



Pseudo-differential operators on Orlicz modulation spaces

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Abstract

We deduce continuity properties for pseudo-differential operators with symbols in quasi-Banach Orlicz modulation spaces when rely on other quasi-Banach Orlicz modulation spaces. In particular we extend some earlier results.

Keywords Orlicz · Quasi-banach · Quasi-young functionals

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0 Introduction

In the paper we deduce continuity properties for pseudo-differential operators when acting on quasi-Banach Orlicz modulation spaces. For example, for a pseudo-differential operator $\text{Op}(a)$ with the symbol a we show that the following is true:

- Suppose that $q_0 \in (0, 1]$, Φ_j are quasi-Young functions which satisfy $\Phi_j(t) \lesssim t^{q_0}$ near origin, and that a belongs to the classical modulation space $M^{\infty, q_0}(\mathbf{R}^{2d})$. Then $\text{Op}(a)$ is continuous on the quasi-Banach Orlicz modulation space $M^{\Phi_1, \Phi_2}(\mathbf{R}^d)$;
- Suppose that Φ is a quasi-Young function which satisfy $t \lesssim \Phi(t)$ near origin, and that a belongs to $M^\Phi(\mathbf{R}^{2d})$. Then $\text{Op}(a)$ is continuous from $M^\infty(\mathbf{R}^d)$ to $M^\Phi(\mathbf{R}^d)$;
- Suppose that Φ_0 is a Young function and Φ_0^* is the complementary Young function, and that a belongs to $M^{\Phi_0}(\mathbf{R}^{2d})$. Then $\text{Op}(a)$ is continuous from $M^{\Phi_0^*}(\mathbf{R}^d)$ to $M^\Phi(\mathbf{R}^d)$.

(We refer to [18] and Sect. 1 for notations).

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More generally, we deduce weighted versions of such continuity results. In particular we extend some continuity properties for pseudo-differential operators when acting on (ordinary) modulation spaces, e.g. in [4, 5, 14, 15, 30, 31, 33].

Essential parts of our analysis are based on [28] by Schnackcrs and Führ concerning Orlicz modulation spaces, and on [37] concerning quasi-Banach Orlicz modulation spaces. In these approaches, general properties and aspects on quasi-Banach Orlicz spaces given in [17] by Harjulehto and Hästö are fundamental. In this respect, we show that for mixed quasi-Banach Orlicz modulation spaces like $M_{(\omega)}^{\Phi_1, \Phi_2}(\mathbf{R}^{2d})$ we have $M_{(\omega)}^{\Phi, \Phi}(\mathbf{R}^{2d}) = M_{(\omega)}^{\Phi}(\mathbf{R}^{2d})$ when Φ, Φ_1 and Φ_2 are quasi-Young functions. This leads to convenient improvement of the style of the continuity results for our pseudo-differential operators when acting on quasi-Banach Orlicz modulation spaces.

In some situations it might be beneficial to replace Lebesgue norm estimates with more refined Orlicz norm estimates. This may appear when dealing with certain non-linear functionals. For example, in statistics or statistical physics, the entropy applied on probability density functions f on \mathbf{R}^d is given by

$$E(f) = - \int_{\mathbf{R}^d} f(x) \log f(x) dx.$$

When investigating E , it might be more efficient to replace the pair of Lebesgue spaces (L^1, L^∞) by the pair of Orlicz spaces $(L \log(L + 1), L^{\cosh^{-1}})$, where the Young functions are given by

$$\Phi(t) = t \log(1 + t) \quad \text{and} \quad \Phi(t) = \cosh(t) - 1,$$

respectively. We also observe that the Zygmund space $L \log^+ L$ is an Orlicz space related to Hardy-Littlewood maximal functions (see [21, 22] and the references therein).

Such questions are also relevant when investigating localized Fourier transforms like short-time Fourier transforms $V_\phi f$ because of the entropy conditions

$$E(|V_\phi f|^2) \geq C,$$

for some constant C , when

$$\|f\|_{L^2} = \|g\|_{L^2} = \|\phi\|_{L^2} = 1.$$

(See [19].) We remark that such refined Fourier transforms are indispensable tools within time-frequency, signal processing and certain parts of quantum mechanics.

In time-frequency analysis and signal processing, non-stationary filters can be modelled by pseudo-differential operators $f \mapsto \text{Op}(a)f$, where the symbols a are determined by time and frequency varying filters, the target functions f are the original signals and $\text{Op}(a)f$ are the reflected signals. In such situations it is suitable to discuss continuity properties by means of certain types of time-frequency invariant (quasi-)Banach spaces. This leads to modulation spaces.

The classical modulation spaces is a family of function and distribution spaces, introduced by Feichtinger in [6]. Here the modulation spaces are defined by imposing a weighted mixed Lebesgue norm estimate on the short-time Fourier transforms of the involved functions and distributions. The theory has thereafter been extended and generalized, especially by Feichtinger and Gröchenig in [8, 9], where the theory of (Banach) modulation spaces was put into the context of coorbit space theory. A less abstract extension of the classical modulation spaces is performed in [7], where Feichtinger replaces the mixed Lebesgue norm estimates in [6] with more general translation invariant norms of solid Banach function spaces.

Some extensions to the quasi-Banach case have thereafter been performed in e.g. [10, 26, 27, 32, 35].

In [28], Führ and Schnacker study Orlicz modulation spaces of the form M^{Φ_1, Φ_2} , where Φ_1 and Φ_2 are Young functions. That is, they consider modulation spaces in [7], where the solid Banach function spaces are Orlicz spaces, a naturally generalization of L^p spaces which contain certain Sobolev spaces as subspaces. In particular their investigations also include the classical modulation spaces in [6], since these spaces are obtained by choosing

$$\Phi_j(t) = t^p \quad \text{or} \quad \Phi_j(t) = \begin{cases} 0, & t \leq 1, \\ \infty, & t > 1. \end{cases}$$

The analysis in [28] is extended in [37] to quasi-Banach weighted Orlicz modulation spaces, $M_{(\omega)}^{\Phi_1, \Phi_2}(\mathbf{R}^d)$, where Φ_1, Φ_2 are quasi-Young functions of certain degrees and ω is a suitable weight function on \mathbf{R}^{2d} . In particular, it is here allowed to let $\Phi_j(t) = t^p$ for every $p > 0$ (instead of $p \geq 1$ as in [28]), which implies that any modulation space $M_{(\omega)}^{p, q}(\mathbf{R}^d)$ for $p, q \in (0, \infty]$ are included in the studies in [37].

In the paper, our deduced continuity for pseudo-differential operators, are based on the various properties of quasi-Banach Orlicz modulation spaces, obtained in [37].

1 Preliminaries

In this section we recall some facts for Gelfand-Shilov spaces, Orlicz spaces, Orlicz modulation spaces and pseudo-differential operators. First we discuss some useful properties of Gelfand-Shilov spaces. Thereafter we recall some classes of weight functions which are used later on in the definition of Orlicz modulation spaces. In Sects. 1.3 and 1.4 we define and present some properties for Orlicz spaces and Orlicz modulation spaces. We conclude the section by discussing Gabor analysis for Orlicz modulation spaces and pseudo-differential operators.

1.1 Gelfand–Shilov spaces

We start by discussing Gelfand-Shilov spaces and their properties. Let $0 < s \in \mathbf{R}$ be fixed. Then the (Fourier invariant) Gelfand-Shilov space $\mathcal{S}_s(\mathbf{R}^d)$ ($\Sigma_s(\mathbf{R}^d)$) of Roumieu

type (Beurling type) with parameter s consists of all $f \in C^\infty(\mathbf{R}^d)$ such that

$$\sup \left(\frac{x^\beta \partial^\alpha f(x)}{h^{|\alpha+\beta|}(\alpha! \beta!)^s} \right) \tag{1.1}$$

is finite for some $h > 0$ (for every $h > 0$). Here the supremum should be taken over all $\alpha, \beta \in \mathbf{N}^d$ and $x \in \mathbf{R}^d$. We equip $\mathcal{S}_s(\mathbf{R}^d)$ ($\Sigma_s(\mathbf{R}^d)$) by the canonical inductive limit topology (projective limit topology) with respect to $h > 0$, induced by the semi-norms in (1.1).

For any $s, s_0 > 0$ such that $\frac{1}{2} \leq s_0 < s$ we have

$$\begin{aligned} \mathcal{S}_{s_0}(\mathbf{R}^d) &\hookrightarrow \Sigma_s(\mathbf{R}^d) \hookrightarrow \mathcal{S}_s(\mathbf{R}^d) \hookrightarrow \mathcal{S}(\mathbf{R}^d), \\ \mathcal{S}'(\mathbf{R}^d) &\hookrightarrow \mathcal{S}'_s(\mathbf{R}^d) \hookrightarrow \Sigma'_s(\mathbf{R}^d) \hookrightarrow \mathcal{S}'_{s_0}(\mathbf{R}^d), \end{aligned} \tag{1.2}$$

with dense embeddings. Here $A \hookrightarrow B$ means that the topological spaces A and B satisfy $A \subseteq B$ with continuous embeddings. The space $\Sigma_s(\mathbf{R}^d)$ is a Fréchet space with seminorms $\|\cdot\|_{\mathcal{S}_{s,h}}$, $h > 0$. Moreover, $\Sigma_s(\mathbf{R}^d) \neq \{0\}$, if and only if $s > 1/2$, and $\mathcal{S}_s(\mathbf{R}^d) \neq \{0\}$, if and only if $s \geq 1/2$.

The Gelfand-Shilov distribution spaces $\mathcal{S}'_s(\mathbf{R}^d)$ and $\Sigma'_s(\mathbf{R}^d)$ are the (strong) dual spaces of $\mathcal{S}_s(\mathbf{R}^d)$ and $\Sigma_s(\mathbf{R}^d)$, respectively. As for the Gelfand-Shilov spaces there is a canonical projective limit topology (inductive limit topology) for $\mathcal{S}'_s(\mathbf{R}^d)$ ($\Sigma'_s(\mathbf{R}^d)$) (cf. [11, 23, 24]).

From now on we let \mathcal{F} be the Fourier transform which takes the form

$$(\mathcal{F} f)(\xi) = \widehat{f}(\xi) \equiv (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} f(x) e^{-i\langle x, \xi \rangle} dx$$

when $f \in L^1(\mathbf{R}^d)$. Here $\langle \cdot, \cdot \rangle$ denotes the usual scalar product on \mathbf{R}^d . The map \mathcal{F} extends uniquely to homeomorphisms on $\mathcal{S}'(\mathbf{R}^d)$, on $\mathcal{S}'_s(\mathbf{R}^d)$ and on $\Sigma'_s(\mathbf{R}^d)$. Furthermore, \mathcal{F} restricts to homeomorphisms on $\mathcal{S}(\mathbf{R}^d)$, on $\mathcal{S}_s(\mathbf{R}^d)$ and on $\Sigma_s(\mathbf{R}^d)$, and to a unitary operator on $L^2(\mathbf{R}^d)$.

Gelfand-Shilov spaces can in convenient ways be characterized in terms of estimates of the functions and their Fourier transforms. More precisely, in [3] it is proved that if $f \in \mathcal{S}'(\mathbf{R}^d)$ and $s > 0$, then $f \in \mathcal{S}_s(\mathbf{R}^d)$ ($f \in \Sigma_s(\mathbf{R}^d)$), if and only if

$$|f(x)| \lesssim e^{-r|x|^{\frac{1}{s}}} \quad \text{and} \quad |\widehat{f}(\xi)| \lesssim e^{-r|\xi|^{\frac{1}{s}}}, \tag{1.3}$$

for some $r > 0$ (for every $r > 0$). Here $r_1(\theta) \lesssim r_2(\theta)$ means that $r_1(\theta) \leq c \cdot r_2(\theta)$ holds uniformly for all θ in the intersection of the domains of r_1 and r_2 for some constant $c > 0$. We write $r_1 \asymp r_2$ when $r_1 \lesssim r_2 \lesssim r_1$.

Let $\phi \in \mathcal{S}_s(\mathbf{R}^d)$ be fixed. Then the short-time Fourier transform $V_\phi f$ of $f \in \mathcal{S}'_s(\mathbf{R}^d)$ with respect to the window function ϕ is the Gelfand-Shilov distribution on \mathbf{R}^{2d} , defined by

$$V_\phi f(x, \xi) = \mathcal{F}(f \overline{\phi(\cdot - x)})(\xi). \tag{1.4}$$

If $f, \phi \in \mathcal{S}_s(\mathbf{R}^d)$, then it follows that

$$V_\phi f(x, \xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} f(y) \overline{\phi(y-x)} e^{-i\langle y, \xi \rangle} dy.$$

We recall that Gelfand-Shilov spaces and their distribution spaces can also be characterized by estimates of short-time Fourier transforms, (see e.g. [16, 35]).

1.2 Weight functions

A *weight* or *weight function* on \mathbf{R}^d is a positive function $\omega \in L_{loc}^\infty(\mathbf{R}^d)$ such that $1/\omega \in L_{loc}^\infty(\mathbf{R}^d)$. The weight ω is called *moderate*, if there is a positive weight v on \mathbf{R}^d such that

$$\omega(x+y) \lesssim \omega(x)v(y), \quad x, y \in \mathbf{R}^d. \quad (1.5)$$

If ω and v are weights on \mathbf{R}^d such that (1.5) holds, then ω is also called *v-moderate*. We note that (1.5) implies that ω fulfills the estimates

$$v(-x)^{-1} \lesssim \omega(x) \lesssim v(x), \quad x \in \mathbf{R}^d. \quad (1.6)$$

We let $\mathcal{P}_E(\mathbf{R}^d)$ be the set of all moderate weights on \mathbf{R}^d .

It can be proved that if $\omega \in \mathcal{P}_E(\mathbf{R}^d)$, then ω is v -moderate for some $v(x) = e^{r|x|}$, provided the positive constant r is large enough (cf. [13]). That is, (1.5) implies

$$\omega(x+y) \lesssim \omega(x)e^{r|y|} \quad (1.7)$$

for some $r > 0$. In particular, (1.6) shows that for any $\omega \in \mathcal{P}_E(\mathbf{R}^d)$, there is a constant $r > 0$ such that

$$e^{-r|x|} \lesssim \omega(x) \lesssim e^{r|x|}, \quad x \in \mathbf{R}^d.$$

We say that v is *submultiplicative* if v is even and (1.5) holds with $\omega = v$. In the sequel, v and v_j for $j \geq 0$, always stand for submultiplicative weights if nothing else is stated.

We let $\mathcal{P}_E^0(\mathbf{R}^d)$ be the set of all $\omega \in \mathcal{P}_E(\mathbf{R}^d)$ such that (1.7) holds for every $r > 0$. We also let $\mathcal{P}(\mathbf{R}^d)$ be the set of all $\omega \in \mathcal{P}_E(\mathbf{R}^d)$ such that

$$\omega(x+y) \lesssim \omega(x)(1+|y|)^r$$

for some $r > 0$. Evidently,

$$\mathcal{P}(\mathbf{R}^d) \subseteq \mathcal{P}_E^0(\mathbf{R}^d) \subseteq \mathcal{P}_E(\mathbf{R}^d).$$

1.3 Orlicz spaces

In this subsection we provide an overview of some basic definitions and state some technical results that will be needed.

First we recall some facts concerning Young functions and Orlicz spaces (see [17, 25]).

Definition 1.1 A function $\Phi : \mathbf{R} \rightarrow \mathbf{R} \cup \{\infty\}$ is called *convex* if

$$\Phi(s_1 t_1 + s_2 t_2) \leq s_1 \Phi(t_1) + s_2 \Phi(t_2)$$

when $s_j, t_j \in \mathbf{R}$ satisfy $s_j \geq 0$ and $s_1 + s_2 = 1, j = 1, 2$.

We observe that Φ might not be continuous, because we permit ∞ as function value. For example,

$$\Phi(t) = \begin{cases} c, & \text{when } t \leq a \\ \infty, & \text{when } t > a \end{cases}$$

is convex but discontinuous at $t = a$.

Definition 1.2 Let $r_0 \in (0, 1]$, Φ_0 and Φ be functions from $[0, \infty)$ to $[0, \infty]$. Then Φ_0 is called a *Young function* if

- (1) Φ_0 is convex,
- (2) $\Phi_0(0) = 0$,
- (3) $\lim_{t \rightarrow \infty} \Phi_0(t) = +\infty$.

The function Φ is called *r_0 -Young function* or *quasi-Young function of order r_0* , if $\Phi(t) = \Phi_0(t^{r_0}), t \geq 0$, for some Young function Φ_0 .

It is clear that Φ in Definition 1.2 is non-decreasing, because if $0 \leq t_1 \leq t_2$ and $s \in [0, 1]$ is chosen such that $t_1 = s t_2$ and Φ_0 is the same as in Definition 1.2, then

$$\Phi(t_1) = \Phi_0(s^{r_0} t_2^{r_0} + (1 - s^{r_0})0) \leq s^{r_0} \Phi_0(t_2^{r_0}) + (1 - s^{r_0})\Phi_0(0) \leq \Phi(t_2),$$

since $\Phi(0) = \Phi_0(0) = 0$ and $s \in [0, 1]$.

Definition 1.3 Let (Ω, Σ, μ) be a Borel measure space, with $\Omega \subseteq \mathbf{R}^d$, Φ_0 be a Young function and let $\omega_0 \in \mathcal{P}_E(\mathbf{R}^d)$.

- (1) $L_{(\omega_0)}^{\Phi_0}(\mu)$ consists of all μ -measurable functions $f : \Omega \rightarrow \mathbf{C}$ such that

$$\|f\|_{L_{(\omega_0)}^{\Phi_0}(\mu)} = \inf \left\{ \lambda > 0; \int_{\Omega} \Phi_0 \left(\frac{|f(x) \cdot \omega_0(x)|}{\lambda} \right) d\mu(x) \leq 1 \right\}$$

is finite. Here f and g in $L_{(\omega_0)}^{\Phi_0}(\mu)$ are equivalent if $f = g$ a.e.

(2) Let Φ be a quasi-Young function of order $r_0 \in (0, 1]$, given by $\Phi(t) = \Phi_0(t^{r_0})$, $t \geq 0$, for some Young function Φ_0 . Then $L_{(\omega_0)}^\Phi(\mu)$ consists of all μ -measurable functions $f : \Omega \rightarrow \mathbf{C}$ such that

$$\|f\|_{L_{(\omega_0)}^\Phi(\mu)} = (\| |f \cdot \omega_0|^{r_0} \|_{L^{\Phi_0}(\mu)})^{1/r_0}$$

is finite.

Remark 1.4 Let Φ , Φ_0 and ω_0 be the same as in Definition 1.2. Then it follows by straight-forward computation that

$$\|f\|_{L_{(\omega_0)}^\Phi(\mu)} = \inf \left\{ \lambda > 0; \int_{\Omega} \Phi_0 \left(\frac{|f(x) \cdot \omega_0(x)|^{r_0}}{\lambda^{r_0}} \right) d\mu(x) \leq 1 \right\}.$$

Definition 1.5 Let $(\Omega_j, \Sigma_j, \mu_j)$ be Borel measure spaces, with $\Omega_j \subseteq \mathbf{R}^d$, $r_0 \in (0, 1]$, Φ_j be r_0 -Young functions, $j = 1, 2$ and let $\omega \in \mathcal{P}_E(\mathbf{R}^{2d})$. Then the mixed quasi-norm Orlicz space $L_{(\omega)}^{\Phi_1, \Phi_2} = L_{(\omega)}^{\Phi_1, \Phi_2}(\mu_1 \otimes \mu_2)$ consists of all $\mu_1 \otimes \mu_2$ -measurable functions $f : \Omega_1 \times \Omega_2 \rightarrow \mathbf{C}$ such that

$$\|f\|_{L_{(\omega)}^{\Phi_1, \Phi_2}} \equiv \|f_{1, \omega}\|_{L^{\Phi_2}},$$

is finite, where

$$f_{1, \omega}(x_2) = \|f(\cdot, x_2)\omega(\cdot, x_2)\|_{L^{\Phi_1}}.$$

If $r_0 = 1$ in Definition 1.5, then $L_{(\omega)}^{\Phi_1, \Phi_2}(\mu_1 \otimes \mu_2)$ is a Banach space and is called a mixed norm Orlicz space.

Remark 1.6 Suppose Φ_j are quasi-Young functions of order $q_j \in (0, 1]$, $j = 1, 2$. Then both Φ_1 and Φ_2 are quasi-Young functions of order $r_0 = \min(q_1, q_2)$.

Let $\Lambda \subseteq \mathbf{R}^d$ be a lattice, i.e., Λ is given by

$$\Lambda = \{n_1 e_1 + \dots + n_d e_d; (n_1, \dots, n_d) \in \mathbf{Z}^d\}$$

for some basis e_1, \dots, e_d of \mathbf{R}^d . Then $\ell'_0(\Lambda)$ is the set of all formal sequences

$$\{a(n)\}_{n \in \Lambda} = \{a(n); n \in \Lambda\} \subseteq \mathbf{C}.$$

Let $\ell_0(\Lambda)$ be the set of all sequences $\{a(n)\}_{n \in \Lambda}$ such that $a(n) \neq 0$ for at most finite numbers of n . We observe that

$$\Lambda^2 = \Lambda \times \Lambda = \{(x, \xi); x, \xi \in \Lambda\}$$

is a lattice in $\mathbf{R}^{2d} \simeq \mathbf{R}^d \times \mathbf{R}^d$.

Remark 1.7 Let $\Lambda \subseteq \mathbf{R}^d$ be a lattice, Φ, Φ_1 and Φ_2 be r_0 -Young functions, $\omega_0, v_0 \in \mathcal{P}_E(\mathbf{R}^d)$ and $\omega, v \in \mathcal{P}_E(\mathbf{R}^{2d})$ be such that ω_0 and ω are v_0 - respectively v -moderate (in the sequel it is understood that all lattices contain 0). Then we set

$$L_{(\omega_0)}^\Phi(\mathbf{R}^d) = L_{(\omega_0)}^\Phi(\mu) \quad \text{and} \quad L_{(\omega_0)}^{\Phi_1, \Phi_2}(\mathbf{R}^{2d}) = L_{(\omega_0)}^{\Phi_1, \Phi_2}(\mu \otimes \mu),$$

when μ is the Lebesgue measure on \mathbf{R}^d . If instead μ is the standard (Haar) measure on Λ , i.e. $\mu(n) = 1, n \in \Lambda$, then we set

$$\ell_{(\omega_0)}^\Phi(\Lambda) = \ell_{(\omega_0)}^\Phi(\mu) \quad \text{and} \quad \ell_{(\omega_0)}^{\Phi_1, \Phi_2}(\Lambda \times \Lambda) = \ell_{(\omega_0)}^{\Phi_1, \Phi_2}(\mu \otimes \mu).$$

Evidently, $\ell_{(\omega_0)}^{\Phi_1, \Phi_2}(\Lambda \times \Lambda) \subseteq \ell'_{(\omega_0)}(\Lambda \times \Lambda)$.

Lemma 1.8 Let Φ, Φ_j be Young functions, $j = 1, 2$, $\omega_0, v_0 \in \mathcal{P}_E(\mathbf{R}^d)$ and $\omega, v \in \mathcal{P}_E(\mathbf{R}^{2d})$ be such that ω_0 is v_0 -moderate and ω is v -moderate. Then $L_{(\omega_0)}^\Phi(\mathbf{R}^d)$ and $L_{(\omega_0)}^{\Phi_1, \Phi_2}(\mathbf{R}^{2d})$ are invariant under translations, and

$$\|f(\cdot - x)\|_{L_{(\omega_0)}^\Phi} \lesssim \|f\|_{L_{(\omega_0)}^\Phi} v_0(x), \quad f \in L_{(\omega_0)}^\Phi(\mathbf{R}^d), \quad x \in \mathbf{R}^d,$$

and

$$\|f(\cdot - (x, \xi))\|_{L_{(\omega_0)}^{\Phi_1, \Phi_2}} \lesssim \|f\|_{L_{(\omega_0)}^{\Phi_1, \Phi_2}} v(x, \xi), \quad f \in L_{(\omega_0)}^{\Phi_1, \Phi_2}(\mathbf{R}^{2d}), \quad (x, \xi) \in \mathbf{R}^{2d}.$$

Proof We only prove the assertion for $L_{(\omega_0)}^{\Phi_1, \Phi_2}(\mathbf{R}^{2d})$. The other part follows by similar arguments and is left for the reader.

We have $\Phi_j(t) = \Phi_{0,j}(t^{r_0}), t \geq 0$, for some Young functions $\Phi_{0,j}, j = 1, 2$. This gives

$$\begin{aligned} \|f(\cdot - (x, \xi))\|_{L_{(\omega_0)}^{\Phi_1, \Phi_2}} &= \left(\| |f(\cdot - (x, \xi))| \omega^{r_0} \|_{L^{\Phi_{0,1}, \Phi_{0,2}}} \right)^{\frac{1}{r_0}} \\ &\lesssim \left(\| |f(\cdot - (x, \xi))| \omega(\cdot - (x, \xi)) v(x, \xi)^{r_0} \|_{L^{\Phi_{0,1}, \Phi_{0,2}}} \right)^{\frac{1}{r_0}} \\ &= \left(\| |f \cdot \omega|^{r_0} \|_{L^{\Phi_{0,1}, \Phi_{0,2}}} \right)^{\frac{1}{r_0}} \cdot v(x, \xi) = \|f\|_{L_{(\omega_0)}^{\Phi_1, \Phi_2}} \cdot v(x, \xi). \end{aligned}$$

Here the inequality follows from the fact that ω is v -moderate, and the last two relations follow from the definitions. □

We refer to [17, 25, 28] for more facts about Orlicz spaces.

1.4 Orlicz modulation spaces

The definitions of classical modulation spaces and Orlicz modulation spaces are the following (cf. [6, 7, 28, 37]).

Definition 1.9 Let $\phi(x) = \pi^{-\frac{d}{4}} e^{-\frac{|x|^2}{2}}$, $x \in \mathbf{R}^d$, $p, q \in (0, \infty]$ and $\omega \in \mathcal{P}_E(\mathbf{R}^{2d})$. Then the modulation spaces $M_{(\omega)}^{p,q}(\mathbf{R}^d)$ is set of all $f \in \mathcal{S}'_{1/2}(\mathbf{R}^d)$ such that $V_\phi f \in L_{(\omega)}^{p,q}(\mathbf{R}^{2d})$. We equip these spaces with the quasi-norm

$$\|f\|_{M_{(\omega)}^{p,q}} \equiv \|V_\phi f\|_{L_{(\omega)}^{p,q}}.$$

Also let Φ, Φ_1, Φ_2 be quasi-Young functions. Then the Orlicz modulation spaces $M_{(\omega)}^\Phi(\mathbf{R}^d)$ and $M_{(\omega)}^{\Phi_1, \Phi_2}(\mathbf{R}^d)$ are given by

$$M_{(\omega)}^\Phi(\mathbf{R}^d) = \{f \in \mathcal{S}'_{1/2}(\mathbf{R}^d); V_\phi f \in L_{(\omega)}^\Phi(\mathbf{R}^{2d})\} \tag{1.8}$$

and

$$M_{(\omega)}^{\Phi_1, \Phi_2}(\mathbf{R}^d) = \{f \in \mathcal{S}'_{1/2}(\mathbf{R}^d); V_\phi f \in L_{(\omega)}^{\Phi_1, \Phi_2}(\mathbf{R}^{2d})\}. \tag{1.9}$$

The quasi-norms on $M_{(\omega)}^\Phi(\mathbf{R}^d)$ and $M_{(\omega)}^{\Phi_1, \Phi_2}(\mathbf{R}^d)$ are given by

$$\|f\|_{M_{(\omega)}^\Phi} = \|V_\phi f\|_{L_{(\omega)}^\Phi} \tag{1.10}$$

and

$$\|f\|_{M_{(\omega)}^{\Phi_1, \Phi_2}} = \|V_\phi f\|_{L_{(\omega)}^{\Phi_1, \Phi_2}}. \tag{1.11}$$

For conveniency we set

$$M^{p,q} = M_{(\omega)}^{p,q}, \quad M^\Phi = M_{(\omega)}^\Phi \quad \text{and} \quad M^{\Phi_1, \Phi_2} = M_{(\omega)}^{\Phi_1, \Phi_2} \quad \text{when} \quad \omega(x, \xi) = 1,$$

and $M^p = M^{p,p}$ and $M_{(\omega)}^p = M_{(\omega)}^{p,p}$.

We notice that (1.10) and (1.11) are norms when Φ, Φ_1 and Φ_2 are Young functions. If $\omega \in \mathcal{P}_E(\mathbf{R}^{2d})$ as in Definition 1.9, then the conditions

$$\|V_\phi f\|_{L_{(\omega)}^{\Phi_1, \Phi_2}} < \infty \quad \text{and} \quad \|V_\phi f\|_{L_{(\omega)}^\Phi} < \infty$$

are independent of the choices of ϕ in $\Sigma_1(\mathbf{R}^d) \setminus 0$ and that different ϕ give rise to equivalent quasi-norms (see e.g. [37, Sect. 5]).

Later on we need the following proposition.

Proposition 1.10 Let Φ, Φ_j be Young functions, $j = 1, 2$, $\omega_0 \in \mathcal{P}_E(\mathbf{R}^d)$ and $\omega \in \mathcal{P}_E(\mathbf{R}^{2d})$. Then

$$\begin{aligned} \mathcal{S}(\mathbf{R}^d) &\subseteq L^\Phi(\mathbf{R}^d) \subseteq \mathcal{S}'(\mathbf{R}^d), \quad \mathcal{S}(\mathbf{R}^{2d}) \subseteq L^{\Phi_1, \Phi_2}(\mathbf{R}^{2d}) \subseteq \mathcal{S}'(\mathbf{R}^{2d}), \\ \Sigma_1(\mathbf{R}^d) &\subseteq L_{(\omega_0)}^\Phi(\mathbf{R}^d) \subseteq \Sigma'_1(\mathbf{R}^d), \quad \Sigma_1(\mathbf{R}^{2d}) \subseteq L_{(\omega)}^{\Phi_1, \Phi_2}(\mathbf{R}^{2d}) \subseteq \Sigma'_1(\mathbf{R}^{2d}). \end{aligned}$$

Proof Let $v_0 \in \mathcal{P}_E(\mathbf{R}^d)$ and $v \in \mathcal{P}_E(\mathbf{R}^{2d})$ be chosen such that ω_0 is v_0 -moderate and ω is v -moderate. Since $L_{(\omega_0)}^\Phi(\mathbf{R}^d)$ and $L_{(\omega)}^{\Phi_1, \Phi_2}(\mathbf{R}^{2d})$ are invariant under translation and modulation, we have

$$M_{(v_0)}^1(\mathbf{R}^d) \subseteq L_{(\omega_0)}^\Phi(\mathbf{R}^d) \subseteq M_{(1/v_0)}^\infty(\mathbf{R}^d),$$

and

$$M_{(v)}^1(\mathbf{R}^{2d}) \subseteq L_{(\omega)}^{\Phi_1, \Phi_2}(\mathbf{R}^{2d}) \subseteq M_{(1/v)}^\infty(\mathbf{R}^{2d})$$

(see [12, 36, 37]). The result now follows from well-known inclusions between modulation spaces, Schwartz spaces, Gelfand-Shilov spaces, and their duals. \square

The next result gives some information about the roles that Φ_1 and Φ_2 play for M^{Φ_1, Φ_2} . We omit the proof since it can be found in [37]. See also [28] for the Banach case.

Proposition 1.11 *Let Φ_j and Ψ_j , $j = 1, 2$, be quasi-Young functions, Λ be a lattice in \mathbf{R}^d and $\omega \in \mathcal{P}_E(\mathbf{R}^{2d})$. Then the following conditions are equivalent:*

- (1) $M_{(\omega)}^{\Phi_1, \Phi_2}(\mathbf{R}^d) \subseteq M_{(\omega)}^{\Psi_1, \Psi_2}(\mathbf{R}^d)$;
- (2) $\ell_{(\omega)}^{\Phi_1, \Phi_2}(\Lambda) \subseteq \ell_{(\omega)}^{\Psi_1, \Psi_2}(\Lambda)$;
- (3) $\Psi_j(t) \lesssim \Phi_j(t)$ for every $t \in [0, t_0]$, for some $t_0 > 0$.

1.5 Gabor frames

Definition 1.12 Let $\omega, v \in \mathcal{P}_E(\mathbf{R}^{2d})$ be such that ω is v -moderate, $\phi, \psi \in M_{(v)}^1(\mathbf{R}^d)$, $\varepsilon > 0$ and let $\Lambda \subseteq \mathbf{R}^d$ be a lattice.

- (1) The *analysis operator* $C_\phi^{\varepsilon, \Lambda}$ is the operator from $M_{(\omega)}^\infty(\mathbf{R}^d)$ to $\ell_{(\omega)}^\infty(\varepsilon\Lambda^2)$, given by

$$C_\phi^{\varepsilon, \Lambda} f \equiv \{V_\phi f(j, \iota)\}_{j, \iota \in \varepsilon\Lambda}.$$

- (2) The *synthesis operator* $D_\psi^{\varepsilon, \Lambda}$ is the operator from $\ell_{(\omega)}^\infty(\varepsilon\Lambda^2)$ to $M_{(\omega)}^\infty(\mathbf{R}^d)$, given by

$$D_\psi^{\varepsilon, \Lambda} c \equiv \sum_{j, \iota \in \varepsilon\Lambda'} c(j, \iota) e^{i\langle \cdot, \iota \rangle} \psi(\cdot - j).$$

- (3) The *Gabor frame operator* $S_{\phi, \psi}^{\varepsilon, \Lambda}$ is the operator on $M_{(\omega)}^\infty(\mathbf{R}^d)$, given by $D_\psi^{\varepsilon, \Lambda} \circ C_\phi^{\varepsilon, \Lambda}$, i.e.

$$S_{\phi, \psi}^{\varepsilon, \Lambda} f \equiv \sum_{j, \iota \in \varepsilon\Lambda'} V_\phi f(j, \iota) e^{i\langle \cdot, \iota \rangle} \psi(\cdot - j).$$

The next result shows that it is possible to find suitable ϕ and ψ in the previous definition.

Lemma 1.13 *Let $\Lambda \subseteq \mathbf{R}^d$ be a lattice, $v \in \mathcal{P}_E(\mathbf{R}^{2d})$ be submultiplicative and $\phi \in M_{(v)}^1(\mathbf{R}^d) \setminus 0$. Then there is an $\varepsilon > 0$ and $\psi \in M_{(v)}^1(\mathbf{R}^d) \setminus 0$ such that*

$$\{\phi(x - j)e^{i\langle x, \iota \rangle}\}_{j, \iota \in \varepsilon\Lambda} \quad \text{and} \quad \{\psi(x - j)e^{i\langle x, \iota \rangle}\}_{j, \iota \in \varepsilon\Lambda} \tag{1.12}$$

are dual frames to each others.

Remark 1.14 There are several ways to achieve dual frames (1.12). In fact, let $v, v_0 \in \mathcal{P}_E(\mathbf{R}^{2d})$ be submultiplicative such that ω is v -moderate and $L_{(v_0)}^1(\mathbf{R}^{2d}) \subseteq L^r(\mathbf{R}^{2d})$, $r \in (0, 1]$. Then Lemma 1.13 guarantees that for some choice of $\phi, \psi \in M_{(v_0 v)}^1(\mathbf{R}^d) \subseteq M_{(v)}^r(\mathbf{R}^d)$ and lattice Λ , the set in (1.12) are dual frames to each others, and that $\psi = (S_{\phi, \phi}^\Lambda)^{-1}\phi$. (Cf. [33, Proposition 1.5 and Remark 1.6].)

Lemma 1.15 *Let $\Lambda \subseteq \mathbf{R}^d$ be a lattice, $v \in \mathcal{P}_E(\mathbf{R}^{4d})$ be submultiplicative, $\phi_1, \phi_2 \in \Sigma_1(\mathbf{R}^d) \setminus 0$ and*

$$\varphi(x, \xi) = \phi_1(x)\overline{\widehat{\phi_2}(\xi)}e^{-i\langle x, \xi \rangle}.$$

Then there is an $\varepsilon > 0$ such that

$$\{\varphi(x - j, \xi - \iota)e^{i\langle (x, \kappa) + (k, \xi) \rangle}\}_{j, \iota, k, \kappa \in \varepsilon\Lambda}$$

is a Gabor frame with canonical dual frame

$$\{\psi(x - j, \xi - \iota)e^{i\langle (x, \kappa) + (k, \xi) \rangle}\}_{j, \iota, k, \kappa \in \varepsilon\Lambda}$$

where $\psi = (S_{\phi, \phi}^{\Lambda^2 \times \Lambda^2})^{-1}\varphi$ belongs to $M_{(v)}^r(\mathbf{R}^{2d})$ for every $r > 0$.

The next result shows that Gabor theory is suitable when dealing with Orlicz modulation spaces. We omit the proof since the result follows from [37, Theorem 4.7]. See also [28] for the Banach case.

Proposition 1.16 *Let $\Lambda \subseteq \mathbf{R}^d$ be a lattice, $v \in \mathcal{P}_E(\mathbf{R}^{4d})$ be submultiplicative, Φ_1, Φ_2 be quasi-Young functions of order $r_0 \in (0, 1]$, $\omega, v \in \mathcal{P}_E(\mathbf{R}^{2d})$ be such that ω is v -moderate and let $\phi, \psi \in M_{(v)}^{r_0}(\mathbf{R}^d)$ and $\varepsilon > 0$ be chosen such that*

$$\{e^{i\langle \cdot, \kappa \rangle}\phi(\cdot - k)\}_{k, \kappa \in \varepsilon\Lambda} \quad \text{and} \quad \{e^{i\langle \cdot, \kappa \rangle}\psi(\cdot - k)\}_{k, \kappa \in \varepsilon\Lambda} \tag{1.13}$$

are dual frames to each others. If $f \in M_{(\omega)}^{\Phi_1, \Phi_2}(\mathbf{R}^d)$, then

$$f = \sum_{k, \kappa \in \varepsilon\Lambda} (V_\psi f)(k, \kappa)e^{i\langle \cdot, \kappa \rangle}\phi(\cdot - k)$$

with unconditionally convergence in $M_{(\omega)}^{\Phi_1, \Phi_2}(\mathbf{R}^d)$ when $\mathcal{S}(\mathbf{R}^{2d})$ is dense in $L^{\Phi_1, \Phi_2}(\mathbf{R}^{2d})$, and with convergence in $M_{(\omega)}^\infty(\mathbf{R}^d)$ with respect to the weak* topology otherwise. It holds

$$\begin{aligned} \|\{(V_\phi f)(k, \kappa)\}_{k, \kappa \in \varepsilon\Lambda}\|_{\ell_{(\omega)}^{\Phi_1, \Phi_2}} &\asymp \|\{(V_\psi f)(k, \kappa)\}_{k, \kappa \in \varepsilon\Lambda}\|_{\ell_{(\omega)}^{\Phi_1, \Phi_2}} \\ &\asymp \|f\|_{M_{(\omega)}^{\Phi_1, \Phi_2}}. \end{aligned} \tag{1.14}$$

We also recall that the previous result was heavily based on the following consequence of Theorems 4.5 and 4.6 in [37]. The proof is therefore omitted.

Proposition 1.17 *Let $\Lambda \subseteq \mathbf{R}^d$ be a lattice, $\varepsilon > 0$, $\phi, \psi \in \Sigma_1(\mathbf{R}^d)$, Φ_1, Φ_2 be quasi-Young functions of order $r_0 \in (0, 1]$, and let $\omega, v \in \mathcal{P}_E(\mathbf{R}^{2d})$ be such that ω is v -moderate. Then the the following is true:*

- (1) *The analysis operator $C_\phi^{\varepsilon, \Lambda}$ is continuous from $M_{(v)}^{\Phi_1, \Phi_2}(\mathbf{R}^d)$ into $\ell_{(\omega)}^{\Phi_1, \Phi_2}(\varepsilon\Lambda^2)$, and*

$$\|C_\phi^{\varepsilon, \Lambda} f\|_{\ell_{(\omega)}^{\Phi_1, \Phi_2}} \lesssim \|f\|_{M_{(v)}^{\Phi_1, \Phi_2}}, \quad f \in M_{(v)}^{\Phi_1, \Phi_2}(\mathbf{R}^d);$$

- (2) *The synthesis operator $D_\psi^{\varepsilon, \Lambda}$ is continuous from $\ell_{(\omega)}^{\Phi_1, \Phi_2}(\varepsilon\Lambda^2)$ into $M_{(\omega)}^{\Phi_1, \Phi_2}(\mathbf{R}^d)$, and*

$$\|D_\psi^{\varepsilon, \Lambda} c\|_{M_{(\omega)}^{\Phi_1, \Phi_2}} \lesssim \|c\|_{\ell_{(\omega)}^{\Phi_1, \Phi_2}}, \quad c \in \ell_{(\omega)}^{\Phi_1, \Phi_2}(\varepsilon\Lambda^2).$$

1.6 Pseudo-differential operators

Let $\mathbf{M}(d, \Omega)$ be the set of all $d \times d$ -matrices with entries in the set Ω , and let $s \geq 1/2$, $a \in \mathcal{S}_s(\mathbf{R}^{2d})$ and $A \in \mathbf{M}(d, \mathbf{R})$ be fixed. Then the pseudo-differential operator $\text{Op}_A(a)$ is the linear and continuous operator on $\mathcal{S}_s(\mathbf{R}^d)$, given by

$$(\text{Op}_A(a)f)(x) = (2\pi)^{-d} \iint a(x - A(x - y), \xi) f(y) e^{i(x-y, \xi)} dy d\xi, \tag{1.15}$$

when $f \in \mathcal{S}_s(\mathbf{R}^d)$. For general $a \in \mathcal{S}'_s(\mathbf{R}^{2d})$, the pseudo-differential operator $\text{Op}_A(a)$ is defined as the linear and continuous operator from $\mathcal{S}_s(\mathbf{R}^d)$ to $\mathcal{S}'_s(\mathbf{R}^d)$ with distribution kernel given by

$$K_{a,A}(x, y) = (2\pi)^{-d/2} (\mathcal{F}_2^{-1} a)(x - A(x - y), x - y). \tag{1.16}$$

Here $\mathcal{F}_2 F$ is the partial Fourier transform of $F(x, y) \in \mathcal{S}'_s(\mathbf{R}^{2d})$ with respect to the y variable. This definition makes sense, since the mappings

$$\mathcal{F}_2 \quad \text{and} \quad F(x, y) \mapsto F(x - A(x - y), x - y) \tag{1.17}$$

are homeomorphisms on $\mathcal{S}'_s(\mathbf{R}^{2d})$. In particular, the map $a \mapsto K_{a,A}$ is a homeomorphism on $\mathcal{S}'_s(\mathbf{R}^{2d})$.

An important special case appears when $A = t \cdot I$, with $t \in \mathbf{R}$. Here and in what follows, $I \in \mathbf{M}(d, \mathbf{R})$ denotes the $d \times d$ identity matrix. In this case we set

$$\text{Op}_t(a) = \text{Op}_{t \cdot I}(a).$$

The normal or Kohn-Nirenberg representation, $a(x, D)$, is obtained when $t = 0$, and the Weyl quantization, $\text{Op}^w(a)$, is obtained when $t = \frac{1}{2}$. That is,

$$a(x, D) = \text{Op}_0(a) \quad \text{and} \quad \text{Op}^w(a) = \text{Op}_{1/2}(a).$$

For any $K \in \mathcal{S}'_s(\mathbf{R}^{d_1+d_2})$, we let T_K be the linear and continuous mapping from $\mathcal{S}_s(\mathbf{R}^{d_1})$ to $\mathcal{S}'_s(\mathbf{R}^{d_2})$, defined by the formula

$$(T_K f, g)_{L^2(\mathbf{R}^{d_2})} = (K, g \otimes \bar{f})_{L^2(\mathbf{R}^{d_1+d_2})}. \tag{1.18}$$

It is well-known that if $A \in \mathbf{M}(d, \mathbf{R})$, then it follows from Schwartz kernel theorem that $K \mapsto T_K$ and $a \mapsto \text{Op}_A(a)$ are bijective mappings from $\mathcal{S}'(\mathbf{R}^{2d})$ to the set of linear and continuous mappings from $\mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}'(\mathbf{R}^d)$ (cf. e.g. [18]). Furthermore, by e.g. [20, Theorem 2.2] it follows that the same holds true if each \mathcal{S} and \mathcal{S}' are replaced by \mathcal{S}_s and \mathcal{S}'_s , respectively, or by Σ_s and Σ'_s , respectively.

In particular, for every $a_1 \in \mathcal{S}'_s(\mathbf{R}^{2d})$ and $A_1, A_2 \in \mathbf{M}(d, \mathbf{R})$, there is a unique $a_2 \in \mathcal{S}'_s(\mathbf{R}^{2d})$ such that $\text{Op}_{A_1}(a_1) = \text{Op}_{A_2}(a_2)$. The following result explains the relations between a_1 and a_2 .

Proposition 1.18 *Let $a_1, a_2 \in \mathcal{S}'_{1/2}(\mathbf{R}^{2d})$ and $A_1, A_2 \in \mathbf{M}(d, \mathbf{R})$. Then*

$$\text{Op}_{A_1}(a_1) = \text{Op}_{A_2}(a_2) \iff e^{i(A_2 D_\xi, D_x)} a_2(x, \xi) = e^{i(A_1 D_\xi, D_x)} a_1(x, \xi). \tag{1.19}$$

In [32], a proof of the previous proposition is given, which is similar to the proof of the case $A = t \cdot I$ in [18, 29, 38].

Let $a \in \mathcal{S}'_s(\mathbf{R}^{2d})$ be fixed. Then a is called a rank-one element with respect to $A \in \mathbf{M}(d, \mathbf{R})$, if $\text{Op}_A(a)$ is an operator of rank-one, i.e.

$$\text{Op}_A(a)f = (f, f_2)f_1, \quad f \in \mathcal{S}_s(\mathbf{R}^d), \tag{1.20}$$

for some $f_1, f_2 \in \mathcal{S}'_s(\mathbf{R}^d)$. By straight-forward computations it follows that (1.20) is fulfilled if and only if $a = (2\pi)^{\frac{d}{2}} W_{f_1, f_2}^A$, where W_{f_1, f_2}^A is the A -Wigner distribution, defined by the formula

$$W_{f_1, f_2}^A(x, \xi) \equiv \mathcal{F}(f_1(x + A \cdot) \overline{f_2(x + (A - I) \cdot)})(\xi), \tag{1.21}$$

which takes the form

$$W_{f_1, f_2}^A(x, \xi) = (2\pi)^{-\frac{d}{2}} \int f_1(x + Ay) \overline{f_2(x + (A - I)y)} e^{-i\langle y, \xi \rangle} dy,$$

when $f_1, f_2 \in \mathcal{S}_s(\mathbf{R}^d)$. By combining these facts with (1.19), it follows that

$$e^{i\langle A_2 D_\xi, D_x \rangle} W_{f_1, f_2}^{A_2} = e^{i\langle A_1 D_\xi, D_x \rangle} W_{f_1, f_2}^{A_1}, \tag{1.22}$$

for every $f_1, f_2 \in \mathcal{S}'_s(\mathbf{R}^d)$ and $A_1, A_2 \in \mathbf{M}(d, \mathbf{R})$. Since the Weyl case is particularly important, we set $W_{f_1, f_2}^A = W_{f_1, f_2}$ when $A = \frac{1}{2}I$, i.e. W_{f_1, f_2} is the usual (cross-) Wigner distribution of f_1 and f_2 .

For future references we note the link

$$\begin{aligned} (\text{Op}_A(a) f, g)_{L^2(\mathbf{R}^d)} &= (2\pi)^{-d/2} (a, W_{g, f}^A)_{L^2(\mathbf{R}^{2d})}, \\ a \in \mathcal{S}'_s(\mathbf{R}^{2d}) \quad \text{and} \quad f, g \in \mathcal{S}_s(\mathbf{R}^d) \end{aligned} \tag{1.23}$$

between pseudo-differential operators and Wigner distributions, which follows by straight-forward computations (see e.g. [34] and the references therein).

For any $A \in \mathbf{M}(d, \mathbf{R})$, the A -product, $a \#_A b$ between $a \in \mathcal{S}'_s(\mathbf{R}^{2d})$ and $b \in \mathcal{S}'_s(\mathbf{R}^{2d})$ is defined by the formula

$$\text{Op}_A(a \#_A b) = \text{Op}_A(a) \circ \text{Op}_A(b), \tag{1.24}$$

provided the right-hand side makes sense as a continuous operator from $\mathcal{S}_s(\mathbf{R}^d)$ to $\mathcal{S}'_s(\mathbf{R}^d)$.

2 More general Orlicz modulation spaces

In this section we analyse more general Orlicz modulation spaces, parameterized with more quasi-Young functions, compared to what is introduced in Sect. 1. We prove that if two consecutive quasi-Young functions are the same, then the Orlicz modulation space remains the same if one of these parameterizing quasi-Young functions are removed. In particular it follows $M_{(\omega)}^{\Phi, \Phi} = M_{(\omega)}^{\Phi}$ for the Orlicz modulation spaces considered in Sect. 1.

Definition 2.1 Let μ_j be (Borel) measures on \mathbf{R}^{d_j} , $\mu = \mu_1 \otimes \dots \otimes \mu_N$, Φ_j be quasi-Young functions, $j = 1, \dots, N$, ω be a weight function and f be measurable on $\mathbf{R}^{d_1 + \dots + d_N}$. Then $\|f\|_{L_{(\omega)}^{\Phi_1, \dots, \Phi_N}(\mu)} = \|f_{N-1, \omega}\|_{L^{\Phi_N}(\mu)}$ where $f_{k, \omega}, k = 1, \dots, N - 1$ are inductively defined by

$$\begin{aligned} f_{1, \omega}(x_2, \dots, x_N) &= \|f(\cdot, x_2, \dots, x_N) \omega(\cdot, x_2, \dots, x_N)\|_{L^{\Phi_1}(\mu_1)} \\ f_{k+1, \omega}(x_{k+2}, \dots, x_N) &= \|f_{k, \omega}(\cdot, x_{k+2}, \dots, x_N)\|_{L^{\Phi_{k+1}}(\mu_{k+1})}, \quad k = 1, \dots, N - 2. \end{aligned}$$

The space $L_{(\omega)}^{\Phi_1, \dots, \Phi_N}(\mu)$ consists of all measurable functions f on $\mathbf{R}^{d_1 + \dots + d_N}$ such that $\|f\|_{L_{(\omega)}^{\Phi_1, \dots, \Phi_N}(\mu)}$ is finite, and the topology of $L_{(\omega)}^{\Phi_1, \dots, \Phi_N}(\mu)$ is induced by the quasi-norm $\|\cdot\|_{L_{(\omega)}^{\Phi_1, \dots, \Phi_N}(\mu)}$.

Let

$$I_{d,N} = \{ (d_1, \dots, d_N) \in \mathbf{Z}_+^N; d_1 + \dots + d_N = d \}.$$

For $\vec{d} = (d_1, \dots, d_N) \in I_{d,N}$, let

$$L_{\vec{d},(\omega)}^{\Phi_1, \dots, \Phi_N}(\mathbf{R}^d) = L_{(\omega)}^{\Phi_1, \dots, \Phi_N}(\mu),$$

with $\mu = dx_1 \otimes \dots \otimes dx_N$ with $x_j \in \mathbf{R}^{d_j}$.

If $\Lambda_j \subseteq \mathbf{R}^{d_j}$ are lattices and μ_j is the standard discrete or Haar measure on Λ_j , then we set

$$\ell_{(\omega)}^{\Phi_1, \dots, \Phi_N}(\Lambda) = \ell_{\vec{d},(\omega)}^{\Phi_1, \dots, \Phi_N}(\Lambda) \equiv L_{\vec{d},(\omega)}^{\Phi_1, \dots, \Phi_N}(\mu), \quad \Lambda = \Lambda_1 \times \dots \times \Lambda_N,$$

as usual.

When discussing modulation spaces, it is suitable that \vec{d} should belong to $I_{2d,N}^0$, which consists of all $(d_1, \dots, d_N) \in I_{2d,N}$ such that

$$d_1 + \dots + d_k = d \tag{2.1}$$

for some $k \in \{1, \dots, N - 1\}$, when $N \geq 2$. We observe that (2.1) implies

$$d_{k+1} + \dots + d_N = d.$$

We observe that $I_{2d,1} = \{2d\}$, and for convenience, we put $I_{2d,1}^0 = \{2d\}$.

Now suppose that $\vec{d} \in I_{2d,N}^0$, $\Lambda_j = \varepsilon \mathbf{Z}^{d_j}$, k is chosen such that (2.1) holds, and let

$$\Lambda = \Lambda_1 \times \dots \times \Lambda_k = \Lambda_{k+1} \times \dots \times \Lambda_N = \varepsilon \mathbf{Z}^d.$$

Then we write $\Lambda^2 = \Lambda \times \Lambda$ and

$$\ell_{\vec{d},(\omega)}^{\Phi_1, \dots, \Phi_N}(\Lambda^2) = \ell_{\vec{d},(\omega)}^{\Phi_1, \dots, \Phi_N}(\varepsilon \mathbf{Z}^{2d}) = \ell_{(\omega)}^{\Phi_1, \dots, \Phi_N}(\Lambda_1 \times \dots \times \Lambda_N).$$

Let Φ_j be quasi-Young functions, $j = 1, \dots, N$, $\omega \in \mathcal{P}_E(\mathbf{R}^{2d})$, $\vec{d} \in I_{2d,N}$ and $\phi \in \Sigma_1(\mathbf{R}^d) \setminus 0$. Then the Orlicz modulation space

$$M_{\vec{d},(\omega)}^{\Phi_1, \dots, \Phi_N}(\mathbf{R}^d)$$

consists of all $f \in \Sigma'_1(\mathbf{R}^d)$ such that

$$\|f\|_{M_{\vec{d},(\omega)}^{\Phi_1, \dots, \Phi_N}} \equiv \|V_\phi f\|_{L_{\vec{d},(\omega)}^{\Phi_1, \dots, \Phi_N}}$$

is finite. By similar arguments as in [37] it follows that $M_{\vec{d},(\omega)}^{\Phi_1,\dots,\Phi_N}(\mathbf{R}^d)$ is a quasi-Banach space with quasi-norm $\|\cdot\|_{M_{\vec{d},(\omega)}^{\Phi_1,\dots,\Phi_N}}$, which is a Banach space and norm, respectively, when Φ_j is a Young function for every $j \in \{1, \dots, N\}$.

A common situation is when $\vec{d} = (d_0, \dots, d_0)$ for some integer $d_0 \geq 1$, and then we put

$$M_{(\omega)}^{\Phi_1,\dots,\Phi_N} = M_{\vec{d},(\omega)}^{\Phi_1,\dots,\Phi_N}.$$

Remark 2.2 For future references we observe that Proposition 1.16 carry over to Orlicz modulation spaces of the form $M_{\vec{d},(\omega)}^{\Phi_1,\dots,\Phi_N}(\mathbf{R}^d)$ when Φ_j are quasi-Young functions, $j = 1, \dots, N$, $\omega \in \mathcal{P}_E(\mathbf{R}^{2d})$ and $\vec{d} = (d_1, \dots, d_N) \in I_{2d,N}^0$. In particular it follows that (1.14) takes the form

$$\begin{aligned} \|\{(V_\phi f)(k, \kappa)\}_{k,\kappa \in \varepsilon\Lambda}\|_{\ell_{\vec{d},(\omega)}^{\Phi_1,\dots,\Phi_N}} &\asymp \|\{(V_\psi f)(k, \kappa)\}_{k,\kappa \in \varepsilon\Lambda}\|_{\ell_{\vec{d},(\omega)}^{\Phi_1,\dots,\Phi_N}} \\ &\asymp \|f\|_{M_{\vec{d},(\omega)}^{\Phi_1,\dots,\Phi_N}}. \end{aligned} \tag{1.14}'$$

Proposition 2.3 Let $N, j_0 \in \{1, \dots, N-1\}$, $\vec{d} = (d_1, \dots, d_N) \in I_{d,N}$, Λ_j be lattices in \mathbf{R}^{d_j} , $j = 1, \dots, N$, and let

$$\vec{d}_0 = (d_1, \dots, d_{j_0-1}, d_{j_0} + d_{j_0+1}, d_{j_0+2}, \dots, d_N) \in I_{d,N-1}.$$

Also let ω be a weight on \mathbf{R}^d and $\Phi_j, \Psi_k, j = 1, \dots, N, k = 1, \dots, N-1$, be quasi-Young functions such that

$$\Phi_{j_0+1} = \Phi_{j_0} \quad \text{and} \quad \Psi_j = \begin{cases} \Phi_j, & j \leq j_0, \\ \Phi_{j+1}, & j \geq j_0 + 1. \end{cases} \tag{2.2}$$

If $\Lambda = \Lambda_1 \times \dots \times \Lambda_N$, then

$$\ell_{\vec{d}_0,(\omega)}^{\Psi_1,\dots,\Psi_{N-1}}(\Lambda) = \ell_{\vec{d},(\omega)}^{\Phi_1,\dots,\Phi_N}(\Lambda),$$

and

$$\|a\|_{\ell^{\Psi_1, \dots, \Psi_{N-1}}_{d_0, (\omega)}} \asymp \|a\|_{\ell^{\Phi_1, \dots, \Phi_N}_{d, (\omega)}}, \quad a \in \ell'_0(\Lambda).$$

For the proof we recall that for the sequence a on $\mathbf{Z}^{d_1+d_2}$ it holds

$$a \in \ell^\Phi(\mathbf{Z}^{d_1+d_2}) \Leftrightarrow \sum_{j_1, j_2} \Phi(c \cdot a(j_1, j_2)) < \infty \tag{2.3}$$

for some $c > 0$. This implies that

$$a \in \ell^{\Phi, \Phi}(\mathbf{Z}^{d_1+d_2}) \Leftrightarrow \sum_{j_2} \Phi(c_1 \sum_{j_1} \Phi(c_2(j_2)a(j_1, j_2))) < \infty \tag{2.4}$$

for some $c_1 > 0$ and a positive sequence c_2 on \mathbf{Z}^{d_2} .

Proof We only prove the result in the case $N = 2$ and for $\Lambda_j = \mathbf{Z}^{d_j}$. The general case follows by these arguments and induction, and is left for the reader.

Let $r_0 \in (0, 1]$ be chosen such that $\Phi_{0,j}(t) = \Phi_j(t^{\frac{1}{r_0}})$ are Young functions. Then

$$\|a\|_{\ell^{\Phi_1, \Phi_2}_{(\omega)}} \asymp (\| |a \cdot \omega|^{r_0} \|_{\ell^{\Phi_{0,1}, \Phi_{0,2}}})^{1/r_0}.$$

Furthermore, $\|a\|_{\ell^{\Phi_1 \Phi_2}} = \|a\|_{\ell^{\Phi_1 \Phi_2}}$. This reduce the result to the case when Φ is a Young function, $\omega = 1$ and $a \geq 0$.

The result is obviously true when $\Phi = 0$ near origin. In fact for such Φ ,

$$\ell^\Phi(\mathbf{Z}^{d_1+d_2}) = \ell^\infty(\mathbf{Z}^{d_1+d_2}) = \ell^{\infty, \infty}(\mathbf{Z}^{d_1+d_2}) = \ell^{\Phi, \Phi}(\mathbf{Z}^{d_1+d_2})$$

in view of Proposition 1.11. In the same way, If $\lim_{t \rightarrow 0^+} \left(\frac{\Phi(t)}{t}\right) > 0$, then

$$\ell^\Phi = \ell^1 = \ell^{1,1} = \ell^{\Phi, \Phi},$$

and the result follows in this case as well (see, e.g. [28, 37]). It remains to consider the case when $\Phi(t) > 0$, for $t > 0$, and when $\lim_{t \rightarrow 0^+} \left(\frac{\Phi(t)}{t}\right) = 0$. Since ℓ^Φ and $\ell^{\Phi, \Phi}$ do not change when $\Phi(t)$ is replaced by an increasing convex function which is equal to $c \cdot \Phi(t)$ near $t = 0$, where $c > 0$ is a constant, it follows from Proposition 1.11 that we may assume that $\Phi(t) \leq t$ and that Φ is increasing.

This gives

$$\sum_{j_2} \Phi \left(c_1 \sum_{j_1} \Phi(c_2 a(j_1, j_2)) \right) \leq c_1 \sum_{j_1, j_2} \Phi(c_2 a(j_1, j_2))$$

when $c_1, c_2 > 0$ are constants.

Hence if

$$\sum_{j_1, j_2} \Phi(c \cdot a(j_1, j_2)) < \infty$$

for some constant $c > 0$, then

$$\sum_{j_2} \Phi \left(c_1 \sum_{j_1} \Phi(c_2 \cdot a(j_1, j_2)) \right) < \infty$$

for some constants $c_1, c_2 > 0$. By (2.3) and (2.4) we get

$$\ell^\Phi(\mathbf{Z}^{d_1+d_2}) \hookrightarrow \ell^{\Phi, \Phi}(\mathbf{Z}^{d_1+d_2}). \tag{2.5}$$

We need to deduce the reversed inclusion in (2.5).

First we assume that a has finite support, i.e. $a(j_1, j_2) \neq 0$ for at most finite numbers of (j_1, j_2) . Since $\Phi(t) > 0$ when $t > 0$ and $\lim_{t \rightarrow 0^+} \left(\frac{\Phi(t)}{t} \right) = 0$, it follows that the complementary Young function Φ^* to Φ fulfills the same properties.

By Propositions 3 and 4 in Section 3.3 in [25], we have

$$\|a\|_{\ell^\Phi} \asymp \sup_{\|b\|_{\ell^{\Phi^*}} \leq 1} |(a, b)_{\ell^2}|$$

and

$$\|a\|_{\ell^{\Phi, \Phi}} \asymp \sup_{\|b\|_{\ell^{\Phi^*, \Phi^*}} \leq 1} |(a, b)_{\ell^2}|.$$

By a combination of these relations and (2.5) we get

$$\|a\|_{\ell^\Phi} \asymp \sup_{\|b\|_{\ell^{\Phi^*}} \leq 1} |(a, b)_{\ell^2}| \lesssim \sup_{\|b\|_{\ell^{\Phi^*, \Phi^*}} \leq 1} |(a, b)_{\ell^2}| \asymp \|a\|_{\ell^{\Phi, \Phi}},$$

and the searched estimate follows for sequences with finite support.

For general $a \geq 0$, let $a_j, j \geq 1$ be sequences such that

$$a_j \leq a_{j+1} \quad \text{and} \quad \lim_{j \rightarrow \infty} a_j = a. \tag{2.6}$$

Then Beppo-Levi's theorem gives

$$\|a\|_{\ell^\Phi} = \lim_{j \rightarrow \infty} \|a_j\|_{\ell^\Phi} \lesssim \lim_{j \rightarrow \infty} \|a_j\|_{\ell^{\Phi, \Phi}} = \|a\|_{\ell^{\Phi, \Phi}}.$$

For general a , we may split up a into positive and negative real and imaginary parts and use (2.6) to get

$$\|a\|_{\ell^\Phi} \lesssim \|a\|_{\ell^{\Phi,\Phi}}.$$

This implies $\ell^{\Phi,\Phi}(\mathbf{Z}^{d_1+d_2}) \hookrightarrow \ell^\Phi(\mathbf{Z}^{d_1+d_2})$ and the result follows. □

By combining Propositions 1.16, 2.3 and Remark 2.2 we get the following. The details are left for the reader.

Theorem 2.4 *Let N and j_0 be positive integers such that $1 \leq j_0 \leq N - 1$, Φ_j and Ψ_k , $j = 1, \dots, N, k = 1, \dots, N - 1$, be quasi-Young functions such that (2.2) holds and let $\omega \in \mathcal{P}_E(\mathbf{R}^{2d})$. Also let*

$$\vec{d} = (d_1, \dots, d_N) \in I_{2d,N}^0$$

and

$$\vec{d}_0 = (d_1, \dots, d_{j_0-1}, d_{j_0} + d_{j_0+1}, d_{j_0+2}, \dots, d_N) \in I_{2d,N-1}^0.$$

Then

$$M_{\vec{d},(\omega)}^{\Phi_1,\dots,\Phi_N}(\mathbf{R}^d) = M_{\vec{d}_0,(\omega)}^{\Psi_1,\dots,\Psi_{N-1}}(\mathbf{R}^d)$$

and

$$\|f\|_{M_{\vec{d},(\omega)}^{\Phi_1,\dots,\Phi_N}} \asymp \|f\|_{M_{(\omega)}^{\Psi_1,\dots,\Psi_{N-1}}}, \quad f \in \Sigma'_1(\mathbf{R}^d).$$

Corollary 2.5 *Let N be a positive integer, $\vec{d} \in I_{2d,N}^0$, Φ be a quasi-Young function and $\omega \in \mathcal{P}_E(\mathbf{R}^{2d})$. Then*

$$M_{\vec{d},(\omega)}^{\Phi,\dots,\Phi}(\mathbf{R}^d) = M_{(\omega)}^\Phi(\mathbf{R}^d)$$

and

$$\|f\|_{M_{\vec{d},(\omega)}^{\Phi,\dots,\Phi}} \asymp \|f\|_{M_{(\omega)}^\Phi}, \quad f \in \Sigma'_1(\mathbf{R}^d).$$

3 Continuity of pseudo-differential operators on Orlicz modulation spaces

In this section we deduce continuity properties of pseudo-differential operators when acting on Orlicz modulation spaces. The main results are Theorems 3.7 and 3.10 which deal with such operators with symbols in $M_{(\omega)}^{\infty,r_0}(\mathbf{R}^{2d})$ and $M_{(\omega)}^{\Phi,\Phi}(\mathbf{R}^{2d})$, respectively, where $r_0 \in (0, 1]$ and Φ is a quasi-Young functions.

In the first part we deduce related continuity results for suitable matrix operators. In the second part we combine these results and Gabor analysis results from the previous section to establish the continuity results for the pseudo-differential operators.

In the following definition we recall some matrix classes, considered in [33]. Here we observe that we may identify $\Lambda \times \Lambda$ matrices with sequences on $\Lambda \times \Lambda$, when Λ is a lattice in \mathbf{R}^d .

Definition 3.1 Let $p, q \in (0, \infty]$, Φ_1, Φ_2 be quasi-Young functions, $\omega \in \mathcal{P}_E(\mathbf{R}^{2d})$, Λ be a lattice in \mathbf{R}^d and let T be the map on $\ell'_0(\Lambda \times \Lambda)$, given by

$$(Ta)(j, k) = a(j, j - k), \quad a \in \ell'_0(\Lambda \times \Lambda), \quad j, k \in \Lambda.$$

- (1) The set $\mathbb{U}'_0(\Lambda \times \Lambda)$ consists of all (formal) matrices

$$A = (a(j, k))_{j, k \in \Lambda} \tag{3.1}$$

with entries $a(j, k)$ in \mathbf{C} , and $\mathbb{U}_0(\Lambda \times \Lambda)$ consists of all A in (3.1) such that at most finite numbers of $a(j, k)$ are nonzero.

- (2) The set $\mathbb{U}^{p,q}_{(\omega)}(\Lambda \times \Lambda)$ consists of all matrices $A = (a(j, k))_{j, k \in \Lambda}$ such that

$$\|A\|_{\mathbb{U}^{p,q}_{(\omega)}} \equiv \|T(a \cdot \omega)\|_{\ell^{p,q}},$$

is finite.

- (3) The set $\mathbb{U}^{\Phi_1, \Phi_2}_{(\omega)}(\Lambda \times \Lambda)$ consists of all matrices $A = (a(j, k))_{j, k \in \Lambda}$ such that

$$\|A\|_{\mathbb{U}^{\Phi_1, \Phi_2}_{(\omega)}} \equiv \|T(a \cdot \omega)\|_{\ell^{\Phi_1, \Phi_2}},$$

is finite.

Remark 3.2 Let $p \in (0, \infty]$, Φ be a quasi-Young function and $\omega \in \mathcal{P}_E(\mathbf{R}^{2d})$. Then it follows from Proposition 2.3 and straight-forward changes of variables that the following is true. The details are left for the reader.

- (1) If $A_0 = (a(j, k))_{j, k \in \mathbf{Z}^d}$ is a matrix, then $A_0 \in \mathbb{U}^{p,p}_{(\omega)}(\mathbf{Z}^{2d})$, if and only if $a \in \ell^{p,p}_{(\omega)}(\mathbf{Z}^{2d}) = \ell^p_{(\omega)}(\mathbf{Z}^{2d})$, and

$$\|A_0\|_{\mathbb{U}^{p,p}_{(\omega)}} = \|a\|_{\ell^{p,p}_{(\omega)}} = \|a\|_{\ell^p_{(\omega)}}.$$

- (2) If $A_0 = (a(j, k))_{j, k \in \mathbf{Z}^d}$ is a matrix, then $A_0 \in \mathbb{U}^{\Phi, \Phi}_{(\omega)}(\mathbf{Z}^{2d})$, if and only if $a \in \ell^{\Phi, \Phi}_{(\omega)}(\mathbf{Z}^{2d}) = \ell^\Phi_{(\omega)}(\mathbf{Z}^{2d})$, and

$$\|A_0\|_{\mathbb{U}^{\Phi, \Phi}_{(\omega)}} = \|a\|_{\ell^{\Phi, \Phi}_{(\omega)}} = \|a\|_{\ell^\Phi_{(\omega)}}.$$

Next we discuss continuity for certain matrix operators when acting on discrete Orlicz spaces. We recall that if $\Lambda \subseteq \mathbf{R}^d$ is a lattice, $\omega_1, \omega_2 \in \mathcal{P}_E(\mathbf{R}^{2d})$ and $\omega \in \mathcal{P}_E(\mathbf{R}^{4d})$ are such that

$$\frac{\omega_2(j)}{\omega_1(k)} \leq \omega(j, k), \quad j, k \in \Lambda^2, \tag{3.2}$$

$r_0 \in (0, 1]$ and $p, q \in [r_0, \infty]$, then [33, Theorem 2.3] shows that A_0 from $\ell_0(\Lambda^2)$ to $\ell'_0(\Lambda^2)$ is uniquely extendable to a continuous map from $\ell^{p,q}_{(\omega_1)}(\Lambda^2)$ to $\ell^{p,q}_{(\omega_2)}(\Lambda^2)$. The following result extends this result to discrete Orlicz spaces.

Theorem 3.3 *Let $\varepsilon > 0$, $N \geq 1$ be an integer, $\vec{d} \in I^0_{2d,N}$, Φ_1, \dots, Φ_N be quasi Young functions of order $r_0 \in (0, 1]$, $\omega_1, \omega_2 \in \mathcal{P}_E(\mathbf{R}^{2d})$, $\omega \in \mathcal{P}_E(\mathbf{R}^{4d})$ be such that (3.2) holds. If $A \in \mathbb{U}^{\infty,r_0}_{(\omega)}(\varepsilon\mathbf{Z}^{4d})$, then A from $\ell^{\infty}_{(\omega_1)}(\varepsilon\mathbf{Z}^{2d})$ to $\ell^{\infty}_{(\omega_2)}(\varepsilon\mathbf{Z}^{2d})$ restricts to a continuous map from $\ell^{\Phi_1, \dots, \Phi_N}_{\vec{d}, (\omega_1)}(\varepsilon\mathbf{Z}^{2d})$ to $\ell^{\Phi_1, \dots, \Phi_N}_{\vec{d}, (\omega_2)}(\varepsilon\mathbf{Z}^{2d})$ and*

$$\|Af\|_{\ell^{\Phi_1, \dots, \Phi_N}_{\vec{d}, (\omega_2)}} \leq \|A\|_{\mathbb{U}^{\infty,r_0}_{(\omega)}} \|f\|_{\ell^{\Phi_1, \dots, \Phi_N}_{\vec{d}, (\omega_1)}}, \quad f \in \ell^{\Phi_1, \dots, \Phi_N}_{\vec{d}, (\omega_1)}(\varepsilon\mathbf{Z}^{2d}). \tag{3.3}$$

We need the following lemma for the proof of Theorem 3.3. We omit the proof since the result is a consequence of [37, Lemma 3.1].

Lemma 3.4 *Let $\Lambda \subseteq \mathbf{R}^d$ be a lattice, $\mathcal{B} \subseteq \ell'_0(\Lambda)$ be a quasi-Banach space of order $r_0 \in (0, 1]$, with quasi-norm $\|\cdot\|_{\mathcal{B}}$. If*

$$\|f(\cdot - j)\|_{\mathcal{B}} = \|f\|_{\mathcal{B}}, \quad f \in \mathcal{B}, j \in \Lambda,$$

*then the discrete convolution map $(f, g) \mapsto f *_{\Lambda} g$ from $\ell^{r_0}(\Lambda) \times \ell^{r_0}(\Lambda)$ to $\ell^{r_0}(\Lambda)$ extends uniquely to a continuous map from $\mathcal{B} \times \ell^{r_0}(\Lambda)$ to \mathcal{B} , and*

$$\|f * g\|_{\mathcal{B}} \leq \|f\|_{\mathcal{B}} \|g\|_{\ell^{r_0}(\Lambda)}, \quad f \in \mathcal{B}, g \in \ell^{r_0}(\Lambda).$$

Proof of Theorem 3.3 We only prove the result in the case $N = 2$. For general N , the result follows by similar arguments, and is left for the reader. Let $f \in \ell^{\Phi_1, \Phi_2}_{(\omega_1)}(\varepsilon\mathbf{Z}^{2d})$ and set $g = Af$.

First we consider the case when $A \in \mathbb{U}_0(\varepsilon\mathbf{Z}^{4d})$ and let

$$a_{\omega}(j, k) = |a(j, k)\omega(j, k)|, \quad f_{\omega_1}(k) = |f(k)\omega_1(k)|$$

and

$$g_{\omega_2}(j) = |g(j)\omega_2(j)|.$$

We get

$$g_{\omega_2}(j) = |Af(j)\omega_2(j)|$$

$$\begin{aligned} &\leq \sum_{k \in \varepsilon \mathbf{Z}^{2d}} |a(j, k) f(k) \omega_1(k) \omega(j, k)| \\ &= \sum_{k \in \varepsilon \mathbf{Z}^{2d}} |a_\omega(j, j - k) f_{\omega_1}(j - k)| \\ &\leq \sum_{k \in \varepsilon \mathbf{Z}^{2d}} h_\omega(k) f_{\omega_1}(j - k) = (h_\omega * f_{\omega_1})(j), \end{aligned}$$

where $h_\omega(k) = \sup_{j \in \varepsilon \mathbf{Z}^{2d}} a_\omega(j, j - k)$.

By Lemma 3.4 we get

$$\begin{aligned} \|Af\|_{\ell_{(\omega_2)}^{\Phi_1, \Phi_2}} &= \|g_{\omega_2}\|_{\ell^{\Phi_1, \Phi_2}} \leq \|h_\omega * f_{\omega_1}\|_{\ell^{\Phi_1, \Phi_2}} \leq \|h_\omega\|_{\ell^{r_0}} \|f_{\omega_1}\|_{\ell^{\Phi_1, \Phi_2}} \\ &= \|A\|_{\mathbb{U}_{(\omega)}^{\infty, r_0}} \|f\|_{\ell_{(\omega_1)}^{\Phi_1, \Phi_2}}, \end{aligned}$$

and the result follows in this case.

For general $A \in \mathbb{U}_{(\omega)}^{\infty, r_0}(\varepsilon \mathbf{Z}^{4d})$ we decompose A and f into

$$A = A_1 - A_2 + i(A_3 - A_4) \quad \text{and} \quad f = f_1 - f_2 + i(f_3 - f_4), \tag{3.4}$$

where A_j and f_k only have non-negative entries, chosen as small as possible. By Beppo-Levi's theorem and the estimates above it follows that $A_j f_k$ is uniquely defined as an element in $\ell_{(\omega_2)}^{\Phi_1, \Phi_2}(\varepsilon \mathbf{Z}^{2d})$. It also follows from these estimates that (3.3) holds. \square

Remark 3.5 Let

$$A = (a(j, k))_{j, k \in \varepsilon \mathbf{Z}^{2d}} \in \mathbb{U}'_0(\varepsilon \mathbf{Z}^{4d}) \quad \text{and} \quad f = \{f(j)\}_{j \in \varepsilon \mathbf{Z}^{2d}} \in \ell'_0(\varepsilon \mathbf{Z}^{2d}).$$

Then A_n in (3.4) are given by

$$A_n = (a_n(j, k))_{j, k \in \varepsilon \mathbf{Z}^{2d}}, \quad n = 1, 2, 3, 4,$$

where

$$\begin{aligned} a_1(j, k) &= \max(\operatorname{Re}(a(j, k)), 0), & a_2(j, k) &= \min(\operatorname{Re}(a(j, k)), 0), \\ a_3(j, k) &= \max(\operatorname{Im}(a(j, k)), 0), & a_4(j, k) &= \min(\operatorname{Im}(a(j, k)), 0), \end{aligned}$$

and $f_n = \{f_n(j)\}_{j \in \varepsilon \mathbf{Z}^{2d}}$, are obtained in the same way after each $a_n(j, k)$ and $a(j, k)$ are replaced by $f_n(j)$ and $f(j)$, respectively.

Before we discuss continuity properties of pseudo-differential operators on Orlicz modulation spaces, we have the following result concerning operator classes

$$\{\operatorname{Op}_A(a) ; a \in M_{(\omega)}^{\Phi_1, \Phi_2}(\mathbf{R}^{2d})\}$$

of continuous operators from $\Sigma_1(\mathbf{R}^d)$ to $\Sigma'_1(\mathbf{R}^d)$. Here recall [36, Proposition 1.9] for analogous relations for pseudo-differential operators with symbols in (ordinary) modulation spaces. Here and in what follow, A^* denotes the transpose of the matrix A .

Proposition 3.6 *Let $N \geq 1$ be an integer, $\vec{d} \in I_{4d,N}^0$, $A \in \mathbf{M}(d, \mathbf{R})$, Φ_1, \dots, Φ_N be quasi-Young functions, $\omega \in \mathcal{P}_E(\mathbf{R}^{4d})$ and let*

$$\omega_A(x, \xi, \eta, y) = \omega(x - Ay, \xi - A^*\eta, \eta, y).$$

Then the following is true:

- (1) *The map $e^{i\langle AD_\xi, D_x \rangle}$ from $M_{(\omega)}^\infty(\mathbf{R}^{2d})$ to $M_{(\omega_A)}^\infty(\mathbf{R}^{2d})$ restricts to a homeomorphism from $M_{\vec{d},(\omega)}^{\Phi_1, \dots, \Phi_N}(\mathbf{R}^{2d})$ to $M_{\vec{d},(\omega_A)}^{\Phi_1, \dots, \Phi_N}(\mathbf{R}^{2d})$;*
- (2) *The set*

$$\{ \text{Op}_A(a) ; a \in M_{\vec{d},(\omega_A)}^{\Phi_1, \dots, \Phi_N}(\mathbf{R}^{2d}) \}$$

of operators from $\Sigma_1(\mathbf{R}^d)$ to $\Sigma'_1(\mathbf{R}^d)$ is independent of $A \in \mathbf{M}(d, \mathbf{R})$.

Proof We only prove the result in the case $N = 2$. For general N , the result follows by similar arguments and is left for the reader.

It suffices to prove (1) in view of Proposition 1.18.

Let $a \in M_{(\omega)}^{\Phi_1, \Phi_2}(\mathbf{R}^{2d})$, $\phi \in \Sigma_1(\mathbf{R}^d)$, $\psi = e^{i\langle AD_\xi, D_x \rangle} \phi$ and $b = e^{i\langle AD_\xi, D_x \rangle} a$. Then it follows from Theorem 3.1 and (3.1) in [1] that $\psi \in \Sigma_1(\mathbf{R}^d)$ and

$$\begin{aligned} & |V_\psi b(x, \xi, \eta, y) \omega_A(x, \xi, \eta, y)| \\ &= |V_\phi a(x - Ay, \xi - A^*\eta, \eta, y) \omega(x - Ay, \xi - A^*\eta, \eta, y)|. \end{aligned}$$

By applying the L^{Φ_1} quasi-norm with respect to the (x, ξ) variables we obtain

$$\begin{aligned} & \|V_\psi b(\cdot, \eta, y) \omega_A(\cdot, \eta, y)\|_{L^{\Phi_1}} \\ &= \|V_\phi a(\cdot - (Ay, A^*\eta), \eta, y) \omega(\cdot - (Ay, A^*\eta), \eta, y)\|_{L^{\Phi_1}} \\ &= \|V_\phi a(\cdot, \eta, y) \omega(\cdot, \eta, y)\|_{L^{\Phi_1}}, \end{aligned}$$

and applying the L^{Φ_2} quasi-norm with respect to the (y, η) variable on the last equality gives

$$\|V_\psi b \cdot \omega_A\|_{L^{\Phi_1, \Phi_2}} = \|V_\phi a \cdot \omega\|_{L^{\Phi_1, \Phi_2}}.$$

This gives

$$\|b\|_{M_{(\omega_A)}^{\Phi_1, \Phi_2}} = \|a\|_{M_{(\omega)}^{\Phi_1, \Phi_2}},$$

and the result follows. □

We have now the following continuity result for pseudo-differential operators acting on Orlicz modulation spaces. Here the involved weight functions should satisfy

$$\frac{\omega_2(x, \xi)}{\omega_1(y, \eta)} \lesssim \omega(x + A(y - x), \eta + A^*(\xi - \eta), \xi - \eta, y - x). \tag{3.5}$$

Theorem 3.7 *Let $A \in M(d, \mathbf{R})$, Φ_1, Φ_2 be quasi Young functions of order $r_0 \in (0, 1]$, $\omega \in \mathcal{P}_E(\mathbf{R}^{4d})$ and $\omega_1, \omega_2 \in \mathcal{P}_E(\mathbf{R}^{2d})$ be such that (3.5) holds, and let $a \in M_{(\omega)}^{\infty, r_0}(\mathbf{R}^{2d})$. Then $\text{Op}_A(a)$ from $\Sigma_1(\mathbf{R}^d)$ to $\Sigma'_1(\mathbf{R}^d)$ is uniquely extendable to a continuous map from $M_{(\omega_1)}^{\Phi_1, \Phi_2}(\mathbf{R}^d)$ to $M_{(\omega_2)}^{\Phi_1, \Phi_2}(\mathbf{R}^d)$, and*

$$\| \text{Op}_A(a) \|_{M_{(\omega_1)}^{\Phi_1, \Phi_2}(\mathbf{R}^d) \rightarrow M_{(\omega_2)}^{\Phi_1, \Phi_2}(\mathbf{R}^d)} \lesssim \| a \|_{M_{(\omega)}^{\infty, r_0}}. \tag{3.6}$$

The previous result can be generalized into the following.

Theorem 3.8 *Let $A \in M(d, \mathbf{R})$, N be a positive integer, $\vec{d} \in I_{d, N}$, $\Phi_j, j = 1, \dots, N$, be quasi-Young functions of order $r_0 \in (0, 1]$, $\omega \in \mathcal{P}_E(\mathbf{R}^{4d})$ and $\omega_1, \omega_2 \in \mathcal{P}_E(\mathbf{R}^{2d})$ be such that (3.5) holds, and let $a \in M_{(\omega)}^{\infty, r_0}(\mathbf{R}^{2d})$. Then $\text{Op}_A(a)$ from $\Sigma_1(\mathbf{R}^d)$ to $\Sigma'_1(\mathbf{R}^d)$ is uniquely extendable to a continuous map from $M_{\vec{d}, (\omega_1)}^{\Phi_1, \dots, \Phi_N}(\mathbf{R}^d)$ to $M_{\vec{d}, (\omega_2)}^{\Phi_1, \dots, \Phi_N}(\mathbf{R}^d)$, and*

$$\| \text{Op}_A(a) \|_{M_{\vec{d}, (\omega_1)}^{\Phi_1, \dots, \Phi_N}(\mathbf{R}^d) \rightarrow M_{\vec{d}, (\omega_2)}^{\Phi_1, \dots, \Phi_N}(\mathbf{R}^d)} \lesssim \| a \|_{M_{(\omega)}^{\infty, r_0}}. \tag{3.7}$$

We only prove Theorem 3.7. Theorem 3.8 follows by similar arguments and is left for the reader.

We need some preparations for the proof of Theorem 3.7. First we have the following extension of [33, Lemma 3.3] to the case of Orlicz modulation spaces.

Lemma 3.9 *Let $\Lambda, \phi_1, \phi_2, \varphi, \psi$ and $\varepsilon > 0$ be as in Lemma 1.15. Also let $v \in \mathcal{P}_E(\mathbf{R}^{4d})$, $a \in M_{(1/v)}^{\infty}(\mathbf{R}^{2d})$,*

$$c_0(\mathbf{j}, \mathbf{k}) \equiv (V_{\psi} a)(j, \kappa, l - \kappa, k - j) e^{i(k - j \cdot \kappa)},$$

where $\mathbf{j} = (j, l) \in \varepsilon \Lambda^2$, $\mathbf{k} = (k, \kappa) \in \varepsilon \Lambda^2$,

and let A_a be the matrix $A_a = (c_0(\mathbf{j}, \mathbf{k}))_{\mathbf{j}, \mathbf{k} \in \varepsilon \Lambda^2}$. Then the following is true:

(1) *If Φ_1, Φ_2 are quasi-Young functions and $\omega, \omega_0 \in \mathcal{P}_E(\mathbf{R}^{4d})$ satisfy*

$$\omega(x, \xi, y, \eta) \asymp \omega_0(x, \eta, \xi - \eta, y - x), \tag{3.8}$$

then $a \in M_{(\omega_0)}^{\Phi_1, \Phi_2}(\mathbf{R}^{2d})$, if and only if $A_a \in \mathbb{U}_{(\omega)}^{\Phi_1, \Phi_2}(\varepsilon(\Lambda^2 \times \Lambda^2))$, and then

$$\| a \|_{M_{(\omega_0)}^{\Phi_1, \Phi_2}} \asymp \| A_a \|_{\mathbb{U}_{(\omega)}^{\Phi_1, \Phi_2}(\varepsilon(\Lambda^2 \times \Lambda^2))};$$

(2) $\text{Op}(a)$ as map from $\Sigma_1(\mathbf{R}^d)$ to $\Sigma'_1(\mathbf{R}^d)$ is given by

$$\text{Op}(a) = D_{\phi_1}^{\varepsilon, \Lambda} \circ A_a \circ C_{\phi_2}^{\varepsilon, \Lambda}. \quad (3.9)$$

Proof We have

$$|c_0(\mathbf{j}, \mathbf{j} - \mathbf{k})| = |(V_{\Psi}a)(j, \iota - \kappa, \kappa, -k)|.$$

Hence, Proposition 1.16 gives

$$\|A_a\|_{\mathbb{U}_{(\omega)}^{\Phi_1, \Phi_2}(\varepsilon(\Lambda^2 \times \Lambda^2))} = \|V_{\Psi}a\|_{\ell_{(\omega_0)}^{\Phi_1, \Phi_2}(\varepsilon(\Lambda^2 \times \Lambda^2))} \asymp \|a\|_{M_{(\omega_0)}^{\Phi_1, \Phi_2}},$$

and (1) follows.

The assertion (2) is the same as assertion (2) in [33, Lemma 3.3]. The proof is therefore omitted. \square

Proof of Theorem 3.7 By Proposition 3.6 we may assume that $A = 0$.

Let a, A_a, ϕ_1 and ϕ_2 be the same as in Proposition 1.17 and Lemma 3.9. Then by Proposition 1.17, Theorem 3.3 and Lemma 3.9 we get

$$\|\text{Op}(a)\|_{M_{(\omega_1)}^{\Phi_1, \Phi_2} \rightarrow M_{(\omega_2)}^{\Phi_1, \Phi_2}} \lesssim J_1 \cdot J_2 \cdot J_3,$$

where

$$J_1 = \|D_{\phi_1}\|_{\ell_{(\omega_2)}^{\Phi_1, \Phi_2} \rightarrow M_{(\omega_2)}^{\Phi_1, \Phi_2}} < \infty, \quad (3.10)$$

$$J_2 = \|A_a\|_{\ell_{(\omega_2)}^{\Phi_1, \Phi_2} \rightarrow \ell_{(\omega_2)}^{\Phi_1, \Phi_2}} < \infty \quad (3.11)$$

and

$$J_3 = \|C_{\phi_2}\|_{M_{(\omega_1)}^{\Phi_1, \Phi_2} \rightarrow \ell_{(\omega_1)}^{\Phi_1, \Phi_2}} < \infty. \quad (3.12)$$

This gives the asserted continuity. The uniqueness follows from the facts that

$$M_{(\omega_j)}^{\Phi_1, \Phi_2}(\mathbf{R}^d) \subseteq M_{(\omega_j)}^{\infty}(\mathbf{R}^d),$$

in view of Proposition 1.11 and that $\text{Op}(a)$ is uniquely defined as a continuous operator from $M_{(\omega_1)}^{\infty}(\mathbf{R}^d)$ to $M_{(\omega_2)}^{\infty}(\mathbf{R}^d)$, in view of [33, Theorem 3.1]. \square

We have also the following.

Theorem 3.10 Let $A \in M(d, \mathbf{R})$, Φ_0 be a Young function, Φ_0^* the complementary Young function of Φ_0 , Φ be a quasi-Young function such that

$$\lim_{t \rightarrow 0^+} \frac{t}{\Phi(t)} \quad (3.13)$$

is finite and let $\omega \in \mathcal{P}_E(\mathbf{R}^{4d})$ and $\omega_1, \omega_2 \in \mathcal{P}_E(\mathbf{R}^{2d})$ be such that (3.5) holds. Then the following is true:

- (1) if $a \in M_{(\omega)}^{\Phi_0}(\mathbf{R}^{2d})$, then $\text{Op}_A(a)$ from $M_{(\omega_1)}^1(\mathbf{R}^d)$ to $M_{(\omega_2)}^\infty(\mathbf{R}^d)$ is extendable to a continuous map from $M_{(\omega_1)}^{\Phi_0^*}(\mathbf{R}^d)$ to $M_{(\omega_2)}^{\Phi_0}(\mathbf{R}^d)$ and

$$\|\text{Op}_A(a)\|_{M_{(\omega_1)}^{\Phi_0^*}(\mathbf{R}^d) \rightarrow M_{(\omega_2)}^{\Phi_0}(\mathbf{R}^d)} \lesssim \|a\|_{M_{(\omega)}^{\Phi_0}(\mathbf{R}^{2d})};$$

- (2) if $a \in M_{(\omega)}^\Phi(\mathbf{R}^{2d})$, then $\text{Op}_A(a)$ from $M_{(\omega_1)}^1(\mathbf{R}^d)$ to $M_{(\omega_2)}^\infty(\mathbf{R}^d)$ is uniquely extendable to a continuous map from $M_{(\omega_1)}^\infty(\mathbf{R}^d)$ to $M_{(\omega_2)}^\Phi(\mathbf{R}^d)$, and

$$\|\text{Op}_A(a)\|_{M_{(\omega_1)}^\infty(\mathbf{R}^d) \rightarrow M_{(\omega_2)}^\Phi(\mathbf{R}^d)} \lesssim \|a\|_{M_{(\omega)}^\Phi(\mathbf{R}^{2d})}.$$

Proof By Proposition 3.6 we may assume that $A = 0$.

Let $\Lambda \subseteq \mathbf{R}^d$ be a lattice, $A_0 = (a(j, k))_{j, k \in \Lambda} \in \mathbb{U}_{(\omega)}^{\Phi_0, \Phi_0}(\Lambda \times \Lambda)$ and $f \in \ell_{(\omega_1)}^{\Phi_0^*}(\Lambda)$ be such that $a(j, k) \geq 0$ and $f(j) \geq 0$ for every $j, k \in \Lambda$. We have

$$0 \leq (A_0 f)(j) \omega_2(j) = (a(j, \cdot), f) \omega_2(j) \lesssim \|a(j, \cdot) \omega(j, \cdot)\|_{\ell^{\Phi_0}} \|f \cdot \omega_1\|_{\ell^{\Phi_0^*}}.$$

By applying the ℓ^{Φ_0} norm and using Remark 3.2 we get

$$\|A_0 f\|_{\ell_{(\omega_2)}^{\Phi_0}} \lesssim \|a\|_{\ell_{(\omega)}^{\Phi_0, \Phi_0}} \|f\|_{\ell_{(\omega_1)}^{\Phi_0^*}} \asymp \|A_0\|_{\mathbb{U}_{(\omega)}^{\Phi_0, \Phi_0}} \|f\|_{\ell_{(\omega_1)}^{\Phi_0^*}}, \tag{3.14}$$

which implies that $A_0 f$ makes sense as an element in $\ell_{(\omega_2)}^{\Phi_0}(\Lambda)$.

For general $A_0 = (a(j, k))_{j, k \in \Lambda} \in \mathbb{U}_{(\omega)}^{\Phi_0, \Phi_0}(\Lambda \times \Lambda)$ and $f \in \ell_{(\omega_1)}^{\Phi_0^*}(\Lambda)$, we define $A_0 f$ in similar ways as in the proof of Theorem 3.3, by splitting up A_0 and f into positive and negative parts of their real and imaginary parts. By (3.14) we obtain

$$\|A_0\|_{\ell_{(\omega_1)}^{\Phi_0^*}(\Lambda) \rightarrow \ell_{(\omega_2)}^{\Phi_0}(\Lambda)} \lesssim \|A_0\|_{\mathbb{U}_{(\omega)}^{\Phi_0, \Phi_0}}. \tag{3.15}$$

Now let $a \in M_{(\omega)}^{\Phi_0}(\mathbf{R}^{2d})$ and $f \in M_{(\omega_1)}^{\Phi_0^*}(\mathbf{R}^d)$. Then we define $\text{Op}(a) f$ by (3.9). The asserted continuity in (1) now follows from Proposition 2.3, (3.10), (3.12) and (3.15).

Next let Φ be as in (2) and let $A_0 = (a(j, k))_{j, k \in \Lambda} \in \mathbb{U}_{(\omega)}^{\Phi, \Phi}(\Lambda \times \Lambda)$ and $f \in \ell_{(\omega_1)}^\infty(\Lambda)$ be such that $a(j, k) \geq 0$ and $f(j) \geq 0$ for every $j, k \in \Lambda$. Then

$$\begin{aligned} 0 \leq (A_0 f)(j) \omega_2(j) &= (a(j, \cdot), f) \omega_2(j) \\ &\lesssim \|a(j, \cdot) \omega(j, \cdot)\|_{\ell^1} \|f \cdot \omega_1\|_{\ell^\infty} \lesssim \|a(j, \cdot) \omega(j, \cdot)\|_{\ell^\Phi} \|f\|_{\ell_{(\omega_1)}^\infty}, \end{aligned}$$

where the last inequality follows from Proposition 1.11 and (3.13). By applying the ℓ^Φ quasi-norm and splitting up general $A_0 = (a(j, k))_{j, k \in \Lambda} \in \mathbb{U}_{(\omega)}^{\Phi, \Phi}(\Lambda \times \Lambda)$ and

$f \in \ell^\infty_{(\omega_1)}(\Lambda)$ into positive and negative real and imaginary parts, we obtain

$$\|A_0\|_{\ell^\infty_{(\omega_1)}(\Lambda) \rightarrow \ell^\Phi_{(\omega_2)}(\Lambda)} \lesssim \|A_0\|_{\mathbb{U}_{(\omega)}^{\Phi,\Phi}}. \tag{3.16}$$

The asserted continuity in (2) now follows by combining Proposition 2.3, (3.10), (3.12) and (3.16).

The asserted uniqueness follows from the fact that if $a \in M_{(\omega)}^\Phi(\mathbf{R}^{2d})$, then $a \in M_{(\omega)}^1(\mathbf{R}^{2d})$ in view of Proposition 1.11 and (3.13). Hence, if $f \in M_{(\omega_1)}^\infty(\mathbf{R}^d)$, then $\text{Op}(a)f$ is uniquely defined as an element in $M_{(\omega_2)}^1(\mathbf{R}^d)$ (see e.g. [33, Theorem 3.1]). This in turn implies that $\text{Op}(a)f$ is uniquely defined as an element in $M_{(\omega_2)}^\Phi(\mathbf{R}^d)$, and the result follows. \square

4 Symbol product estimates on Orlicz modulation spaces

In this section we show that if ω_j are suitable weights, $j = 0, 1, 2$, Φ_1, Φ_2 are quasi-Young functions of order $r_0 \in (0, 1]$, $a_1 \in M_{(\omega_1)}^{\Phi_1, \Phi_2}$ and $a_2 \in M_{(\omega_2)}^{\infty, r_0}$, then $\text{Op}_A(a_1) \circ \text{Op}_A(a_2)$ equals $\text{Op}_A(b)$ for some $a_1 \in M_{(\omega_0)}^{\Phi_1, \Phi_2}$.

An essential condition on the weight functions is

$$\omega_0(T_A(Z, X)) \lesssim \omega_1(T_A(Y, X))\omega_2(T_A(Z, Y)), \quad X, Y, Z \in (\mathbf{R}^{2d}), \tag{4.1}$$

where

$$\begin{aligned} T_A(X, Y) &= (y + A(x - y), \xi + A^*(\eta - \xi), \eta - \xi, x - y), \\ X &= (x, \xi) \in \mathbf{R}^{2d}, Y = (y, \eta) \in \mathbf{R}^{2d}. \end{aligned} \tag{4.2}$$

Theorem 4.1 *Let $A \in M(d, \mathbf{R})$ and suppose that $\omega_k \in \mathcal{P}_E(\mathbf{R}^{4d})$, $k = 0, 1, 2$, satisfy (4.1) and (4.2). Let Φ_1, Φ_2 be quasi Young functions of order $r_0 \in (0, 1]$. Then the map $(a_1, a_2) \mapsto a_1 \#_A a_2$ from $\Sigma_1(\mathbf{R}^{2d}) \times \Sigma_1(\mathbf{R}^{2d})$ to $\Sigma_1(\mathbf{R}^{2d})$ is uniquely extendable to a continuous map from $M_{(\omega_1)}^{\Phi_1, \Phi_2}(\mathbf{R}^{2d}) \times M_{(\omega_2)}^{\infty, r_0}(\mathbf{R}^{2d})$ to $M_{(\omega_0)}^{\Phi_1, \Phi_2}(\mathbf{R}^{2d})$, and*

$$\|a_1 \#_A a_2\|_{M_{(\omega_0)}^{\Phi_1, \Phi_2}} \lesssim \|a_1\|_{M_{(\omega_1)}^{\Phi_1, \Phi_2}} \|a_2\|_{M_{(\omega_2)}^{\infty, r_0}}. \tag{4.3}$$

We need some preparations for the proof. By [2, Proposition 3.2] it follows that the map $(A_1, A_2) \mapsto A_1 \circ A_2$ is uniquely defined and continuous from $\mathbb{U}_{(\omega_1)}^{\infty, \infty}(\Lambda \times \Lambda) \times \mathbb{U}_{(\omega_2)}^{\infty, r_0}(\Lambda \times \Lambda)$ to $\mathbb{U}_{(\omega)}^{\infty, \infty}(\Lambda \times \Lambda)$ when $\Lambda \subseteq \mathbf{R}^d$ is a lattice, $r_0 \in (0, 1]$ and $\omega, \omega_1, \omega_2 \in \mathcal{P}_E(\mathbf{R}^{2d})$ satisfy

$$\omega(x, z) \leq \omega_1(x, y)\omega_2(y, z), \quad x, y, z \in \mathbf{R}^d. \tag{4.4}$$

The following lemma extends certain parts of this continuity to matrix classes satisfying Orlicz estimates.

Lemma 4.2 *Let $\Lambda \subseteq \mathbf{R}^d$ be a lattice, Φ_1, Φ_2 be quasi Young functions of order $r_0 \in (0, 1]$, and let $\omega, \omega_1, \omega_2 \in \mathcal{P}_E(\mathbf{R}^{2d})$ satisfy (4.4). Then $(A_1, A_2) \mapsto A_1 \circ A_2$ from $\mathbb{U}_{(\omega_1)}^{\infty, \infty}(\Lambda \times \Lambda) \times \mathbb{U}_{(\omega_2)}^{\infty, r_0}(\Lambda \times \Lambda)$ to $\mathbb{U}_{(\omega)}^{\infty, \infty}(\Lambda \times \Lambda)$ restricts to a continuous map from $\mathbb{U}_{(\omega_1)}^{\Phi_1, \Phi_2}(\Lambda \times \Lambda) \times \mathbb{U}_{(\omega_2)}^{\infty, r_0}(\Lambda \times \Lambda)$ to $\mathbb{U}_{(\omega)}^{\Phi_1, \Phi_2}(\Lambda \times \Lambda)$ and*

$$\|A_1 \circ A_2\|_{\mathbb{U}_{(\omega)}^{\Phi_1, \Phi_2}} \lesssim \|A_1\|_{\mathbb{U}_{(\omega_1)}^{\Phi_1, \Phi_2}} \|A_2\|_{\mathbb{U}_{(\omega_2)}^{\infty, r_0}}. \tag{4.5}$$

Proof Let $A_1 = (\mathbf{a}_1(j, k))_{j, k \in \Lambda}$, $A_2 = (\mathbf{a}_2(j, k))_{j, k \in \Lambda}$ be matrices, let the matrix elements of $B = A_1 \circ A_2$ be denoted by $\mathbf{b}(j, k)$, and set

$$a_m(j, k) \equiv |\mathbf{a}_m(j, j - k)|\omega_m(j, j - k), \quad m = 1, 2,$$

and

$$b(j, k) \equiv |\mathbf{b}(j, j - k)|\omega(j, j - k).$$

Then

$$\begin{aligned} \|A_1\|_{\mathbb{U}_{(\omega_1)}^{\Phi_1, \Phi_2}} &= \|a_1\|_{\ell^{\Phi_1, \Phi_2}}, & \|A_2\|_{\mathbb{U}_{(\omega_2)}^{\infty, r_0}} &= \|a_2\|_{\ell^{\infty, r_0}}, \\ \|B\|_{\mathbb{U}_{(\omega)}^{\Phi_1, \Phi_2}} &= \|b\|_{\ell^{\Phi_1, \Phi_2}} \end{aligned}$$

and

$$b(j, k) \leq \sum_{m \in \mathbf{Z}^d} a_1(j, m)a_2(j - m, k - m). \tag{4.6}$$

By a similar application of Beppo-Levi’s theorem, and splitting up A_j as in Remark 3.5, the result follows if we prove

$$\|b\|_{\ell^{\Phi_1, \Phi_2}} \leq \|a_1\|_{\ell^{\Phi_1, \Phi_2}} \|a_2\|_{\ell^{\infty, r_0}},$$

when $a_1, a_2 \in \mathbb{U}_0(\Lambda \times \Lambda)$ have non-negative entries.

Let $\Phi_{0,j}$ be Young functions such that $\Phi_j(t) = \Phi_{0,j}(t^{r_0})$, $t \geq 0$, $j = 1, 2$, and let

$$c_1(m) = \|a_1(\cdot, m)\|_{\ell^{\Phi_1}} \quad \text{and} \quad c_2(k) = \sup_{j \in \Lambda} a_2(j, k)^{r_0}.$$

By (4.6) and the fact that $\Phi_{0,1}$ is convex we get

$$\begin{aligned} \sum_{j \in \Lambda} \Phi_{0,1} \left(\frac{|b(j, k)|^{r_0}}{\lambda^{r_0}} \right) &\leq \sum_{j \in \Lambda} \Phi_{0,1} \left(\frac{1}{\lambda^{r_0}} \sum_{m \in \Lambda} a_1(j, m)^{r_0} a_2(j - m, k - m)^{r_0} \right) \\ &\leq \sum_{j \in \Lambda} \Phi_{0,1} \left(\frac{1}{\lambda^{r_0}} \sum_{m \in \Lambda} a_1(j, m)^{r_0} c_2(k - m)^{r_0} \right) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{j \in \Lambda} \Phi_{0,1} \left(\sum_{m \in \Lambda} \frac{a_1(j, k - m)^{r_0}}{\lambda^{r_0}} c_2(m)^{r_0} \right) \\
 &\leq \sum_{j \in \Lambda} \left(\sum_{m \in \Lambda} \Phi_{0,1} \left(\frac{a_1(j, k - m)^{r_0}}{\lambda^{r_0}} \right) c_2(m)^{r_0} \right) \\
 &= \sum_{m \in \Lambda} c_2(m)^{r_0} \sum_{j \in \Lambda} \Phi_{0,1} \left(\frac{a_1(j, k - m)^{r_0}}{\lambda^{r_0}} \right).
 \end{aligned}$$

This gives

$$\|b(\cdot, k)\|_{\ell^{\Phi_1}}^{r_0} \leq \sum_{m \in \Lambda} c_2(m)^{r_0} \|a_1(\cdot, k - m)\|_{\ell^{\Phi_1}}^{r_0} = (c_1^{r_0} * c_2^{r_0})(k), \tag{4.7}$$

in view of the definition of ℓ^{Φ_1, Φ_2} norm.

By (4.7) we get

$$\begin{aligned}
 \|B\|_{\mathbb{U}_{(\omega)}}^{\Phi_1, \Phi_2} &= \|b\|_{\ell^{\Phi_1, \Phi_2}} \leq (\|c_1^{r_0} * c_2^{r_0}\|_{\ell^{\Phi_{0,2}}})^{\frac{1}{r_0}} \\
 &\leq (\|c_1^{r_0}\|_{\ell^{\Phi_{0,2}}} \|c_2^{r_0}\|_{\ell^1})^{\frac{1}{r_0}} = \|a_1\|_{\ell^{\Phi_1, \Phi_2}} \|a_2\|_{\ell^{\infty, r_0}} \\
 &= \|A_1\|_{\mathbb{U}_{(\omega_1)}^{\Phi_1, \Phi_2}} \|A_2\|_{\mathbb{U}_{(\omega_2)}^{\infty, r_0}}.
 \end{aligned}$$

□

Proof of Theorem 4.1 By Proposition 3.6 we may assume that $A = 0$.

Let $\varepsilon > 0$, ϕ_1, ϕ_2 and Λ be the same as in the proofs of Theorem 3.7 and Lemma 3.9, $a_1 \in M_{(\omega_1)}^{\Phi_1, \Phi_2}(\mathbf{R}^{2d})$ and $a_2 \in M_{(\omega_2)}^{\infty, r_0}(\mathbf{R}^{2d})$. By Theorem 2.17 we have

$$\begin{aligned}
 \|a_1\|_{M_{(\omega_1)}^{\Phi_1, \Phi_2}} &\asymp \|A_1\|_{\mathbb{U}_{(\phi_1)}^{\Phi_1, \Phi_2}}, \quad \|a_2\|_{M_{(\omega_2)}^{\infty, r_0}} \asymp \|A_2\|_{\mathbb{U}_{(\phi_2)}^{\infty, r_0}} \\
 \text{Op}(a_1) &= D_{\phi_1}^{\varepsilon, \Lambda} \circ A_1 \circ C_{\phi_2}^{\varepsilon, \Lambda} \quad \text{and} \quad \text{Op}(a_2) = D_{\phi_1}^{\varepsilon, \Lambda} \circ A_2 \circ C_{\phi_2}^{\varepsilon, \Lambda}, \tag{4.8}
 \end{aligned}$$

where

$$A_m = (\mathbf{a}_m(\mathbf{j}, \mathbf{k}))_{\mathbf{j}, \mathbf{k} \in \varepsilon \Lambda^2},$$

$$\mathbf{a}_m(\mathbf{j}, \mathbf{k}) \equiv e^{i\langle k-j, \kappa \rangle} V_{\varphi} a_m(j, \kappa, \iota - \kappa, k - j), \quad \mathbf{j} = (j, \iota) \in \varepsilon \Lambda^2, \quad \mathbf{k} = (k, \kappa) \in \varepsilon \Lambda^2$$

and

$$\vartheta_m(x, \xi, y, \eta) = \omega_m(x, \eta, \xi - \eta, y - x).$$

The condition (4.1) means for the weights ϑ_m , $m = 0, 1, 2$,

$$\vartheta_0(X, Y) \lesssim \vartheta_1(X, Z) \vartheta_2(Z, Y), \quad X, Y, Z \in \mathbf{R}^{2d}. \tag{4.9}$$

Pick $v_1 \in \mathcal{P}_E(\mathbf{R}^d)$ such that ω_2 is v_2 -moderate, where

$$v_2 = v_1 \otimes v_1 \otimes v_1 \otimes v_1 \in \mathcal{P}_E(\mathbf{R}^{4d}).$$

Also let $v = v_1^2 \otimes v_1^2 \in \mathcal{P}_E(\mathbf{R}^{2d})$ and

$$v_0(X, Y) = v(X - Y) \in \mathcal{P}_E(\mathbf{R}^{4d}), \quad X, Y \in \mathbf{R}^{2d}.$$

Then

$$\vartheta_2(X, Y) \lesssim v_0(X, Z)\vartheta_2(Z, Y), \quad X, Y, Z \in \mathbf{R}^{2d}. \tag{4.10}$$

By (1.23) and (1.24) we get

$$\text{Op}(a_1) \circ \text{Op}(a_2) = D_{\phi_1}^{\varepsilon, \Lambda} \circ A \circ C_{\phi_2}^{\varepsilon, \Lambda},$$

where

$$A = A_1 \circ C \circ A_2$$

and $C = C_{\phi_2}^{\varepsilon, \Lambda} \circ D_{\phi_1}^{\varepsilon, \Lambda}$ is a matrix of the form $(\mathbf{c}(\mathbf{j}, \mathbf{k}))_{\mathbf{j}, \mathbf{k} \in \varepsilon\Lambda^2}$ with matrix elements $\mathbf{c}(\mathbf{j}, \mathbf{k}), \mathbf{j}, \mathbf{k} \in \varepsilon\Lambda^2$.

By [2, Lemma 3.3] we get

$$\begin{aligned} \|C\|_{\mathbb{U}_{(v_0)}^{\infty, r_0}} &= \left(\sum_{\mathbf{k} \in \varepsilon\Lambda^2} \left(\sup_{\mathbf{j} \in \varepsilon\Lambda^2} |\mathbf{c}(\mathbf{j}, \mathbf{j} - \mathbf{k})v(\mathbf{k})|^{r_0} \right) \right)^{\frac{1}{r_0}} \\ &= \left(\sum_{\mathbf{k} \in \varepsilon\Lambda^2} |V_{\phi_2}\phi_1(\mathbf{k})v(\mathbf{k})|^{r_0} \right)^{\frac{1}{r_0}} \asymp \|\phi_1\|_{M_{(v)}^{r_0}} < \infty. \end{aligned}$$

Thus

$$C \in \bigcap_{r_0 > 0} \mathbb{U}_{(v_0)}^{\infty, r_0}(\varepsilon\Lambda^2 \times \varepsilon\Lambda^2).$$

Then we obtain from Lemmas 3.9 and 4.2

$$\begin{aligned} \|a_1 \#_0 a_2\|_{M_{(w_0)}^{\Phi_1, \Phi_2}} &\asymp \|A_1 \circ C \circ A_2\|_{\mathbb{U}_{(\vartheta_0)}^{\Phi_1, \Phi_2}} \\ &\leq \|A_1 \circ C\|_{\mathbb{U}_{(\vartheta_1)}^{\Phi_1, \Phi_2}} \|A_2\|_{\mathbb{U}_{(\vartheta_2)}^{\infty, r_0}} \leq \|A_1\|_{\mathbb{U}_{(\vartheta_1)}^{\Phi_1, \Phi_2}} \|C\|_{\mathbb{U}_{(v_0)}^{\infty, r_0}} \|A_2\|_{\mathbb{U}_{(\vartheta_2)}^{\infty, r_0}} \\ &\lesssim \|A_1\|_{\mathbb{U}_{(\vartheta_1)}^{\Phi_1, \Phi_2}} \|A_2\|_{\mathbb{U}_{(\vartheta_2)}^{\infty, r_0}} \asymp \|a_1\|_{M_{(w_1)}^{\Phi_1, \Phi_2}} \|a_2\|_{M_{(w_2)}^{\infty, r_0}}. \end{aligned}$$

□

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