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Some further classes of pseudo-differential operators in the *p*-adic context and their applications

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Abstract
The purpose of this article is to study new non-Archimedean pseudo-differential operators whose symbols are determined from the behavior of two functions defined on the p-adic numbers. Thanks to the characteristics of our symbols, we can find connections between these operators and new types of non-homogeneous differential equations, Feller semigroups, contraction semigroups and strong Markov processes.

Keywords Pseudo-differential operators · Heat kernel · Markov processes · Feller semigroups · Transition functions · p-Adic analysis

1 Introduction

The study of archimedean pseudo-differential operators constitutes a classical area of research due to their connections with Feller semigroups and Markov processes, see e.g. [14, 20–22, 34, 35], and the references therein.

In the mid 80s, the use of p-adic numbers $\mathbb{Q}_p$ was proposed in order to study physical problems at very small distances such as the Planck's length, see e.g. [32, 33, 43, 45, 46], and the references therein. Later Avetisov, Bikulov, Kozyrev, Osipov in [5–8], introduce a family of parabolic-type p-adic pseudo-differential equations of

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the form

$$\frac{\partial u\,(x,t)}{\partial t} = \int_{\mathbb{Q}_p} j\left(|x-y|_p\right)\{u\,(y,t) - u\,(x,t)\}\,dy, \quad t \geq 0, \tag{1.1}$$

with the aim of studying the relaxation of certain complex systems. Since then, the study of non-archimedean pseudo-differential operators (in particular, p-adic pseudo-differential operators) has strengthened due to its usefulness in numerous application in mathematical physics, cellular neural networks, human memory retrieval, probability theory, Sobolev spaces and the formation of oil reservoirs, see e.g., [3, 4, 7, 8, 11, 12, 15–19, 23, 25, 26, 28, 30, 31, 38–42, 47, 48], and references therein.

In this article, for $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$, we introduce new types of p-adic pseudo-differential operators of the form

$$\begin{aligned}(\mathcal{B}^{\alpha}_{\boldsymbol{h}_1,\boldsymbol{h}_2}\varphi)(x) &:= -\mathcal{F}^{-1}_{\xi\to x}\left(\left[\max\{|\boldsymbol{h}_1(||\xi||_p)|, |\boldsymbol{h}_2(||\xi||_p)|\}\right]^{\alpha}\widehat{\varphi}(\xi)\right),\\ &= -\int\limits_{\mathbb{Q}_p^n}\chi_p(-x\cdot\xi)\left[\max\{|\boldsymbol{h}_1(||\xi||_p)|, |\boldsymbol{h}_2(||\xi||_p)|\}\right]^{\alpha}\widehat{\varphi}(\xi)d^n\xi, \quad x \in \mathbb{Q}_p^n,\end{aligned}$$

where $\mathcal{F}^{-1}_{\xi\to x}$ corresponds to the inverse Fourier transform, $\widehat{\varphi}$ is the Fourier transform of φ, α is a positive real number, and the function $[\max\{|\boldsymbol{h}_1(\cdot)|, |\boldsymbol{h}_2(\cdot)|\}]^{\alpha} : \mathbb{Q}_p^n \to \mathbb{C}$ is the symbol of the operator.

With the implementation of these symbols it is intended to introduce new classes of p-adic pseudo-differential equations connected in a natural way with our pseudo-differential operators. Furthermore, a significant contribution to the theory of parabolic p-adic pseudo-differential equations and their applications to p-adic theoretical physics is sought. It is important to mention that due to the nature of our symbols, the results obtained require a more demanding mathematical rigor than the other classes of pseudo-differential operators previously studied.

From a mathematical perspective, the p-adic heat equation (or Cauchy problem) corresponding to $\mathcal{B}^{\alpha}_{\boldsymbol{h}_1,\boldsymbol{h}_2}$ take the form

$$\begin{cases}\frac{\partial u}{\partial t}(x,t) = -\mathcal{B}^{\alpha}_{\boldsymbol{h}_1,\boldsymbol{h}_2}(u(x,t)),\, t \in (0,\infty)\,, \; x \in \mathbb{Q}_p^n,\\ \\ u(x,0) = u_0(x) \in \mathcal{D}(\mathbb{Q}_p^n),\end{cases} \tag{1.2}$$

with fundamental solution given by

$$Z_t(x) = Z(x,t) := \int_{\mathbb{Q}_p^n}\chi_p\,(-x\cdot\xi)\,e^{-t\left[\max\{|\boldsymbol{h}_1(||\xi||_p)|,|\boldsymbol{h}_2(||\xi||_p)|\}\right]^{\alpha}}d^n\xi.$$

Now, from the physics point of view, the heat kernel $Z_t(x)$ controls the evolution of temperature in the position x and at time t. An important fact to highlight is that we show that the fundamental solutions $Z_t(\cdot)$ determine "explicitly" Feller semigroups $\{T_t\}_{t\geq 0}$ on $\mathbb{Q}_p^n$ and transition functions, $p_t(x,\cdot)$, of strong Markov process $\mathfrak{X}(t,w)$

with space state $\mathbb{Q}_p^n$ and whose paths are right continuous and have no discontinuities other than jumps, see Theorems 2 and 3, respectively. We also introduce other Markov processes X with sample paths in $D_{\mathbb{Q}_p^n}[0,\infty)$ ($D_{\mathbb{Q}_p^n}[0,\infty)$ is the space of right continuous functions $f:[0,\infty)\to\mathbb{Q}_p^n$ with left limits). Moreover, X is a strong Markov processes with respect to a certain filtration, see Theorem 4.

In this article, we also study new classes of contraction semigroups, $(P_t)_{t\geq 0}$, in the Hilbert space $L^2(\mathbb{Q}_p^n)$. Moreover, via the theory of m-dissipative operators we will show that our pseudo-differential operators are the generators of these semigroups, see Theorem 6. The above enables us to guarantee unique solutions for the inhomogeneous initial value problem

$$\begin{cases} u(x,\cdot)\in C([0,T],\mathcal{D}(\mathbb{Q}_p^n))\cap C^1([0,T],L^2(\mathbb{Q}_p^n)),\\ \\ \frac{\partial u}{\partial t}(x,t)=\mathcal{B}^{\alpha}_{\boldsymbol{h}_1,\boldsymbol{h}_2}u(x,t)+g(x,t),\quad t\in[0,T],\quad x\in\mathbb{Q}_p^n,\\ \\ u_0(x)=h(x)\in L^2(\mathbb{Q}_p^n).\end{cases}$$

where $T>0$ and $g:\mathbb{Q}_p^n\times[0,T]\to L^2(\mathbb{Q}_p^n)$ be a function such that

$$g\in L^1((0,T),L^2(\mathbb{Q}_p^n))\cap C([0,T],L^2(\mathbb{Q}_p^n)),$$

see Theorem 7.

On the other hand, in the last three years and thanks to its topology, p-adic numbers have become a natural structure to study new mathematical models related to the spread of infectious diseases (say COVID-19) over certain types of population groups, see for example, [1, 24, 27].

Suppose that $B_r^n, r\in\mathbb{Z}$, is the ultrametric ball that mathematically represent a social cluster in a situation of extreme social isolation and 100% free from an infectious or contagious disease at time $t=0$. Let $\boldsymbol{h}_1,\boldsymbol{h}_2:\mathbb{Q}_p^n\to\mathbb{C}$ be continuous negative definite and radial functions. Moreover, suppose that $|\boldsymbol{h}_1(||\xi||_p)|$ represents the degree of contagion of a person in the ξ position and $|\boldsymbol{h}_2(||\xi||_p)|$ represents the ability to respond of the body's immune system of a person in the ξ position. Since $|\boldsymbol{h}_1|$ and $|\boldsymbol{h}_2|$ are radial functions and their values at each point depends only on the p-adic distance between point and the origin, we have that $|\boldsymbol{h}_1|$ and $|\boldsymbol{h}_2|$ are increasing functions with respect to $||\cdot||_p$. Moreover, if the environment of the isolated population has a high degree of contamination and taking into account the speed with which infectious or contagious disease attacks and deteriorates the immune system of people, then (under a serious risk of contagion) $|\boldsymbol{h}_1(||\xi||_p)|\leq|\boldsymbol{h}_2(||\xi||_p)|$ if and only if $\xi\in B_r^n$. Motivated by the above, from the point of view of the applications of non-archimedean analysis to physics, an interesting open problem consists in determine if our p-adic equations can be applied to study the spread of an infectious or contagious disease.

The article is organized as follows: In Sect. 2, we will collect some basic results on the p-adic analysis and fix the notation that we will use through the article. In Sect. 3, we introduce a new class of non-archimedean pseudo-differential operators (which we will denote by $\mathcal{B}^{\alpha}_{\boldsymbol{h}_1,\boldsymbol{h}_2}$). Moreover, we also study some properties of the heat Kernel

$Z(x,t)$, $x \in \mathbb{Q}_p^n$, $t > 0$, associated with this type of operators. In Sect. 4, we will show the existence of Feller semigroups, transition functions and Strong Markov processes on $\mathbb{Q}_p^n$ associated with the heat Kernel $Z(x,t)$. In Sect. 5, new types of contraction semigroups, $(P_t)_{t\geq 0}$, are introduced. Also, we will prove that our operators are m-dissipative, see Theorem 5. Next, we will show that our pseudo-differential operators are the generators of these semigroups. Finally, we propose and solve new types of inhomogeneous initial value problem.

2 *p*-adic analysis: essential ideas

Let p be a fixed prime number. By $\mathbb{Q}_p$ we denote the field of p-adic numbers,

$$\mathbb{Q}_p = \left\{ \sum_{i=k}^{\infty} a_i p^i : k \in \mathbb{Z},\ a_i \in \{0, 1, 2, \ldots, p-1\},\ a_0 \neq 0 \right\}.$$

$\mathbb{Q}_p$ is the completion of the field $\mathbb{Q}$ of rational numbers with respect to the p-adic absolute value $|\cdot|_p$ given by

$$|x|_p = \begin{cases} 0, & \text{if } x = 0 \\ \\ p^{-\gamma}, & \text{if } x = p^{\gamma}\frac{a}{b}, \end{cases}$$

where a and b are integers no divided by p. The integer γ is called the *p-adic order of x*, denoted $ord_p(x)$, with $ord_p(0) := +\infty$. Note that the p-adic norm $|\cdot|_p$ takes the discrete set of values $\{p^{\gamma} : \gamma \in \mathbb{Z}\} \bigcup \{0\}$.

For $\mathbb{Q}_p \ni x = \sum_{i=k}^{\infty} a_i p^i$, we denote and define *the fractional part of x* as follows:

$$\{x\}_p := \begin{cases} 0, & \text{if } x = 0 \text{ or } ord_p(x) \geq 0 \\ \\ \sum_{i=k}^{-ord_p(x)-1} a_i p^i, & \text{if } ord_p(x) < 0. \end{cases}$$

The space $\mathbb{Q}_p^n := \mathbb{Q}_p \times \mathbb{Q}_p \times \ldots \times \mathbb{Q}_p$ consists of points $x = (x_1, x_2, \ldots, x_n)$, where $x_i \in \mathbb{Q}_p$, $i = 1, 2, \ldots, n$, $n \geq 2$. The norm in $\mathbb{Q}_p^n$ is denoted and defined as

$$||x||_p := \max_{1 \leq i \leq n} |x_i|_p, \ \text{ for } x = (x_1, x_2, \ldots, x_n) \in \mathbb{Q}_p^n.$$

Let fixed $y = (y_1, y_2, \ldots, y_n) \in \mathbb{Q}_p^n$ and $m \in \mathbb{Z}$. *The ball* (respectively, *the sphere*) of radius p^m and center in y is the set $B_m^n(y) = \{x \in \mathbb{Q}_p^n : ||x-y||_p \leq p^m\}$ (respectively, $S_m^n(y) = \{x \in \mathbb{Q}_p^n : ||x-y||_p = p^m\}$). For simplicity we will write $B_m^n(0) := B_m^n$ and $S_m^n(0) := S_m^n$. The set $B_0^n = \mathbb{Z}_p^n$ is the ring of p-adic integers.

The balls and spheres are compact subsets in $\mathbb{Q}_p^n$. Moreover, as a topological space $\left(\mathbb{Q}_p^n, ||\cdot||_p\right)$ is a totally disconnected and locally compact topological space. $\mathbb{Q}_p^n$ admits a Haar measure $d^n x$ normalized such that $\int_{\mathbb{Z}_p^n} d^n x = 1$.

By $L^{\rho}\left(\mathbb{Q}_p^n\right)$, $1 \le \rho < \infty$, we denote the space of all the functions $g : \mathbb{Q}_p^n \to \mathbb{C}$ satisfying $\int\limits_{\mathbb{Q}_p^n} |g(x)|^{\rho}\, d^n x < \infty$; and by $C_0(\mathbb{Q}_p^n)$ the space of all the continuous functions $f : \mathbb{Q}_p^n \to \mathbb{C}$ such that $\lim_{||x||_p \to \infty} f(x) = 0$. $C_0(\mathbb{Q}_p^n)$ is a Banach space with the norm $||f|| = sup_{x \in \mathbb{Q}_p^n} |f(x)|$.

A function $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ is called *locally constant* if for any $x \in \mathbb{Q}_p^n$ there exist an integer $m := m(x)$ such that

$$\varphi(x') = \varphi(x) \text{ for all } x' \in B_m^n(x).$$

Let $\varepsilon(\mathbb{Q}_p^n)$ denote the space of all locally constant functions on $\mathbb{Q}_p^n$.

A function $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ is called a *Bruhat-Schwartz function or a test function* if it is locally constant with compact support. The space of Bruhat-Schwartz functions is denoted by $\mathcal{D}(\mathbb{Q}_p^n)$. Let $\mathcal{D}'(\mathbb{Q}_p^n)$ denote the set of all continuous functional (distributions) on $\mathcal{D}(\mathbb{Q}_p^n)$.

Every function $f \in L^1_{loc}(\mathbb{Q}_p^n)$ ($L^1_{loc}(\mathbb{Q}_p^n)$ denotes the space of locally integrable functions) defines a distribution $f \in \mathcal{D}'(\mathbb{Q}_p^n)$ (called regular distribution) by the formula

$$< f, \varphi >:= \int\limits_{\mathbb{Q}_p^n} f(x)\varphi(x) d^n x, \quad \varphi \in \mathcal{D}(\mathbb{Q}_p^n).$$

If $f \in L^1(\mathbb{Q}_p^n)$, its Fourier transform is defined by

$$(\mathcal{F} f)(\xi) = \mathcal{F}_{x \to \xi}(f) := \widehat{f}(\xi) = \int\limits_{\mathbb{Q}_p^n} \chi_p(\xi \cdot x) f(x) d^n x, \quad \xi \in \mathbb{Q}_p^n,$$

where $x \cdot \xi := \sum_{j=1}^{n} x_j \xi_j$ for $x = (x_1, \ldots, x_n), \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{Q}_p^n$, and $\chi_p(\xi \cdot x) = e^{2\pi i \{\xi \cdot x\}_p}$ is a additive character on $\mathbb{Q}_p^n$. The inverse Fourier transform of a function $f \in L^1(\mathbb{Q}_p^n)$ is

$$(\mathcal{F}^{-1} f)(x) = \mathcal{F}^{-1}_{\xi \to x}(f) := \check{f}(\xi) = \int\limits_{\mathbb{Q}_p^n} \chi_p(-x \cdot \xi) f(\xi) d^n \xi, \quad x \in \mathbb{Q}_p^n.$$

The set $L^2(\mathbb{Q}_p^n)$ is the Hilbert space with the scalar product

$$(f, g) = \int\limits_{\mathbb{Q}_p^n} f(x)\overline{g}(x) d^n x, \ f, g \in L^2(\mathbb{Q}_p^n),$$

so that $||f||_{L^2} = \sqrt{(f, f)}$.

If $f \in L^2(\mathbb{Q}_p^n)$, its Fourier transform is defined as

$$(\mathcal{F}f)(\xi) = \lim_{k\to\infty} \int_{||x||\leq p^k} \chi_p(\xi \cdot x) f(x) d^n x, \quad \xi \in \mathbb{Q}_p^n,$$

where the limit is taken in $L^2(\mathbb{Q}_p^n)$. We recall that the Fourier transform is unitary on $L^2(\mathbb{Q}_p^n)$, i.e. $||f||_{L^2(\mathbb{Q}_p^n)} = ||\mathcal{F}f||_{L^2}$ for $f \in L^2(\mathbb{Q}_p^n)$. The Fourier transform $f \to \widehat{f}$ maps $L^2(\mathbb{Q}_p^n)$ onto $L^2(\mathbb{Q}_p^n)$ one-to-one and mutually continuous. Moreover, the Parseval-Steklov equality holds:

$$(f, g) = (\mathcal{F}f, \mathcal{F}g), \quad ||f||_{L^2(\mathbb{Q}_p^n)} = ||\mathcal{F}f||_{L^2(\mathbb{Q}_p^n)}, \quad f, g \in L^2(\mathbb{Q}_p^n).$$

For a more detailed discussion of the p-adic analysis the reader may consult [2, 36, 44].

3 The non-archimedean pseudo-differential operators $\mathcal{B}^{\alpha}_{\boldsymbol{h}_1,\boldsymbol{h}_2}$ and its Heat Kernel

Throughout this section we study new classes of p-adic pseudo-differential operators denoted as $\mathcal{B}^{\alpha}_{\boldsymbol{h}_1,\boldsymbol{h}_2}$ and we will prove some properties of the heat Kernel associated with this type of operators. Along this article, we write $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$.

Definition 1 **(Hypothesis B)** We say that the continuous functions $\boldsymbol{h}_1, \boldsymbol{h}_2 : \mathbb{Q}_p^n \to \mathbb{C}$ satisfies the *Hypothesis B* if the following properties hold.

(i) $\boldsymbol{h}_i$ is a radial function with $\boldsymbol{h}_i(0) = 0$, $i = 1, 2$. We use the notation $\boldsymbol{h}_i(\xi) = \boldsymbol{h}_i(||\xi||_p)$, $i = 1, 2$; $\xi \in \mathbb{Q}_p^n$, to indicate that $\boldsymbol{h}_i$, $i = 1, 2$ are radial functions on $\mathbb{Q}_p^n$.

(ii) $|\boldsymbol{h}_1|$ and $|\boldsymbol{h}_2|$ are increasing functions with respect to $||\cdot||_p$;

(iii) there is $r := r(\boldsymbol{h}_1, \boldsymbol{h}_2) \in \mathbb{Z}$, such that

$$|\boldsymbol{h}_2(||\xi||_p)| \geq |\boldsymbol{h}_1(||\xi||_p)| \text{ if and only if } \xi \in B_r^n;$$

(iv) there exist positive constants $C_1 := C_1(\boldsymbol{h}_1)$, $\beta_1 := \beta_1(\boldsymbol{h}_1)$ such that

$$|\boldsymbol{h}_1(||\xi||_p)| \geq C_1 ||\xi||_p^{\beta_1}, \text{ for all } \xi \in \mathbb{Q}_p^n \setminus B_r^n.$$

Example 1 (i) Taking $\boldsymbol{h}_1(||\xi||_p) := ||\xi||_p^{1/2}$ and $\boldsymbol{h}_2(||\xi||_p) := ||\xi||_p^{1/3}$, $\xi \in \mathbb{Q}_p^n$, we have that $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$ satisfies the Hypothesis B.

(2) Let α_1 and α_2 be positive constants such that $\alpha_1 \geq \alpha_2 > 0$. Then the functions $\boldsymbol{h}_1(||\xi||_p) := e^{||\xi||_p^{\alpha_1}} - 1$ and $\boldsymbol{h}_2(||\xi||_p) := e^{||\xi||_p^{\alpha_2}} - 1$, $\xi \in \mathbb{Q}_p^n$, satisfies the Hypothesis B.

Remark 1 We have that $e^{-t[\max\{|\boldsymbol{h}_1|,|\boldsymbol{h}_2|\}]^\alpha} \in L^\rho(\mathbb{Q}_p^n)$, $1 \leq \rho < \infty$, for all fixed $t > 0$ and $\alpha > 0$. Indeed, $\int_{\mathbb{Q}_p^n} e^{-t\rho[\max\{|\boldsymbol{h}_1(||\xi||_p)|,|\boldsymbol{h}_2(||\xi||_p)|\}]^\alpha} d^n\xi$ is exactly

$$\int_{B_r^n} e^{-t\rho|\boldsymbol{h}_2(||\xi||_p)|^\alpha} d^n\xi + \int_{\mathbb{Q}_p^n \setminus B_r^n} e^{-t\rho|\boldsymbol{h}_1(||\xi||_p)|^\alpha} d^n\xi = I_1 + I_2.$$

Since B_r^n is a compact set and $e^{-t\rho|\boldsymbol{h}_2(\cdot)|^\alpha}$ is a continuous function on B_r^n, we have that $I_1 < \infty$. Moreover,

$$\begin{aligned} I_2 &= \sum_{j=r+1}^{\infty} e^{-t\rho|\boldsymbol{h}_1(p^j)|^\alpha} \int_{||\xi||_p = p^j} d^n\xi \\ &\leq (1 - p^{-n}) \sum_{j=r+1}^{\infty} e^{-t\rho C_1^\alpha p^{j\alpha\beta_1}} p^{nj} \\ &< \infty. \end{aligned}$$

From now on, we assume that $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$ are functions satisfying the Hypothesis B. Moreover, for fixed $\alpha > 0$ we define the non-archimedean pseudo-differential operator

$$\begin{aligned} (\mathcal{B}^\alpha_{\boldsymbol{h}_1,\boldsymbol{h}_2}\varphi)(x) &:= -\mathcal{F}^{-1}_{\xi\to x}\left(\left[\max\{|\boldsymbol{h}_1(||\xi||_p)|, |\boldsymbol{h}_2(||\xi||_p)|\}\right]^\alpha \widehat{\varphi}(\xi)\right) \\ &= -\int_{\mathbb{Q}_p^n} \chi_p(-x\cdot\xi)\left[\max\{|\boldsymbol{h}_1(||\xi||_p)|, |\boldsymbol{h}_2(||\xi||_p)|\}\right]^\alpha \widehat{\varphi}(\xi) d^n\xi, \quad x \in \mathbb{Q}_p^n, \end{aligned}$$

where $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$ and the function $\left[\max\{|\boldsymbol{h}_1(||\xi||_p)|, |\boldsymbol{h}_2(||\xi||_p)|\}\right]^\alpha : \mathbb{Q}_p^n \to \mathbb{R}_+$, is the symbol of the operator $\mathcal{B}^\alpha_{\boldsymbol{h}_1,\boldsymbol{h}_2}$.

Remark 2 By [36, (1.3), p. 118] we have that $\mathcal{D}(\mathbb{Q}_p^n)$ is dense in $L^\rho(\mathbb{Q}_p^n)$, $1 \leq \rho < \infty$ and in $C_0(\mathbb{Q}_p^n)$.

Since $[\max\{|\boldsymbol{h}_1(\cdot)|, |\boldsymbol{h}_2(\cdot)|\}]^\alpha$ is a continuous function on $\mathbb{Q}_p^n$ and $\widehat{\varphi} \in \mathcal{D}(\mathbb{Q}_p^n)$ for $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$, we have that $[\max\{|\boldsymbol{h}_1(\cdot)|, |\boldsymbol{h}_2(\cdot)|\}]^\alpha\,\widehat{\varphi} \in L^1(\mathbb{Q}_p^n)$.

Therefore, by Riemann–Lebesgue Theorem (see [36, Theorem 1.6, p. 24]) and [36, (3.8), p. 38], we have that the operator $\mathcal{B}^\alpha_{\boldsymbol{h}_1,\boldsymbol{h}_2} : \mathcal{D}(\mathbb{Q}_p^n) \to C_0(\mathbb{Q}_p^n) \cap L^\rho(\mathbb{Q}_p^n)$, $1 \leq \rho < \infty$, is a well-defined pseudo-differential operator.

We define the heat Kernel or fundamental solution attached to operator $\mathcal{B}^\alpha_{\boldsymbol{h}_1,\boldsymbol{h}_2}$ as

$$Z_t(x) = Z(x,t) := \int_{\mathbb{Q}_p^n} \chi_p(-x\cdot\xi)\, e^{-t\left[\max\{|\boldsymbol{h}_1(||\xi||_p)|,|\boldsymbol{h}_2(||\xi||_p)|\}\right]^\alpha} d^n\xi, \quad x \in \mathbb{Q}_p^n, \quad t > 0. \tag{3.1}$$

The notation $Z_t(x)$ means that $Z(x,t)$ is for fixed $t > 0$ a function of x with $x \in \mathbb{Q}_p^n$.

Remark 3 (i) By [2, Example 4.9.1] we have for $x \in \mathbb{Q}_p^n$ that

$$\lim_{t \to 0^+} Z_t(x) = \delta(x),$$

where δ is the Dirac delta function.

(ii) Since $e^{-t[\max\{|\boldsymbol{h}_1(\cdot)|,|\boldsymbol{h}_2(\cdot)|\}]^\alpha} \in \varepsilon(\mathbb{Q}_p^n)$, then by [2, (4.4.3), p. 60] we have that $e^{-t[\max\{|\boldsymbol{h}_1(\cdot)|,|\boldsymbol{h}_2(\cdot)|\}]^\alpha}$ determines a regular distribution.
Therefore, by [2, Proposition 4.9.1] and the fact that $e^{-t[\max\{|\boldsymbol{h}_1(\cdot)|,|\boldsymbol{h}_2(\cdot)|\}]^\alpha}$ is a radial function, we have that $\mathcal{F}(Z_t(\cdot)) = e^{-t[\max\{|\boldsymbol{h}_1(\cdot)|,|\boldsymbol{h}_2(\cdot)|\}]^\alpha}$, for all $t > 0$.

Theorem 1 *The function $Z_t(\cdot)$ satisfies the following properties:*

(i) $Z(x,t) \geq 0$, *for all* $x \in \mathbb{Q}_p^n$ *and* $t > 0$.
(ii) For any $x \in \mathbb{Q}_p^n \backslash \{0\}$ *and* $t > 0$ *we have that* $Z(x,t) \leq t||x||_p^{-n}$.
(iii) $Z_{t+s}(x) = (Z_t * Z_s)(x)$, *for all* $t, s > 0$ *and* $x \in \mathbb{Q}_p^n$.
(iv) $\int_{\mathbb{Q}_p^n} Z(x,t) d^n x = 1$, *i.e.,* $Z(\cdot, t)$ *is a probability measure on* $\mathbb{Q}_p^n$, *for all* $t > 0$.

Proof (i) Since in the cases $x = 0$ the assertion is clear for all $t > 0$, we consider the case $x \in \mathbb{Q}_p^n \backslash \{0\}$.
We first note that

$$\begin{aligned}
&\int_{B_r^n} \chi_p(-x \cdot \xi) e^{-t|\boldsymbol{h}_2(||\xi||_p)|^\alpha} d^n\xi + \int_{\mathbb{Q}_p^n \backslash B_r^n} \chi_p(-x \cdot \xi) e^{-t|\boldsymbol{h}_1(||\xi||_p)|^\alpha} d^n\xi \\
&= \sum_{j=-\infty}^{r} e^{-t|\boldsymbol{h}_2(p^j)|^\alpha} \int\limits_{||p^j \xi||_p = 1} \chi_p(-x \cdot \xi) d^n\xi \\
&\quad + \sum_{j=r+1}^{\infty} e^{-t|\boldsymbol{h}_1(p^j)|^\alpha} \int\limits_{||p^j \xi||_p = 1} \chi_p(-x \cdot \xi) d^n\xi \\
&= \sum_{j=-\infty}^{r} e^{-t|\boldsymbol{h}_2(p^j)|^\alpha} p^{nj} \int\limits_{||w||_p = 1} \chi_p(-p^{-j} x \cdot w) d^n w \\
&\quad + \sum_{j=r+1}^{\infty} e^{-t|\boldsymbol{h}_1(p^j)|^\alpha} p^{nj} \int\limits_{||w||_p = 1} \chi_p(-p^{-j} x \cdot w) d^n w.
\end{aligned}$$

Therefore, for $x \in \mathbb{Q}_p^n \backslash \{0\}$ and $t > 0$, $Z(x,t)$ is exactly

$$\begin{aligned}
&\sum_{j=-\infty}^{r} e^{-t|\boldsymbol{h}_2(p^j)|^\alpha} p^{nj} \int\limits_{||w||_p = 1} \chi_p(-p^{-j} x \cdot w) d^n w \\
&\quad + \sum_{j=r+1}^{\infty} e^{-t|\boldsymbol{h}_1(p^j)|^\alpha} p^{nj} \int\limits_{||w||_p = 1} \chi_p(-p^{-j} x \cdot w) d^n w. \qquad (3.2)
\end{aligned}$$

For $x \in \mathbb{Q}_p^n \backslash \{0\}$ with $||x||_p = p^{-\gamma}$, $\gamma \in \mathbb{Z}$, and by using the formula

$$\int_{||w||_p=1} \chi_p\left(-p^{-j}x \cdot w\right) d^n w = \begin{cases} 1-p^{-n}, & \text{if} \quad j \leq \gamma, \\ -p^{-n}, & \text{if} \quad j = \gamma+1, \\ 0, & \text{if} \quad j \geq \gamma+2, \end{cases} \tag{3.3}$$

we consider the following cases for γ:
If $\gamma \leq r-1$, then by (3.2) and (3.3) we have that

$$\begin{aligned} Z(x,t) &= (1-p^{-n}) \sum_{j=-\gamma}^{\infty} e^{-t|\boldsymbol{h}_2(p^{-j})|^\alpha} p^{-nj} - e^{-t|\boldsymbol{h}_2(p^{\gamma+1})|^\alpha} p^{n\gamma} \\ &\geq e^{-t|\boldsymbol{h}_2(p^{\gamma})|^\alpha} \sum_{j=-\gamma}^{\infty} (p^{-nj} - p^{-n(j+1)}) - e^{-t|\boldsymbol{h}_2(p^{\gamma+1})|^\alpha} p^{n\gamma} \\ &= ||x||_p^{-n} \left\{ e^{-t|\boldsymbol{h}_2(||x||_p^{-1})|^\alpha} - e^{-t|\boldsymbol{h}_2(||x||_p^{-1}p)|^\alpha} \right\} \geq 0. \end{aligned}$$

On the other hand, if $\gamma \geq r$, then considering (3.2), (3.3) and applying mathematical induction on $\gamma - r$, we have:
If $\gamma - r = 0$ (or equivalently $\gamma = r$), then

$$\begin{aligned} Z(x,t) &= (1-p^{-n}) \sum_{j=-\gamma}^{\infty} e^{-t|\boldsymbol{h}_2(p^{-j})|^\alpha} p^{-nj} - e^{-t|\boldsymbol{h}_1(p^{\gamma+1})|^\alpha} p^{n\gamma} \\ &\geq e^{-t|\boldsymbol{h}_2(p^{\gamma})|^\alpha} \sum_{j=-\gamma}^{\infty} (p^{-nj} - p^{-n(j+1)}) - e^{-t|\boldsymbol{h}_1(p^{\gamma+1})|^\alpha} p^{n\gamma} \\ &= ||x||_p^{-n} \left\{ e^{-t|\boldsymbol{h}_2(||x||_p^{-1})|^\alpha} - e^{-t|\boldsymbol{h}_1(||x||_p^{-1}p)|^\alpha} \right\} \\ &\geq ||x||_p^{-n} \left\{ e^{-t|\boldsymbol{h}_2(||x||_p^{-1}p)|^\alpha} - e^{-t|\boldsymbol{h}_1(||x||_p^{-1}p)|^\alpha} \right\} \geq 0. \end{aligned}$$

Suppose that the assertion is true for $\gamma - r = m$ (or equivalently $\gamma = r + m$), for some $m \geq 1$, i.e.,

$$\begin{aligned} Z(x,t) = (1-p^{-n}) \sum_{j=-\infty}^{r} e^{-t|\boldsymbol{h}_2(p^{j})|^\alpha} p^{nj} + (1-p^{-n}) \sum_{j=r+1}^{r+m} e^{-t|\boldsymbol{h}_1(p^{j})|^\alpha} p^{nj} \\ - p^{-n} e^{-t|\boldsymbol{h}_1(p^{r+m+1})|^\alpha} p^{n(r+m+1)} \geq 0. \end{aligned}$$

Let's see if the hypothesis is met for $\gamma - r = m+1$ (or equivalently $\gamma = r+m+1$). Then

$$
\begin{aligned}
Z(x,t) &= (1-p^{-n}) \sum_{j=-\infty}^{r} e^{-t|\boldsymbol{h}_2(p^j)|^{\alpha}} p^{nj} + (1-p^{-n}) \sum_{j=r+1}^{r+m+1} e^{-t|\boldsymbol{h}_1(p^j)|^{\alpha}} p^{nj} \\
&\quad - p^{-n} e^{-t|\boldsymbol{h}_1(p^{r+m+2})|^{\alpha}} p^{n(r+m+2)} \\
&= (1-p^{-n}) \sum_{j=-\infty}^{r} e^{-t|\boldsymbol{h}_2(p^j)|^{\alpha}} p^{nj} + (1-p^{-n}) \sum_{j=r+1}^{r+m} e^{-t|\boldsymbol{h}_1(p^j)|^{\alpha}} p^{nj} \\
&\quad + (1-p^{-n}) e^{-t|\boldsymbol{h}_1(p^{r+m+1})|^{\alpha}} p^{n(r+m+1)} - p^{-n} e^{-t|\boldsymbol{h}_1(p^{r+m+2})|^{\alpha}} p^{n(r+m+2)} \\
&\geq (1-p^{-n}) \sum_{j=-\infty}^{r} e^{-t|\boldsymbol{h}_2(p^j)|^{\alpha}} p^{nj} + (1-p^{-n}) \sum_{j=r+1}^{r+m} e^{-t|\boldsymbol{h}_1(p^j)|^{\alpha}} p^{nj} \\
&\quad + (1-p^{-n}) e^{-t|\boldsymbol{h}_1(p^{r+m+1})|^{\alpha}} p^{n(r+m+1)} - e^{-t|\boldsymbol{h}_1(p^{r+m+1})|^{\alpha}} p^{n(r+m+1)} \\
&= (1-p^{-n}) \sum_{j=-\infty}^{r} e^{-t|\boldsymbol{h}_2(p^j)|^{\alpha}} p^{nj} + (1-p^{-n}) \sum_{j=r+1}^{r+m} e^{-t|\boldsymbol{h}_1(p^j)|^{\alpha}} p^{nj} \\
&\quad - p^{-n} e^{-t|\boldsymbol{h}_1(p^{r+m+1})|^{\alpha}} p^{n(r+m+1)} \geq 0.
\end{aligned}
$$

(ii) For $x = p^{\gamma} x_0 \neq 0$ such that $\gamma \in \mathbb{Z}$ and $||x_0||_p = 1$, and making the change of variable $z = p^{\gamma}\xi$, we have that

$$
\begin{aligned}
Z(x,t) &= \int_{\mathbb{Q}_p^n} \chi_p\left(-p^{\gamma} x_0 \cdot \xi\right) e^{-t\left[\max\{|\boldsymbol{h}_1(||\xi||_p)|, |\boldsymbol{h}_2(||\xi||_p)|\}\right]^{\alpha}} d^n\xi \\
&= ||x||_p^{-n} \int_{\mathbb{Q}_p^n} \chi_p(-z\cdot x_0)\, e^{-t\left[\max\{|\boldsymbol{h}_1(p^{\gamma}||z||_p)|, |\boldsymbol{h}_2(p^{\gamma}||z||_p)|\}\right]^{\alpha}} d^n z \\
&= ||x||_p^{-n} \int_{B_r^n} \chi_p(-z\cdot x_0)\, e^{-t|\boldsymbol{h}_2(p^{\gamma}||z||_p)|^{\alpha}} d^n z \\
&\quad + ||x||_p^{-n} \int_{\mathbb{Q}_p^n \setminus B_r^n} \chi_p(-z\cdot x_0)\, e^{-t|\boldsymbol{h}_1(p^{\gamma}||z||_p)|^{\alpha}} d^n z \\
&= ||x||_p^{-n} \{I_1 + I_2\}.
\end{aligned}
$$

Making the change of variable $w = p^j z$, we have that

$$
\begin{aligned}
I_1 &= \sum_{j=-\infty}^{r} e^{-t|\boldsymbol{h}_2(p^{\gamma+j})|^{\alpha}} \int_{||p^j z||_p = 1} \chi_p(-z\cdot x_0)\, d^n z \\
&= \sum_{j=-\infty}^{r} e^{-t|\boldsymbol{h}_2(p^{\gamma+j})|^{\alpha}} p^{nj} \int_{||w||_p=1} \chi_p\left(-p^{-j} x_0 \cdot w\right) d^n w.
\end{aligned}
$$

So

$$I_1 = \sum_{j=-\infty}^{r} e^{-t|\boldsymbol{h}_2(p^{\gamma+j})|^\alpha} p^{nj} \begin{cases} 1-p^{-n}, & \text{if } \ j \le 0, \\ -p^{-n}, & \text{if } \ j=1, \\ 0, & \text{if } \ j \ge 2. \end{cases} \tag{3.4}$$

On the other hand, we have that

$$\begin{aligned} I_2 &= \sum_{j=r+1}^{\infty} e^{-t|\boldsymbol{h}_1(p^{\gamma+j})|^\alpha} \int_{||p^j z||_p=1} \chi_p\left(-z\cdot x_0\right) d^n z \\ &= \sum_{j=r+1}^{\infty} e^{-t|\boldsymbol{h}_1(p^{\gamma+j})|^\alpha} p^{nj} \int_{||w||_p=1} \chi_p\left(-p^{-j}x_0\cdot w\right) d^n w. \end{aligned}$$

So

$$I_2 = \sum_{j=r+1}^{\infty} e^{-t|\boldsymbol{h}_1(p^{\gamma+j})|^\alpha} p^{nj} \begin{cases} 1-p^{-n}, & \text{if } \ j \le 0, \\ -p^{-n}, & \text{if } \ j=1, \\ 0, & \text{if } \ j \ge 2. \end{cases} \tag{3.5}$$

Consider the following cases for r:
If $r=0$, then by (3.4) and (3.5) we have that

$$\begin{aligned} Z(x,t) &= ||x||_p^{-n} \left\{ (1-p^{-n}) \sum_{j=0}^{\infty} e^{-t|\boldsymbol{h}_2(||x||_p^{-1}p^{-j})|^\alpha} p^{-nj} - e^{-t|\boldsymbol{h}_1(||x||_p^{-1}p)|^\alpha} \right\} \\ &\le ||x||_p^{-n} \left\{ \sum_{j=0}^{\infty} (p^{-nj} - p^{-n(j+1)}) - e^{-t|\boldsymbol{h}_1(||x||_p^{-1}p)|^\alpha} \right\} \\ &= ||x||_p^{-n} \left\{ 1 - e^{-t|\boldsymbol{h}_1(||x||_p^{-1}p)|^\alpha} \right\}. \end{aligned}$$

Now, by applying the mean value theorem to the real function $e^{-u|\boldsymbol{h}_1(||x||_p^{-1}p)|^\alpha}$ on $[0,t]$ with $t>0$, we have

$$1 - e^{-t|\boldsymbol{h}_1(||x||_p^{-1}p)|^\alpha} = te^{-\tau|\boldsymbol{h}_1(||x||_p^{-1}p)|^\alpha}$$

for some $\tau \in (0,t)$. So that,

$$Z(x,t) \le t||x||_p^{-n}. \tag{3.6}$$

If $r>0$, then by (3.4) and (3.5) we have that

$$\begin{aligned} Z(x,t) &= ||x||_p^{-n} \left\{ (1-p^{-n}) \sum_{j=0}^{\infty} e^{-t|\boldsymbol{h}_2(||x||_p^{-1}p^{-j})|^\alpha} p^{-nj} - e^{-t|\boldsymbol{h}_2(||x||_p^{-1}p)|^\alpha} \right\} \\ &\le ||x||_p^{-n} \left\{ e^{-t|\boldsymbol{h}_2(||x||_p^{-1})|^\alpha} - e^{-t|\boldsymbol{h}_2(||x||_p^{-1}p)|^\alpha} \right\} \end{aligned}$$

$$\leq ||x||_p^{-n} \left\{1 - e^{-t|\boldsymbol{h}_2(||x||_p^{-1} p)|^\alpha}\right\}.$$

Proceeding analogously to the previous case we obtain (3.6).

If $r < 0$, then by (3.4) and (3.5) we have that

$$Z(x,t) = ||x||_p^{-n} \left\{(1 - p^{-n}) \sum_{j=-\infty}^{r} e^{-t|\boldsymbol{h}_2(||x||_p^{-1} p^j)|^\alpha} p^{nj} \right.$$
$$\left. + (1 - p^{-n}) \sum_{j=r+1}^{0} e^{-t|\boldsymbol{h}_1(||x||_p^{-1} p^j)|^\alpha} p^{nj} - e^{-t|\boldsymbol{h}_1(||x||_p^{-1} p)|^\alpha}\right\}. \tag{3.7}$$

The last equality allows us consider the following cases for $||x||_p^{-1}$.

Case 1. $||x||_p^{-1} \leq p^r$. Then, by (3.7) we have that

$$Z(x,t) \leq ||x||_p^{-n} \left\{(1 - p^{-n}) \sum_{j=-\infty}^{r} e^{-t|\boldsymbol{h}_1(||x||_p^{-1} p^j)|^\alpha} p^{nj} \right.$$
$$\left. + (1 - p^{-n}) \sum_{j=r+1}^{0} e^{-t|\boldsymbol{h}_1(||x||_p^{-1} p^j)|^\alpha} p^{nj} - e^{-t|\boldsymbol{h}_1(||x||_p^{-1} p)|^\alpha}\right\},$$

since $||x||_p^{-1} p^j \leq p^r$ for all $-\infty < j \leq r$. Consequently,

$$Z(x,t) \leq ||x||_p^{-n} \left\{(1 - p^{-n}) \sum_{j=0}^{\infty} e^{-t|\boldsymbol{h}_1(||x||_p^{-1} p^{-j})|^\alpha} p^{-nj} - e^{-t|\boldsymbol{h}_1(||x||_p^{-1} p)|^\alpha}\right\}$$
$$\leq ||x||_p^{-n} \left\{e^{-t|\boldsymbol{h}_1(||x||_p^{-1})|^\alpha} \sum_{j=0}^{\infty}(p^{-nj} - p^{-n(j+1)}) - e^{-t|\boldsymbol{h}_1(||x||_p^{-1} p)|^\alpha}\right\}$$
$$\leq ||x||_p^{-n} \left\{1 - e^{-t|\boldsymbol{h}_1(||x||_p^{-1} p)|^\alpha}\right\}.$$

Now, proceeding analogously to the case when $r = 0$ we obtain (3.6).

Case 2. $||x||_p^{-1} > p^r$. Then, by (3.7) we have that

$$
\begin{aligned}
Z(x,t) &= ||x||_p^{-n}\left\{(1-p^{-n})\sum_{j=-r}^{\infty} e^{-t|\boldsymbol{h}_2(||x||_p^{-1}p^{-j})|^{\alpha}}p^{-nj}\right.\\
&\qquad\left. + (1-p^{-n})\sum_{j=0}^{-r-1} e^{-t|\boldsymbol{h}_1(||x||_p^{-1}p^{-j})|^{\alpha}}p^{-nj} - e^{-t|\boldsymbol{h}_1(||x||_p^{-1}p)|^{\alpha}}\right\}\\
&\le ||x||_p^{-n}\left\{(1-p^{-n})\sum_{j=0}^{\infty} e^{-t|\boldsymbol{h}_2(||x||_p^{-1}p^{-j})|^{\alpha}}p^{-nj}\right.\\
&\qquad\left. + (1-p^{-n})\sum_{j=0}^{\infty} e^{-t|\boldsymbol{h}_1(||x||_p^{-1}p^{-j})|^{\alpha}}p^{-nj} - e^{-t|\boldsymbol{h}_1(||x||_p^{-1}p)|^{\alpha}}\right\}\\
&= ||x||_p^{-n}\left\{e^{-t|\boldsymbol{h}_2(||x||_p^{-1})|^{\alpha}} + e^{-t|\boldsymbol{h}_1(||x||_p^{-1})|^{\alpha}} - e^{-t|\boldsymbol{h}_1(||x||_p^{-1}p)|^{\alpha}}\right\}.
\end{aligned}
$$

Since $e^{t|\boldsymbol{h}_2(||x||_p^{-1})|^{\alpha}} > 1$, $e^{t|\boldsymbol{h}_1(||x||_p^{-1})|^{\alpha}} > 1$ and $m + p \le 2mp$ if $m, p \ge 1$, we have that

$$
Z(x,t) \le ||x||_p^{-n}\left\{1 - e^{-t|\boldsymbol{h}_1(||x||_p^{-1}p)|^{\alpha}}\right\}.
$$

Now, proceeding analogously to the case when $r = 0$ we obtain (3.6).

(iii) For $t, s > 0$ and $x \in \mathbb{Q}_p^n$ we have by Remark 3-(ii) that

$$
\begin{aligned}
Z_{t+s}(x) &= \int_{\mathbb{Q}_p^n} \chi_p(-x\cdot\xi)\, e^{-(t+s)\left[\max\{|\boldsymbol{h}_1(||\xi||_p)|,|\boldsymbol{h}_2(||\xi||_p)|\}\right]^{\alpha}} d^n\xi\\
&= \int_{\mathbb{Q}_p^n} \chi_p(-x\cdot\xi)\, e^{-t\left[\max\{|\boldsymbol{h}_1(||\xi||_p)|,|\boldsymbol{h}_2(||\xi||_p)|\}\right]^{\alpha}} \widehat{Z}_s(\xi) d^n\xi\\
&= (Z_t * Z_s)(x).
\end{aligned}
$$

(iv) By Remark 3-(ii) we have that $\mathcal{F}(Z_t(x)) = e^{-t\left[\max\{|\boldsymbol{h}_1(||x||_p)|,|\boldsymbol{h}_2(||x||_p)|\}\right]^{\alpha}}$, for all $x \in \mathbb{Q}_p^n$. Therefore, by Definition 1-(i) we have that $\mathcal{F}(Z(0,t)) = 1$. On the other hand, $\mathcal{F}(Z(x,t)) = \int_{\mathbb{Q}_p^n} \chi_p(\xi\cdot x)Z(x,t)d^n x$ and $\mathcal{F}(Z(0,t)) = \int_{\mathbb{Q}_p^n} Z(x,t)d^n x$. Therefore, $\int_{\mathbb{Q}_p^n} Z(x,t)d^n x = 1$, for all $t > 0$.

□

Remark 4 The following affirmations are satisfied:

(i) By Remark 1, the previous theorem and [2, Theorem 5.3.1] we have that $Z_t(\cdot) \in L^1(\mathbb{Q}_p^n) \cap L^2(\mathbb{Q}_p^n)$, for all $t > 0$.

(ii) A function $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ is called negative definite, if

$$
\sum\nolimits_{i,j=1}^{m}\left(\varphi(x_i) + \overline{\varphi(x_j)} - \varphi(x_i - x_j)\right)\lambda_i\overline{\lambda_j} \ge 0
$$

for all $m \in \mathbb{N}\backslash\{0\}$, $x_1, \ldots, x_m \in \mathbb{Q}_p^n$, $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$. If $\varphi : \mathbb{Q}_p^n \to \mathbb{R}$ is a negative definite function, then by [9] we have that $\varphi(x) \geq \varphi(0) \geq 0$ for all $x \in \mathbb{Q}_p^n$. By the previous theorem and [9, Theorem 8.3] we have that the function $\left[\max\{|\boldsymbol{h}_1(||\cdot||_p)|, |\boldsymbol{h}_2(||\cdot||_p)|\}\right]^\alpha$ is negative definite on $\mathbb{Q}_p^n$.

Remark 5 Consider the following Cauchy problem:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = -\mathcal{B}^\alpha_{\boldsymbol{h}_1,\boldsymbol{h}_2}(u(x,t)),\, t \in (0,\infty)\,,\ \ x \in \mathbb{Q}_p^n \\ \\ u(x,0) = u_0(x) \in \mathcal{D}(\mathbb{Q}_p^n). \end{cases} \tag{3.8}$$

By proceeding as in [41], we have that

$$\begin{aligned} u(x,t) &= \int_{\mathbb{Q}_p^n} \chi_p\left(-x\cdot\xi\right) e^{-t\left[\max\{|\boldsymbol{h}_1(||\xi||_p)|, |\boldsymbol{h}_2(||\xi||_p)|\}\right]^\alpha} \widehat{u_0}(\xi) d^n\xi,\ x \in \mathbb{Q}_p^n,\ t \geq 0, \\ &= Z_t(x) * u_0(x) \end{aligned}$$

is a classical solution of (3.8), where $Z_t(x)$ is the heat Kernel defined in (3.1). Moreover,

(i) $u(x,t)$ satisfies the principles of *mass conservation* and *comparison* i.e., $u(x,t)$, $x \in \mathbb{Q}_p^n$, $t \geq 0$, satisfies, respectively, $\int_{\mathbb{Q}_p^n} u(x,t) d^n x = \int_{\mathbb{Q}_p^n} u_0(x) d^n x$ and if $u_0(x) \geq v_0(x)$ for all $x \in \mathbb{Q}_p^n$, then $u(x,t) \geq v(x,t)$.
(ii) By Theorem 1 we have that there exists a constant $C := C(u_0)$ such that $|u(x,t)| \leq Ct||x||_p^{-n}$, for all $x \in \mathbb{Q}_p^n\backslash\{0\}$ and $t > 0$.

4 Feller semigroups, transition functions and Strong Markov processes

In this section, we will show the existence of Feller semigroups, transition functions and Strong Markov processes on $\mathbb{Q}_p^n$ associated with the heat Kernel $Z_t(x)$. In what follows we denote by $\mathcal{B}(\mathbb{Q}_p^n)$ the σ-algebra of all Borel sets in $\mathbb{Q}_p^n$. For the basic results on Feller semigroups, transition functions and Strong Markov processes, the reader may consult [37].

We define for $u \in C_0(\mathbb{Q}_p^n)$, the heat Kernel given at (3.1) and $x \in \mathbb{Q}_p^n$, the operators

$$T_t u(x) := \begin{cases} u(x) & \text{if } t = 0, \\ \\ \int_{\mathbb{Q}_p^n} Z_t(x-y)u(y)d^n y = (Z_t * u)(x) & \text{if } t > 0, \end{cases} \tag{4.1}$$

Lemma 1 *$T_t : C_0(\mathbb{Q}_p^n) \to C_0(\mathbb{Q}_p^n)$, $t \geq 0$, are well-defined contraction operators.*

Proof If $t = 0$, the assertion is clear. Let $t > 0$, $u \in C_0(\mathbb{Q}_p^n)$ and $x \in \mathbb{Q}_p^n$. Then, by Theorem 1 we have

$$|T_t u(x)| = \left| \int_{\mathbb{Q}_p^n} Z_t(x-y)u(y)d^n y \right| \leq ||u||_{L^\infty(\mathbb{Q}_p^n)} \int_{\mathbb{Q}_p^n} Z_t(x-y)d^n y = ||u||_{L^\infty(\mathbb{Q}_p^n)}. \tag{4.2}$$

By (3.1), [2, Subsection 4.9] and [2, Subsection 5.2] we have that Z_t is continuous. We now show that $|T_t u(x)| \to 0$ when $||x||_p \to \infty$. Since $u \in C_0(\mathbb{Q}_p^n)$, we can assume without loss of generality taht $Supp(u) \subseteq B_M^n$, $M \in \mathbb{Z}$.

By using the compactness of B_M^n and the ultrametricity of the norm $|| \cdot ||_p$, by Theorem 1 we have for $||x||_p \gg 0$ that

$$\begin{aligned} 0 &\leq |T_t u(x)| \leq ||u||_{L^\infty(\mathbb{Q}_p^n)} \int_{B_M^n} Z_t(x-y)d^n y \leq t||u||_{L^\infty(\mathbb{Q}_p^n)} \int_{B_M^n} ||x-y||_p^{-n} d^n y \\ &= t||u||_{L^\infty(\mathbb{Q}_p^n)} ||x||_p^{-n} Vol(B_M^n) = 0. \end{aligned}$$

Therefore, $T_t : C_0(\mathbb{Q}_p^n) \to C_0(\mathbb{Q}_p^n)$ is a well-defined bounded linear operator. □

By a direct calculation one proves the following result.

Lemma 2 *The operators T_t, $t \geq 0$, satisfies the semigroup property i.e.,*

$$T_t(T_s u)(x) = T_{t+s} u(x), \quad u \in C_0(\mathbb{Q}_p^n), \quad x \in (\mathbb{Q}_p^n), \quad t, s \geq 0.$$

Lemma 3 *For fixed $u \in C_0(\mathbb{Q}_p^n)$, the operators T_t, $t \geq 0$, satisfies the following condition:*

$$\lim_{t \to 0^+} ||T_t u - u||_{L^\infty(\mathbb{Q}_p^n)} = 0.$$

Proof Let fixed $x \in \mathbb{Q}_p^n$. Note that

$$(T_t u - u)(x) = \int_{\mathbb{Q}_p^n} Z_t(x-y) \left[u(y) - u(x) \right] d^n y. \tag{4.3}$$

Since that $u \in C_0(\mathbb{Q}_p^n)$, given any number $\epsilon > 0$, however small, there exists some number $s := s(x, \epsilon) \in \mathbb{Z}$ such that if $||x-y||_p < p^s$ then $||u(y) - u(x)||_{L^\infty(\mathbb{Q}_p^n)} < \epsilon$. Then, by (4.3) and Theorem 1 we have that

$$\begin{aligned} |(T_t u - u)(x)| &\leq \int_{||x-y||_p < p^s} Z_t(x-y) \left| u(y) - u(x) \right| d^n y \\ &\quad + \int_{||x-y||_p \geq p^s} Z_t(x-y) \left| u(y) - u(x) \right| d^n y \\ &\leq \epsilon + 2||u||_{L^\infty(\mathbb{Q}_p^n)} \int_{||w||_p \geq p^s} Z_t(w) d^n w \\ &\leq \epsilon + 2t||u||_{L^\infty(\mathbb{Q}_p^n)} \int_{||w||_p \geq p^s} ||w||_p^{-n} d^n w \end{aligned}$$

Now, since $\int_{||w||_p \geq p^s} ||w||_p^{-n} d^n w = C < \infty$, we have that

$$|(T_t u - u)(x)| \leq \epsilon + 2Ct||u||_{L^\infty(\mathbb{Q}_p^n)}.$$

Therefore, given any $\epsilon > 0$ we have that

$$\lim_{t \to 0^+} \sup |(T_t u - u)(x)| \leq \epsilon,$$

for all $x \in \mathbb{Q}_p^n$. □

Remark 6 By Lemmas 1 and 3 we have that the semigroup $\{T_t\}_{t \geq 0}$ is strongly continuous in t for all $t \geq 0$:

$$\lim_{s \to 0^+} ||T_{t+s} f - T_t f||_{L^\infty(\mathbb{Q}_p^n)} = 0, \ f \in C_0(\mathbb{Q}_p^n).$$

Moreover, by Theorems 1 and (4.1) we have that the operators $\{T_t\}_{t \geq 0}$ is non-negative and contractive on $C_0(\mathbb{Q}_p^n)$:

$$f \in C_0(\mathbb{Q}_p^n), \ 0 \leq f(x) \leq 1 \text{ on } K \Longrightarrow 0 \leq T_t f(x) \leq 1 \text{ on } \mathbb{Q}_p^n.$$

Theorem 2 *The operators T_t, $t \geq 0$, correspond to a Feller semigroup on $\mathbb{Q}_p^n$.*

Proof The result follows from Remark 6, Lemmas 1, 2 and [37], taking into account that

$$(T_t u)(x) \geq 0, \ \text{ for } u \in C_0(\mathbb{Q}_p^n) \text{ with } u \geq 0 \text{ and } t \geq 0.$$

□

Definition 2 For $E \in \mathcal{B}(\mathbb{Q}_p^n)$, we define

$$p_t(x, E) = \begin{cases} Z_t(x) * 1_E(x), & \text{for } t > 0, x \in \mathbb{Q}_p^n \\ \\ 1_E(x), & \text{for } t = 0, x \in \mathbb{Q}_p^n. \end{cases} \tag{4.4}$$

As a direct consequence of [37, Subsections 2.2.4 and 2.2.5] and Theorem 2, we obtain the following result.

Theorem 3 *The function $p_t(x, E)$ satisfies the following conditions:*

(i) $p_t(x, E)$ is a Markov transition function on $\mathbb{Q}_p^n$, i.e.,

(a) $p_t(x, \cdot)$ is a measure on $\mathcal{B}(\mathbb{Q}_p^n)$ and $p_t(x, \mathbb{Q}_p^n) \leq 1$ for all $t \geq 0$ and $x \in \mathbb{Q}_p^n$.
(b) $p_t(\cdot, E)$ is a Borel measurable function for all $t \geq 0$ and $E \in \mathcal{B}(\mathbb{Q}_p^n)$.
(c) $p_0(x, \{x\}) = 1$ for all $x \in \mathbb{Q}_p^n$.
(d) For all $t, s \geq 0$, $x \in \mathbb{Q}_p^n$ and $E \in \mathcal{B}(\mathbb{Q}_p^n)$, we have the equation

$$p_{t+s}(x, E) = \int_{\mathbb{Q}_p^n} p_t(x, d^n y) p_s(y, E).$$

(ii) *For each $s > 0$ and each compact subset $E \subset \mathbb{Q}_p^n$,*

$$\lim_{x \to \infty} \sup_{0 \le t \le s} p_t(x, E) = 0.$$

(iii) *For each $r \in \mathbb{Z}$ and each compact $E \subset \mathbb{Q}_p^n$, we have that*

$$\lim_{t \to 0^+} \sup_{x \in E} [1 - p_t(x, B_r^n(x))] = 0.$$

(iv) *The space $C_0(\mathbb{Q}_p^n)$ is invariant for the operators T_t, $t \ge 0$, i.e.,*

$$f \in C_0(\mathbb{Q}_p^n) \Longrightarrow T_t f \in C_0(\mathbb{Q}_p^n).$$

(v) *For $f \in C_0(\mathbb{Q}_p^n)$ and $t \ge 0$, the formula*

$$T_t f(x) := \int_{\mathbb{Q}_p^n} p_t(x, d^n y) f(y),$$

holds.

(vi) *$p_t(x, \cdot)$ is the transition function of some strong Markov processes $\mathfrak{X}$ with state space $\mathbb{Q}_p^n$ and whose paths are right continuous and have no discontinuities other than jumps.*

We have that $\mathbb{Q}_p^n$ is locally compact and separable, see e.g. [2, 44]. We consider $\mathcal{P}(\mathbb{Q}_p^n)$ as the family of Borel probability measures on $\mathbb{Q}_p^n$ and use the terminology and notation of [13]. A collection $\{\mathcal{F}_t\} \equiv \{\mathcal{F}_t, t \in \mathcal{I}\}$ of σ-algebras of sets in $\mathcal{F}$ is a filtration if $\mathcal{F}_t \subset \mathcal{F}_{t+s}$, for $t, s \in \mathcal{I}$. Intuitively $\mathcal{F}_t$ corresponds to the information known to an observer at time t. A process $\mathfrak{X}$ is adapted to a filtration $\mathcal{F}_t$ (or simply $\{\mathcal{F}_t\}$-adapted) if $\mathfrak{X}(t)$ is $\mathcal{F}_t$-measurable for each $t \ge 0$.

Let $\{X(T)\}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$ with values in $\mathbb{Q}_p^n$, and let $\mathcal{F}_t^X = \sigma(X(s) : s \le t)$. Then X is a Markov process if

$$P\left\{X(t+s) \in \Gamma : \mathcal{F}_t^X\right\} = P\left\{X(t+s) \in \Gamma : X(t)\right\} \tag{4.5}$$

for all $s, t \ge 0$ and $\Gamma \in \mathcal{B}(\mathbb{Q}_p^n)$. If $\mathcal{G}_t$ is a filtration with $\mathcal{F}_t^X \subset \mathcal{G}_t$, $t \ge 0$, then X is a Markov process with respect to $\{\mathcal{G}_t\}$ if (4.5) holds with $\mathcal{F}_t^X$ replaced by $\mathcal{G}_t$.

For the following result we will consider the Feller semigroup $\{T_t\}_{t \ge 0}$ obtained in the Theorem 2.

Theorem 4 *For each $\nu \in \mathcal{P}(\mathbb{Q}_p^n)$, there exists a Markov process X corresponding to $\{T_t\}_{t \ge 0}$ with initial distribution ν and sample paths in $D_{\mathbb{Q}_p^n}[0, \infty)$ ($D_{\mathbb{Q}_p^n}[0, \infty)$ is the space of right continuous functions $f : [0, \infty) \to \mathbb{Q}_p^n$ with left limits). Moreover, X is strong Markov with respect to the filtration $\mathcal{G}_t = \mathcal{F}_{t^+}^X = \bigcap_{\epsilon > 0} \mathcal{F}_{t+\epsilon}^X$.*

Proof The result follow from [13, Theorem 2.7, p. 169]. $\square$

5 Contraction semigroups on $L^2(\mathbb{Q}_p^n)$ and inhomogeneous initial value problem

In this section, new types of contraction semigroups and inhomogeneous initial value problem are introduced.

Definition 3 For $f \in L^2(\mathbb{Q}_p^n)$, we define

$$P_t f(x) := \begin{cases} (Z_t * f)(x), & \text{for } t > 0, x \in \mathbb{Q}_p^n \\ \\ f, & \text{for } t = 0 \text{ , } x \in \mathbb{Q}_p^n, \end{cases}$$

where $(Z_t)_{t>0}$ is the function given by (3.1).

Lemma 4 *The family of operators $(P_t)_{t\geq 0}$ is a contraction semigroup in $L^2(\mathbb{Q}_p^n)$, i.e., $(P_t)_{t\geq 0}$ satisfies the following conditions:*

(i) $||P_t||_{L^2(\mathbb{Q}_p^n)} \leq 1$ for all $t \geq 0$;
(ii) $P_0 = I$;
(iii) $P_{t+s} = P_t P_s$ for all $s, t \geq 0$;
(iv) for all $f \in L^2(\mathbb{Q}_p^n)$, the function $t \mapsto P_t f$ belongs to $C_0([0, \infty), L^2(\mathbb{Q}_p^n))$.

Proof (i) If $t = 0$ the assertion is clear. Let fixed $f \in L^2(\mathbb{Q}_p^n)$ and $t > 0$. Then by Theorem 1-(iv) we have that

$$\begin{aligned} ||P_t f||^2_{L^2(\mathbb{Q}_p^n)} &= ||Z_t * f||^2_{L^2(\mathbb{Q}_p^n)} \\ &= \int_{\mathbb{Q}_p^n} |Z_t(x-y) f(y)|^2 \, d^n y \\ &\leq ||f||_{L^2(\mathbb{Q}_p^n)}. \end{aligned}$$

Therefore, for all $t \geq 0$, $P_t : L^2(\mathbb{Q}_p^n) \to L^2(\mathbb{Q}_p^n)$ is a well-defined bounded linear operator with $||P_t||_{L^2(\mathbb{Q}_p^n)} \leq 1$.

(ii) This is a direct consequence of the definition of P_t.

(iii) We consider the case $t, s > 0$, since in the other cases the assertion is clear. For all $s, t > 0$, $f \in L^2(\mathbb{Q}_p^n)$ and by using Theorem 1-(iii), we have that

$$\begin{aligned} P_{t+s} f(x) &= (Z_{t+s} * f)(x) \\ &= (Z_t * (Z_s * f))(x) \\ &= (Z_t * P_s f)(x) \\ &= P_t P_s f(x). \end{aligned}$$

Then, $P_{t+s} = P_t P_s$ for all $s, t \geq 0$.

(iv) Is a direct consequence of (i), (ii) and Theorem 1.

□

Remark 7 An operator A in a Banach space X, endowed with the norm $||\cdot||$, is called dissipative if $||u-\lambda Au|| \geq ||u||$, for all $u \in D(A)$ and all $\lambda > 0$. Moreover, if A is dissipative and for all $\lambda > 0$ and $f \in X$, there exists $u \in D(A)$ such that $u - \lambda Au = f$, then A is called m-dissipative.

Now, if X is a Hilbert space and $(\cdot,\cdot)$ its scalar product, then the operator A is dissipative in X if and only if $(Au, u) \leq 0$, for all $u \in D(A)$. For more details, the reader may consult [10, 29], and the references therein.

Lemma 5 *The pseudo-differential operator $\mathcal{B}^{\alpha}_{\boldsymbol{h}_1,\boldsymbol{h}_2}$ is dissipative in $L^2(\mathbb{Q}_p^n)$.*

Proof Let fixed $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$. Then by [2, Theorem 5.3.1] we have that

$$\begin{aligned}\left(\mathcal{B}^{\alpha}_{\boldsymbol{h}_1,\boldsymbol{h}_2}\varphi,\varphi\right) &= -\left(\mathcal{F}^{-1}_{\xi\to x}\left([\max\{|\boldsymbol{h}_1|,|\boldsymbol{h}_2|\}]^{\alpha}\,\widehat{\varphi}\right),\varphi\right)\\ &= -\left([\max\{|\boldsymbol{h}_1|,|\boldsymbol{h}_2|\}]^{\alpha}\,\widehat{\varphi},\widehat{\varphi}\right)\\ &= -\int_{\mathbb{Q}_p^n}[\max\{|\boldsymbol{h}_1|,|\boldsymbol{h}_2|\}]^{\alpha}\,|\widehat{\varphi}(\xi)|^2 d^n\xi \leq 0.\end{aligned}$$

Therefore, the pseudo-differential operator $\mathcal{B}^{\alpha}_{\boldsymbol{h}_1,\boldsymbol{h}_2}$ is dissipative in $L^2(\mathbb{Q}_p^n)$. $\square$

Remark 8 [10, Subsection 2.4] If A is a linear operator in the Hilbert space $(X, \langle\cdot,\cdot\rangle)$ with dense domain, then

$$G(A^*) := \{(v,\varphi) \in X \times X : \langle\varphi, u\rangle = \langle v, f\rangle \text{ for all } (u,f) \in G(A)\},$$

defines a linear operator A^* (the adjoint of A). The domain of A^* is

$$D(A^*) := \{v \in X : \exists C < \infty, |\langle Au, v\rangle| \leq C||u||, \forall u \in D(A)\},$$

and A^* satisfies

$$\left\langle A^* v, u\right\rangle = \langle v, Au\rangle, \forall u \in D(A).$$

We say that A is self-adjoin if $A^* = A$.

By application of Parseval–Steklov equality, we obtain the following result.

Lemma 6 *The pseudo-differential operator $\mathcal{B}^{\alpha}_{\boldsymbol{h}_1,\boldsymbol{h}_2}$ is self-adjoint, i.e. for $f, g \in \mathcal{D}(\mathbb{Q}_p^n)$ we have that*

$$\left\langle\mathcal{B}^{\alpha}_{\boldsymbol{h}_1,\boldsymbol{h}_2} f, g\right\rangle = \left\langle f, \mathcal{B}^{\alpha}_{\boldsymbol{h}_1,\boldsymbol{h}_2} g\right\rangle.$$

Proof For $f, g \in \mathcal{D}(\mathbb{Q}_p^n)$,

$$\begin{aligned}\left(\mathcal{B}^{\alpha}_{\boldsymbol{h}_1,\boldsymbol{h}_2} f, g\right) &= \left(-\mathcal{F}^{-1}_{\xi\to x}\left([\max\{|\boldsymbol{h}_1|,|\boldsymbol{h}_2|\}]^{\alpha}\,\widehat{f}\right), g\right)\\ &= -\int_{\mathbb{Q}_p^n}\widehat{f}(\xi)\overline{\left[\max\{|\boldsymbol{h}_1(||\xi||_p)|,|\boldsymbol{h}_2(||\xi||_p)|\}\right]^{\alpha}\widehat{g}}(\xi)d^n\xi\end{aligned}$$

$$= \left(f, -\mathcal{F}^{-1}_{\xi \to x} \left([\max\{|\boldsymbol{h}_1|, |\boldsymbol{h}_2|\}]^{\alpha} \widehat{g} \right) \right)$$
$$= \left(f, \mathcal{B}^{\alpha}_{\boldsymbol{h}_1, \boldsymbol{h}_2} g \right).$$

□

Theorem 5 *The pseudo-differential operator $\mathcal{B}^{\alpha}_{\boldsymbol{h}_1,\boldsymbol{h}_2}$ is $m-$dissipative in $L^2(\mathbb{Q}_p^n)$.*

Proof The result follows from Lemma 5, Lemma 6 and [10, Corollary 2.4.8]. □

Remark 9 The generator of $(P_t)_{t\geq 0}$ is the linear operator L defined by

$$D(L) = \left\{ f \in L^2(\mathbb{Q}_p^n) : \frac{P_t f - f}{h} \text{ has a limit in } L^2(\mathbb{Q}_p^n) \text{ as } h \to 0^+ \right\},$$

and

$$Lf = \lim_{h \to 0^+} \frac{P_t f - f}{h},$$

for all $f \in D(L)$.

Theorem 6 *The pseudo-differential operator $\mathcal{B}^{\alpha}_{\boldsymbol{h}_1,\boldsymbol{h}_2}$ is the generator of the contraction semigroup $(P_t)_{t\geq 0}$ obtained in the Lemma 4.*

Proof The result follows from Remark 2, Theorem 5 and [10, Theorem 3.4.4]. □

Considering the contraction semigroup $(P_t)_{t\geq 0}$ obtained in the Lemma 4 we will study the initial value problem

$$\begin{cases} u(x,\cdot) \in C([0,T], \mathcal{D}(\mathbb{Q}_p^n)) \cap C^1([0,T], L^2(\mathbb{Q}_p^n)), \\ \frac{\partial u}{\partial t}(x,t) = \mathcal{B}^{\alpha}_{\boldsymbol{h}_1,\boldsymbol{h}_2} u(x,t) + g(x,t), \quad t \in [0,T], \quad x \in \mathbb{Q}_p^n, \\ u_0(x) = h(x) \in L^2(\mathbb{Q}_p^n). \end{cases} \tag{5.1}$$

where $T > 0$ and $g : \mathbb{Q}_p^n \times [0,T] \to L^2(\mathbb{Q}_p^n)$ be a function such that

$$g \in L^1((0,T), L^2(\mathbb{Q}_p^n)) \cap C([0,T], L^2(\mathbb{Q}_p^n)).$$

Theorem 7 *With the above hypotheses, $u(x,t)$ given by*

$$u(x,t) := P_t h(x) + \int_0^t P_{t-s} g(x,s) ds,$$

belongs to the space $C([0,T], L^2(\mathbb{Q}_p^n)) \cap C^1([0,T], L^2(\mathbb{Q}_p^n))$ and is the unique solution of the inhomogeneous problem (5.1).

Proof The result follows from [10, Section 4.1], Theorems 5 and 6. □

We should mention that the inhomogeneous initial value problem introduced in this article are a complement to the studied in [38].

Author contributions All authors reviewed the manuscript.

Declarations

Conflict of interest The authors declare no competing interests.

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