 \langle\xi\rangle_\gamma^{-1})^{(|\mu+\nu|-1)/2} \\ &\quad + \omega \langle\xi\rangle_\gamma^{-1/2} M^{-\epsilon(\alpha',\beta')} \langle\xi\rangle_\gamma^{(|\alpha'|-|\beta'|)/2} (M^4 \langle\xi\rangle_\gamma^{-1})^{(|\mu+\nu|-2)/2} \langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}\end{aligned}$$

where $1 - |\mu + \nu| \leq -\epsilon(\alpha', \beta')$ and $\langle\xi\rangle_\gamma^{-1} \leq \omega \langle\xi\rangle_\gamma^{-1/2}$ are used. Thus the case $s = 2$ is proved. Since $\langle\xi\rangle_\gamma^{-1/2} \leq \omega$ it is obvious $\omega^2 \in S(\omega^2, g_\epsilon)$. The estimates for general $\omega^s = (\omega^2)^{s/2}$ follows from those of ω^2. □

Lemma 4.4 *We have $\phi \in S(\phi, g_\epsilon)$.*

Proof Using (4.5) we write

$$\partial_x^\alpha \partial_\xi^\beta \phi = \frac{-\partial_x^\alpha \partial_\xi^\beta \psi}{\omega} \phi + \frac{\partial_x^\alpha \partial_\xi^\beta \langle\xi\rangle_\gamma^{-1}}{2\omega} = \phi_{\alpha\beta} \phi + \psi_{\alpha\beta}, \quad |\alpha + \beta| = 1. \tag{4.6}$$

Since $\omega^{-1} \in S(\omega^{-1}, g_\epsilon)$ by Lemma 4.3 then

$$\begin{aligned}|\partial_x^\mu \partial_\xi^\nu (\psi_{\alpha\beta})| &\lesssim \omega^{-1} \langle\xi\rangle_\gamma^{-1} M^{-\epsilon(\mu,\nu)} \langle\xi\rangle_\gamma^{(|\alpha+\mu|-|\beta+\nu|)/2} \langle\xi\rangle_\gamma^{-1/2} \\ &\lesssim \phi M^{-\epsilon(\alpha+\mu,\beta+\nu)} \langle\xi\rangle_\gamma^{(|\alpha+\mu|-|\beta+\nu|)/2}\end{aligned} \tag{4.7}$$

in view of $\langle\xi\rangle_\gamma^{-1/2} \leq M^{-1}$ and (4.4). On the other hand Lemma 4.2 shows

$$\phi_{\alpha\beta} \in S(\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}, g_\epsilon), \quad |\alpha + \beta| \geq 1. \tag{4.8}$$

Hence differentiating (4.6) the assertion is proved by induction on $|\alpha + \beta|$ noting (4.7) and (4.8). □

Proposition 4.1 *We have $\omega^s \in S(\omega^s, g_\epsilon)$ and $\phi^s \in S(\phi^s, g_\epsilon)$. For $|\alpha + \beta| \geq 1$*

$$\begin{aligned}\partial_x^\alpha \partial_\xi^\beta \omega^s &\in S(\omega^{-1} \langle\xi\rangle_\gamma^{-1/2} \langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2} \omega^s, g), \\ \partial_x^\alpha \partial_\xi^\beta \phi^s &\in S(\omega^{-1} \langle\xi\rangle_\gamma^{-1/2} \langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2} \phi^s, g).\end{aligned}$$

Proof It remains to prove the assertion for ϕ. Let $\phi_{\alpha\beta}$, $\psi_{\alpha\beta}$ be those in (4.6). Note $\phi_{\alpha\beta} \in S(\omega^{-1} \langle\xi\rangle_\gamma^{-1/2} \langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}, g_\epsilon)$ for $|\alpha + \beta| \geq 1$ by Lemma 4.2, while $\psi_{\alpha\beta} \in S(\omega^{-1} \langle\xi\rangle_\gamma^{-1/2} \phi \langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}, g_\epsilon)$ for $|\alpha + \beta| \geq 1$ because of (4.7) and (4.4). Hence the assertion for $s = 1$ follows from (4.6). The estimate for general $s \in \mathbb{R}$ follows from the estimate for the case $s = 1$. □

Proposition 4.2 *ω and ϕ are g_ϵ admissible weights (Definition 6.1).*

Proof It suffices to show

$$\omega(z+w) \le C\omega(z)(1+g_{\epsilon,z}(w)), \quad \phi(z+w) \le C\phi(z)(1+g_{\epsilon,z}(w))^2. \tag{4.9}$$

If $|\eta| \ge \langle\xi\rangle_\gamma/2$ noting $\langle\xi\rangle_\gamma^{-1/2} \le \omega \le CM^{-1}$ one has

$$g_{\epsilon,z}(w) \ge M^{-2}\langle\xi\rangle_\gamma^{-1}|\eta|^2 \ge M^{-2}\langle\xi\rangle_\gamma/4 \ge \big(M^{-2}\langle\xi\rangle_\gamma^{1/2}\big)\langle\xi\rangle_\gamma^{1/2}/4 \ge \langle\xi\rangle_\gamma^{1/2}/4.$$

Thus in view of (4.4) one sees

$$\begin{aligned} &\omega(z+w) \le CM^{-1} \le CM^{-1}\langle\xi\rangle_\gamma^{1/2}\omega(z) \le C\omega(z)(1+g_{\epsilon,z}(w)),\\ &\phi(z+w) \le CM^{-1} \le CM^{-2}\langle\xi\rangle_\gamma\phi(z) \le C\phi(z)(1+g_{\epsilon,z}(w))^2. \end{aligned} \tag{4.10}$$

Assume $|\eta| < \langle\xi\rangle_\gamma/2$. Set $f = t - \psi$ and $h = \langle\xi\rangle_\gamma^{-1/2}$ so that $\omega^2 = f^2 + h^2$. Since $|f(z+w) + f(z)|/|\omega(z+w) + \omega(z)|$ and $|h(z+w) + h(z)|/|\omega(z+w) + \omega(z)|$ are bounded by 2 we have

$$\begin{aligned} |\omega(z+w) - \omega(z)| &= |\omega^2(z+w) - \omega^2(z)|/|\omega(z+w) + \omega(z)|\\ &\le 2|f(z+w) - f(z)| + 2|h(z+w) - h(z)|. \end{aligned} \tag{4.11}$$

Noting $|f(z+w) - f(z)| = |\psi(z+w) - \psi(z)|$ the estimate

$$\begin{aligned} |f(z+w) - f(z)| &\le C(M^{-\delta_{\epsilon a}}|y| + M^{-\delta_{\epsilon b}}\langle\xi + s\eta\rangle_\gamma^{-1}|\eta|)\\ &\le C\langle\xi\rangle_\gamma^{-1/2}(M^{-\delta_{\epsilon a}}\langle\xi\rangle_\gamma^{1/2}|y| + M^{-\delta_{\epsilon b}}\langle\xi\rangle_\gamma^{-1/2}|\eta|)\\ &\le C\omega(z)g_{\epsilon,z}^{1/2}(w). \end{aligned} \tag{4.12}$$

follows from Lemma 4.2 and (6.28). Similarly noting $g_{\epsilon,z}^{1/2}(w) \ge M^{-1}\langle\xi\rangle_\gamma^{-1/2}|\eta|$ one has $|h(z+w) - h(z)| \le C\langle\xi\rangle_\gamma^{-3/2}|\eta| \le CM\langle\xi\rangle_\gamma^{-1}g_{\epsilon,z}^{1/2}(w) \le C\omega(z)g_{\epsilon,z}^{1/2}(w)$ hence (4.11) gives

$$|\omega(z+w) - \omega(z)| \le C\omega(z)g_{\epsilon,z}^{1/2}(w). \tag{4.13}$$

Together with (4.10) one concludes that ω is g_ϵ admissible weight. Turn to ϕ. Since $\phi = \omega + f$ one can write $\phi(z+w) - \phi(z)$ as

$$\frac{(f(z+w) - f(z))(\phi(z+w) + \phi(z)) + h^2(z+w) - h^2(z)}{\omega(z+w) + \omega(z)} \tag{4.14}$$

where $|f(z+w) - f(z)| \le C\langle\xi\rangle_\gamma^{-1/2} g_{\epsilon,z}^{1/2}(w)$ by (4.12) and $|h^2(z+w) - h^2(z)| \le CM\langle\xi\rangle_\gamma^{-3/2} g_{\epsilon,z}^{1/2}(w)$ is easy. The insertion of these estimates into (4.14) yields

$$|\phi(z+w) - \phi(z)| \le C\Big(\frac{\langle\xi\rangle_\gamma^{-1/2}}{\omega(z+w)+\omega(z)}(\phi(z+w)+\phi(z)) + \frac{M\langle\xi\rangle_\gamma^{-3/2}}{\omega(z+w)+\omega(z)}\Big) g_{\epsilon,z}^{1/2}(w). \tag{4.15}$$

From $\phi(z) \ge M\langle\xi\rangle_\gamma^{-1}/C$ by (4.4) it follows that

$$|\phi(z+w) - \phi(z)| \le C(\phi(z+w) + 2\phi(z))\frac{\langle\xi\rangle_\gamma^{-1/2}}{\omega(z+w)+\omega(z)}\, g_{\epsilon,z}^{1/2}(w).$$

If $C\langle\xi\rangle_\gamma^{-1/2}\, g_{\epsilon,z}^{1/2}(w)\big/(\omega(z+w)+\omega(z)) < 1/3$ then $\big|\phi(z+w)/\phi(z) - 1\big| \le (\phi(z+w)/\phi(z)+2)/3$ and hence

$$2\phi(z+w)/5 \le \phi(z) \le 4\,\phi(z+w). \tag{4.16}$$

If $C\langle\xi\rangle_\gamma^{-1/2}\, g_{\epsilon,z}^{1/2}(w)\big/(\omega(z+w)+\omega(z)) \ge 1/3$ then $C^2 g_{\epsilon,z}(w) \ge \langle\xi\rangle_\gamma(\omega(z+w)+\omega(z))^2/9 \ge 2\langle\xi\rangle_\gamma\omega(z+w)\omega(z)/9$ hence noting $\phi(z) \ge \langle\xi\rangle_\gamma^{-1}/(2\omega(z))$ and using an obvious inequality $2\,\omega(z+w) \ge \phi(z+w)$ one obtains

$$18C^2(1 + g_{\epsilon,z}(w)) \ge \phi(z+w)\big/\phi(z)$$

which together with (4.10) proves that ϕ is g_ϵ admissible weight. □

4.2 Weighted energy estimates

With $\bar\lambda$ which we have fixed in the previous section we write $\hat P(t,x,\tau,\xi)$ as

$$\hat P(t,x,\tau,\xi) = -\tau^2 + \ell^2(t,x,\xi) + (q(t,x,\xi) + \bar\lambda\langle\xi\rangle_\gamma) + (a_1(t,x,\xi) - \bar\lambda)\langle\xi\rangle_\gamma + a_0(t,x,\xi)\tau.$$

Let us denote

$$\hat P = \mathrm{op}(\hat P(t,x,\tau,\xi)), \quad L = \mathrm{op}(\ell),$$
$$Q = \mathrm{op}(q + \bar\lambda\langle\xi\rangle_\gamma), \quad \sqrt{Q} = \mathrm{op}((q+\bar\lambda\langle\xi\rangle_\gamma)^{1/2}) = \mathrm{op}(b).$$

In what follows $\hat P$ and $\hat P(t,x,\tau,\xi)$ stands for operator and its symbol respectively. Since $\ell \in S(M^{-1}\langle\xi\rangle_\gamma, G)$ hence $\partial_x^\alpha\partial_\xi^\beta\ell \in S(M\langle\xi\rangle_\gamma^{1-|\beta|}, g_\epsilon)$ for $|\alpha+\beta| = 2$, Theorem

6.1 shows $\ell\#\ell - \ell^2 \in S(M^2, g_\epsilon)$ so that $\mathrm{op}(\ell^2) = L^2 + \mathrm{op}(r)$ with $r \in S(M^2, g_\epsilon)$. Thus $\hat{P}$ can be written

$$\hat{P} = -D_t^2 + L^2 + Q + B_0 D_t + B_1, \quad B_i = \mathrm{op}(\tilde{a}_i), \quad \tilde{a}_i \in S(\langle\xi\rangle_\gamma^i, g_\epsilon). \tag{4.17}$$

for $M^2 \le \langle\xi\rangle_\gamma^{1/2}$. Let $\theta > 0$ be a parameter we consider $\hat{P}_\theta = e^{-\theta t}\hat{P}e^{\theta t}$. Noting $(D_t - i\theta) = e^{-\theta t}D_t e^{\theta t}$ one can write $\hat{P}_\theta$ as

$$\hat{P}_\theta = -A^2 + L^2 + Q + B_0 A + B_1, \quad A = D_t - i\theta. \tag{4.18}$$

Here we define several weights for energy estimates.

Definition 4.1 Define $\Phi_n^{k\sharp} = \mathrm{op}(\omega^{-k/2}\phi^{-n})$, $\Psi_n^{k\sharp} = \mathrm{op}(\omega^{1-k/2}\langle\xi\rangle_\gamma\phi^{-n})$, $k = 0, 1, 2, 3$. We denote $\Phi_n^{0\sharp}, \Phi_n^{1\sharp}$ simply by $\Phi_n, \Phi_n^\sharp$. We apply the same abbreviation for $\Psi_n^{k\sharp}$. For simplicity we will write $\Phi^{k\sharp}, \Psi^{k\sharp}$ dropping the parameter n, but it should be reminded that they include parameters n, M and γ.

Throughout the section, small letters such as $c, \hat{c}, \bar{c}, c_i$ denote constants *independent of* n, M, γ *and* θ, while capital letter C, may change from line to line, denotes constants which may *depend on n but independent of* M, γ *and* θ.

Lemma 4.5 *If* $K^* = K$ *then*

$$\begin{aligned} 2\mathrm{Im}(\Phi K u, \Phi A u) = {} & \partial_t(K\Phi u, \Phi u) + 2\theta(K\Phi u, \Phi u) \\ & + 2\mathrm{Im}([\Phi, K]u, \Phi A u) + 2\mathrm{Im}(K\Phi u, [\Phi, A]u) \\ & - \mathrm{Re}((\partial_t K)\Phi u, \Phi u) \end{aligned} \tag{4.19}$$

and we have

$$\begin{aligned} 2\mathrm{Im}(\Phi K^2 u, \Phi A u) = {} & \partial_t\|\Phi K u\|^2 + 2\theta\|\Phi K u\|^2 \\ & + 2\mathrm{Im}(\Phi[A, K]u, \Phi K u) + 2\mathrm{Im}([A, \Phi]K u, \Phi K u) \\ & + 2\mathrm{Im}([\Phi, K]A u, \Phi K u) + 2\mathrm{Im}(\Phi A u, [K, \Phi]K u). \end{aligned} \tag{4.20}$$

Proof To see the first equality it is enough to write

$$(\Phi K u, \Phi A u) = ([\Phi, K]u, \Phi A u) + (K\Phi u, [\Phi, A]u) + (K\Phi u, A\Phi u)$$

and note $2\mathsf{Im}(K\Phi u, A\Phi u) = \partial_t(K\Phi u, \Phi u) + 2\theta(K\Phi u, \Phi u) - \mathsf{Re}((\partial_t K)\Phi u, \Phi u)$ for $\partial_t = iA - \theta$. To see the second equality write

$$
\begin{aligned}
(\Phi K^2 u, \Phi Au) &= ([\Phi, K]Ku, \Phi Au) + (K\Phi Ku, \Phi Au)\\
&= ([\Phi, K]Ku, \Phi Au) + (\Phi Ku, [K, \Phi]Au) + (\Phi Ku, \Phi KAu)\\
&= ([\Phi, K]Ku, \Phi Au) + (\Phi Ku, [K, \Phi]Au)\\
&\quad + (\Phi Ku, \Phi[K, A]u) + (\Phi Ku, \Phi AKu)\\
&= ([\Phi, K]Ku, \Phi Au) + (\Phi Ku, [K, \Phi]Au) + (\Phi Ku, \Phi[K, A]u)\\
&\quad + (\Phi Ku, [\Phi, A]Ku) + (\Phi Ku, A\Phi Ku)
\end{aligned}
$$

where the twice of the imaginary part of the first 4 terms on the right-hand side coincide with the last 4 terms on the right-hand side of (4.20). Thus it suffices to show $2\mathsf{Im}(\Phi Ku, A\Phi Ku) = \partial_t\|\Phi Ku\|^2 + 2\theta\|\Phi Ku\|^2$ which is clear. □

We aim to estimate $2\mathsf{Im}(\Phi\hat{P}_\theta u, \Phi Au)$. Start with $2\mathsf{Im}(\Phi L^2 u, \Phi Au)$. Consider $2\mathsf{Im}([A, \Phi]Lu, \Phi Lu)$. Since $\partial_t\phi = \omega^{-1}\phi$ then $[A, \Phi] = in\,\mathrm{op}(\omega^{-1}\phi^{-n})$ hence

$$
2\mathsf{Im}([A, \Phi]Lu, \Phi Lu) = 2n\mathsf{Re}(\mathrm{op}(\omega^{-1}\phi^{-n})Lu, \mathrm{op}(\phi^{-n})Lu).
$$

Noting $\phi^{-n}\#(\omega^{-1}\phi^{-n}) - \omega^{-1}\phi^{-2n} \in S(M^{-1}\omega^{-1}\phi^{-2n}, g_\epsilon)$ we have from Corollary 6.4 and Lemma 6.11 that

$$
2\mathsf{Im}([A, \Phi]Lu, \Phi Lu) \ge 2n(1 - CM^{-1})\|\Phi^\sharp Lu\|^2. \tag{4.21}
$$

Next estimate $2\mathsf{Im}(\Phi Au, [L, \Phi]Lu)$. One can write

$$
\begin{aligned}
&\phi^{-n}\#(\ell\#\phi^{-n} - \phi^{-n}\#\ell) = -n\{\ell, \psi\}\omega^{-1}\phi^{-2n} + r_1 + r_2,\\
&r_1 \in S(M^{-1}\omega^{-1}\phi^{-2n}, g_\epsilon), \quad r_2 \in S(\phi^{-2n}, g_\epsilon).
\end{aligned} \tag{4.22}
$$

In fact since $\partial_x^\alpha\partial_\xi^\beta\ell \in S(M^2\langle\xi\rangle_\gamma^{1-|\beta|}, g_\epsilon)$ for $|\alpha+\beta| = 3$, Theorem 6.1 and Lemma 4.4 show $(\ell\#\phi^{-n} - \phi^{-n}\#\ell) + i\{\ell, \phi^{-n}\} \in S(\phi^{-n}, g_\epsilon)$. On the other hand one sees $\{\ell, \phi^{-n}\} = -in\,\omega^{-1}\{\ell, \psi\}\phi^{-n} + in\,\omega^{-1}\{\ell, \langle\xi\rangle_\gamma^{-1}\}\phi^{-n-1}$ in view of (4.5) and $\omega^{-1}\{\ell, \langle\xi\rangle_\gamma^{-1}\}\phi^{-n-1} \in S(\phi^{-n}, g_\epsilon)$ by (4.4). Since $\{\ell, \psi\} \in S(1, g_\epsilon)$ Proposition 3.2 and Theorem 6.1 prove (4.22). Therefore from Lemma 6.11 we have

$$
\begin{aligned}
\big|(\Phi Au, [L, \Phi]Lu)\big| &\le n\|\mathrm{op}(\{\ell, \psi\})\Phi^\sharp Au\|\|\Phi^\sharp Lu\|\\
&\quad + CM^{-1}\|\Phi^\sharp Au\|\|\Phi^\sharp Lu\| + C\|\Phi Au\|\|\Phi Lu\|.
\end{aligned}
$$

Since $\|(\mathrm{op}(\{\ell, \psi\})v\| \le (|\kappa| + CM^{-1/2})\|v\|$ by Proposition 3.2 and Corollary 6.6 one obtains

$$
|(\Phi Au, [L, \Phi]Lu)| \le n(|\kappa| + CM^{-1/2})\|\Phi^\sharp Au\|\|\Phi^\sharp Lu\| + C\|\Phi Au\|\|\Phi Lu\|.
$$

Since the term $|([\Phi, L]Au, \Phi Lu)|$ is estimated similarly one concludes

$$\begin{aligned}&2|(\Phi Au, [L, \Phi]Lu)| + 2|([\Phi, L]Au, \Phi Lu)|\\&\quad\le 2n(|\kappa| + CM^{-1/2})\big(\|\Phi^{\sharp}Au\|^2 + \|\Phi^{\sharp}Lu\|^2\big)\\&\qquad + C\big(\|\Phi Au\|^2 + \|\Phi Lu\|^2\big).\end{aligned}\tag{4.23}$$

From $[A, L] = -i\,\mathrm{op}(\partial_t\ell)$ and $\partial_t\ell \in S(\langle\xi\rangle_\gamma, g_\epsilon)$ it follows that $\phi^{-n}\#\phi^{-n}\#(\partial_t\ell) = (\partial_t\ell)\phi^{-2n} + r$ with $r \in S(M^{-1}\langle\xi\rangle_\gamma\phi^{-2n}, g_\epsilon)$ then the estimate

$$\begin{aligned}&2|(\Phi[A, L]u, \Phi Lu)| \le 2\|\mathrm{op}(\partial_t\ell\langle\xi\rangle_\gamma^{-1})\Psi^{\sharp}u\|\|\Phi^{\sharp}Lu\|\\&\quad + CM^{-1}\|\Psi^{\sharp}u\|\|\Phi^{\sharp}Lu\| \le (c_0 + CM^{-1})\big(\|\Psi^{\sharp}u\|^2 + \|\Phi^{\sharp}Lu\|^2\big)\end{aligned}\tag{4.24}$$

follows from Lemma 6.11. Thus (4.20), (4.21), (4.23) and (4.24) give

Lemma 4.6 *We have*

$$\begin{aligned}&2\mathsf{Im}(\Phi L^2u, \Phi Au) \ge \partial_t\|\Phi Lu\|^2 + (2\theta - C)\|\Phi Lu\|^2\\&\quad+2n(1 - |\kappa| - c_0/2n - CM^{-1/2})\|\Phi^{\sharp}Lu\|^2 - 2n(|\kappa| + CM^{-1/2})\|\Phi^{\sharp}Au\|^2\\&\quad-(c_0 + CM^{-1/2})\|\Psi^{\sharp}u\|^2 - C\|\Phi Au\|^2.\end{aligned}$$

Turn to $-2\mathsf{Im}(\Phi A^2u, \Phi Au)$. Choosing $K = I$ and $L = I$ in (4.19) and (4.21) respectively one has

$$\begin{aligned}&- 2\mathsf{Im}(\Phi Au, \Phi u) = \partial_t\|\Phi u\|^2 + 2\theta\|\Phi u\|^2 + 2\mathsf{Im}([A, \Phi]u, \Phi u)\\&\quad\ge \partial_t\|\Phi u\|^2 + 2\theta\|\Phi u\|^2 + 2n(1 - CM^{-1})\|\Phi^{\sharp}u\|^2.\end{aligned}\tag{4.25}$$

Replacing Φ by $\Phi^{2\sharp}$ a repetition of a similar argument shows

$$-2\mathsf{Im}(\Phi^{2\sharp}Au, \Phi^{2\sharp}u) \ge \partial_t\|\Phi^{2\sharp}u\|^2 + 2\theta\|\Phi^{2\sharp}u\|^2 + 2n(1 - CM^{-1})\|\Phi^{3\sharp}u\|^2.$$

Since the left-hand side is bounded as

$$\begin{aligned}2|(\Phi^{2\sharp}Au, \Phi^{2\sharp}u)| &\le 2(1 + CM^{-1})\|\Phi^{\sharp}Au\|\|\Phi^{3\sharp}u\|\\&\le n^{-1}\|\Phi^{\sharp}Au\|^2 + n(1 + CM^{-1})\|\Phi^{3\sharp}u\|^2\end{aligned}$$

we conclude

$$\|\Phi^{\sharp}Au\|^2 \ge n\partial_t\|\Phi^{2\sharp}u\|^2 + 2\theta n\|\Phi^{2\sharp}u\|^2 + n^2(1 - CM^{-1})\|\Phi^{3\sharp}u\|^2. \tag{4.26}$$

Replacing u by Au in (4.25) one has

$$-2\mathsf{Im}(\Phi A^2u, \Phi Au) \ge \partial_t\|\Phi Au\|^2 + 2\theta\|\Phi Au\|^2 + 2n(1 - CM^{-1})\|\Phi^{\sharp}Au\|^2$$

where we replace $\nu\|\Phi^{\sharp}Au\|^2$ $(0 < \nu < 2)$ by the estimate (4.26) to obtain

Lemma 4.7 *For any* $0 < \nu < 2$ *the following estimate holds.*

$$
\begin{aligned}
-2\mathsf{Im}(\Phi A^2 u, \Phi A u) \geq\ & \partial_t \|\Phi A u\|^2 + 2\theta \|\Phi A u\|^2 \\
& + 2n(1 - \nu/2 - CM^{-1}) \|\Phi^{\sharp} A u\|^2 \\
& + \nu n^2 \partial_t \|\Phi^{2\sharp} u\|^2 + 2\nu\theta n^2 \|\Phi^{2\sharp} u\|^2 \\
& + \nu n^3 (1 - CM^{-1}) \|\Phi^{3\sharp} u\|^2 .
\end{aligned}
\tag{4.27}
$$

Finally we estimate $\mathsf{Im}(\Phi Q u, \Phi A u)$. Study $\mathsf{Im}([\Phi, Q]u, \Phi A u)$. From Proposition 4.1 and Theorem 6.1 it follows that $\phi^{-n}\#(\phi^{-n}\#\langle\xi\rangle_\gamma - \langle\xi\rangle_\gamma\#\phi^{-n}) \in S(\omega^{-1}\phi^{-2n}, g_\epsilon)$ hence Lemma 6.11 shows

$$
|([\Phi, \langle D\rangle_\gamma]u, \Phi A u)| \leq C\|\Phi^{2\sharp} u\| \|\Phi A u\| \leq C(\|\Phi^{2\sharp} u\|^2 + \|\Phi A u\|^2).
$$

To estimate $\mathsf{Im}([\Phi, \mathrm{op}(q)]u, \Phi A u)$ we shall examine that

$$
\begin{aligned}
&\phi^{-n}\#q - q\#\phi^{-n} = -in\omega^{-1}\{\psi, q\}\phi^{-n} + r_1 + r_2, \\
&r_1 \in S(b\phi^{-n}, \bar{g}), \quad r_2 \in S(M\omega^{-1}\phi^{-n}, g_\epsilon).
\end{aligned}
\tag{4.28}
$$

Indeed since $\partial_x^\alpha \partial_\xi^\beta q \in S(M\langle\xi\rangle_\gamma^{2-|\beta|}, g_\epsilon)$ for $|\alpha+\beta| = 3$, Proposition 4.1 and Theorem 6.1 show $\phi^{-n}\#q - q\#\phi^{-n} = -i\{\phi^{-n}, q\} + r$ with $r \in S(M\omega^{-1}\phi^{-n}, g_\epsilon)$. Note $\{\phi^{-n}, q\} = n\omega^{-1}\{\psi, q\}\phi^{-n} - n\omega^{-1}\{\langle\xi\rangle_\gamma^{-1}, q\}\phi^{-n-1}/2$ by (4.5) where the second term on the right-hand is $S(b\phi^{-n}, \bar{g})$ because of Lemma 4.1 and (4.4), hence (4.28). Since $\omega^{-1}\{\phi^{-n}, q\} \in S(M^{-1/2}\omega^{-1}b\phi^{-n}, \bar{g})$ by Lemma 4.1 it follows from Lemmas 6.12, 6.11 and (4.28) that

$$
\begin{aligned}
\big|([\Phi, Q]u, \Phi A u)\big| \leq\ & CM^{-1/2}\big(\|\sqrt{Q}\,\Phi^{\sharp} u\|^2 + \|\Phi^{\sharp} A u\|^2\big) \\
& + C\|\sqrt{Q}\,\Phi u\|^2 + CM\big(\|\Phi A u\|^2 + \|\Phi^{2\sharp} u\|^2\big).
\end{aligned}
\tag{4.29}
$$

Lemma 4.8 *The following estimate holds.*

$$
\begin{aligned}
2\mathsf{Im}(Q\Phi u, [\Phi, A]u) \geq\ & (n - CM^{-1/2})\|\sqrt{Q}\,\Phi^{\sharp} u\|^2 \\
& - CM^{-1/2}\|\Phi^{3\sharp} u\|^2 - C\|\sqrt{Q}\,\Phi u\|^2 - CM\|\Phi^{2\sharp} u\|^2 .
\end{aligned}
$$

Proof Note $2\mathsf{Im}(Q\Phi u, [\Phi, A]u) = 2n\mathsf{Re}\big(Q\mathrm{op}(\phi^{-n})u, \mathrm{op}(\omega^{-1}\phi^{-n})u\big)$ and write $\omega^{-1}\phi^{-n} = \omega^{-1/2}\#(1+k)\#(\omega^{-1/2}\phi^{-n})$ and $\omega^{-1/2}\#\phi^{-n} = (1+\tilde{k})\#(\omega^{-1/2}\phi^{-n})$ with $k, \tilde{k} \in S(M^{-1}, g_\epsilon)$ by use of Lemma 6.10, then one can write

$$
\begin{aligned}
\big(Q\mathrm{op}(\phi^{-n})u, \mathrm{op}(\omega^{-1}\phi^{-n})u\big) =\ & \big(\mathrm{op}(1+\bar{k})Q\mathrm{op}(1+\tilde{k})\Phi^{\sharp} u, \Phi^{\sharp} u\big) \\
& + \big(\mathrm{op}(1+\bar{k})[\mathrm{op}(\omega^{-1/2}), Q]\Phi u, \Phi^{\sharp} u\big)
\end{aligned}
$$

where $[\mathrm{op}(\omega^{-1/2}), Q] = \sum_{i=1}^{3} \mathrm{op}(r_i)$ and

$$
r_1 \in S(M^{-1/2}\omega^{-3/2}b, \bar{g}), \quad r_2 \in S(\omega^{-1/2}b, \bar{g}), \quad r_3 \in S(M\omega^{-3/2}, g_\epsilon). \tag{4.30}
$$

In fact $\omega^{-1/2}\#\langle\xi\rangle_\gamma - \langle\xi\rangle_\gamma\#\omega^{-1/2} \in S(\omega^{-3/2}, g_\epsilon)$ is clear from Proposition 4.1 and Theorem 6.1. Similarly $\omega^{-1/2}\#q - q\#\omega^{-1/2} + i\{\omega^{-1/2}, q\} \in S(M\omega^{-3/2}, g_\epsilon)$, where

$$\{\omega^{-1/2}, q\} = \omega^{-5/2}(t-\psi)\{\psi, q\}/2 - \omega^{-5/2}\{\langle\xi\rangle_\gamma^{-1}, q\}/4. \tag{4.31}$$

The first term on the right-hand is $S(M^{-1/2}\omega^{-3/2}b, \bar{g})$ because of Lemma 4.1 and $(t-\psi) \in S(\omega, g_\epsilon)$ and the second term is $S(\omega^{-1/2}b, \bar{g})$ thanks to Lemma 4.1 and $\omega^{-2} \le \langle\xi\rangle_\gamma$, hence (4.30) is examined. Applying Lemma 6.12 we have

$$\begin{aligned}\big|(\mathrm{op}(1+\bar{k})[\mathrm{op}(\omega^{-1/2}), Q]\Phi u, \Phi^\sharp u)\big| &\le CM^{-1/2}\big(\|\sqrt{Q}\,\Phi^\sharp u\|^2 + \|\Phi^{3\sharp}u\|^2\big)\\ &\quad + C\|\sqrt{Q}\,\Phi u\|^2 + CM\|\Phi^{2\sharp}u\|^2.\end{aligned}$$

Turn to $\big(\mathrm{op}(1+\bar{k})Q\mathrm{op}(1+\tilde{k})\Phi^\sharp u, \Phi^\sharp u\big)$. Since $\bar{k}\#(q+\bar{\lambda}\langle\xi\rangle_\gamma) \in S(M^{-1}b^2, \bar{g})$ from Lemma 6.12 one has $\big|(\mathrm{op}(\bar{k})Q\Phi^\sharp u, \Phi^\sharp u)\big| \le CM^{-1}\|\sqrt{Q}\,\Phi^\sharp u\|^2$. Terms such as $\big|(Q\mathrm{op}(\tilde{k})\Phi^\sharp u, \Phi^\sharp u)\big|$ are estimated similarly. To conclude the proof it suffices to apply Lemma 6.8 to $(Q\Phi^\sharp u, \Phi^\sharp u)$. □

Lemma 4.9 *There exists* $c_1 > 0$ *such that*

$$|((\partial_t Q)\Phi u, \Phi u)| \le (c_1 + CM^{-1/2})(\|\sqrt{Q}\,\Phi^\sharp u\|^2 + \|\Psi^\sharp u\|^2). \tag{4.32}$$

Proof Write $\phi^{-n}\#\partial_t q\#\phi^{-n} = (\omega^{1/2}\langle\xi\rangle_\gamma\phi^{-n})\#r\#(\omega^{-1/2}\phi^{-n})$ with $r \in S(b, g_\epsilon)$ by using Lemmas 4.1, 6.10 then $|((\partial_t Q)\Phi u, \Phi u)| \le \|\mathrm{op}(r)\Phi^\sharp u\|\|\Psi^\sharp u\|$. Write $(\omega^{-1/2}\phi^{-n})\#(1+k)\#(\omega^{1/2}\phi^n) = 1$, $(\omega^{-1/2}\langle\xi\rangle_\gamma^{-1}\phi^n)\#(1+\tilde{k})\#(\omega^{1/2}\langle\xi\rangle_\gamma\phi^{-n}) = 1$ with $k, \tilde{k} \in S(M^{-1}, g_\epsilon)$ by using Lemma 6.10 it is clear

$$r = (\omega^{-1/2}\langle\xi\rangle_\gamma^{-1}\phi^n)\#(1+\tilde{k})\#\phi^{-n}\#(\partial_t q)\#\phi^{-n}\#(1+k)\#(\omega^{1/2}\phi^n).$$

From Theorem 6.1 one sees $\phi^{-n}\#(1+k)\#(\omega^{1/2}\phi^n) - \omega^{1/2} = l \in S(M^{-1}\omega^{1/2}, g_\epsilon)$ and $(\omega^{-1/2}\langle\xi\rangle_\gamma^{-1}\phi^n)\#(1+\tilde{k})\#\phi^{-n} - \omega^{-1/2}\langle\xi\rangle_\gamma^{-1} = \tilde{l} \in S(M^{-1}\omega^{-1/2}\langle\xi\rangle_\gamma^{-1}, g_\epsilon)$ hence $r = (\langle\xi\rangle_\gamma^{-1}\omega^{-1/2} + \tilde{l})\#(\partial_t q)\#(\omega^{1/2} + l) = (\langle\xi\rangle_\gamma^{-1}\omega^{-1/2})\#(\partial_t q)\#\omega^{1/2} + \tilde{r}$ where $\tilde{r} \in S(M^{-1}b, \bar{g})$ by Lemma 4.1. Noting $(\langle\xi\rangle_\gamma^{-1}\omega^{-1/2})\#(\partial_t q)\#\omega^{1/2} \in S(b, \bar{g})$ is independent of n we have $\|\mathrm{op}(r)v\| \le (c_1 + CM^{-1})\|\sqrt{Q}\,v\|$ from Lemma 6.12 with some $c_1 > 0$. Putting $v = \Phi^\sharp u$ we conclude the proof. □

Choosing $K = Q$ in (4.19) it follows from (4.29) and Lemmas 4.8, 4.9 that

$$\begin{aligned}2\mathrm{Im}(\Phi Qu, \Phi Au) &\ge \partial_t(Q\Phi u, \Phi u) + (\theta - C)\|\sqrt{Q}\,\Phi u\|^2\\ &\quad + (n - c_1 - CM^{-1/2})\|\sqrt{Q}\,\Phi^\sharp u\|^2 - (c_1 + CM^{-1/2})\|\Psi^\sharp u\|^2\\ &\quad - CM^{-1/2}\big(\|\Phi^\sharp Au\|^2 + \|\Phi^{3\sharp}u\|^2\big) - CM\big(\|\Phi Au\|^2 + \|\Phi^{2\sharp}u\|^2\big).\end{aligned} \tag{4.33}$$

Writing $\omega^{1-k/2}\phi^{-n}\langle\xi\rangle_\gamma = (\langle\xi\rangle_\gamma\omega)(\omega^{-k/2}\phi^{-n})$ we have from Lemma 6.11

$$(1 - CM^{-1})\|\Psi^{k\sharp}u\| \le \|\mathrm{op}(\langle\xi\rangle_\gamma\omega)\Phi^{k\sharp}u\|. \tag{4.34}$$

Let $\tilde{b} \in S(b^{-1}, \bar{g})$ be given in Proposition 6.1 then $\tilde{b} \in S(\omega^{-1}\langle\xi\rangle_\gamma^{-1}, \bar{g})$ by (4.3). Hence writing $\langle\xi\rangle_\gamma\omega = (\langle\xi\rangle_\gamma\omega)\#\tilde{b}\#b$ with $(\langle\xi\rangle_\gamma\omega)\#\tilde{b} \in S(1, \bar{g})$ there is $\hat{c} > 0$ such that $\|\mathrm{op}(\langle\xi\rangle_\gamma\omega)v\| \le \hat{c}\,\|\sqrt{Q}\,v\|$ thanks to Theorem 6.2. Replacing v by $\Phi^{k\sharp}u$ we have from (4.34) that

Lemma 4.10 *There exist* $\hat{c} > 0$, $c > 0$, $C > 0$ *such that*

$$c(1 - CM^{-1})\|\langle D\rangle_\gamma^{1/2+k/4}\Phi u\| \le (1 - CM^{-1})\|\Psi^{k\sharp}u\| \le \hat{c}\|\sqrt{Q}\,\Phi^{k\sharp}u\| \tag{4.35}$$

for $k = 0, 1, 2$.

Proof It remains to show the left side inequality. Write $\phi^{-n}\langle\xi\rangle_\gamma^{1/2+k/4} = (\omega^{1/2}\langle\xi\rangle_\gamma^{1/4})^{-2+k}(\omega^{1-k/2}\phi^{-n}\langle\xi\rangle_\gamma)$ then from Lemma 6.11 there is $c > 0$ such that $c\|\langle D\rangle_\gamma^{1/2+k/4}\Phi u\| \le (1 + CM^{-1})\|\Psi^{k\sharp}u\|$ for $k \le 2$. □

In (4.33), replacing $\|\Psi^\sharp u\|^2$ by the estimate (4.35) one has

Lemma 4.11 *We have*

$$\begin{aligned} 2\mathrm{Im}(\Phi Qu, \Phi Au) \ge\ & \partial_t(Q\Phi u, \Phi u) + (\theta - C)\|\sqrt{Q}\,\Phi u\|^2 \\ & + n(1 - c_1(1+\hat{c})/n - CM^{-1/2})\|\sqrt{Q}\,\Phi^\sharp u\|^2 \\ & - CM^{-1/2}\big(\|\Phi^\sharp Au\|^2 + \|\Phi^{3\sharp}u\|^2\big) - CM\big(\|\Phi Au\|^2 + \|\Phi^{2\sharp}u\|^2\big). \end{aligned}$$

Finally we estimate the lower order term $B_0A + B_1$. Since $\tilde{a}_j \in S(\langle\xi\rangle_\gamma^j, g_\epsilon)$ Lemma 6.11 shows

$$\begin{aligned} 2\big|(\Phi B_1 u, \Phi Au)\big| &\le 2\|\mathrm{op}(\tilde{a}_1\langle\xi\rangle_\gamma^{-1})\Phi^\sharp Au\|\|\Psi^\sharp u\| \\ &\quad + CM^{-1}\|\Phi^\sharp Au\|\|\Psi^\sharp u\| \\ &\le (\bar{c} + CM^{-1})(\|\Phi^\sharp Au\|^2 + \|\Psi^\sharp u\|^2). \end{aligned} \tag{4.36}$$

Similarly $2|(\Phi B_0Au, \Phi Au)| \le C\|\Phi Au\|^2$. Then from Lemmas 4.6, 4.7, 4.11 and the estimates of lower order term one has

Proposition 4.3 *We have*

$$\begin{aligned} 2\mathrm{Im}(\Phi\hat{P}_\theta u, \Phi Au) \ge\ & \partial_t\big\{\|\Phi Lu\|^2 + \|\Phi Au\|^2 + (Q\Phi u, \Phi u) \\ & + \nu n^2\|\Phi^{2\sharp}u\|^2\big\} + (\theta - CM)\big(\|\Phi Lu\|^2 + \|\Phi Au\|^2 + \|\sqrt{Q}\,\Phi u\|^2\big) \\ & + 2n(1 - |\kappa| - \nu/2 - (c_0+\bar{c})/2n - CM^{-1/2}) \\ & \big(\|\Phi^\sharp Au\|^2 + \|\Phi^\sharp Lu\|^2\big) \\ & + n(1 - \hat{c}(c_0+\bar{c})/n - c_1(1+\hat{c})/n - CM^{-1/2})\|\sqrt{Q}\,\Phi^\sharp u\|^2 \\ & + (2\nu\theta n^2 - CM)\|\Phi^{2\sharp}u\|^2 + (\nu n^3 - CM^{-1/2})\|\Phi^{3\sharp}u\|^2. \end{aligned}$$

Lemma 4.12 *For $n \ge 1$ there is $C > 0$ such that*

$$\|\Phi u\| \le C\|\langle D\rangle_\gamma^n u\|, \quad \|u\| \le C\|\Phi u\|, \quad \|\langle D\rangle_\gamma u\| \le C\|\sqrt{Q}\,\Phi u\|.$$

Proof Since $\phi^{-n} \le C\omega^n\langle\xi\rangle_\gamma^n \le C'\langle\xi\rangle_\gamma^n$ then $\phi^{-n} \in S(\langle\xi\rangle_\gamma^n, g_\epsilon)$ by (4.4) hence the first inequality is clear from Lemma 6.11. Since $\phi^{-n} \ge (2\omega)^{-n} \ge C > 0$ for $\phi \le 2\omega$ hence $1 \in S(\phi^{-n}, g_\epsilon)$ which proves the second inequality. The third inequality follows from $\omega\phi^{-n}\langle\xi\rangle_\gamma \ge C\omega^{1-n}\langle\xi\rangle_\gamma \ge C'\langle\xi\rangle_\gamma$ and Lemma 4.10. □

In Proposition 4.3 we fix $\nu > 0$ such that $1 - |\kappa| - \nu/2 > 0$. Then choose n such that

$$1 - |\kappa| - \nu/2 - (c_0 + \bar{c})/2n > 0, \;\; 1 - \hat{c}(c_0 + \bar{c})/n - c_1(1 + \hat{c})/n > 0 \quad (4.37)$$

and fix such a n. Note that (4.37) is always satisfied for any n greater than such a fixed n. Next, for such fixed n, choose M such that the arguments in this section should be justified, namely the assertions in Sect. 6.3 hold with

$$m,\; m_i = \omega^k\langle\xi\rangle_\gamma^s\phi^l, \quad |k| \le 2,\; |s| \le 1,\; |l| \le n \quad (4.38)$$

and the coefficients of the last four terms in Proposition 4.3 and that of Lemma 4.10 will be positive, and fix such a M then choose γ such that $\gamma \ge M^4$ and $\gamma \ge \bar{\lambda}M^2$ and fix such a γ, while θ is assumed to be free still. Once M and γ are fixed, denoting by g_0 the metric $\underline{g}$ with $\gamma = 1$, there are C, C_s such that

$$\langle\xi\rangle^s/C_s \le \langle\xi\rangle_\gamma^s \le C_s\langle\xi\rangle^s, \quad g_0/C \le G \le Cg_0$$

then $S(\langle\xi\rangle_\gamma^s, G) = S(\langle\xi\rangle^s, g_0) = S^s$. In particular, $\|\langle D\rangle_\gamma^s\cdot\|$ is equivalent to $\|\langle D\rangle^s\cdot\|$. The range of t is consequently fixed if M is fixed by (3.4). As long as γ is fixed, it is allowed to write $\langle\xi\rangle_\gamma$ as $\langle\xi\rangle$. After fixing n, M, γ and taking Lemma 4.10 into account we have

Proposition 4.4 *There exist $c > 0$, $c^* > 0$, $\delta_0 > 0$, $\theta_0 > 0$ such that for $|t| \le \delta_0$, $\theta \ge \theta_0$ one has*

$$\begin{aligned}2\mathrm{Im}(\Phi\hat{P}_\theta u, \Phi Au) \ge\; &\partial_t\big\{\|\Phi Au\|^2 + \|\Phi Lu\|^2 + (Q\Phi u, \Phi u) + c^*\|\Phi^{2\sharp}u\|^2\big\}\\ &+c\,\theta\big(\|\Phi Au\|^2 + \|\Phi Lu\|^2 + \|\sqrt{Q}\,\Phi u\|^2 + \|\langle D\rangle^{1/2}\Phi u\|^2\big)\\ &+c\big(\|\Phi^\sharp Au\| + \|\Phi^\sharp Lu\|^2 + \|\sqrt{Q}\,\Phi^\sharp u\| + \|\langle D\rangle^{3/4}\Phi u\|^2\big).\end{aligned}$$

Definition 4.2 Denote

$$\begin{aligned}\tilde{\mathcal{E}}^2(u) &= \|\Phi Au\|^2 + \|\Phi Lu\|^2 + (Q\Phi u, \Phi u) + c^*\|\Phi^{2\sharp}u\|^2,\\ \tilde{\mathcal{E}}_\sharp^2(u) &= \|\Phi^\sharp Au\|^2 + \|\Phi^\sharp Lu\|^2 + \|\Phi^\sharp\sqrt{Q}\,u\|^2 + \|\langle D\rangle^{3/4}\Phi u\|^2.\end{aligned}$$

Denote the substitution of A with D_t in the definition $\tilde{\mathcal{E}}^2(u)$ and $\tilde{\mathcal{E}}_\sharp^2(u)$ as $\mathcal{E}^2(u)$ and $\mathcal{E}_\sharp^2(u)$ respectively.

To effectively utilize Proposition 4.4, noting that $\ell^2 \in S(M^{-2}\langle\xi\rangle_\gamma^2, G)$ we introduce

$$L^\dagger = \mathrm{op}(b_1) = \mathrm{op}((\ell^2 + \bar\lambda\langle\xi\rangle_\gamma)^{1/2}), \quad b_1^2 = \ell^2 + \bar\lambda\langle\xi\rangle_\gamma$$

where it can be assumed that $\bar\lambda$ is chosen so that both Proposition 6.1 and Lemma 6.8 hold.

Lemma 4.13 *There is $C > 0$ such that*

$$\begin{aligned}&\|\Phi Au\|^2 + \|\Phi L^\dagger u\|^2 + \|\Phi\sqrt{Q}\,u\|^2 + \|\langle D\rangle^{1/2}\Phi u\|^2 \le C\tilde{\mathcal{E}}^2(u)\\ &\quad\le C'(\|\Phi Au\|^2 + \|\Phi Lu\|^2 + \|\Phi\sqrt{Q}\,u\|^2 + \|\langle D\rangle^{1/2}\Phi u\|^2),\\ &\tilde{\mathcal{E}}_\sharp^2(u)/C \le \|\Phi^\sharp Au\|^2 + \|\Phi^\sharp L^\dagger u\|^2 + \|\Phi^\sharp\sqrt{Q}\,u\|^2 + \|\langle D\rangle^{3/4}\Phi u\|^2 \le C\tilde{\mathcal{E}}_\sharp^2(u).\end{aligned}$$

Proof $\|\sqrt{Q}\,\Phi u\|/C \le \|\Phi\sqrt{Q}\,u\| \le C\|\sqrt{Q}\,\Phi u\|$ and $C\|\sqrt{Q}\,\Phi u\|^2 \ge (Q\Phi u, \Phi u)$ follow from Lemma 6.12 and $(Q\Phi u, \Phi u) \ge \|\sqrt{Q}\,\Phi u\|^2/2 \ge \|\langle D\rangle^{1/2}\Phi u\|^2/C$ by Lemmas 6.8 and 4.10. Similarly $\|\Phi L^\dagger u\| \le C(\mathrm{op}(b_1^2)\Phi u, \Phi u) \le C(\|\Phi Lu\| + \|\langle D\rangle^{1/2}\Phi u\|)$. Moreover one has $\|L\Phi u\| \le C(\|\Phi Lu\| + \|\langle D\rangle^{1/2}\Phi u\|)$ thanks to Proposition 4.1 and Theorem 6.1 hence to finish the proof it suffices to note $\omega^{-1}\phi^{-n} \in S(\langle\xi\rangle_\gamma^{1/2}\phi^{-n}, g_\epsilon)$. The second assertion is proved similarly. □

Therefore Proposition 4.4 can be stated as

Proposition 4.5 *There exist $c > 0$, $c^* > 0$, $\delta_0 > 0$, $\theta_0 > 0$ such that for $|t| \le \delta_0$, $\theta \ge \theta_0$ one has* $2\mathrm{Im}(\Phi\hat{P}_\theta u, \Phi Au) \ge \partial_t\tilde{\mathcal{E}}^2(u) + c\,\theta\tilde{\mathcal{E}}^2(u) + c\,\tilde{\mathcal{E}}_\sharp^2(u)$.

4.3 Estimates of higher order derivatives

Recall that n, M, γ are fixed such that the assertions in Sect. 6.3 hold with m or m_i in (4.38). For notational simplicity, we write

$$\tilde{\mathcal{E}}(\langle D\rangle^s u) = \tilde{\mathcal{E}}_s(u), \quad \tilde{\mathcal{E}}_\sharp(\langle D\rangle^s u) = \tilde{\mathcal{E}}_{\sharp s}(u).$$

Lemma 4.14 *There is $C_s > 0$ such that*

$$\begin{aligned}\|\Phi Au\|_s^2 + \|\Phi L^\dagger u\|_s^2 + \|\Phi\sqrt{Q}\,u\|_s^2 + \|\Phi u\|_{s+1/2}^2 \le C_s\tilde{\mathcal{E}}_s^2(u),\\ \|\Phi^\sharp Au\|_s^2 + \|\Phi^\sharp L^\dagger u\|_s^2 + \|\Phi^\sharp\sqrt{Q}\,u\|_s^2 + \|\Phi u\|_{s+3/4}^2 \le C_s\tilde{\mathcal{E}}_{\sharp s}^2(u).\end{aligned}$$

Proof The proof is clear from Lemma 4.13 and Corollaries 6.3, 6.5. □

Estimate $\langle D\rangle^s u$, $s \in \mathbb{R}$. Noting $\langle D\rangle^s\hat{P}_\theta = \hat{P}_\theta\langle D\rangle^s + [\langle D\rangle^s, \hat{P}]$ we consider $|(\Phi[\langle D\rangle^s, \hat{P}]u, \Phi A\langle D\rangle^s u)|$. Write $\hat{P}_\theta$ as

$$\hat{P}_\theta = -A^2 + H + B_0'A + B_1', \quad H = \mathrm{op}(\ell^2 + q) = \mathrm{op}(h) \tag{4.39}$$

where $B_i' = \mathrm{op}(\tilde{a}_i')$, $\tilde{a}_i' \in S^i$. From Theorem 6.1 and Lemma 6.7 we can write

$$\langle\xi\rangle^s \# h - h \# \langle\xi\rangle^s = r_1 + r_2 + \tilde{r}, \quad r_1 \in S(\langle\xi\rangle^s b, \bar{g}), \quad r_2 \in S(\langle\xi\rangle^s b_1, \bar{g}), \quad \tilde{r} \in S^s$$

then it is clear from Corollaries 6.5 and 6.3 that

$$|(\Phi[\langle D\rangle^s, H]u, \Phi A\langle D\rangle^s u)| \le C\|\Phi A u\|_s(\|\Phi L^\dagger u\|_s + \|\Phi\sqrt{Q}u\|_s). \tag{4.40}$$

Similarly it follows from Corollary 6.3 that

$$\begin{aligned} |(\Phi[\langle D\rangle^s, B_0']Au, \Phi A\langle D\rangle^s u)| &\le C\|\Phi A u\|_s^2, \\ |(\Phi[\langle D\rangle^s, B_1']u, \Phi A\langle D\rangle^s u)| &\le C\|\Phi u\|_s\|\Phi A u\|_s. \end{aligned} \tag{4.41}$$

Proposition 4.6 *For any $s \in \mathbb{R}$ there exist $c_s, \theta_s > 0$ such that for $|t| \le \delta_0$, $\theta \ge \theta_s$ one has*

$$2\mathrm{Im}(\Phi\langle D\rangle^s \hat{P}_\theta u, \Phi A\langle D\rangle^s u) \ge \partial_t \tilde{\mathcal{E}}_s^2(u) + c_s\theta\tilde{\mathcal{E}}_s^2(u) + c_s\tilde{\mathcal{E}}_{\sharp s}^2(u).$$

Proof Write $2\mathrm{Im}(\Phi\langle D\rangle^s \hat{P}_\theta u, \Phi A\langle D\rangle^s u)$ as a sum

$$2\mathrm{Im}(\Phi\hat{P}_\theta\langle D\rangle^s u, \Phi A\langle D\rangle^s u) + 2\mathrm{Im}(\Phi[\langle D\rangle^s, \hat{P}]u, \Phi A\langle D\rangle^s u)$$

and apply Proposition 4.4 to the first term. In view of (4.40) and (4.41), taking Lemma 4.14 into account, the term $|(\Phi[\langle D\rangle^s, \hat{P}]u, \Phi A\langle D\rangle^s u)|$ is absorbed in $\theta\tilde{\mathcal{E}}_s^2(u)$ choosing θ large. □

Proposition 4.7 *Let $|\tau| \le \delta_0$. For any $s \in \mathbb{R}$ there are $C_s, C_s' > 0$ such that*

$$\begin{aligned} \sum_{j=0}^{1}\|D_t^j u(t)\|_{s+1-j} &\le C_s\Big(\mathcal{E}_s(u(t)) + \int_\tau^t \mathcal{E}_{\sharp s}(u(t'))dt'\Big) \\ &\le C_s'\Big(\sum_{j=0}^{1}\|D_t^j u(\tau)\|_{s+n+1-j} + \int_\tau^t \|\hat{P}u(t')\|_{n+s}dt'\Big) \end{aligned} \tag{4.42}$$

holds for any $u \in \cap_{j=0}^2 C^j([\tau, \delta_0]; H^{s+n+2-j})$.

Proof Replacing u by $e^{-\theta t}u$ and noting $Ae^{-\theta t} = e^{-\theta t}D_t$, $\hat{P}_\theta e^{-\theta t} = e^{-\theta t}\hat{P}$ it follows from Proposition 4.6 that

$$2e^{-2\theta t}\|\Phi\langle D\rangle^s\hat{P}u\|\|\Phi\langle D\rangle^s D_t u\| \ge \partial_t\big\{e^{-2\theta t}\mathcal{E}_s^2(u(t))\big\} + c_s e^{-2\theta t}\mathcal{E}_{\sharp s}^2(u(t)).$$

If we integrate from τ to t $(-\delta_0 \le \tau < t \le \delta_0)$ and noting Lemma 4.12 we have

$$\mathcal{E}_s^2(u(t)) + \int_\tau^t \mathcal{E}_{\sharp s}^2(u(t'))dt' \le C_s \mathcal{E}_s^2(u(\tau)) \\ + C_s \int_\tau^t \|\hat{P}u(t')\|_{n+s} \|\Phi \langle D\rangle^s D_t u(t')\| dt'.$$

Denoting $K = \sup_{\tau \le t' \le t} \left\{\mathcal{E}_s(u(t')) + \int_\tau^{t'} \mathcal{E}_{\sharp s}(u(t_1))dt_1\right\}$ we see that K^2 is bounded by $C_s' \mathcal{E}_s^2(u(\tau)) + C_s' K \int_\tau^t \|\hat{P}u(t')\|_{n+s} dt'$ hence we have

$$\mathcal{E}_s(u(t)) + \int_\tau^t \mathcal{E}_{\sharp s}(u(t'))dt' \le C_s''\Big(\mathcal{E}_s(u(\tau)) + \int_\tau^t \|\hat{P}u(t')\|_{n+s} dt'\Big).$$

In virtue of Lemmas 4.12, 4.14 there exists $C = C_s$ such that

$$\sum_{j=0}^{1} \|D_t^j u(t)\|_{s+1-j}/C \le \mathcal{E}_s(u(t)) \le C \sum_{j=0}^{1} \|D_t^j u(t)\|_{s+n+1-j} \tag{4.43}$$

from which the proof follows. □

Here consider the adjoint operator $\hat{P}^* = \mathrm{op}(\overline{\hat{P}(t,x,\tau,\xi)})$ of $\hat{P}$ where $\overline{\hat{P}(t,x,\tau,\xi)}$ is obtained from $\hat{P}(t,x,\tau,\xi)$ replacing $a_j(t,x,\xi)$ by $\overline{a_j(t,x,\xi)}$. Therefore replacing n by $-n$ and θ by $-\theta$ the same argument can be repeated to obtain

$$2e^{2\theta t}\|\Phi_{-n}\langle D\rangle^s \hat{P}^* u\| \|\Phi_{-n}\langle D\rangle^s D_t u\| \\ \ge -\partial_t\big\{e^{2\theta t}(\mathcal{E}_s^*)^2(u)\big\} + c_s e^{2\theta t}(\mathcal{E}_{\sharp s}^*)^2(u) \tag{4.44}$$

where we have set

$$(\mathcal{E}_s^*)^2(u) = \|\Phi_{-n}\langle D\rangle^s D_t u\|^2 + \|\Phi_{-n} L \langle D\rangle^s u\|^2 \\ +(Q\Phi_{-n}\langle D\rangle^s u, \Phi_{-n}\langle D\rangle^s u) + c^*\|\Phi_{-n}^{2\sharp}\langle D\rangle^s u\|^2, \\ (\mathcal{E}_{\sharp s}^*)^2(u) = \|\Phi_{-n}^{\sharp} D_t u\|_s^2 + \|\Phi_{-n}^{\sharp} L u\|_s^2 + \|\Phi_{-n}^{\sharp}\sqrt{Q}\, u\|_s^2 + \|\Phi_{-n} u\|_{s+3/4}^2$$

and $\Phi_{-n}^{k\sharp} = \mathrm{op}(\omega^{-k/2}\phi^n)$, $\Phi_{-n}^{0\sharp} = \Phi_{-n}$ and $L^\dagger$ and $\sqrt{Q}$ are as before. It is clear from $\langle\xi\rangle_\gamma^{-1} \le C\phi \le C'$ that

$$\|\langle D\rangle^{-n} u\|/C \le \|\Phi_{-n} u\| \le C\|u\|. \tag{4.45}$$

Since the proof of Lemma 4.10 shows $C\|\Phi_{-n}\sqrt{Q}u\| \ge \|\mathrm{op}(\omega\phi^n\langle\xi\rangle)u\|$ noting $\langle\xi\rangle^{-n} \le (2\omega\phi)^n \le C\omega\phi^n$ by (4.4) one has

$$\|\langle D\rangle^{-n+1} u\| \le C\|\Phi_{-n}\sqrt{Q}\, u\|, \quad n \ge 1. \tag{4.46}$$

Integrating (4.44) over $[t, \tau]$ and repeating the proof of Proposition 4.7 we have

Proposition 4.8 *Let $|\tau| \le \delta_0$. For any $s \in \mathbb{R}$ there exist $C_s, C_s' > 0$ such that*

$$\begin{aligned}\sum_{j=0}^{1} \|D_t^j u(t)\|_{s+1-n-j} &\le C_s\Big(\mathcal{E}_s^*(u(t)) + \int_t^\tau \mathcal{E}_{\sharp s}^*(u(t'))dt'\Big)\\ &\le C_s'\Big(\sum_{j=0}^{1} \|D_t^j u(\tau)\|_{s+1-j} + \int_t^\tau \|\hat{P}^* u(t')\|_s dt'\Big)\end{aligned} \tag{4.47}$$

holds for any $u \in \cap_{j=0}^2 C^j([-\delta_0, \tau]; H^{s+2-j})$.

5 Local existence and uniqueness theorem

In this section, we prove the existence of the solution operator of the localized operator with a finite speed of propagation. Making use of such solution operators we prove the local existence and uniqueness theorem for the original Cauchy problem.

5.1 Local existence theorem

We show the existence and uniqueness of the Cauchy problem for localized $\hat{P}$.

Theorem 5.1 *Let $|\tau| < \delta_0$, $s \in \mathbb{R}$. For any $f \in L^1((\tau, \delta_0); H^{s+n})$ and $\phi_j \in H^{s+n+1-j}$ $(j = 0, 1)$ there exists a unique solution $u \in \cap_{j=0}^1 C^j([\tau, \delta_0]; H^{s+1-j})$ to the Cauchy problem*

$$\begin{cases} \hat{P}u = f, \quad \tau < t < \delta_0, \quad x \in \mathbb{R}^d, \\ D_t^j u(\tau, x) = \phi_j(x), \quad j = 0, 1, \quad x \in \mathbb{R}^d \end{cases} \tag{5.1}$$

and (4.42) holds for this solution.

Proof The uniqueness follows from (4.42). We show the existence of u. Consider the anti-linear from

$$\mathcal{L} : \hat{P}^* \mapsto i(\phi_0, D_t v(\tau)) + i(\phi_1 - B_0(\tau)\phi_0, v(\tau)) + \int_\tau^{\delta_0} (f, v)dt$$

on $\{\hat{P}^* v; v \in C_0^\infty(\{(t, x); t < \delta_0\})\}$ where $B_0 = \mathrm{op}(\tilde{a}_0)$ is given in (4.17). From (4.47) it is seen that $\big|i(\phi_0, D_t v(\tau)) + i(\phi_1 - B_0(\tau)\phi_0, v(\tau))\big|$ is bounded by

$$\begin{aligned} &C(\|\phi_0\|_{s+n+1} + \|\phi_1\|_{s+n})(\|v(\tau)\|_{-s-n} + \|D_t v(\tau)\|_{-s-n-1})\\ &\le C\sum_{j=0}^{1} \|\phi_j\|_{s+n+1-j} \int_\tau^{\delta_0} \|\hat{P}^* v(t)\|_{-s-1} dt\end{aligned}$$

and $\left|\int_\tau^{\delta_0}(f, v)dt\right|$ is estimated by

$$\sup_{\tau\le t\le\delta_0}\|v(t)\|_{-s-n}\int_\tau^{\delta_0}\|f(t)\|_{s+n}dt$$
$$\le C\int_\tau^{\delta_0}\|\hat{P}^*v(t)\|_{-s-1}dt\int_\tau^{\delta_0}\|f(t)\|_{s+n}dt.$$

Using the Hahn-Banach theorem to extend this form we conclude that there is some $u\in L^\infty([\tau,\delta_0];H^{s+1})$ such that

$$i(\phi_0, D_t v(\tau)) + i(\phi_1 - B_0(\tau)\phi_0, v(\tau)) + \int_\tau^{\delta_0}(f, v)dt = \int_\tau^{\delta_0}(u, \hat{P}^*v)dt \tag{5.2}$$

if $v\in C_0^\infty(\{(t,x); t<\delta_0\})$. Thus $\hat{P}u=f$ in $(\tau,\delta_0)\times\mathbb{R}^d$ in the distribution sense. Then $D_t^j u(t)\in L^2([\tau,\delta_0];H^{s+1-j})$, $j=0,1,2$ thanks to [4, Theorem B.2.9] hence $u\in\cap_{j=0}^1 C^j([\tau,\delta_0];H^{s-j})$. Since $v(\tau), D_t v(\tau)\in C_0^\infty(\mathbb{R}^d)$ are arbitrary we conclude $D_t^j u(\tau)=\phi_j$, $j=0,1$. Choose $\phi_{j\nu}\in\mathcal{S}(\mathbb{R}^d)$, $f_\nu\in\mathcal{S}(\mathbb{R}^{1+d})$ so that

$$\|\phi_j-\phi_{j\nu}\|_{s+n+1-j}\to 0,\quad \int_\tau^T\|f-f_\nu\|_{s+n}dt\to 0\quad(\nu\to\infty).$$

There is $u_\nu(t)\in\cap_{j=0}^2 C^j([\tau,\delta_0];H^{s+n+2-j})$ satisfying $\hat{P}u_\nu=f_\nu$ and $D_t^j u_\nu(\tau)=\phi_{j\nu}$ hence u_ν is a Cauchy sequence in $\cap_{j=0}^1 C^j([\tau,\delta_0];H^{s+1-j})$. The limit as $\nu\to\infty$ is the desired solution. Clearly the limit u satisfies (4.42). □

The Cauchy problem for the adjoint operator $\hat{P}^*$ can be treated similarly.

Theorem 5.2 *Let $|\tau|<\delta_0$, $s\in\mathbb{R}$. For any $f\in L^1((-\delta_0,\tau);H^{s+n})$ and $\phi_j\in H^{s+n+1-j}$ $(j=0,1)$ there is a unique solution $u\in\cap_{j=0}^1 C^j([-\delta_0,\tau];H^{s+1-j})$ of*

$$\begin{cases}\hat{P}^*u=f, & -\delta_0<t<\tau,\quad x\in\mathbb{R}^d,\\ D_t^j u(\tau,x)=\phi_j(x), & j=0,1,\quad x\in\mathbb{R}^d\end{cases}\tag{5.3}$$

and (4.47) holds for this solution.

Study the solution operator of the Cauchy problem (5.1) with $\phi_0=\phi_1=0$;

$$\hat{G}: L^1((\tau,\delta_0);H^{s+n})\ni f(t)\mapsto u(t)\in\cap_{j=0}^1 C^j([\tau,\delta_0];H^{s+1-j})$$

where $\hat{P}\hat{G}f=f$ in $(\tau,\delta_0)\times\mathbb{R}^d$ and the following estimate holds

$$\sum_{j=0}^1\|D_t^j\hat{G}f(t)\|_{s+1-j}\le C_s\int_\tau^t\|f(t_1)\|_{n+s}dt_1,\quad \tau\le t\le\delta_0.\tag{5.4}$$

Proposition 5.1 *$\hat{G}$ has a finite speed of propagation, namely $\hat{G}$ satisfies the following Definition 5.1 with $m = 2$.*

A conic set $U \subset \mathbb{R}^d \times (\mathbb{R}^d \backslash 0)$ can be identified with $\{(x, \xi/|\xi|); (x, \xi) \in U\}$, a subset of $\mathbb{R}^d \times S^{d-1}$. The topology for conic sets is induced through this identification. By $\overset{\circ}{U}$ we denote the interior of U and by U^c the complement of U and $U \Subset V$ means that U is relatively compact in $\overset{\circ}{V}$.

Definition 5.1 We say that G has a finite speed of propagation if for any closed conic set U_1 and compact conic set U_2 with $U_1 \cap U_2 = \emptyset$ there exists $\delta > 0$ such that for any $l_i \in \mathbb{R}$ and $h_i(x, \xi) \in S^0(\mathbb{R}^{2d})$ with $\operatorname{supp} h_i \subset U_i$ one can find $C > 0$ such that the estimate

$$\sum_{j=0}^{m-1} \|D_t^j \mathrm{op}(h_2) G \mathrm{op}(h_1) f(t)\|_{l_1 - j} \le C_{l_1, l_2} \int_\tau^t \|f(t_1)\|_{l_2} dt_1 \tag{5.5}$$

holds for any $f \in L^1((\tau, T); H^{l_2})$ and $\tau < t \le \min(\tau + \delta, T)$.

We postpone the proof of Proposition 5.1 to the next section.

Definition 5.2 Let P_i $(i = 1, 2)$ be two operators of the form

$$-D_t^2 + \sum_{j=0}^{1} \mathrm{op}(a_j) D_t^j, \quad a_j(t, x, \xi) \in C^\infty((-T, T); S^{2-j}). \tag{5.6}$$

For $\eta \in \mathbb{R}^d$, $|\eta| \neq 0$ we say $P_1 \equiv P_2$ at $(0, \eta)$ if there are $\delta > 0$ and a conic neighborhood W of $(0, \eta)$ such that one can write

$$P_1 - P_2 = \sum_{j=0}^{1} \mathrm{op}(c_j) D_t^j, \quad c_j(t, x, \xi) \in C^\infty([-\delta, \delta]; S^{2-j} \cap S^{-\infty}(W)).$$

Before going on, we prepare a version of well-known relation on the wave front set under the pullback (e.g. [5, Theorem 8.2.1]). If κ is a diffeomorphism on $\mathbb{R}^d$ and $U \subset \mathbb{R}^d \times (\mathbb{R}^d \backslash 0)$ is a conic set we denote

$$\kappa^* U = \{(x, {}^t\kappa'(x)\eta); (\kappa(x), \eta) \in U\}$$

and $\kappa^* f = f(\kappa(x))$ is the pullback if f is a function on $\mathbb{R}^d$.

Proposition 5.2 *Let κ be a diffeomorphism on $\mathbb{R}^d$ which is a linear transformation outside a compact set. Let U, V be two closed conic sets with $V \cap \kappa^* U = \emptyset$ and $h, k \in S^0$ such that* $\operatorname{supp} h \subset U$, $\operatorname{supp} k \subset V$. *Then for any $p, q \in \mathbb{R}$ there is C such that*

$$\|\mathrm{op}(k) \kappa^* \mathrm{op}(h) v\|_p \le C \|v\|_q, \quad v \in H^q.$$

We give the proof in the Appendix.

Lemma 5.1 *If all critical points* $(0, 0, \tau, \xi)$ *of* $p = 0$ *are effectively hyperbolic then for any* $0 \neq \eta \in \mathbb{R}^d$ *there exists* P_η *of the form* (5.6) *such that* $P_\eta \equiv P$ at $(0, \eta)$ *of whose solution operator has a finite speed of propagation.*

Proof If $p(0, 0, \tau, \eta) = 0$ has a double characteristic root, which is necessarily $\tau = 0$, and $(0, 0, 0, \eta)$ is effectively hyperbolic by assumption. Proposition 2.1 with $\bar{\xi} = \eta$ gives a diffeomorphism on $\mathbb{R}^d$: $x \mapsto \kappa(x)$. Set $(Tu)(t, x) = u(t, \kappa(x))$ and let $\hat{P}$ be the localized operator defined in Sect. 3 and denote $P_\eta = T\hat{P}T^{-1}$. Since $(y(x), \eta(\xi)) = (x, \xi)$ in some neighborhood of $(0, e_d)$ given by (3.3) it is clear that

$$P_\eta \equiv P \quad \text{at} \quad (0, \eta).$$

The solution operator $\hat{G}$ of $\hat{P}$, given in Theorem 5.1, has a finite speed of propagation by Proposition 5.1. Set $G_\eta = T\hat{G}T^{-1}$ then $P_\eta G_\eta = I$ is obvious. We examine that G_η has a finite speed of propagation. Let U_1, U_2 be closed and compact conic set with $U_1 \cap U_2 = \emptyset$. Choose open conic sets V_i and compact conic sets W_i such that $(\kappa^{-1})^* U_2 \Subset V_2 \Subset W_2 \Subset V_1 \Subset W_1$ with $W_1 \cap (\kappa^{-1})^* U_1 = \emptyset$ and $\phi_i \in S^0$ such that $\phi_1 = 1$ on W_1^c with $\operatorname{supp} \phi_1 \subset V_1^c$ and $\phi_2 = 1$ on V_2 with $\operatorname{supp} \phi_2 \subset W_2$. Write $\operatorname{op}(h_2) G_\eta \operatorname{op}(h_1)$ as a sum

$$\begin{aligned}&\operatorname{op}(h_2) T \operatorname{op}(\phi_2) \hat{G} \operatorname{op}(\phi_1) T^{-1} \operatorname{op}(h_1) + \operatorname{op}(h_2) T \hat{G} \operatorname{op}(\phi_1^c) T^{-1} \operatorname{op}(h_1)\\ &+ \operatorname{op}(h_2) T \operatorname{op}(\phi_2^c) \hat{G} \operatorname{op}(\phi_1) T^{-1} \operatorname{op}(h_1), \quad \phi_i^c = 1 - \phi_i.\end{aligned}$$

Since $\operatorname{supp} \phi_1^c \subset W_1$, $W_1 \cap (\kappa^{-1})^* U_1 = \emptyset$ and $\operatorname{supp} \phi_2^c \subset V_2^c$, $U_2 \cap \kappa^* V_2^c = \emptyset$ one can apply Proposition 5.2 to $\operatorname{op}(\phi_1^c) T^{-1} \operatorname{op}(h_1)$ and $\operatorname{op}(h_2) T \operatorname{op}(\phi_2^c)$ to obtain the desired estimates. On the other hand to estimate $\operatorname{op}(\phi_2) \hat{G} \operatorname{op}(\phi_1)$ it suffices to use a finite speed of propagation of $\hat{G}$ for $W_2 \cap V_1^c = \emptyset$.

If $p(0, 0, \tau, \eta) = 0$ has a simple root one can find $\delta > 0$ and a conic neighborhood U of $(0, \eta)$ and real valued $\lambda_i(t, x, \xi) \in C^\infty((-\delta, \delta) \times U)$, homogeneous of degree 1 in ξ, such that $\inf_{(-\delta,\delta)\times U} |\lambda_1(t, x, \xi) - \lambda_2(t, x, \xi)|/|\xi| > 0$ which satisfy

$$p(t, x, \tau, \xi) = -(\tau + \lambda_1(t, x, \xi))\big(\tau + \lambda_2(t, x, \xi)\big). \tag{5.7}$$

Taking Theorem 6.1 into account one can find $\lambda_{ij} \in C^\infty((-\delta, \delta) \times U)$, $j \in \mathbb{N}$, homogeneous of degree $-j$, such that

$$P(t, x, \tau, \xi) = -\left(\tau + \lambda_1 + \sum_{j=0}^{\infty} \lambda_{1j}\right) \# \left(\tau + \lambda_2 + \sum_{j=0}^{\infty} \lambda_{2j}\right)$$

is verified formally. Take a conic neighborhood $V \Subset U$ of $(0, \eta)$ and $\chi \in S^0$ such that $\chi = 1$ in $V \cap \{|\xi| \geq 1\}$ and $\operatorname{supp} \chi \subset U \cap \{|\xi| \geq 1/2\}$. Then there is $\tilde{\lambda}_i \in S^1$ such that $\tilde{\lambda}_i \sim \chi\lambda_i + \sum_{j=0}^{\infty} \chi\lambda_{ij}$ (e.g. [4, Proposition 13.1.3]). If we set $P_i = D_t + \mathrm{op}(\tilde{\lambda}_i)$ it is clear that

$$P \equiv P_1 P_2 \quad \text{at} \quad (0, \eta).$$

Since P_i is a first order operator it is easily checked that there is a solution operator G_i with a finite speed of propagation ($m = 1$ in Definition 5.1) and consequently $G_2 G_1$ has a finite speed of propagation. Then $P_\eta = P_1 P_2$ is the desired one whose solution operator is $G_\eta = G_2 G_1$. □

Theorem 5.3 *If all critical points $(0, 0, \tau, \xi)$ of $p = 0$ are effectively hyperbolic then there are $\delta > 0$, $n > 0$ and a neighborhood Ω of $x = 0$ such that for any $|\tau| < \delta$ and $f \in L^1((\tau, \delta); H^{s+n})$ there exists $u \in \cap_{j=0}^{1} C^j([\tau, \delta]; H^{s+1-j})$ satisfying $Pu = f$ in $(\tau, \delta) \times \Omega$ and*

$$\sum_{j=0}^{1} \|D_t^j u(t)\|_{s+1-j} \le C_s \int_\tau^t \|f(t')\|_{n+s} dt', \quad \tau \le t \le \delta. \tag{5.8}$$

Proof Thanks to Lemma 5.1, for any $|\eta| = 1$ there are $\delta_\eta > 0$, a conic neighborhood W_η of $(0, \eta)$, a second order operator P_η with solution operator G_η with a finite speed of propagation satisfying (5.4) with $n = n_\eta$ and P_η satisfying

$$P - P_\eta = R_\eta = \sum_{j=0}^{1} \mathrm{op}(c_{\eta,j}) D_t^j, \quad c_{\eta,j} \in S^{2-j} \cap S^{-\infty}(W_\eta), \quad |t| \le \delta_\eta. \tag{5.9}$$

Since $\{|\eta| = 1\}$ is compact there are finite number of η_i and a neighborhood Ω of $x = 0$ such that $\cup_i W_{\eta_i} \supset \Omega \times (\mathbb{R}^d \backslash \{0\})$. Note that G_{η_i} satisfies (5.4) with $n = \max_i n_{\eta_i}$. Take open conic coverings $\{U_i\}$, $\{V_i\}$ of $\Omega \times (\mathbb{R}^d \backslash \{0\})$ such that $U_i \Subset V_i \Subset W_{\eta_i}$ and a partition of unity $\{\alpha_i(x, \xi)\}$, $\alpha_i \in S^0$ associated to $\{U_i\}$. If we set $\sum_i \alpha_i(x, \xi) = \alpha(x)$ then $\alpha(x) = 1$ in a neighborhood of $x = 0$ and we may assume that $\alpha(x)$ has a compact support. Define

$$G = \sum_i G_{\eta_i} \mathrm{op}(\alpha_i)$$

then it is clear from (5.9) that

$$PGf = \sum_i \big(P_{\eta_i} + R_{\eta_i}\big) G_{\eta_i} \mathrm{op}(\alpha_i) f = \big(\alpha(x) - \tilde{R}\big) f \tag{5.10}$$

where $\tilde{R} = -\sum_i R_{\eta_i} G_{\eta_i} \mathrm{op}(\alpha_i)$. Set $R = \alpha \tilde{R}$ and we show that there are $\delta_1, \delta' > 0$ such that

$$\|Rf(t)\|_{\tilde{s}} \le C_{\tilde{s}} \int_\tau^t \|f(t')\|_{\tilde{s}} dt', \quad \tau \le t \le \tau + \delta', \quad |\tau| \le \delta_1 \tag{5.11}$$

for any $\tilde{s}$. Take $\chi_i \in S^0$ be 1 on V_i with $\operatorname{supp}\chi_i \subset W_{\eta_i}$ and $\tilde{\alpha} \in C_0^\infty(\mathbb{R}^d)$ be 1 in a neighborhood of $x = 0$ with $\operatorname{supp}\alpha \Subset \{\tilde{\alpha} = 1\}$ and write

$$\alpha R_{\eta_i} G_{\eta_i} \mathrm{op}(\alpha_i) = \alpha R_{\eta_i}(1-\tilde{\alpha}) G_{\eta_i} \mathrm{op}(\alpha_i) \\ +\alpha R_{\eta_i}\tilde{\alpha}(\mathrm{op}(\chi_i) + \mathrm{op}(1-\chi_i)) G_{\eta_i} \mathrm{op}(\alpha_i)$$

where $\alpha(1-\tilde{\alpha}) = 0$. Since one can write $\tilde{\alpha}\#\chi_i = \kappa_i + r_i$, $\operatorname{supp}\kappa_i \subset W_{\eta_i}$, $\kappa_i \in S^0$, $r_i \in S^{-n}$ and $\tilde{\alpha}\#(1-\chi_i) = \tilde{\kappa}_i + \tilde{r}_i$, $\operatorname{supp}\tilde{\kappa}_i \subset V_i^c \cap \operatorname{supp}\tilde{\alpha}$, $\tilde{\kappa}_i \in S^0$, $\tilde{r}_i \in S^{-n}$ it is clear that $\|\alpha R_{\eta_i}(1-\tilde{\alpha}) G_{\eta_i}\mathrm{op}(\alpha_i) f\|_{\tilde{s}}$, $\|\alpha R_{\eta_i}\mathrm{op}(\kappa_i + r_i) G_{\eta_i}\mathrm{op}(\alpha_i) f\|_{\tilde{s}}$ and $\|\alpha R_{\eta_i}\mathrm{op}(\tilde{r}_i) G_{\eta_i}\mathrm{op}(\alpha_i) f\|$ are bounded by

$$C\sum_{j=0}^{1} \|D_t^j G_{\eta_i}\mathrm{op}(\alpha_i) f\|_{\tilde{s}-n+1-j} \le C' \int_\tau^t \|f(t')\|_{\tilde{s}} dt'$$

while $\|\alpha R_{\eta_i}\mathrm{op}(\tilde{\kappa}_i) G_{\eta_i}\mathrm{op}(\alpha_i) f\|_{\tilde{s}} \le C\sum_{j=0}^{1} \|D_t^j \mathrm{op}(\tilde{\kappa}_i) G_{\eta_i}\mathrm{op}(\alpha_i) f\|_{\tilde{s}+2-j}$ to which we apply a finite speed of propagation of G_{η_i} for $(V_i^c \cap \operatorname{supp}\tilde{\alpha}) \cap U_i = \emptyset$. Thus (5.11) is proved.

Multiply (5.11) by $e^{-\theta t}$ $(\theta > 0)$ and integrate from τ $(|\tau| \le \delta_1)$ to t one has

$$\int_\tau^t e^{-\theta t'} \|Rf(t')\|_{\tilde{s}} dt' \le \frac{C_{\tilde{s}}}{\theta}\int_\tau^t e^{-\theta t'}\|f(t')\|_{\tilde{s}} dt', \quad \tau \le t \le \tau + \delta'$$

for any $f \in L^1((\tau, \tau+\delta'); H^{\tilde{s}})$. Choose $\theta = \theta_s$ such that $C_{\tilde{s}}/\theta < 1/2$ then $Sf = \sum_{l=0}^\infty R^l f$ converges in the weighted $L^1((\tau, \tau+\delta'); H^{\tilde{s}})$ with the weight $e^{-\theta t}$ and it yields

$$\int_\tau^t e^{-\theta t'}\|Sf(t')\|_{\tilde{s}} dt' \le 2\int_\tau^t e^{-\theta t'}\|f(t')\|_{\tilde{s}} dt'. \tag{5.12}$$

Let $\beta(x) \in C_0^\infty(\mathbb{R}^d)$ be 1 in a neighborhood of $x = 0$ with $\operatorname{supp}\beta \Subset \{\alpha = 1\}$. Noting $\beta(\alpha - \tilde{R}) = \beta(I - \alpha\tilde{R})$ it is clear $\beta PGSf = \beta(I-R)Sf = \beta f$ hence $P\big(GSf\big) = f$ on $\{\beta(x) = 1\}$. If $f \in L^1((\tau, \tau+\delta'); H^{s+n})$ then $u = GSf \in \cap_{j=0}^1 C^j([\tau, \tau+\delta']; H^{s+1-j})$ and choosing $\tilde{s} = s+n$ in (5.4), (5.12) one obtains

$$e^{-\theta t}\sum_{j=0}^{1}\|D_t^j u(t)\|_{s+1-j} \le C\int_\tau^t e^{-\theta t'}\|Sf(t')\|_{s+n} dt' \\ \le 2C\int_\tau^t e^{-\theta t'}\|f(t')\|_{s+n} dt'$$

which proves (5.8). □

5.2 Finite speed of propagation

Here we shall prove Proposition 5.1. Write $\hat{P}_\theta$ in the form (4.39).

Definition 5.3 $f(t,x,\xi) \in C^\infty((-T,T); S^0)$ is called to be spacelike (for $\hat{P}$) if there exist $0 < \delta_1$, $0 < \kappa < 1$ such that

$$\partial_t f \geq \delta_1, \quad 4\kappa(\partial_t f)^2 h \geq \{h, f\}^2. \tag{5.13}$$

Following [8], for a spacelike f we denote

$$\bar{f}(t,x,\xi) = \begin{cases} \exp\left(1/f(t,x,\xi)\right), & f < 0, \\ 0, & f \geq 0 \end{cases} \tag{5.14}$$

and set

$$\bar{f}_1 = f^{-1}(\partial_t f)^{1/2}\bar{f}, \quad m = f(\partial_t f)^{-1/2}. \tag{5.15}$$

It is clear that $\bar{f}, \bar{f}_1, \partial_t\bar{f}, m \in S^0$ and $\bar{f} - m\#\bar{f}_1 \in S^{-1}$. Take a $\ell \geq 0$ and with $w_\delta = \langle\delta\xi\rangle^{-\ell}$ $(0 < \delta < 1)$ we set

$$F^\delta = \mathrm{op}(w_\delta \bar{f}), \quad F_1^\delta = \mathrm{op}(w_\delta \bar{f}_1).$$

It is easy to see that $|\partial_\xi^\beta w_\delta^{\pm 1}| \leq C_\beta w_\delta^{\pm 1}\langle\xi\rangle^{-|\beta|}$ with some C_β independent of δ. In the following, all arguments are uniform in $0 < \delta < 1$ though we do not mention it.

Definition 5.4 Let $S_i(t,\cdot)$ be two real functionals on $C^2((-T,T); H^{s+n+1})$. We say $S_1 \overset{s}{\sim} S_2$ if for any $\epsilon > 0$ there is $C_\epsilon > 0$ independent of δ such that

$$\left|S_1(t,u(t)) - S_2(t,u(t))\right| \leq C_\epsilon\left(\tilde{\mathcal{E}}^2_{\sharp(s-1/4)}(u) + \tilde{\mathcal{E}}^2_s(F^\delta u)\right) + \epsilon\|\Phi A F_1^\delta u(t)\|_s^2$$

for any $u(t) \in C^2((-T,T); H^{s+n+1})$. We write $S_1 \overset{s}{\lesssim} S_2$ or $S_2 \overset{s}{\gtrsim} S_1$ if $S_1(t,u(t)) - S_2(t,u(t))$ is bounded by the right-hand side.

In the following, all constants c, C may depend on s but not on δ and may change from line to line. The main step to the proof of Proposition 5.1 is to estimate $(\Phi\langle D\rangle^s[F^\delta, \hat{P}_\theta]u, \Phi\langle D\rangle^s A F^\delta u)$.

Lemma 5.2 *Let* $r_i \in S(\langle\xi\rangle^{l_i}\phi^{-n_i}, \bar{g})$ *satisfy* $\partial_t r_i \in S(\langle\xi\rangle^{l_i+1/2}\phi^{-n_i}, \bar{g})$ $(i = 1,2,3,4)$. *With* $R_j = \mathrm{op}(r_j)$ *one has*

$$\begin{aligned}
&\Sigma_{i=1}^2 l_i = 2s+1, \quad \Sigma_{i=1}^2 n_i = 2n \implies (R_1 u, R_2 u) \overset{s}{\sim} 0,\\
&\Sigma_{i=1}^3 l_i = 2s+1/4, \quad \Sigma_{i=1}^3 n_i = 2n \implies (R_1 u, R_2 A R_3 u) \overset{s}{\sim} 0,\\
&\Sigma_{i=1}^4 l_i = 2s-1/2, \quad \Sigma_{i=1}^4 n_i = 2n \implies (R_1 A R_2 u, R_3 A R_4 u) \overset{s}{\sim} 0.
\end{aligned}$$

Proof The proof is immediate from Corollary 6.3. □

Lemma 5.3 *We have*

$$(\Phi\langle D\rangle^s[F^\delta, H]u, \Phi A\langle D\rangle^s F^\delta u) \overset{s}{\sim} -i(\mathrm{op}(\{h, f\}/\partial_t f)\Phi\langle D\rangle^s F_1^\delta u, \Phi\langle D\rangle^s A F_1^\delta u).$$

Proof Since $(w_\delta \bar{f})\#h - h\#(w_\delta \bar{f}) - i\{h, w_\delta \bar{f}\} \in S^0$ it follows from Lemma 5.2

$$(\Phi\langle D\rangle^s[F^\delta, H]u, \Phi A\langle D\rangle^s F^\delta u) \overset{s}{\sim} (\Phi\langle D\rangle^s \mathrm{op}(i\{h, w_\delta \bar{f}\})u, \Phi\langle D\rangle^s A F^\delta u).$$

Write $\{h, w_\delta \bar{f}\} = \{h, \bar{f}\}w_\delta + \{h, w_\delta\}\bar{f}$. Since $w_\delta \bar{f} - m\#(w_\delta \bar{f}_1) \in S^{-1}$ and $\{h, \bar{f}\}w_\delta \in S^1$ then $(\Phi\langle D\rangle^s \mathrm{op}(i\{h, \bar{f}\}w_\delta)u, \Phi\langle D\rangle^s A F^\delta u)$ is

$$\overset{s}{\sim} (\Phi\langle D\rangle^s \mathrm{op}(i\{h, \bar{f}\}w_\delta)u, \Phi \mathrm{op}(m)\langle D\rangle^s A F_1^\delta u). \tag{5.16}$$

Since $r = \langle\xi\rangle^s\#\phi^{-n}\#\phi^{-n}\#m - m\#\langle\xi\rangle^s\#\phi^{-n}\#\phi^{-n} \in S(\langle\xi\rangle^{s-1/2}\phi^{-2n}, \bar{g})$ and $\{h, \bar{f}\}w_\delta \in S^1$ it follows from Corollary 6.3 that for any $\epsilon > 0$ one has

$$\begin{aligned} |(\mathrm{op}(i\{h, \bar{f}\}w_\delta)u, \mathrm{op}(r)\langle D\rangle^s A F_1^\delta u)| &\le C\|\Phi u\|_{s+1/2}\|\Phi A F_1^\delta u\|_s \\ &\le \epsilon\|\Phi A F_1^\delta u\|_s^2 + C_\epsilon\|\Phi u\|_{s+1/2}^2 \end{aligned} \tag{5.17}$$

hence (5.16) $\overset{s}{\sim} (\Phi\langle D\rangle^s \mathrm{op}(m)\mathrm{op}(i\{h, \bar{f}\}w_\delta)u, \Phi\langle D\rangle^s A F_1^\delta u)$. Noting $m\#(\{h, \bar{f}\}w_\delta) + \big(\{h, f\}/\partial_t f\big)\#(w_\delta \bar{f}_1) \in S^0$ we see that (5.16) is

$$\overset{s}{\sim} -i(\Phi\langle D\rangle^s \mathrm{op}(\{h, f\}/\partial_t f)F_1^\delta u, \Phi\langle D\rangle^s A F_1^\delta u)$$

this is still $\overset{s}{\sim} -i(\mathrm{op}(\{h, f\}/\partial_t f)\Phi\langle D\rangle^s F_1^\delta u, \Phi\langle D\rangle^s A F_1^\delta u)$ arguing as (5.17) for $\phi^{-n}\#\langle\xi\rangle^s\#(\{h, f\}/\partial_t f) - (\{h, f\}/\partial_t f)\#\phi^{-n}\#\langle\xi\rangle^s \in S(\langle\xi\rangle^{s+1/2}\phi^{-n}, \bar{g})$.

For $\{h, w_\delta\}\bar{f}$ setting $k = \{h, w_\delta\}w_\delta^{-1} \in S^1$ one sees $\{h, w_\delta\}\bar{f} - k\#(w_\delta \bar{f}) \in S^0$ hence $(\Phi\langle D\rangle^s \mathrm{op}(i\{h, w_\delta\}\bar{f})u, \Phi\langle D\rangle^s A F^\delta u) \overset{s}{\sim} (\Phi\langle D\rangle^s \mathrm{op}(ik)F^\delta u, \Phi\langle D\rangle^s A F^\delta u)$. Since $\phi^{-n}\#\langle\xi\rangle^s\#k - k\#\phi^{-n}\#\langle\xi\rangle^s \in S(\langle\xi\rangle^{s+1/2}\phi^{-n}, \bar{g})$ this is still

$$\overset{s}{\sim} (\mathrm{op}(ik)\Phi\langle D\rangle^s F^\delta u, \Phi\langle D\rangle^s A F^\delta u).$$

Thanks to Lemma 6.7 we have $\{h, w_\delta\}w_\delta^{-1} \in S(b, \bar{g}) + S(b_1, \bar{g})$ then it follows that $(\mathrm{op}(\{h, w_\delta\}w_\delta^{-1})\Phi\langle D\rangle^s F^\delta u, \Phi\langle D\rangle^s A F^\delta u) \overset{s}{\sim} 0$ from Corollary 6.5. □

Turn to $(\Phi\langle D\rangle^s[A^2, F^\delta]u, \Phi\langle D\rangle^s A F^\delta u)$ which is a sum

$$(\Phi\langle D\rangle^s[A, F^\delta]Au, \Phi\langle D\rangle^s A F^\delta u) + (\Phi\langle D\rangle^s A[A, F^\delta]u, \Phi\langle D\rangle^s A F^\delta u). \tag{5.18}$$

Noting $[A, F^\delta] = \mathrm{op}(if^{-2}(\partial_t f)w_\delta \bar{f}) \in S^0$ and $m\#(f^{-2}(\partial_t f)w_\delta \bar{f}) - w_\delta \bar{f}_1 \in S^{-1}$ it follows from a repetition of similar arguments that

$$(\Phi\langle D\rangle^s [A, F^\delta]Au, \Phi\langle D\rangle^s AF^\delta u) \overset{s}{\sim} i\|\Phi\langle D\rangle^s AF_1^\delta u\|^2. \tag{5.19}$$

Lemma 5.4 *We have*

$$\begin{aligned}&\mathrm{Im}(\Phi\langle D\rangle^s A[A, F^\delta]u, \Phi\langle D\rangle^s Av) = -\partial_t \mathrm{Re}(\Phi\langle D\rangle^s [A, F^\delta]u, \Phi\langle D\rangle^s Av)\\ &\quad+\mathrm{Im}(\Phi\langle D\rangle^s [A, F^\delta]u, \Phi\langle D\rangle^s A^2 v) - n\mathrm{Re}(\mathrm{op}(\omega^{-1}\phi^{-n})\langle D\rangle^s [A, F^\delta]u, \Phi\langle D\rangle^s Av)\\ &\quad-n\mathrm{Re}(\Phi\langle D\rangle^s [A, F^\delta]u, \mathrm{op}(\omega^{-1}\phi^{-n})\langle D\rangle^s Av) - 2\theta\mathrm{Re}(\Phi\langle D\rangle^s [A, F^\delta]u, \Phi\langle D\rangle^s Av).\end{aligned}$$

Proof The proof is easy if we note $\partial_t \phi = \omega^{-1}\phi$. □

Noting $A^2 F^\delta = F^\delta A^2 + A[A, F^\delta] + [A, F^\delta]A$ and $\omega^{-1}\phi^{-n} \in S(\langle\xi\rangle^{1/2}\phi^{-n}, \bar{g})$ it follows from Lemma 5.4 with $v = F^\delta u$ that

$$\begin{aligned}&\mathrm{Im}(\Phi\langle D\rangle^s A[A, F^\delta]u, \Phi\langle D\rangle^s AF^\delta u) \overset{s}{\sim} -\partial_t \mathrm{Re}(\Phi\langle D\rangle^s [A, F^\delta]u, \Phi\langle D\rangle^s AF^\delta u)\\ &\quad+\mathrm{Im}(\Phi\langle D\rangle^s [A, F^\delta]u, \Phi\langle D\rangle^s F^\delta A^2 u)\end{aligned}$$

where replacing A^2 by $A^2 = -\hat{P}_\theta + H + B_0' A + B_1'$ this is still

$$\begin{aligned}&\overset{s}{\sim} -\partial_t \mathrm{Re}(\Phi\langle D\rangle^s [A, F^\delta]u, \Phi\langle D\rangle^s AF^\delta u) - \mathrm{Im}(\Phi\langle D\rangle^s [A, F^\delta]u, \Phi\langle D\rangle^s F^\delta \hat{P}_\theta u)\\ &\quad+\mathrm{Im}(\Phi\langle D\rangle^s [A, F^\delta]u, \Phi\langle D\rangle^s F^\delta Hu)\end{aligned}$$

for $B_i' = \mathrm{op}(\tilde{a}_i')$, $\tilde{a}_i' \in S^i$. We check the third term.

Lemma 5.5 $(\Phi\langle D\rangle^s [A, F^\delta]u, \Phi\langle D\rangle^s F^\delta Hu) \overset{s}{\sim} i(H\Phi\langle D\rangle^s F_1^\delta u, \Phi\langle D\rangle^s F_1^\delta u)$.

Proof Since $h \in S^2$ this is $\overset{s}{\sim} (\Phi\langle D\rangle^s [A, F^\delta]u, \Phi H\langle D\rangle^s F^\delta u)$. From Lemma (6.7) and Corollaries 6.5, 6.3 we see that $|(\Phi\langle D\rangle^s [A, F^\delta]u, \mathrm{op}(\{\phi^{-n}, h\})\langle D\rangle^s F^\delta u)|$ is bounded by $C\|\Phi u\|_{s+1/2}\tilde{\mathcal{E}}_s(F^\delta u)$ hence this is still

$$\overset{s}{\sim} (H\Phi\langle D\rangle^s [A, F^\delta]u, \Phi\langle D\rangle^s F^\delta u) \overset{s}{\sim} (H\Phi\langle D\rangle^s [A, F^\delta]u, \Phi\mathrm{op}(m)\langle D\rangle^s F_1^\delta u).$$

If we move $\mathrm{op}(m)$ to in front of $[A, F^\delta]$, any term coming out in the process, is either $S(b^2\langle\xi\rangle^{2s-1/2}\phi^{-2n}, \bar{g})$ or $S(b_1^2\langle\xi\rangle^{2s-1/2}\phi^{-2n}, \bar{g})$ then thanks to Corollary 6.5 such a term is bounded by $\tilde{\mathcal{E}}_{s-1/4}^2(u)$. Finally, applying similar arguments to $(H\Phi\langle D\rangle^s \mathrm{op}(m)[A, F^\delta]u, \Phi$
$\langle D\rangle^s F_1^\delta u)$ we conclude the proof. □

Together with (5.19) we have

Lemma 5.6 *We have*

$$\mathrm{Im}(\Phi\langle D\rangle^s[A^2, F^\delta]u, \Phi\langle D\rangle^s AF^\delta u) \overset{s}{\sim} -\partial_t \mathrm{Re}(\Phi\langle D\rangle^s[A, F^\delta]u, \Phi\langle D\rangle^s AF^\delta u)$$
$$-\mathrm{Im}(\Phi\langle D\rangle^s[A, F^\delta]u, \Phi\langle D\rangle^s F^\delta \hat{P}_\theta u) + \|\Phi\langle D\rangle^s AF_1^\delta u\|^2$$
$$+\mathrm{Re}\,(H\Phi\langle D\rangle^s F_1^\delta u, \Phi\langle D\rangle^s F_1^\delta u).$$

Let κ be the constant in Definition 5.3. It follows from Lemma 5.3 that

$$\big|(\Phi\langle D\rangle^s[F^\delta, H]u, \Phi A\langle D\rangle^s F^\delta u)\big|$$
$$\overset{s}{\lesssim} \kappa\|\Phi\langle D\rangle^s AF_1^\delta u\|^2 + 4^{-1}\kappa^{-1}\|\mathrm{op}(\{h, f\}/\partial_t f)\Phi\langle D\rangle^s F_1^\delta u\|^2$$

where, noting $(\{h, f\}/\partial_t f)\#(\{h, f\}/\partial_t f) - (\{h, f\}/\partial_t f)^2 \in S^0$, the second term on the right-hand side is $\overset{s}{\lesssim} 4^{-1}\kappa^{-1}(\mathrm{op}((\{h, f\}/\partial_t f)^2)\Phi\langle D\rangle^s F_1^\delta u, \Phi\langle D\rangle^s F_1^\delta u)$. In view of (5.13) Corollary 6.2 proves

$$4^{-1}\kappa^{-1}\|\mathrm{op}(\{h, f\}/\partial_t f)\Phi\langle D\rangle^s F_1^\delta u\|^2 \overset{s}{\lesssim} \mathrm{Re}\,(H\Phi\langle D\rangle^s F_1^\delta u, \Phi\langle D\rangle^s F_1^\delta u).$$

Then using Lemma 5.6 we obtain

$$\begin{aligned}&\mathrm{Im}(\Phi\langle D\rangle^s[F^\delta, \hat{P}_\theta]u, \Phi\langle D\rangle^s AF^\delta u)\\ &\overset{s}{\gtrsim} -\partial_t \mathrm{Re}(\Phi\langle D\rangle^s[A, F^\delta]u, \Phi\langle D\rangle^s AF^\delta u)\\ &\quad - \mathrm{Im}(\Phi\langle D\rangle^s[A, F^\delta]u, \Phi\langle D\rangle^s F^\delta \hat{P}_\theta u) + (1-\kappa)\|\Phi AF_1^\delta u\|_s^2.\end{aligned} \tag{5.20}$$

From Proposition 4.6 and (5.20) there are $c > 0$, $C > 0$ such that

$$\begin{aligned}&2\mathrm{Im}(\Phi\langle D\rangle^s F^\delta \hat{P}_\theta u, \Phi A\langle D\rangle^s F^\delta u)\\ &\geq \partial_t \tilde{\mathcal{E}}_s^2(F^\delta u) + c\,\theta\tilde{\mathcal{E}}_s^2(F^\delta u) + c\,\tilde{\mathcal{E}}_{\sharp s}^2(F^\delta u)\\ &\quad -2\partial_t \mathrm{Re}(\Phi\langle D\rangle^s[A, F^\delta]u, \Phi\langle D\rangle^s AF^\delta u)\\ &\quad -2\mathrm{Im}(\Phi\langle D\rangle^s[A, F^\delta]u, \Phi\langle D\rangle^s F^\delta \hat{P}_\theta u) - C_\epsilon\big(\tilde{\mathcal{E}}_{\sharp(s-1/4)}^2(u) + \tilde{\mathcal{E}}_s^2(F^\delta u)\big)\\ &\quad +(2(1-\kappa)-\epsilon)\|\Phi AF_1^\delta u\|_s^2.\end{aligned}$$

Choosing $\epsilon > 0$ and θ such that $\epsilon \leq 2(1-\kappa)$, $c\,\theta \geq C_\epsilon$ we have

$$\begin{aligned}&C\|\Phi\langle D\rangle^s F^\delta \hat{P}_\theta u\|\big(\|\Phi A\langle D\rangle^s F^\delta u\| + \|\Phi\langle D\rangle^s u\|\big) + C\tilde{\mathcal{E}}_{\sharp(s-1/4)}^2(u)\\ &\quad \geq \partial_t \tilde{\mathcal{E}}_s^2(F^\delta u) + c\,\tilde{\mathcal{E}}_{\sharp s}^2(F^\delta u) - 2\partial_t \mathrm{Re}(\Phi\langle D\rangle^s[A, F^\delta]u, \Phi\langle D\rangle^s AF^\delta u).\end{aligned} \tag{5.21}$$

Assume now $\lim_{t\downarrow\tau}\sum_{j=0}^{1}\|D_t^j u(t)\|_{l+1-j} = 0$ with some l. Since $\mathcal{E}_s(u(t)) \leq C\sum_{j=0}^{1}\|D_t^j u(t)\|_{n+s+1-j}$ choosing ℓ so that $\ell \geq n+s-l$ we have

$$\lim_{t\downarrow\tau}\mathcal{E}_s(F^\delta u(t)) = 0, \quad \lim_{t\downarrow\tau}(\Phi\langle D\rangle^s[A, F^\delta]u(t), \Phi\langle D\rangle^s AF^\delta u(t)) = 0$$

for any $\delta > 0$. Since $|(\Phi\langle D\rangle^s[A, F^\delta]u, \Phi\langle D\rangle^s AF^\delta u)| \leq C\tilde{\mathcal{E}}_{s-1/4}^2(u)$ integrating (5.21) from τ to t ($|t| \leq \delta_0$) we have

$$\begin{aligned}
&C\int_\tau^t \|F^\delta \hat{P}_\theta u(t_1)\|_{n+s}\big(\|\Phi\langle D\rangle^s AF^\delta u(t_1)\| + \|\Phi\langle D\rangle^s u(t_1)\|\big)dt_1 \\
&\quad + C\int_\tau^t \tilde{\mathcal{E}}_{\sharp(s-1/4)}^2(u(t_1))dt_1 + C\tilde{\mathcal{E}}_{s-1/4}^2(u(t)) \\
&\geq \tilde{\mathcal{E}}_s^2(F^\delta u(t)) + c\int_\tau^t \tilde{\mathcal{E}}_{\sharp s}^2(F^\delta u(t_1))dt_1.
\end{aligned} \tag{5.22}$$

One can replace $\tilde{\mathcal{E}}_s(\cdot)$, $\tilde{\mathcal{E}}_{\sharp s}(\cdot)$ and $\hat{P}_\theta$ by $\mathcal{E}_s(\cdot)$, $\mathcal{E}_{\sharp s}(\cdot)$ and $\hat{P}$ in (5.22) if we replace u by $e^{-\theta t}u$. Denote

$$\mathcal{N}_s^2(u; t) = \sup_{\tau\leq t'\leq t}\left\{\mathcal{E}_s^2(u(t')) + \int_\tau^{t'} \mathcal{E}_{\sharp s}^2(u(t_1))dt_1\right\}. \tag{5.23}$$

Since $\|\Phi\langle D\rangle^s D_t F^\delta u\| + \|\Phi\langle D\rangle^s u\| \leq C(\mathcal{E}_s^{1/2}(F^\delta u) + \mathcal{E}_{s-1/4}^{1/2}(u))$ it follows that

$$\begin{aligned}
\mathcal{N}_s^2(F^\delta u; t) \leq C\Big\{&\mathcal{N}_{s-1/4}^2(u; t) + \big(\mathcal{N}_s(F^\delta u; t) + \mathcal{N}_{s-1/4}(u; t)\big) \\
&\times\int_\tau^t \|F^\delta \hat{P}u(t_1)\|_{n+s}dt_1\Big\}
\end{aligned}$$

from which we obtain

$$\mathcal{N}_s(F^\delta u; t) \leq C\left(\mathcal{N}_{s-1/4}(u; t) + \int_\tau^t \|F^\delta \hat{P}u(t_1)\|_{n+s}dt_1\right).$$

Letting $\delta \downarrow 0$ we have

Proposition 5.3 *Assume f is spacelike and $u \in \cap_{j=0}^{1}C^j([\tau, \delta_0]; H^{l+1-j})$ verifies $\lim_{t\downarrow\tau}\sum_{j=0}^{1}\|D_t^j u(t)\|_{l+1-j} = 0$. Then if $\mathcal{N}_{s-1/4}(u; t) < +\infty$, $\tau \leq t \leq \delta_0$ and $F\hat{P}u \in L^1([\tau, \delta_0]; H^{n+s})$ with $F = \mathrm{op}(\bar{f})$ one has $\mathcal{N}_s(Fu; t) < +\infty$ for $\tau_1 \leq t \leq \delta_0$ and*

$$\mathcal{N}_s(Fu; t) \leq C\left(\mathcal{N}_{s-1/4}(u; t) + \int_\tau^t \|F\hat{P}u(t_1)\|_{n+s}dt_1\right). \tag{5.24}$$

Let $\chi(s) \in C^\infty(\mathbb{R})$ be nondecreasing such that $\chi(s) = s$ for $|s| \le 1$ and $|\chi(s)| = 2$ for $|s| \ge 2$ and let $0 \le \tilde{\chi}(\xi) \in C^\infty(\mathbb{R}^d)$ be 0 in a neighborhood of the origin and $\tilde{\chi} = 1$ for $|\xi| \ge 1$. For $w = (y, \eta) \in \mathbb{R}^d \times (\mathbb{R}^d \backslash 0)$ we set

$$d_\epsilon(x, \xi; w) = \big\{\Sigma_{j=1}^d \chi^2(x_j - y_j) + \tilde{\chi}(\xi)\big|\xi/|\xi| - \eta/|\eta|\big|^2 + \epsilon^2\big\}^{1/2}.$$

Note that $d_\epsilon^2(x, \xi; w) \ge \min\{1, |x - y|^2\} + |\xi/|\xi| - \eta/|\eta||^2 + \epsilon^2$ for $|\xi| \ge 1$. We often write $d_\epsilon(x, \xi)$ for $d_\epsilon(x, \xi; w)$ dropping w. It is clear that $d_\epsilon \in S^0$ if $\epsilon \ne 0$. For $\nu > 0$ we define

$$f_\epsilon(t, x, \xi; w) = t - \tau - 2\nu\epsilon + \nu d_\epsilon(x, \xi; w). \tag{5.25}$$

It is easy to see that $|\partial_x^\alpha \partial_\xi^\beta d_\epsilon| \le C\langle\xi\rangle^{-|\beta|}$ ($|\alpha + \beta| = 1$) with $C > 0$ independent of $\epsilon > 0$. From $0 \le h \in G(M^{-2}\langle\xi\rangle^2, G)$ one has $\{h, \nu d_\epsilon\}^2 \le C\nu^2 q$ in virtue of the Glaeser inequality then there is $\nu_0 > 0$ such that f_ϵ is spacelike for any $0 < \nu \le \nu_0$ and $\epsilon > 0$. We fix such a $\nu > 0$ and denote $F_\epsilon = \mathrm{op}(\bar{f}_\epsilon)$.

Lemma 5.7 *Assume that* $u \in \cap_{j=0}^1 C^j([\tau, \delta_0]; H^{l+1-j})$, $\hat{P}u \in L^1([\tau, \delta_0]; H^{l'})$ *and* $\lim_{t\downarrow\tau} \sum_{j=0}^1 \|D_t^j u(t)\|_{l+1-j} = 0$ *with some* $l, l' \in \mathbb{R}$. *If* $F_{\epsilon_0}\hat{P}u \in L^1([\tau, \delta_0]; H^{s_0+n})$ *with some* $\epsilon_0 > 0$, $s_0 \in \mathbb{R}$ *then for any* $0 < \epsilon < \epsilon_0$ *and* $s \le s_0 - 1/4$ *one has* $F_\epsilon u \in \cap_{j=0}^1 C^j([\tau, \delta_0]; H^{s+1-j})$ *and*

$$\sum_{j=0}^1 \|D_t^j F_\epsilon u(t)\|_{s+1-j} \le C\int_\tau^t \|F_{\epsilon_0}\hat{P}u\|_{n+s_0} dt_1 + C\Big(R_l(u; t) + \int_\tau^t \|\hat{P}u\|_{l'} dt_1\Big)$$

where $R_l(u; t) = \sup_{\tau \le t' \le t} \sum_{j=0}^1 (\|D_t^j u(t')\|_{l+1-j} + \int_\tau^{t'} \|D_t^j u(t_1)\|_{l+1-j} dt_1)$.

Proof Choose a strictly decreasing sequence $\epsilon < \epsilon_j < \epsilon_0$ converging to ϵ as $j \to \infty$. Denoting $F_j = F_{\epsilon_j}$, $f_j = f_{\epsilon_j}$ we shall prove

$$\begin{aligned} \mathcal{N}_{l+j/4}(F_j u; t) \le C R_l(u; t) \\ + C\int_\tau^t \big\{\|F_0\hat{P}u(t_1)\|_{n+s_0} + \|\hat{P}u(t_1)\|_{l'}\big\} dt_1 \end{aligned} \tag{5.26}$$

for j with $l + j/4 \le s_0$ by induction on j.Take $g_j \in S^0$ such that $\mathrm{supp}\, g_j \subset \{f_j < 0\}$ and $\{f_{j+1} < 0\} \subset \{g_j = 1\}$. Write $F_{j+1}P\mathrm{op}(g_j)u = F_{j+1}\mathrm{op}(g_j)Pu + F_{j+1}[P, \mathrm{op}(g_j)]u$ then it is seen that $\|F_{j+1}\hat{P}\mathrm{op}(g_j)u\|_{l+n+(j+1)/4}$ is bounded by

$$C\big\{\|F_0\hat{P}u\|_{l+n+(j+1)/4} + \sum_{j=0}^1 \|D_t^j u\|_{l+1-j} + \|\hat{P}u\|_{l'}\big\}$$

hence an application of Proposition 5.3 with $s = l + (j+1)/4 \le s_0$, $F = F_{j+1}$, $u = \mathrm{op}(g_j)u$ gives

$$\mathcal{N}_{l+(j+1)/4}(F_{j+1}\mathrm{op}(g_j)u;t) \le C\mathcal{N}_{l+j/4}(\mathrm{op}(g_j)u;t) + C\int_\tau^t \big\{\|F_0\hat{P}u(t_1)\|_{n+s_0}dt_1 + \|\hat{P}u(t_1)\|_{l'}\big\}dt_1 + CR_l(u;t). \tag{5.27}$$

Repeating similar arguments one has $\mathcal{N}_{l+j/4}(\mathrm{op}(g_j)u;t) \le C\big\{\mathcal{N}_{l+j/4}(F_ju;t) + R_l(u;t)\big\}$ and $\mathcal{N}_{l+(j+1)/4}(F_{j+1}u;t) \le C\big\{\mathcal{N}_{l+(j+1)/4}(F_{j+1}\mathrm{op}(g_j)u;t) + R_l(u;t)\big\}$. Estimating $\mathcal{N}_{l+j/4}(\mathrm{op}(g_j)u;t)$ by use of the inductive hypothesis we conclude that (5.26) holds for maximal $j = j_0$ satisfying $l + j/4 \le s_0$. Since $\epsilon < \epsilon_{j_0}$ one can write $\bar{f}_\epsilon - k\#\bar{f}_{j_0} \in S^{-\infty}$ with some $k \in S^0$ the assertion follows. □

Proof of Proposition 5.1 Take $0 < \epsilon < 1/4$ such that $16\epsilon^2 \le |x-\tilde{x}|^2 + |\xi/|\xi| - \tilde{\xi}/|\tilde{\xi}||^2$ holds for any $(x,\xi) \in U_1$ and $(\tilde{x},\tilde{\xi}) \in U_2$.Fix a $0 < \nu_1 < \nu$ then there are finite many $w_i = (y_i, \eta_i) \in U_2$, $i = 1, \ldots, n$ such that

$$U_2 \subset \cup_{i=1}^n \{f_\epsilon(\tau + \nu_1\epsilon, x, \xi; w_i) = -(2\nu - \nu_1)\epsilon + \nu d_\epsilon(x,\xi;w_i) < 0\}.$$

Write $f_{i,\epsilon} = f_\epsilon(\tau, x, \xi; w_i)$, $F_{i,\epsilon} = \mathrm{op}(\bar{f}_{i,\epsilon})$ then it is clear that $\sum_i f_{i,\epsilon} < 0$ on $[\tau, \tau + \nu_1\epsilon] \times U_2$, while $\{f_{i,2\epsilon}(t,x,\xi) = t - \tau - 4\nu\epsilon + \nu d_{2\epsilon}(x,\xi;w_i) < 0\}$ does not intersect with $[\tau, \tau+\nu_1\epsilon] \times (U_1 \cap \{|\xi| \ge 1\})$. Therefore it follows that $\int_\tau^t \|F_{i,2\epsilon}\mathrm{op}(h_1)f\|_p dt_1 \le C\int_\tau^t \|f\|_{l_2}dt_1$ for any $p \in \mathbb{R}$.Here we apply Lemma 5.7 with $u = \hat{G}\mathrm{op}(h_1)f \in \cap_{j=0}^1 C^j([\tau, \delta_0]; H^{l_2-n+1-j})$, $F_{\epsilon_0} = F_{i,2\epsilon}$, $F_\epsilon = F_{i,\epsilon}$, $l = l_2 - n$, $l' = l_2$ to obtain

$$\sum_{j=0}^1 \|D_t^j F_{i,\epsilon}\hat{G}\mathrm{op}(h_1)f\|_{s+1-j} \le C\int_\tau^t \big\{\|F_{i,2\epsilon}\mathrm{op}(h_1)f\|_p + \|\mathrm{op}(h_1)f\|_{l_2}\big\}dt_1 + R_{l_2-n}(\hat{G}\mathrm{op}(h_1)f;t) \tag{5.28}$$

for any $p, s \in \mathbb{R}$, $s \le p - n - 1/4$. From (5.4) one has $R_{l_2-n}(\hat{G}\mathrm{op}(h_1)f;t) \le C\int_\tau^t \|\mathrm{op}(h_1)f\|_{l_2}dt_1$ hence $\sum_{j=0}^1 \|D_t^j F_{i,\epsilon}\hat{G}\mathrm{op}(h_1)f\|_{l_1-j} \le C\int_\tau^t \|f\|_{l_2}dt_1$ choosing $s = l_1 - 1$ and $l_1 \le p - n + 3/4$ in (5.28). Then Proposition 5.1 is proved if we remark

$$\sum_{j=0}^1 \|D_t^j \mathrm{op}(h_2)v\|_{l_1-j} \le C\sum_{j=0}^1\sum_i \|D_t^j F_{i,\epsilon}v\|_{l_1-j} + C\sum_{j=0}^1 \|D_t^j v\|_{l_2-n+1-j}$$

and take $v = \hat{G}\mathrm{op}(h_1)f$ there. □

For the solution operator of the Cauchy problem (5.3) with $\phi_0 = \phi_1 = 0$;

$$\hat{G}^* : L^1((-\delta_0, \tau) : H^{s+n}) \ni f(t) \mapsto u(t) \in \cap_{j=0}^1 C^j([-\delta_0, \tau]; H^{s+1-j})$$

the same argument for $\hat{G}$ proves that $\hat{G}^*$ has a finite speed of propagation. Then repeating the proof of the local existence theorem for P one obtains

Theorem 5.4 *If all critical points $(0, 0, \tau, \xi)$ of $p = 0$ are effectively hyperbolic then there are $\delta > 0$, $n > 0$ and a neighborhood Ω of $x = 0$ such that for any $|\tau| < \delta$ and $f \in L^1((-\delta, \tau); H^{s+n})$ there exists $u \in \cap_{j=0}^1 C^j([-\delta, \tau]; H^{s+1-j})$ satisfying $P^*u = f$ in $(-\delta, \tau) \times \Omega$ and*

$$\sum_{j=0}^{1} \|D_t^j u(t)\|_{s+1-j} \le C_s \int_t^\tau \|f(t')\|_{n+s} dt', \quad -\delta \le t \le \tau.$$

5.3 Local uniqueness theorem

Consider the second order differential operator

$$P = \mathrm{op}(-\tau^2 + \sum_{j+|\alpha|\le 2,\, j<2} a_{j,\alpha}(t, x)\xi^\alpha \tau^j) \tag{5.29}$$

with the principal symbol

$$p(t, x, \tau, \xi) = -\tau^2 + \sum_{j+|\alpha|=2,\, j<2} a_{j,\alpha}(t, x)\xi^\alpha \tau^j$$

where $a_{j,\alpha}(t, x)$ are C^∞ functions defined in a neighborhood of $(t, x) = (0, 0) \in \mathbb{R}^{1+d}$. For notational convenience we write x_0, ξ_0 instead of t, τ and denote $x = (x_0, x_1, \ldots, x_d) = (x_0, x')$, $\xi = (\xi_0, \xi_1, \ldots, \xi_d) = (\xi_0, \xi')$. Let $y = \kappa(x)$, $\kappa(0) = 0$ be a change of local coordinates x then, in y coordinates, the principal symbol $\tilde{p}(y, \eta)$ of P is $p(\kappa^{-1}(y), {}^t\kappa'(x)\eta)$.The following lemma is a special case of a well-known fact (e.g. [14]).

Lemma 5.8 *If $(0, \bar{\xi})$ is effectively hyperbolic characteristic of p then $(0, \bar{\eta})$, $\bar{\xi} = {}^t\kappa'(0)\bar{\eta}$ is effectively hyperbolic characteristic of $\tilde{p}$ and vice versa.*

Proof Denote $\kappa^{-1}(y) = \lambda(y)$ and $\kappa(x) = (\kappa_0(x), \kappa_1(x), \ldots, \kappa_d(x))$.If Q is the quadratic form associated with the Hessian of p then we have

$$\begin{aligned}\tilde{p}(\epsilon y, \bar{\eta} + \epsilon\eta) &= p(\epsilon\lambda'(0)y + O(\epsilon^2), \bar{\xi} + \epsilon(Cy + {}^t\kappa'(0)\eta) + O(\epsilon^2)) \\ &= \epsilon^2 Q(\lambda'(0)y, Cy + {}^t\kappa'(0)\eta) + O(\epsilon^3) \quad (\epsilon \to 0)\end{aligned}$$

where $C = (c_{ij})$ is the $(d+1) \times (d+1)$ matrix

$$c_{ij} = \sum_{0\le k,\ell\le d} \big(\partial^2\kappa_\ell(0)/\partial x_k \partial x_i\big)\big(\partial\lambda_k(0)/\partial y_j\big)\bar{\eta}_\ell.$$

Therefore denoting by $\tilde{Q}$ the corresponding quadratic form of $\tilde{p}$ at $(0,\bar{\eta})$ one has

$$\tilde{Q} = {}^tKQK, \quad K = \begin{pmatrix} \lambda'(0) & O \\ C & {}^t\kappa'(0) \end{pmatrix}.$$

Checking that $C\kappa'(0)$ is symmetric one concludes that $F_{\tilde{p}}(0,\bar{\eta}) = K^{-1}F_p(0,\bar{\xi})K$ hence the assertion. □

Next, consider a new system of local coordinates y such that

$$y_0 = x_0 + \epsilon\sum_{j=1}^{d} x_j^2, \quad y_j = x_j, \quad j = 1, 2, \dots, d \tag{5.30}$$

which is so called Holmgren transform (e.g. [15]) where $\epsilon > 0$ is a small positive constant that will be fixed later. It is clear that

$$\tilde{p}(y,\eta) = p(y_0 - \epsilon|y'|^2, y', \eta_0, \eta' + 2\epsilon\eta_0 y'). \tag{5.31}$$

The following lemma is also well-known (e.g. [21]).

Lemma 5.9 *If $p(x,\xi_0,\xi') = 0$ has only real root in ξ_0 for any x in a neighborhood of the origin of $\mathbb{R}^{1+d}$ and $\xi' \in \mathbb{R}^d$ then there exist $r > 0$ and $\epsilon_0 > 0$ such that for any $|\epsilon| \le \epsilon_0$, $\tilde{p}(y,\eta_0,\eta') = 0$ has only real root in η_0 for any $|y| \le r$ and $\eta' \in \mathbb{R}^d$.*

Lemma 5.10 *One can find a neighborhood Ω of the origin of $\mathbb{R}^{1+d}$ and $\bar{\epsilon} > 0$, $\epsilon > 0$ such that for any $f(x) \in C_0^\infty(\Omega)$ with $\mathrm{supp} f \subset \{x; x_0 \le \bar{\epsilon} - \epsilon|x'|^2\}$ there exists $v(x) \in C^2(\Omega)$ with $\mathrm{supp}\, v \subset \{x; x_0 \le \bar{\epsilon} - \epsilon|x'|^2\}$ satisfying $P^*v = f$ in Ω.*

Proof Since $P^* = \mathrm{op}(p + \bar{P}_1 + \bar{P}_0)$ then P^* in the local coordinates y is given by $P^* = \mathrm{op}(\tilde{p} + P_1' + P_0'')$.Thanks to Lemmas 5.8 and 5.9 one can apply Theorem 5.4 to conclude the assertion. □

Now prove the local uniqueness theorem. Assume that $u(x) \in C^2(\Omega)$ verifies $Pu = 0$ in $\Omega \cap \{x_0 > \tau\}$ and $D_0^j u(\tau, x') = 0$, $j = 0, 1$ on $\Omega \cap \{x_0 = \tau\}$ $(|\tau| \le \bar{\epsilon})$.For $f \in C_0^\infty(\Omega)$ with $\mathrm{supp} f \subset \{x; x_0 \le \bar{\epsilon} - \epsilon|x'|^2\}$ take $v(x)$ in Lemma 5.10 then one has

$$0 = \int_\tau^{\bar{\epsilon}}\int_{\mathbb{R}^d} Pu \cdot v dx_0 dx' = \int_\tau^{\bar{\epsilon}}\int_{\mathbb{R}^d} u \cdot P^* v dx_0 dx' = \int_\tau^{\bar{\epsilon}}\int_{\mathbb{R}^d} u \cdot f dx_0 dx'.$$

Since f is arbitrary we conclude $u = 0$ in $\{x; \tau < x_0 < \bar{\epsilon} - \epsilon|x'|^2\}$. Returning to the original notation $x_0 = t$, $(x_0, x') = (t, x)$ the assertion can be stated as

Theorem 5.5 *Assume that all critical points $(0,0,\tau,\xi)$ of $p = 0$ are effectively hyperbolic. Then there are a neighborhood ω of the origin and $\epsilon > 0$ such that if $u \in C^2(\omega)$ satisfies $(|\tau| \le \epsilon)$*

$$\begin{cases} Pu = 0, & \omega \cap \{t > \tau\}, \\ D_t^j u(\tau, x) = 0, & j = 0, 1, \quad x \in \omega \cap \{t = \tau\} \end{cases}$$

then $u = 0$ *in* $\omega \cap \{t > \tau\}$.

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Declarations

Conflict of interest The authors declare no competing interests.

6 Appendix

In this appendix, we summarize the properties of the pseudodifferential operators used in this paper and also give the proof of Proposition 5.2.

6.1 Pseudodifferential operators, composition, L^2 continuity and inverse

In this paper, all metrics are supposed to be of the form

$$g_z(w) = |y|^2/\phi(z)^2 + |\eta|^2/\psi(z)^2, \quad z = (x, \xi), \ w = (y, \eta) \in \mathbb{R}^d \times \mathbb{R}^d \tag{6.1}$$

(Beals-Fefferman metric [1]) where $\phi(z)$, $\psi(z)$ are positive functions on $\mathbb{R}^{2d}$ depending on positive parameters γ, M constrained by

$$\gamma \geq M^4 \geq 1.$$

For notational simplicity, we omit to write parameters in ϕ, ψ, and all constants assumed to be *independent of parameters* γ, M in what follows, although we do not mention this. Recall several notions related to Weyl-Hörmander calculus from Hörmander's book [4, Chapter XVIII]. For a positive function $m(z)$ we define $S(m, g)$ the set of all $a \in C^\infty(\mathbb{R}^{2d})$ such that for every $k \in \mathbb{N}$

$$\sup_{z\in\mathbb{R}^{2d},\alpha\in\mathbb{N}^{2d},|\alpha|\leq k} \left|\partial_z^\alpha a(z)\right|/m(z) \prod g_z^{1/2}(t_i) < \infty, \quad \partial_z^\alpha = \prod \partial_z^{t_i}, \quad |t_i| = 1. \tag{6.2}$$

The left hand is denoted by $|a|^{(k)}_{S(m,g)}$ which induces the topology in $S(m, g)$ as a Fréchet space. Denote

$$h^2(z) = \sup_{w\in\mathbb{R}^{2n}} g_z(w)/g_z^\sigma(w) \le 1, \quad z \in \mathbb{R}^{2d} \tag{6.3}$$

where $g_z^\sigma(w) = \sup_{0\neq v\in\mathbb{R}^{2n}} |\sigma(w, v)|^2/g_z(v)$. For a metric (6.1) it is easy to see

$$g_z^\sigma(w) = \psi(z)^2\phi(z)^2 g_z(w), \quad g_z/g_z^\sigma = \psi^{-2}(z)\phi^{-2}(z). \tag{6.4}$$

A metric (6.1) is σ temperate (see [4, Definition 18.5.1]) if there are positive constants c, C, N such that

$$g_z(w - z) < c \Longrightarrow 1/C \le \phi(z)/\phi(w),\ \psi(z)/\psi(w) \le C, \tag{6.5}$$

$$\phi(z)/\phi(w) + \psi(z)/\psi(w) \le C(1 + g_w^\sigma(z - w))^N, \quad z, w \in \mathbb{R}^{2d}. \tag{6.6}$$

Note that (6.6) implies

$$g_z^\sigma(w - z) \le C(1 + g_w^\sigma(w - z))^{N+1}, \quad z, w \in \mathbb{R}^{2d} \tag{6.7}$$

which is symmetric with respect to z, w. Let g be σ temperate metric. A positive function $m(z)$ is called σ, g temperate weight (see [4, Definition 18.5.1]) if there are positive constants c, C, N such that

$$g_z(w - z) < c \Longrightarrow m(z)/C \le m(w) \le Cm(z), \quad w, z \in \mathbb{R}^{2d}, \tag{6.8}$$

$$m(w) \le Cm(z)\big(1 + g_w^\sigma(w - z)\big)^N, \quad w, z \in \mathbb{R}^{2d}. \tag{6.9}$$

Note that (6.9) is equivalent to $m(w) \le C'm(z)(1 + g_z^\sigma(w - z))^{N'}$ because of (6.7). This paper uses more restricted weights than σ, g temperate weights.

Definition 6.1 Let g be σ temperate metric. A positive function m is called g admissible weight if there are positive constants C, N such that

$$m(w) \le Cm(z)\big(1 + \max\{g_w(w - z), g_z(w - z)\}\big)^N, \quad w, z \in \mathbb{R}^{2d}. \tag{6.10}$$

It is clear from the definition that if m is g admissible weight then m is also $\tilde{g}$ admissible weight for any σ temperate metric $\tilde{g} \ge g$.

Lemma 6.1 *Let g be σ temperate and satisfy* (6.3). *If m is g admissible weight then m is σ, g temperate weight.*

Proof If $g_z(w - z) < c$ one has $m(w) \le C'(1 + c\,C)m(z)$ in view of (6.5) and (6.10). Since $\max\{g_w(w - z), g_z(w - z)\}$ is symmetric for w, z one concludes (6.8). Noting $\max\{g_w(w - z), g_z(w - z)\} \le \max\{g_w^\sigma(w - z), g_z^\sigma(w - z)\}$ by (6.3) we have (6.9) from (6.7). □

Lemma 6.2 *If m is g admissible weight so is m^s for any $s \in \mathbb{R}$. If m_i $(i = 1, 2)$ are g admissible weights so is $m_1 m_2$.*

Proof Since $1/m$ is g admissible weight by (6.10) then the first assertion is clear. The second assertion is also clear by (6.10). □

In this paper we work with more restricted metrics (6.1) which satisfies with some $0 < \delta < 1$ and $c > 0$ that

$$\langle\xi\rangle_\gamma^{-\delta} \lesssim \phi \lesssim 1, \quad \psi \lesssim \langle\xi\rangle_\gamma, \quad \phi\psi \geq 1, \tag{6.11}$$

$$|\xi - \eta|/\langle\xi\rangle_\gamma < c \Longrightarrow \phi(z) \approx \phi(w), \quad \psi(z) \approx \psi(w). \tag{6.12}$$

Lemma 6.3 *A metric (6.1) satisfying (6.11), (6.12) is σ temperate and satisfies (6.3).*

Proof If $g_z(z - w) < c_1^2$ then $|\xi - \eta| < c_1\psi(z) \leq c_1 C\langle\xi\rangle_\gamma$ so (6.5) is immediate by (6.12) choosing $c_1 C \leq c$. If $\langle\eta\rangle_\gamma \leq \langle\xi\rangle_\gamma/2\sqrt{2}$ then $|\xi - \eta| \geq (\gamma + |\xi|) - (\gamma + |\eta|) \geq \langle\xi\rangle_\gamma - \sqrt{2}\langle\eta\rangle_\gamma \geq \langle\xi\rangle_\gamma/2$ which gives $|\xi - \eta| \geq c\langle\xi\rangle_\gamma^{1-\delta}\langle\eta\rangle_\gamma^\delta$ with some $c > 0$. On the other hand if $\langle\eta\rangle_\gamma \geq 2\sqrt{2}\langle\xi\rangle_\gamma$ then $|\xi - \eta| \geq \langle\eta\rangle_\gamma/2$ hence $|\xi - \eta| \geq c\langle\eta\rangle_\gamma^{1-\delta}\langle\eta\rangle_\gamma^\delta$. Therefore there is C such that

$$\langle\xi\rangle_\gamma/\langle\eta\rangle_\gamma + \langle\eta\rangle_\gamma/\langle\xi\rangle_\gamma \leq C\big(1 + \langle\eta\rangle_\gamma^{-\delta}|\xi - \eta|\big)^{1/(1-\delta)}, \quad \xi, \eta \in \mathbb{R}^d. \tag{6.13}$$

Note that $g_w^\sigma(z - w) \geq \phi^2(w)|\xi - \eta|^2 \geq \langle\eta\rangle_\gamma^{-2\delta}|\xi - \eta|^2/C$ which proves (6.6) while (6.3) is obvious by (6.4) and (6.11). □

Lemma 6.4 *All metrics $\underline{g}$, g_ϵ, $\bar{g}$ in (4.1) satisfy (6.11), (6.12) and $\langle\xi\rangle_\gamma^s$, $s \in \mathbb{R}$ is $\underline{g}$, g_ϵ, $\bar{g}$ admissible weight.*

Proof Indeed (6.11) is verified with $\delta = 1/2$. If $|\xi - \eta| < c\langle\xi\rangle_\gamma$. Here we remark that

$$|\xi - \eta| < c\langle\xi\rangle_\gamma \Longrightarrow (1 - c)\langle\xi\rangle_\gamma/\sqrt{2} \leq \langle\eta\rangle_\gamma \leq \sqrt{2}(1 + c)\langle\xi\rangle_\gamma. \tag{6.14}$$

which proves (6.12). Next since $|\partial_\xi^\alpha\langle\xi\rangle_\gamma| \leq C$ for $|\alpha| = 1$ we see that

$$|\langle\xi + \eta\rangle_\gamma - \langle\xi\rangle_\gamma| \leq C|\eta| \leq C\langle\xi\rangle_\gamma(\langle\xi\rangle_\gamma^{-1}|\eta|) \leq C\langle\xi\rangle_\gamma\, \underline{g}_z^{1/2}(w) \tag{6.15}$$

hence $\langle\xi + \eta\rangle_\gamma \leq C\langle\xi\rangle_\gamma(1 + \underline{g}_z(w))^{1/2}$ which shows $\langle\xi\rangle_\gamma$ is $\underline{g}$ admissible weight hence the assertion because $\underline{g} \leq g_\epsilon \leq \bar{g}$. □

We state the main theorem of the Weyl-Hörmander calculus [4, Theorem 18.5.4] for the present case.

Theorem 6.1 *Let g satisfy (6.11), (6.12) and m_i be g admissible weights and $a_i \in S(m_i, g)$. Then the oscillatory integral*

$$\pi^{-2d}\int e^{-2i\sigma(v,w)}a_1(z + v)a_2(z + w)dvdw \tag{6.16}$$

defines $c(z) \in S(m_1 m_2, g)$*. Denoting* $c(z)$ *by* $a_1 \# a_2$ *one has*

$$\mathrm{op}(a_1)\mathrm{op}(a_2)u = \mathrm{op}(a_1 \# a_2)u, \quad \forall u \in \mathcal{S}$$

and for every $l \in \mathbb{N}$ *there are* C, l' *such that*

$$\big|a_1 \# a_2\big|^{(l)}_{S(m_1 m_2, g)} \le C |a_1|^{(l')}_{S(m_1, g)} |a_2|^{(l')}_{S(m_2, g)}. \tag{6.17}$$

Moreover if $\partial_x^\alpha \partial_\xi^\beta a_i \in S(m^\beta_{i,\alpha}, g)$ *for* g *admissible weights* $m^\beta_{i,\alpha}$ *for* $|\alpha + \beta| = l$ *we have*

$$a_1 \# a_2 - \sum_{|\alpha+\beta|<l} \frac{(-1)^{|\alpha|}}{(2i)^{|\alpha+\beta|}\alpha!\beta!} \partial_\xi^\beta \partial_x^\alpha a_1 \, \partial_\xi^\alpha \partial_x^\beta a_2 \in \sum_{|\alpha+\beta|=l} S\big(m^\beta_{1,\alpha} m^\alpha_{2,\beta}, g\big).$$

In particular with $l = 3$ *one has*

$$a_1 \# a_2 - a_2 \# a_1 + i\{a_1, a_2\} \in \sum_{|\alpha+\beta|=3} S\big(m^\beta_{1,\alpha} m^\alpha_{2,\beta}, g\big).$$

The theorem can be proved in a naive way (repeated use of integration by parts) taking into account the special features of such restricted metrics satisfying (6.11), (6.12) and weights given by Definition 6.1, or keeping that the "structural constants" of the metrics and weights are independent of parameters γ, M in mind, it suffices to follow the general proof in [4, Theorem 18.5.4] or [13, Theorem 2.3.7].

Corollary 6.1 *Set* $h^2(z) = \sup g_z / g_z^\sigma$ *then for any* $N \in \mathbb{N}$ *we have*

$$a_1 \# a_2 - \sum_{|\alpha+\beta|<N} \frac{(-1)^{|\alpha|}}{(2i)^{|\alpha+\beta|}\alpha!\beta!} \partial_\xi^\beta \partial_x^\alpha a_1 \, \partial_\xi^\alpha \partial_x^\beta a_2 \in S(h^N m_1 m_2, g). \tag{6.18}$$

In particular we have

$$\begin{cases} a_1 \# a_2 - a_2 \# a_1 - \{a_1, a_2\}/i \in S(h^3 m_1 m_2, g), \\ a_1 \# a_2 + a_2 \# a_1 - 2\, a_1 a_2 \in S(h^2 m_1 m_2, g), \\ a_1 \# a_2 \# a_1 - a_2 a_1^2 \in S(h^2 m_2 m_1^2, g). \end{cases} \tag{6.19}$$

Noting again that "structural constants" of the metric $\bar{g}$ is independent of γ it follows from the proof of [4, Theorem 18.6.3] or [13, Theorem 2.5.1] that

Theorem 6.2 *The operator* $\mathrm{op}(a)$ *is* L^2 *bounded for every* $a \in S(1, \bar{g})$*. Namely there exist* $C > 0$, $\ell \in \mathbb{N}$ *depending only on the dimension* d *such that*

$$\|\mathrm{op}(a)u\| \le C |a|^{(\ell)}_{S(1,\bar{g})} \|u\|, \quad u \in \mathcal{S}.$$

Similarly, following the proofs of [13, Lemma 2.6.26] and [13, Theorem 2.6.27] or [12, Theorem I.1] we have

Theorem 6.3 *There exist $C > 0$, $l^0 \in \mathbb{N}$ such that if $a(x,\xi) \in S(1,\bar{g})$ satisfies $|a|^{(l^0)}_{S(1,\bar{g})} \le C^{-1}$ then*

$$b(x,\xi) = \sum_{j=0}^{\infty} \overbrace{a\#\cdots\#a}^{j} = \sum_{j=0}^{\infty} a^{\#j}$$

converges in $S(1,\bar{g})$ and satisfies $(1-a)\#b = b\#(1-a) = 1$. Moreover for any l there are C_l, l' such that

$$|b|^{(l)}_{S(1,\bar{g})} \le C_l |a|^{(l')}_{S(1,\bar{g})}. \tag{6.20}$$

6.2 Admissible weight related to nonnegative symbols

Let $0 \le a(x,\xi) \in S(M^{-2}\langle\xi\rangle_\gamma^2, G)$ where G is given in (3.5) and M, γ are constrained by (3.1). Since $M^{|\alpha+\beta|}\langle\xi\rangle_\gamma^{-|\beta|} \le (M^2\langle\xi\rangle_\gamma^{-1})^{|\alpha+\beta|/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}$ it is clear that $S(m,G) \subset S(m,\bar{g})$. Introducing a parameter $\lambda \ge 1$ which is constrained by

$$\lambda M^2 \le \gamma, \quad \lambda \ge 1 \tag{6.21}$$

we consider an approximate square root of a;

$$b(x,\xi) = (a(x,\xi) + \lambda\langle\xi\rangle_\gamma)^{1/2}. \tag{6.22}$$

Lemma 6.5 *We have $\partial_x^\alpha \partial_\xi^\beta b^{\pm 1} \in S(\lambda^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2} b^{\pm 1}, \bar{g})$ for $|\alpha+\beta| \ge 1$. In particular $b^{\pm 1} \in S(b^{\pm 1}, \bar{g})$.*

Proof Set $\bar{a} = a(x,\xi)\langle\xi\rangle_\gamma^{-2}$ and $\bar{b} = (\bar{a} + \lambda\langle\xi\rangle_\gamma^{-1})^{1/2}$ so that $b = \bar{b}\langle\xi\rangle_\gamma$. In the proof we often use $(\bar{b}\lambda^{-1/2})^k \ge \langle\xi\rangle_\gamma^{-k/2}$ $(k \ge 0)$ which follows from $CM^{-1} \ge \bar{b} \ge \lambda^{1/2}\langle\xi\rangle_\gamma^{-1/2}$. Since $\bar{a} \in S(M^{-2}, G)$ the Glaeser inequality shows

$$|\partial_x^\alpha \partial_\xi^\beta \bar{a}| \lesssim \langle\xi\rangle_\gamma^{-|\beta|}\sqrt{\bar{a}} \lesssim \langle\xi\rangle_\gamma^{-1/2}\sqrt{\bar{a}}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}, \quad |\alpha+\beta| = 1 \tag{6.23}$$

while it is clear that

$$|\partial_x^\alpha \partial_\xi^\beta \lambda\langle\xi\rangle_\gamma^{-1}| \lesssim \lambda\langle\xi\rangle_\gamma^{-3/2}\langle\xi\rangle_\gamma^{-(|\alpha+\beta|-1)/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}. \tag{6.24}$$

Noting $\sqrt{\bar{a}} \le \bar{b}$ it follows from (6.23) and (6.24) that

$$|\partial_x^\alpha \partial_\xi^\beta \bar{b}| \lesssim |\partial_x^\alpha \partial_\xi^\beta (\bar{a} + \lambda\langle\xi\rangle_\gamma^{-1})/\bar{b}| \lesssim \lambda^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}\,\bar{b}, \quad |\alpha+\beta| = 1. \tag{6.25}$$

Assume (6.25) holds for $1 \le |\alpha+\beta| \le l$. Since $\bar{b}^2 = \bar{a} + \lambda\langle\xi\rangle_\gamma^{-1}$ then for $|\alpha+\beta| \ge l+1 \ge 2$ we see

$$\bar{b}\partial_x^\alpha \partial_\xi^\beta \bar{b} = \sum_{0<|\alpha'+\beta'|<l} C_{\dots}\partial_x^{\alpha'}\partial_\xi^{\beta'}\bar{b}\cdot \partial_x^{\alpha''}\partial_\xi^{\beta''}\bar{b} + \partial_x^\alpha\partial_\xi^\beta \bar{a} + \partial_x^\alpha\partial_\xi^\beta \lambda\langle\xi\rangle_\gamma^{-1} \tag{6.26}$$

where the second term on the right-hand side of (6.26) is estimated as

$$\begin{aligned} |\partial_x^\alpha \partial_\xi^\beta \bar{a}| &\lesssim M^{-2+|\alpha+\beta|}\langle\xi\rangle_\gamma^{-|\beta|} \\ &\lesssim (M^2\langle\xi\rangle_\gamma^{-1})^{(|\alpha+\beta|-2)/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2} \lesssim \bar{b}^2\lambda^{-1}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}. \end{aligned} \tag{6.27}$$

To estimate the third term it suffices to apply (6.24). Therefore we conclude from (6.26) that (6.25) holds for $|\alpha+\beta| = l+1$ and hence for any α, β. The assertion for b follows immediately from (6.25). The estimate for b^{-1} can be obtained from those of b by differentiating $bb^{-1} = 1$. □

Lemma 6.6 *b is $\bar{g}$ admissible weight and $b^{\pm 1} \in S(b^{\pm 1}, \bar{g})$.*

Proof Since $\langle\xi\rangle_\gamma$ is $\bar{g}$ admissible weight it is enough to prove that $\bar{b}$ is $\bar{g}$ admissible weight. Note that $|\partial_x^\alpha\partial_\xi^\beta \bar{b}| \lesssim \langle\xi\rangle_\gamma^{-|\beta|}$ for $|\alpha+\beta| = 1$ in view of (6.23) and (6.24). If $|\eta| < c\langle\xi\rangle_\gamma$ then from (6.14) there is $C > 0$ such that

$$\langle\xi + s\eta\rangle_\gamma / C \le \langle\xi\rangle_\gamma \le C\langle\xi + s\eta\rangle_\gamma, \quad |s| \le 1 \tag{6.28}$$

hence one has

$$|\bar{b}(z+w) - \bar{b}(z)| \le C(|y| + \langle\xi\rangle_\gamma^{-1}|\eta|) \le C\langle\xi\rangle_\gamma^{-1/2}\bar{g}_z^{1/2}(w) \le C\bar{b}(z)\bar{g}_z^{1/2}(w)$$

which proves

$$\bar{b}(z+w) \le C\bar{b}(z)(1 + \bar{g}_z(w))^{1/2} \tag{6.29}$$

when $|\eta| \le c\langle\xi\rangle_\gamma$. If $|\eta| \ge c\langle\xi\rangle_\gamma$ then $\bar{g}_z(w) \ge c^2\langle\xi\rangle_\gamma$ one has

$$\bar{b}(z+w) \le C \le C\bar{b}(z)\langle\xi\rangle_\gamma^{1/2} \le C'\bar{b}(z)(1 + \bar{g}_z(w))^{1/2}$$

so that (6.29) holds. Thus $\bar{b}$ is $\bar{g}$ admissible weight. □

Proposition 6.1 *One can find $\lambda_1 \ge 1$ independent of M and γ such that for $\lambda \ge \lambda_1$ there exists $\tilde{b} \in S(b^{-1}, \bar{g})$ satisfying $b\#\tilde{b} = \tilde{b}\#b = 1$.*

Proof Since $b^{\pm 1} \in S(b^{\pm 1}, \bar{g})$ and $\partial_x^\alpha\partial_\xi^\beta b^{\pm 1} \in S(\lambda^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2} b^{\pm 1}, \bar{g})$ for $|\alpha + \beta| = 1$ by Lemma 6.5 and $b^{\pm 1}$ is $\bar{g}$ admissible weight then thanks to Theorem 6.1 one has $b\#b^{-1} = 1 - r$ with $r \in S(\lambda^{-1/2}, \bar{g})$. Therefore there is λ_1 such that for $\lambda \geq \lambda_1$ one can apply Theorem 6.3 to obtain

$$\tilde{r}(x,\xi) = \sum_{j=0}^{\infty} r^{\# j} \in S(1, \bar{g})$$

and that $b\#(b^{-1}\#\tilde{r}) = 1$. Similarly there exists $\hat{r} \in S(1, \bar{g})$ such that $\hat{r}\#b^{-1}\#b = 1$ which proves $(b^{-1}\#\tilde{r})\#b = 1$. Thus $\tilde{b} = b^{-1}\#\tilde{r} \in S(b^{-1}, \bar{g})$ is a desired one. □

Lemma 6.7 *We have $a \in S(b^2, \bar{g})$ and $\partial_x^\alpha \partial_\xi^\beta a \in S(b\langle\xi\rangle_\gamma^{1-|\beta|}, \bar{g})$ for $|\alpha + \beta| = 1$.*

Proof Since $\langle\xi\rangle_\gamma \in S(b^2, \bar{g})$ is clear for $\langle\xi\rangle_\gamma \leq b^2$ the first assertion is obvious. Note that if $\tilde{a} \in S(M^{-1-k}\langle\xi\rangle_\gamma^\ell, G)$ noting $b \geq \langle\xi\rangle_\gamma^{1/2}$ one has

$$\begin{aligned} |\partial_x^\alpha \partial_\xi^\beta \tilde{a}| &\lesssim M^{-1-k+|\alpha+\beta|}\langle\xi\rangle_\gamma^{\ell-|\beta|} \lesssim M^{-k}\big(M^2\langle\xi\rangle_\gamma^{-1}\big)^{(|\alpha+\beta|-1)/2} \\ &\times \langle\xi\rangle_\gamma^{\ell-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2} \lesssim M^{-k} b\langle\xi\rangle_\gamma^{\ell-1}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}, \quad |\alpha+\beta| \geq 1. \end{aligned} \tag{6.30}$$

For $|\alpha + \beta| = 1$ we have $|\partial_x^\alpha \partial_\xi^\beta a| \leq Cb\langle\xi\rangle_\gamma^{1-|\beta|}$ from (6.23). For $|\alpha + \beta| \geq 2$ it is enough to apply (6.30) to $\tilde{a} = \partial_x^{\alpha'} \partial_\xi^{\beta'} a \in S(M^{-1}\langle\xi\rangle_\gamma^{2-|\beta'|}, G)$, $|\alpha' + \beta'| = 1$. □

Lemma 6.8 *There exists $\lambda_2 \geq \lambda_1$ independent of M and γ such that for $\lambda \geq \lambda_2$ one has*

$$(\mathrm{op}(a + \lambda\langle\xi\rangle_\gamma)v, v) = (\mathrm{op}(b^2)v, v) \geq \|\mathrm{op}(b)v\|^2/2, \quad v \in \mathcal{S}.$$

Proof Noting $\partial_x^\alpha \partial_\xi^\beta b \in S(\lambda^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2} b, \bar{g})$ for $|\alpha + \beta| = 2$ in virtue of Lemma 6.5 it follows from Theorem 6.1 that $b\#b = b^2 + r$, $r \in S(\lambda^{-1}b^2, \bar{g})$. Taking $\tilde{b} \in S(b^{-1}, \bar{g})$ in Proposition 6.1 we set $r = b\#(\tilde{b}\#r\#\tilde{b})\#b$. Since $\tilde{b}\#r\#\tilde{b} \in S(\lambda^{-1}, \bar{g})$, thanks to Theorem 6.2, there exists $C > 0$ independent of λ such that $\|\mathrm{op}(\tilde{b}\#r\#\tilde{b})v\| \leq C\lambda^{-1}\|v\|^2$, hence $|(\mathrm{op}(r)v, v)| \leq C\lambda^{-1}\|\mathrm{op}(b)v\|^2$. Then we see

$$(\mathrm{op}(b^2)v, v) = \|\mathrm{op}(b)v\|^2 - (\mathrm{op}(r)v, v) \geq (1 - C\lambda^{-1})\|\mathrm{op}(b)v\|^2.$$

It is enough to choose $\lambda_2 \geq \lambda_1$ so that $1 - C\lambda_2 \leq 1/2$. □

Let $\langle\xi\rangle$ stand for $\langle\xi\rangle_\gamma$ with $\gamma = 1$. The following inequality is called sharp Gårding inequality ([2]).

Corollary 6.2 *If $0 \leq a(x,\xi) \in S^2_{1,0}$ there is $C > 0$ such that*

$$\mathrm{Re}\,(\mathrm{op}(a)u, u) \geq -C\|\langle D\rangle^{1/2}u\|^2, \quad u \in \mathcal{S}.$$

Proof If we fix $M = 1$ and $\gamma \geq \lambda_2$ then $0 \leq a \in S(\langle \xi \rangle_\gamma^2, G)$ hence one can apply Lemma 6.8 with $\lambda = \lambda_2$ to get

$$(\mathrm{op}(a)v, v) \geq \|\mathrm{op}((a + \lambda_2 \langle \xi \rangle_\gamma)^{1/2})v\|^2/2 - \lambda_2 \|\langle D \rangle_\gamma^{1/2} v\|^2 \geq -C\|\langle D \rangle^{1/2} v\|^2$$

which is the assertion. □

6.3 Pseudodifferential operators associated with metrics related to localization

In this subsection we study pseudodifferential operators associated with metrics g satisfying (6.11), (6.12) and

$$g/g^\sigma \lesssim M^{-2}, \quad M^{-2}\bar{g} \lesssim g \lesssim \bar{g}. \tag{6.31}$$

Lemma 6.9 *Let m be g admissible weight such that $m \in S(m, g)$. Then there exist $M_0 > 0$ and $k \in S(M^{-1}, g)$ $(M > M_0)$ such that*

$$m\#m^{-1}\#(1 + k) = 1, \quad (1 + k)\#m\#m^{-1} = 1, \quad m^{-1}\#(1 + k)\#m = 1.$$

Proof Since m^{-1} is g admissible weight and $m^{-1} \in S(m^{-1}, g)$ one has $m\#m^{-1} = 1 - r$ with $r \in S(M^{-1}, g) \subset S(M^{-1}, \bar{g})$. Thanks to Theorem 6.3 there is M_0 such that $\sum_{l=1}^\infty r^{\#l}$ converges in $S(1, \bar{g})$ to some $k \in S(1, \bar{g})$ for $M > M_0$ so that $(1 - r)\#(1 + k) = (1 + k)\#(1 - r) = 1$ which shows the first two equalities. It remains to prove $k \in S(M^{-1}, g)$. It suffices to show

$$\partial_z^\alpha k \in S(M^{-1} \prod g_z^{1/2}(t_i), \bar{g}), \quad \partial_z^\alpha = \prod \partial_z^{t_i}, \quad |t_i| = 1, \quad \alpha \in \mathbb{N}^{2d}. \tag{6.32}$$

From (6.20) one sees that $k \in S(M^{-1}, \bar{g})$ so that (6.32) holds when $|\alpha| = 0$. Suppose that (6.32) holds for $|\alpha| \leq l$ and consider the case $|\alpha| = l + 1$. Since k verifies $k = r + r\#k$ one has

$$\partial_z^\alpha k = \partial_z^\alpha r + \sum_{\alpha' + \alpha'' = \alpha} C_{\alpha', \alpha''} (\partial_z^{\alpha'} r)\#(\partial_z^{\alpha''} k).$$

When $|\alpha''| = |\alpha| = l + 1$ we have $\partial_z^\alpha k \in S(M^{-1} \bar{g}_z^{1/2}(s) \prod g_z^{1/2}(t_j), \bar{g})$ where $\partial_z^\alpha = \partial_z^s \prod \partial_z^{t_i}$, $|s| = 1$ and $M^{-1}\bar{g}_z^{1/2}(s) \lesssim g_z^{1/2}(s)$ by assumption. Thus we have $r\#(\partial_z^\alpha k) \in S(M^{-1} \prod g_z^{1/2}(t_i), \bar{g})$ with $\partial_z^\alpha = \prod \partial_z^{t_i}$, $|t_i| = 1$. When $|\alpha''| \leq l$ the assumption (6.32) shows that $(\partial_z^{\alpha'} r)\#(\partial_z^{\alpha''} k) \in S(M^{-2} \prod g_z^{1/2}(t_i), \bar{g})$ with $\partial_z^\alpha = \prod \partial_z^{t_i}$. Therefore (6.32) holds for $|\alpha| = l + 1$ and hence $k \in S(M^{-1}, g)$ by induction on $|\alpha|$. Similarly there is $\tilde{k} \in S(M^{-1}, g)$ such that $(1 + \tilde{k})\#m^{-1}\#m = 1$ $(M > M_0)$ which proves the third equality. □

Lemma 6.10 *Let m_i $(i = 1, 2)$ be g admissible weights such that $m_i \in S(m_i, g)$. If $\bar{m}$ is $\bar{g}$ admissible weight and $a \in S(\bar{m}m_1m_2, \bar{g})$ then there exist $M_0 > 0$ and $\tilde{a} \in S(\bar{m}, \bar{g})$ $(M > M_0)$ such that one has $a = m_1\#\tilde{a}\#m_2$. Moreover if $\bar{m}$ is g admissible weight then $\tilde{a} \in S(\bar{m}, g)$ is given by $(m_1m_2)^{-1}a + r$ with $r \in S(M^{-1}\bar{m}, g)$.*

Proof Since m_i^{-1} are g admissible weights such that $m_i^{-1} \in S(m_i^{-1}, g)$, Lemma 6.9 gives $k, \tilde{k} \in S(M^{-1}, g)$ $(M > M_0)$ verifying $m_1\#(1 + k)\#m_1^{-1} = 1$ and $m_2^{-1}\#(1 + \tilde{k})\#m_2 = 1$. Then $\tilde{a} = (1+k)\#m_1^{-1}\#a\#m_2^{-1}\#(1+\tilde{k}) \in S(\bar{m}, \bar{g})$ is a desired one. If $\bar{m}$ is g admissible weight it follows from Corollary 6.1 that $\tilde{a} - (m_1m_2)^{-1}a \in S(M^{-1}\bar{m}, g)$. □

Lemma 6.11 *Let m_i $(i = 1, 2)$ be g admissible weights such that $m_i \in S(m_i, g)$. If $\bar{m}$ is $\bar{g}$ admissible weight and $a \in S(\bar{m}m_1m_2, \bar{g})$ or $a \in S(\bar{m}m_1, \bar{g})$ there are M_0 and $\tilde{a} \in S(\bar{m}, \bar{g})$ $(M > M_0)$ such that the followings hold for $M > M_0$*

$$\begin{aligned}\big|(\mathrm{op}(a)u, v)\big| &\le \|\mathrm{op}(\tilde{a})\mathrm{op}(m_1)u\|\,\|\mathrm{op}(m_2)v\|,\\ \|\mathrm{op}(a)u\| &\le \|\mathrm{op}(\tilde{a})\mathrm{op}(m_1)u\|.\end{aligned}$$

Moreover if $\bar{m}$ is g admissible weight such that $\bar{m} \in S(\bar{m}, g)$ then with $\tilde{a} = (m_1m_2)^{-1}a \in S(\bar{m}, g)$ or $\tilde{a} = m_1^{-1}a \in S(\bar{m}, g)$ the following estimates hold

$$\begin{aligned}|(\mathrm{op}(a)u, v)| \le\; &\|\mathrm{op}(\tilde{a})\mathrm{op}(m_1)u\|\,\|\mathrm{op}(m_2)v\|\\ &+CM^{-1}\|\mathrm{op}(\bar{m})\mathrm{op}(m_1)u\|\,\|\mathrm{op}(m_2)v\|,\\ \|\mathrm{op}(a)u\| \le\; &\|\mathrm{op}(\tilde{a})\mathrm{op}(m_1)u\| + CM^{-1}\|\mathrm{op}(\bar{m})\mathrm{op}(m_1)u\|\end{aligned}$$

for $M > M_0$.

Proof The first two assertions are direct consequences of Lemma 6.10. If $\bar{m}$ is g admissible weight with $\bar{m} \in S(\bar{m}, g)$ one can write $a = m_2\#(\tilde{a} + r)\#m_1$ with $r \in S(M^{-1}\bar{m}, g)$ by Lemma 6.10 from which it follows $\big|(\mathrm{op}(a)u, v)\big| \le \|\mathrm{op}(\tilde{a} + r)\mathrm{op}(m_1)u\|\,\|\mathrm{op}(m_2)v\|$. Writing $r = \tilde{r}\#\bar{m}$, $\tilde{r} \in S(M^{-1}, g)$ with use of Lemma 6.10 one has $\|\mathrm{op}(\tilde{a} + r)v\| \le \|\mathrm{op}(\tilde{a})v\| + CM^{-1}\|\mathrm{op}(\bar{m})v\|$ thanks to Theorem 6.2. Taking $m_2 = 1$ in this proof one obtains the last assertion. □

Corollary 6.3 *If $a \in S(\langle\xi\rangle_\gamma^s m_1m_2, \bar{g})$ and $s_1 + s_2 = s$ then*

$$|(\mathrm{op}(a)u, v)| \le C\|\langle D\rangle_\gamma^{s_1}\mathrm{op}(m_1)u\|\,\|\langle D\rangle_\gamma^{s_2}\mathrm{op}(m_2)v\|.$$

Proof Write $a = \langle\xi\rangle_\gamma^{s_2}\#\tilde{a}\#\langle\xi\rangle_\gamma^{s_1}$ with $\tilde{a} = \langle\xi\rangle_\gamma^{-s_2}\#a\#\langle\xi\rangle_\gamma^{-s_1} \in S(m_1m_2, \bar{g})$ and apply Lemma 6.11 to $\tilde{a}$ to get $|(\mathrm{op}(a)u, v)| \le C\|\mathrm{op}(m_1)\langle D\rangle_\gamma^{s_1}u\|\,\|\mathrm{op}(m_2)\langle D\rangle_\gamma^{s_2}v\|$. The right hand-side is bounded by $C\|\langle D\rangle_\gamma^{s_1}\mathrm{op}(m_1)u\|\,\|\langle D\rangle_\gamma^{s_2}\mathrm{op}(m_2)v\|$ with use of Lemma 6.11 again. □

Corollary 6.4 *Let m be g admissible weight with $m \in S(m, g)$. Then there exists $C > 0$ such that $(\mathrm{op}(m)u, u) \ge (1 - CM^{-1})\|\mathrm{op}(\sqrt{m})u\|^2$.*

Proof Since $\sqrt{m}$ is g admissible weight such that $\sqrt{m} \in S(\sqrt{m}, g)$ one can write $m = \sqrt{m}\#(1+r)\#\sqrt{m}$ with $r \in S(M^{-1}, g)$ from Lemma 6.10 and the rest of the proof is clear. □

Lemma 6.12 *Let m_i be g admissible weights with $m_i \in S(m_i, g)$ $(i = 1, 2)$. Let w be $\bar{g}$ admissible weight with $w \in S(w, \bar{g})$ for which there exists $\tilde{w} \in S(w^{-1}, \bar{g})$ such that $\tilde{w}\#w = w\#\tilde{w} = 1$. If $\bar{m}$ is $\bar{g}$ admissible weight and $a \in S(\bar{m}m_1m_2w, \bar{g})$ there exist M_0 and $\hat{a} \in S(\bar{m}, \bar{g})$ $(M > M_0)$ such that the following estimates hold*

$$\big|(\mathrm{op}(a)u, v)\big| \le \|\mathrm{op}(w)\mathrm{op}(m_1)u\|\|\mathrm{op}(\hat{a})\mathrm{op}(m_2)v\|,$$
$$\|\mathrm{op}(a)u\| \le \|\mathrm{op}(\hat{a})\mathrm{op}(m_2)\mathrm{op}(w)\mathrm{op}(m_1)u\|$$

for $M > M_0$. If $a \in S(m_1m_2w^2, \bar{g})$ one has

$$\big|(\mathrm{op}(a)u, v)\big| \le C\|\mathrm{op}(w)\mathrm{op}(m_1)u\|\|\mathrm{op}(w)\mathrm{op}(m_2)v\|, \quad M > M_0.$$

Proof In virtue of Lemma 6.10 one can write $a = m_2\#\tilde{a}\#m_1$ with $\tilde{a} \in S(\bar{m}w, \bar{g})$. Write $\tilde{a} = (\tilde{a}\#\tilde{w})\#w$ with use of $\tilde{w} \in S(w^{-1}, \bar{g})$ where $\tilde{a}\#\tilde{w} \in S(\bar{m}, \bar{g})$ hence the first assertion is proved. Noting $m_2\#(\tilde{a}\#\tilde{w}) \in S(\bar{m}m_2, \bar{g})$ this can be written as $\hat{a}\#m_2$ with $\hat{a} \in S(\bar{m}, \bar{g})$ thanks to Lemma 6.10 which proves the second estimate. The last estimate can be obtained by taking $\bar{m} = w$ in the first estimate and applying the second estimate to it. □

Corollary 6.5 *Let $a \in S(\langle\xi\rangle_\gamma^s m_1m_2w, \bar{g})$ or $a \in S(\langle\xi\rangle_\gamma^s m_1m_2w^2, \bar{g})$ and $s_1 + s_2 = s$ then one has*

$$|(\mathrm{op}(a)u, v)| \le C\|\langle D\rangle_\gamma^{s_1}\mathrm{op}(w)\mathrm{op}(m_1)u\|\|\langle D\rangle_\gamma^{s_2}\mathrm{op}(m_2)v\|,$$
$$|(\mathrm{op}(a)u, v)| \le C\|\langle D\rangle_\gamma^{s_1}\mathrm{op}(w)\mathrm{op}(m_1)u\|\|\langle D\rangle_\gamma^{s_2}\mathrm{op}(w)\mathrm{op}(m_2)v\|.$$

Proof Writing $a = \langle\xi\rangle_\gamma^{s_2}\#\tilde{a}\#\langle\xi\rangle_\gamma^{s_1}$ it suffices to apply Lemma 6.12 to $\tilde{a} \in S(m_1m_2w, \bar{g})$ or $\tilde{a} \in S(m_1m_2w^2, \bar{g})$ and repeat the proof of Corollary 6.3. □

Next, consider pseudodifferential operators associated with the metric G.

Lemma 6.13 *If $a \in S(1, G)$ satisfies $a \ge c$ with some c then there is $C > 0$ such that*

$$\big(\mathrm{op}(a)u, u\big) \ge (c - CM^{-1})\|u\|^2. \tag{6.33}$$

Proof Considering $a - c$ one may assume $c = 0$. From $0 \le a \in S(1, G)$ it follows that $|\partial_x^\alpha a| \le CM^2$ and $|\partial_\xi^\beta a| \le CM^2\langle\xi\rangle_\gamma^{-2}$ for $|\alpha| = |\beta| = 2$ then we have $|\partial_x^\alpha\partial_\xi^\beta a| \le CM\langle\xi\rangle_\gamma^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}\sqrt{a} \le CM^{-1}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}\sqrt{a}$ for $|\alpha + \beta| = 1$ by the Glaeser inequality. With $b(x, \xi) = (a(x, \xi) + M^{-1})^{1/2} \ge M^{-1/2}$ we see

$$|\partial_x^\alpha\partial_\xi^\beta b| \le CM^{-1}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}\sqrt{a}/b \le CM^{-1/2}\langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2}b, \quad |\alpha + \beta| = 1.$$

Differentiating $b^2 = a + M^{-1}$ one has

$$|b\partial_x^\alpha \partial_\xi^\beta b| \lesssim \sum_{0<|\alpha'+\beta'|<|\alpha+\beta|} |\partial_x^{\alpha'}\partial_\xi^{\beta'} b||\partial_x^{\alpha''}\partial_\xi^{\beta''} b| + |\partial_x^\alpha \partial_\xi^\beta a|.$$

Noting $M^{-1} \le b^2$ one can prove by the induction on $|\alpha+\beta|$ that

$$|\partial_x^\alpha \partial_\xi^\beta b| \le C_{\alpha\beta} M^{-|\alpha+\beta|/2} \langle\xi\rangle_\gamma^{(|\alpha|-|\beta|)/2} b. \tag{6.34}$$

We now show that b is $\bar{g}$ admissible weight. To do so it suffices to repeat the proof of Lemma 6.5, namely when $|\eta| < \langle\xi\rangle_\gamma/2$ one has

$$|b(z+w) - b(z)| \le CM^{-1/2}\bar{g}_z^{1/2}(w) \le C\, b(z)(1+\bar{g}_z(w))^{1/2} \tag{6.35}$$

and if $|\eta| \ge \langle\xi\rangle_\gamma/2$ noting $\bar{g}_z(w) \ge \langle\xi\rangle_\gamma/4 \ge M^4/4$ one has

$$|b(z+w)| \le C \le CM^{-1/2}M^{1/2} \le C\, b(z)(1+\bar{g}_z(w))^{1/8}. \tag{6.36}$$

Thus one can write $a + M^{-1} = b\#b + r$ with $r \in S(M^{-2}b^2, \bar{g}) \subset S(M^{-2}, \bar{g})$ in virtue of (6.34) and Theorem 6.1. Applying Theorem 6.2 to $\mathrm{op}(r)$ to obtain

$$(\mathrm{op}(a+M^{-1})u, u) = \|\mathrm{op}(b)u\|^2 + (\mathrm{op}(r)u, u) \ge -CM^{-2}\|u\|^2$$

which proves the assertion. □

Corollary 6.6 *If $a \in S(1, G)$ there is $C > 0$ such that*

$$\|\mathrm{op}(a)u\| \le \big(\sup|a| + CM^{-1/2}\big)\|u\|.$$

Proof Note that $\|\mathrm{op}(a)u\|^2 = (\mathrm{op}(\bar{a}\#a)u, u)$ and $\bar{a}\#a = |a|^2 + r$ with $r \in S(M^2\langle\xi\rangle_\gamma^{-1}, \bar{g})$ by Theorem 6.1. Since $M^2\langle\xi\rangle_\gamma^{-1} \le M^{-2}$ it suffices to consider $(\mathrm{op}(|a|^2)u, u)$. Applying Lemma 6.13 to $(\sup|a|)^2 - |a|^2 \ge 0$ to get

$$(\mathrm{op}(|a|^2)u, u) \le ((\sup|a|)^2 + CM^{-1})\|u\|^2 \le (\sup|a| + CM^{-1/2})^2\|u\|^2$$

which ends the proof. □

6.4 Proof of Proposition 5.2

For a conic set $U \subset \mathbb{R}^d \times (\mathbb{R}^d\backslash 0)$ we denote $\pi(U) = \{x \in \mathbb{R}^d; (x,\xi) \in U\}$ and $U_{\bar{x}} = \{\xi \in \mathbb{R}^d\backslash 0; (\bar{x},\xi) \in U\}$.It is clear that

$${}^t\kappa'(x)U_y = \big(\kappa^* U\big)_x \quad (y = \kappa(x)). \tag{6.37}$$

We may assume that $\kappa(x) = Ax$ for $|x| \geq R > 2$ with a nonsingular $d \times d$ matrix A.Let $\chi(x) \in C_0^\infty(\mathbb{R}^d)$ be 1 for $|x| \leq R$ and 0 for $|x| \geq R+1$.Write $\mathrm{op}(k)\kappa^*\mathrm{op}(h) = \mathrm{op}(k)(\chi + (1-\chi))\kappa^*\mathrm{op}(h)$ then $\mathrm{op}(k)(1-\chi)\kappa^*\mathrm{op}(h) = \mathrm{op}(k)(1-\chi)\mathrm{op}(h_A)\kappa^*$ where $h_A = h(Ax, {}^t\!A^{-1}\xi)$.Since $k\#(1-\chi)\#h_A \in S^{-\infty}$ it suffices to consider $\mathrm{op}(k\#\chi)\kappa^*\mathrm{op}(h)$. Theorem 6.1 gives $k_{N_1} \in S^0$ with compact support such that $k\#\chi - k_{N_1} \in S^{-N_1}$ for any N_1 hence it is enough to consider $\mathrm{op}(k_{N_1})\kappa^*\mathrm{op}(h)$.With $\tilde{\chi}(x) = \chi(\kappa^{-1}(x))$ and writing $\mathrm{op}(k_{N_1})\kappa^*\mathrm{op}(h) = \mathrm{op}(k_{N_1})\kappa^*(\tilde{\chi} + (1-\tilde{\chi}))\mathrm{op}(h)$ it follows from the same argument that it suffices to consider $\mathrm{op}(k_{N_1})\kappa^*\tilde{\chi}\mathrm{op}(h_{N_2})$ with $h_{N_2} \in S^0$ with compact support such that $\tilde{\chi}\#h - h_{N_2} \in S^{-N_2}$.Thus one can assume that U, V are compact conic sets.

Lemma 6.14 *Let U be a closed conic set and $\Gamma \subset \mathbb{R}^d\backslash 0$ be a closed cone with $U_{\bar{x}} \cap \Gamma = \emptyset$.Then there exist a neighborhood ω of $\bar{x}$ and a closed cone $\tilde{\Gamma} \Supset \Gamma$ such that for any $\alpha(x) \in C_0^\infty(\omega)$ and $h \in S^0$ with $\mathrm{supp}\, h \subset U$ and $p, q \in \mathbb{R}$ there is C such that*

$$\big|\mathcal{F}(\alpha\,\mathrm{op}(h)v)(\xi)\big|,\ \big|\mathcal{F}(\mathrm{op}(h)\alpha v)(\xi)\big| \leq C(1+|\xi|)^p\|v\|_q,\quad \xi \in \tilde{\Gamma},\ v \in H^q. \tag{6.38}$$

Proof One can take a compact neighborhood ω of $\bar{x}$ and a closed cone $\tilde{\Gamma} \Supset \Gamma$ with $(\omega \times \tilde{\Gamma}) \cap U = \emptyset$ such that the following holds with some $\epsilon > 0$

$$\xi \in \tilde{\Gamma},\quad (x,\eta) \in \pi^{-1}(\omega) \cap U \Longrightarrow |\xi - \eta| \geq \epsilon(|\xi| + |\eta|). \tag{6.39}$$

Let $\alpha(x) \in C_0^\infty(\omega)$. For any $m \in \mathbb{N}$ there is $h_m \in S^0$ with $\mathrm{supp}\, h_m \subset \pi^{-1}(\omega) \cap U$ such that $\alpha\#h - h_m = r_m \in S^{-m}$. Since

$$\mathcal{F}(\mathrm{op}(h_m)v)(\xi) = \pi^{-d}\int e^{-2ix(\xi-\eta)}h_m(x,\eta)\hat{v}(2\eta - \xi)dxd\eta \tag{6.40}$$

using $e^{-2ix(\xi-\eta)} = \langle\xi - \eta\rangle^{-2N}\langle D_x/2\rangle^{2N}e^{-2ix(\xi-\eta)}$ integration by parts shows

$$\big|\mathcal{F}(\mathrm{op}(h_m)v)(\xi)\big| \leq C\int\langle\xi-\eta\rangle^{-2N}|\langle D_x/2\rangle^{2N}h_m(x,\eta)||\hat{v}(2\eta-\xi)|dxd\eta$$

where the right-hand side is bounded by that of (6.38) if $\xi \in \tilde{\Gamma}$ because of (6.39).Replacing h_m by r_m in (6.40) and noting $|\langle D_x/2\rangle^{2N}r_m(x,\eta)| \leq C(1+|x|)^{-d-1}(1+|\eta|)^{-m}$ it follows from integration by parts

$$\begin{aligned}\big|\mathcal{F}(\mathrm{op}(r_m)v)(\xi)\big| &\leq C_m\int(1+|\xi-\eta|^2)^{-N}(1+|\eta|)^{-m}|\hat{v}(2\eta-\xi)|d\eta\\ &\leq C_m\int(1+|\xi-\eta|+|\eta|)^{-\min\{2N,m\}}|\hat{v}(2\eta-\xi)|d\eta.\end{aligned}$$

Since N, m is arbitrary the right-hand side is bounded by that of (6.38) (for any ξ). Thus we conclude the assertion. For $\mathcal{F}(\mathrm{op}(h)\alpha v)(\xi)$ the proof is similar. □

Take compact conic sets W, Z such that $U \Subset W$, $V \Subset Z$ and $Z \cap \kappa^* W = \emptyset$.Denote $\tilde{\Gamma}_y = \overset{\circ}{W}_y$, $\Gamma_x = \overset{\circ}{Z}_x$ then by Lemma 6.14 there is a neighborhood Ω_y of y such that for any $\alpha \in C_0^\infty(\Omega_y)$ and $p, q \in \mathbb{R}$ one has

$$\left|\mathcal{F}(\alpha\, \mathrm{op}(h)v)(\eta)\right| \le C(1+|\eta|)^p \|v\|_q, \quad \eta \in \tilde{\Gamma}_y^c, \quad v \in H^q. \tag{6.41}$$

Similarly there exist a neighborhood ω_x of x and a closed cone $\hat{\Gamma}_x \Supset \Gamma_x^c$ such that for any $\beta \in C_0^\infty(\omega_x)$ and $p, q \in \mathbb{R}$ we have

$$\left|\mathcal{F}(\mathrm{op}(k)\beta u)(\xi)\right| \le C(1+|\xi|)^p \|u\|_q, \quad \xi \in \hat{\Gamma}_x, \quad u \in H^q. \tag{6.42}$$

Shrinking ω_x if necessary one may assume $\kappa(\omega_x) \Subset \Omega_y$ $(y = \kappa(x))$.Note that $\pi(Z)$ can be covered by a finite number of ω_{x_i}. We denote $\omega_{x_i} = \omega_i$ and $\Omega_i = \Omega_{y_i}$, $\Gamma_i = \Gamma_{x_i}$, $\tilde{\Gamma}_i$, $\hat{\Gamma}_i$ so on.Take $\beta_i \in C_0^\infty(\omega_i)$ such that $\sum_i \beta_i = 1$ on $\pi(Z)$.Since $k\#(1 - \sum \beta_i) \in S^{-\infty}$ it is enough to consider $\sum_i \mathrm{op}(k)\beta_i$.Similarly taking $\alpha_i \in C_0^\infty(\Omega_i)$ which is 1 on $\kappa(\omega_i)$ it suffices to consider $\sum_i \mathrm{op}(k)\beta_i\kappa^*\alpha_i$.Denoting $u = \alpha_i \mathrm{op}(h)v$ and using $\kappa^* u = (2\pi)^{-d} \int e^{i\langle\kappa(x),\eta\rangle}\hat{u}(\eta)d\eta$ one sees

$$\mathcal{F}(\beta_i\kappa^* u) = (2\pi)^{-d} \int \beta_i(x) e^{i(\langle\kappa(x),\eta\rangle - \langle x,\xi\rangle)}\hat{u}(\eta)d\eta dx = \int I(\xi,\eta)\hat{u}(\eta)d\eta$$

where

$$I(\xi,\eta) = (2\pi)^{-d}\int \beta_i(x) e^{i(\langle\kappa(x),\eta\rangle - \langle x,\xi\rangle)}dx.$$

Since $d(\langle\kappa(x),\eta\rangle - \langle x,\xi\rangle) = \langle dx, {}^t\kappa'(x)\eta - \xi\rangle$ we have $|{}^t\kappa'(x)\eta - \xi| \ge |\xi|/2$ for $|\xi| \ge 2B|\eta|$ with some $B > 0$, while if $|\xi| \le 2B|\eta|$ it is obvious $|I(\xi,\eta)| \le C \le C(2B)^N(1+|\xi|)^{-N}(1+|\eta|)^N$. Thus for any $N \in \mathbb{N}$ the following estimate holds

$$\left|I(\xi,\eta)\right| \le C_N(1+|\xi|)^{-N}(1+|\eta|)^N, \quad \xi,\eta \in \mathbb{R}^d. \tag{6.43}$$

Next, one can assume that ω_i is chosen such that

$$x \in \omega_i \quad \eta \in \tilde{\Gamma}_i, \quad \xi \in \Gamma_i \Longrightarrow |{}^t\kappa'(x)\eta - \xi| \ge \epsilon(|\xi| + |\eta|) \tag{6.44}$$

holds with some $\epsilon > 0$. Using (6.44) a repetition of integration by parts gives $\left|I(\xi,\eta)\right| \le C_N(1+|\xi|+|\eta|)^{-N}$ for $\xi \in \Gamma_i$ and $\eta \in \tilde{\Gamma}_i$. Summarizing we conclude

$$\begin{aligned}\left|\mathcal{F}(\beta_i\kappa^* u)(\xi)\right| \le C_N' \Bigg(&\int_{\tilde{\Gamma}_i} |\hat{u}(\eta)|(1+|\xi|+|\eta|)^{-N} d\eta \\ &+ (1+|\xi|)^{-N}\int_{\tilde{\Gamma}_i^c} |\hat{u}(\eta)|(1+|\eta|)^N d\eta\Bigg), \quad \xi \in \Gamma_i.\end{aligned} \tag{6.45}$$

From (6.41) it follows that $|\hat{u}(\eta)| \le C(1+|\eta|)^p \|v\|_q$ for any p if $\eta \in \tilde{\Gamma}_i^c$ which together with (6.45) gives $|\mathcal{F}(\beta_i \kappa^* u)(\xi)| \le C\langle\xi\rangle^{-N}\|v\|_q$ for any $\xi \in \Gamma_i$ and $N \in \mathbb{N}$.

Lemma 6.15 *If the support of $h \in S^0$ is contained in a compact conic set and $u \in H^q$ satisfies $\hat{u}(\xi) = O(|\xi|^{-N})$ for any N in an open cone Γ then for any N and open cone $\Gamma' \Subset \Gamma$ there is C such that*

$$\big|\mathcal{F}(\mathrm{op}(h)u)(\xi)\big| \le C\langle\xi\rangle^{-N}\big\{\sup_{\eta\in\Gamma}(1+|\eta|)^{N+(d+1)/2}|\hat{u}(\eta)| + \|u\|_q\big\}, \quad \xi \in \Gamma'.$$

Proof Using (6.40) write $\mathcal{F}(\mathrm{op}(h)u)(\xi)$ as

$$\pi^d\mathcal{F}(\mathrm{op}(h)u)(\xi) = \int_{|\xi-\eta|<c|\xi|} + \int_{|\xi-\eta|\ge c|\xi|} = \int_{|\xi-\eta|<c|\xi|}$$
$$+ \int_{|\eta|\ge c|\xi|} e^{2ix\eta}h(x,\eta+\xi)\hat{u}(2\eta+\xi)dxd\eta = I_1(\xi) + I_2(\xi).$$

Choose $0 < c < 1$ such that $2\eta - \xi = \xi + 2(\eta - \xi) \in \Gamma$ if $|\xi - \eta| < c|\xi|$ and $\xi \in \Gamma'$ then a repetition of integration by parts gives

$$\begin{aligned}|I_1(\xi)| &\le \int_{|\xi-\eta|<c|\xi|}\langle\xi-\eta\rangle^{-2N}|\langle D_x/2\rangle^{2N}h(x,\eta)||\hat{u}(2\eta-\xi)|dxd\eta\\ &\le C\sup_{\eta\in\Gamma}(1+|\eta|)^{2N}|\hat{u}(\eta)|\int\langle\xi-\eta\rangle^{-2N}(1+|2\eta-\xi|)^{-2N}d\eta\\ &\le C'(1+|\xi|)^{-2N+d+1}\sup_{\eta\in\Gamma}(1+|\eta|)^{2N}|\hat{u}(\eta)|, \quad \xi \in \Gamma'\end{aligned}$$

for $1+|\xi|+|\eta| \le 3(1+|\xi-\eta|)(1+|2\eta-\xi|)$. For $I_2(\xi)$ noting $|2\eta+\xi| \le (2+c^{-1})|\eta|$ if $|\eta| \ge c|\xi|$ integration by parts proves

$$\begin{aligned}|I_2(\xi)| &\le \int_{|\eta|\ge c|\xi|}\langle\eta\rangle^{-2N}|\langle D_x/2\rangle^{2N}h(x,\eta+\xi)||\hat{u}(2\eta+\xi)|dxd\eta\\ &\le C\langle\xi\rangle^{-2N+|q|+(d+1)/2}\int\langle\eta\rangle^{-(d+1)/2}\langle 2\eta+\xi\rangle^q|\hat{u}(2\eta+\xi)|d\eta\\ &\le C'\langle\xi\rangle^{-2N+|q|+(d+1)/2}\|u\|_q.\end{aligned}$$

Since N is arbitrary the proof is completed. □

Here apply Lemma 6.15 with $\Gamma = \Gamma_i$, $\Gamma' = \hat{\Gamma}_i^c$ to obtain

$$\big|\mathcal{F}(\mathrm{op}(k)\beta_i\kappa^*\alpha_i\mathrm{op}(h)v)(\xi)\big| \le C\langle\xi\rangle^{-N}\big(\|v\|_q + \|\alpha_i\mathrm{op}(h)v\|_q\big) \le C'\langle\xi\rangle^{-N}\|v\|_q$$

for $\xi \in \hat{\Gamma}_i^c$. If $\xi \in \hat{\Gamma}_i$ (6.42) shows that for any p one has

$$\big|\mathcal{F}(\mathrm{op}(k)\beta_i\kappa^*\alpha_i\mathrm{op}(h)v)(\xi)\big| \le C\langle\xi\rangle^p\|\kappa^*\alpha_i\mathrm{op}(h)v\|_q \le C'\langle\xi\rangle^p\|v\|_q.$$

Combining these two estimates we complete the proof of the proposition. □

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