Pseudo-fractional differential equations and generalized g -Laplace transform

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Abstract

In this article, we introduce a generalized g -Laplace transform and discuss some essential results of integral transform theory, in particular, involving a ψ -Hilfer pseudo-fractional derivative and function convolution. In this sense, we investigated the existence and uniqueness of known solutions for a pseudo-fractional differential equation.

Keywords Pseudo-fractional operator · Existence and uniqueness · g -Laplace transform

Mathematics Subject Classification 34A08 · 34A12 · 47G30 · 44A10

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1 Introduction and motivation

Over the years, one of the issues discussed in the scientific community, is what is fractional calculus? What is its importance to the academic community? When the first ideas about fractional calculus were discussed, there was no dimension of the importance and relevance that their theory and applications in physics, chemistry, biology, engineering, medicine, among other areas of knowledge, [1–8], and recently, fractional derivatives have been used to generalize and refine models that describe the epidemic disease of coronavirus [9,10]. Nowadays we have countless definitions of derivatives and fractional integrals [11–16].

In 2018 when Sousa and Oliveira [13], introduced the fractional derivative ψ -Hilfer motivated by solving the vast number of fractional derivatives in a single operator, it was already known, which would be a possible version for the Laplace transform with respect to another function, that is, ψ . Thus, in 2019 Jarad and Abdeljawad [17], introduced a version for the Laplace transform with ψ . However, the reverse version was still missing. In 2020 Fahad et al. [18], introduced the inverse of the Laplace transform with ψ . In this sense, from the Laplace transform with ψ , it became possible to discuss properties such as existence, uniqueness, stability, controllability of mild solutions of fractional differential equations, involving the fractional derivative ψ -Hilfer. Some works involving the fractional derivative ψ -Hilfer, can be consulted in [19–23].

On the other hand, Endre Pap [24,25] being one of the main researchers in the area of g -calculus, with numerous cutting-edge works and applications, started to discuss problems involving partial differential equations via pseudo-analysis. Many works in the area come from Pap works. Recently, some researchers began to discuss more closely, the idea of unifying the g -calculus theory, pseudo-analysis with fractional calculation, since it is still a new field and because there are open questions [14,26–29]. There are some works on the g -calculus theory with fractional calculus, in particular, involving inequalities, but it still needs more research and future contributions to the development and growth of the area [1,27,30–32]. Aiming to unify these two areas more, in 2020 Sousa et al. [14], extended the fractional derivative ψ -Hilfer to pseudo-operators and discussed some basic properties.

In 2005, Pap [25] discussed a theory of generalized functions in analogy to Mikusinski's operators, which allows the construction of a generalized solution of the Burgers equation. Considering the extensions of operations $\oplus$ and $\odot$ for non-commutative and non-associative cases, some non-linear partial differential equations were addressed using the pseudo-linear superposition principle. Other works in the same top, involving different equations via pseudo-analysis, can be obtained in the following works [28,29,33–36].

In 2020 Sousa et al. [36], investigated the existence and uniqueness of global solutions of the Cauchy problem associated with data (t_0, x_0) any solution $(I := [a, b], x)$ is given by

$$\begin{cases} \frac{d^\oplus}{dt}x(t) = F(t, x) \\ x(t_0) = x_0 \end{cases}$$

with $t_0 \in I$.

On the other hand, also in 2020 Sousa et al. [29], discussed the reachability of linear and non-linear systems in the sense of the ψ -Hilfer pseudo-fractional derivative in g -calculus by means of the Mittag-Leffler functions (one and two parameters) with the form

$$\begin{aligned}\mathbb{H}_{\oplus, \odot, t_0+}^{\alpha, \beta; \psi} x(t) &= \mathbf{A}x(t) \oplus \mathbf{B}u(t), \quad t \in J := [t_0, t_1] \\ \mathbb{I}_{\oplus, \odot, t_0+}^{1-\gamma; \psi} x(t_0) &= 0\end{aligned}\quad (1)$$

and

$$\begin{aligned}\mathbb{H}_{\oplus, \odot, t_0+}^{\alpha, \beta; \psi} x(t) &= \mathbf{A}x(t) \oplus \mathbf{B}u(t) \oplus f(t, x(t), u(t)), \quad t \in J \\ \mathbb{I}_{\oplus, \odot, t_0+}^{1-\gamma; \psi} x(t_0) &= 0\end{aligned}\quad (2)$$

where $\mathbb{H}_{\oplus, \odot, t_0+}^{\alpha, \beta; \psi}(\cdot)$ is the ψ -Hilfer pseudo-fractional derivative of order $0 < \alpha < 1$ and type $0 \leq \beta \leq 1$, $\gamma = \alpha + \beta(1 - \alpha)$, $\mathbb{I}_{\oplus, \odot, t_0+}^{1-\gamma; \psi}(\cdot)$ is the Riemann-Liouville pseudo-fractional integral with respect to another function $1 - \gamma$, the state vector $x \in \mathbb{R}^n$, the control vector $u \in \mathbb{R}^m$ and $\mathbf{A}$ and $\mathbf{B}$ are the constant matrices of dimension $n \times n$ and $n \times m$ respectively and the nonlinear function $f : J \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is continuous, respectively.

To date, the studies Sousa et al. [29,36], are the first results in this area involving the reachability approach of pseudo-differential equations involving ψ -Hilfer pseudo-fractional derivative. Motivated by the above works and by issues that are still open in the theory of seven-sector and quasi-sector operators, we will briefly describe the main contributions of this article, in order to make it clear throughout the article. as a result, we have:

1. The generalized g -Laplace transform, is given by

$$\mathcal{L}_{\psi}^{\oplus}(f(t)) = g^{-1}(\mathcal{L}_{\psi}(g(f(t)))) . \quad (3)$$

2. We discuss some properties of Eq. (3), specially related to the ψ -convolution product. In addition, we discussed the calculation of the pseudo-fractional derivative ψ -Hilfer and the pseudo-fractional derivative ψ -Riemann-Liouville.

Finally, we consider the following ψ -Hilfer pseudo-fractional differential equation given by

$$\begin{cases} \mathbb{H}_{\oplus, \odot, t_0+}^{\alpha, \beta; \psi} x(t) = \mathbf{A}x(t) \oplus f(t, x(t)), \quad t \in J \\ \mathbb{I}_{\oplus, \odot, t_0+}^{1-\gamma; \psi} x(t_0) = x_0 \end{cases} \quad (4)$$

in which $\mathbb{H}_{\oplus, \odot, t_0+}^{\alpha, \beta; \psi}(\cdot)$ is the pseudo-fractional ψ -Hilfer derivative with order $0 < \alpha \leq 1$ and type $0 \leq \beta \leq 1$, $\mathbb{I}_{\oplus, \odot, t_0+}^{1-\gamma; \psi}(\cdot)$ is the pseudo-fractional integral with order $1 - \gamma$ ($\gamma = \alpha - \beta(1 - \alpha)$), $n \times n$ matrix $\mathbf{A}$ and $f : [t_0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous function.

The second main objective of this article is to investigate the existence and uniqueness of the solution of Eq. (4), given by the following Theorems 1 and 2:

Theorem 1 *Let $f : [t_0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be continuous function satisfying the Lipschitz condition*

$$\|f(t, x_1(t)) \ominus f(t, x_2(t))\|_g \leq_g L \odot \|x_1(t) \ominus x_2(t)\|_g, t \in J := [t_0, t_1], L > 0.$$

Then, the initial value Eq. (4) has a unique solution whenever

$$M \odot L \odot g^{-1} \left(\frac{(\psi(t) - \psi(t_0))^\alpha}{\alpha} \right) < 1$$

Theorem 2 *Let $f : [t_0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous function such that there exist a positive constant L , such that*

$$\|f(t, x(t))\|_g \leq_g L \odot \|x(t)\|_g, t \in J.$$

Then, the initial value Eq. (4) has a solution on J .

The paper is organized as follows: In Sect. 2, we present some definitions and essential results for the good development of the article. In Sect. 3, we discuss the first main result of the article, that is, we introduce a new extension for the g -Laplace transform with respect to the ψ function. From this result, some results were discussed, in particular, involving the pseudo-fractional derivative ψ -Hilfer. In Sect. 4, we discussed the convolution of functions. Finally, the second main result of this article is discussed in Sect. 5, that is, we investigate the existence and uniqueness of mild solutions for Eq. (4) via Theorem 1 and Theorem 2. We conclude the article with open questions and problems.

2 Preliminaries

In this section, we will discuss some concepts and results, essential for the development of this paper.

The Mittag-Leffler function $\mathbb{E}_{\alpha, \beta}(z)$ is a complex function which depends on two complex parameters, and it is defined by [13]

$$\mathbb{E}_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \alpha, \beta > 0. \quad (5)$$

The function $\mathbb{E}_{\alpha, \beta}(\cdot)$ converges for all values of the argument z . For a $n \times n$ matrix $\mathbf{A}$, the matrix extension of the above Mittag-Leffler function is

$$\mathbb{E}_{\alpha, \beta}(\mathbf{A}) = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{\Gamma(\alpha k + \beta)}. \quad (6)$$

Definition 1 [12,24,25,27] A binary operator $\oplus$ on $\mathbf{J}$ is pseudo-addition if it is commutative, non-decreasing, with respect to $\preceq$, continuous; associative, and with a zero (neutral) element denoted by 0. Let $\mathbf{J}_+ = \{x | x \in [a, b], 0 \leq x\}$ for $a, b \in \mathbb{R}^+$.

Definition 2 [12,24,25,27] A binary operation $\odot$ on $\mathbf{J}$ is pseudo-multiplication if it is commutative, positively non-decreasing, i.e., $x \leq y$ implies $x \odot z \leq y \odot z$ for all $z \in [a, b]_+$, associative and with a unit element $1 \in [a, b]$, i.e., for each $x \in [a, b]$, $1 \odot x = x$. Also, $0 \odot x = 0$ and that $\odot$ is distributive over $\oplus$, i.e., $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$.

The structure $(\mathbf{J}, \oplus, \odot)$ is a semiring [12,24,25,27].

Definition 3 [12,24,25,27] An important class of pseudo-operations $\oplus$ and $\odot$ is when these are defined by a monotone and continuous function $g : \mathbf{J} \mapsto [0, \infty]$, i.e., pseudo-operations $\oplus$ and $\odot$ are given by

$$x \oplus y = g^{-1}(g(x) + g(y)) \text{ and } x \odot y = g^{-1}(g(x)g(y)). \quad (7)$$

Definition 4 [12,24,25,27] Let X be a non-empty set and $\mathcal{A}$ be a σ -algebra of subsets of a set X . A set function $\mu : \mathcal{A} \mapsto [a, b]$ is called a σ - $\oplus$ -measure if the following conditions are satisfy:

1. $\mu(\emptyset) = 0$;
2. $\mu\left(\bigcup_{i=1}^{\infty} \mathcal{A}_i\right) = \bigoplus_{i=1}^{\infty} \mu(\mathcal{A}_i)$

for any sequence $\{\mathcal{A}_i\}_{i \in \mathbb{N}}$ of pairwise disjoint sets from $\mathcal{A}$.

Definition 5 [12,24,25,27] Let pseudo-operations $\oplus$ and $\odot$ be defined a monotone and continuous function $g : \mathbf{J} \mapsto [0, \infty]$.

1. The g -integral for a measurable function $f : [c, d] \rightarrow \mathbf{J}$ is given by

$$\int_{[c,d]}^{\oplus} f \odot dx = g^{-1} \left(\int_c^d g(f(x)) dx \right).$$

2. The g -Laplace of a function f is defined by

$$\mathcal{L}^{\oplus}[f(x)] = g^{-1}(\mathcal{L}[g(f(x))]).$$

Definition 6 [12,24,25,27] Let g be the additive generator of the strict-pseudo-addition $\oplus$ on $\mathbf{J}$ such that g is continuous differentiable on (a, b) . The corresponding pseudo-multiplication $\odot$ will always be defined as $u \odot v = g^{-1}(g(u) \cdot g(v))$. If the function f is differentiable on (c, d) and has the same monotonicity as the function g , then the g -derivative of f at the point $x \in (c, d)$ is defined by

$$\frac{d^{\oplus}}{dx} f(x) = g^{-1} \left(\frac{d}{dx} g(f(x)) \right).$$

Also, if there exists the n - g -derivative of f , then

$$\frac{d^{(n)\oplus}}{dx} f(x) = g^{-1} \left(\frac{d^n}{dx^n} g(f(x)) \right).$$

Definition 7 [12,24,25,27] Let g be a generator of a pseudo-addition $\oplus$ on interval $[-\infty, +\infty]$. Binary operations $\ominus$ and $\oslash$ on $[-\infty, +\infty]$ are defined by the expressions

$$x \ominus y = g^{-1}(g(x) - g(y)) \quad \text{and} \quad x \oslash y = g^{-1} \left(\frac{g(x)}{g(y)} \right).$$

If the expressions $g(x) - g(y)$ and $\frac{g(x)}{g(y)}$ have sense are said to be the pseudo-subtraction and pseudo-division consistent with the pseudo-addition $\oplus$.

Definition 8 [12,24,25,27] Let $g : [-\infty, +\infty] \mapsto [-\infty, +\infty]$ be a continuous, strictly increasing and odd function such that $g(0) = 0$, $g(1) = 1$ and $g(+\infty) = +\infty$. The system of pseudo-arithmetical operations $\{\oplus, \oslash, \odot, \ominus\}$ generated by these functions is said to be the consistent system.

Definition 9 [13,37] Let $\mathbf{J} := [a, b]$ ($-\infty \leq a < b \leq \infty$) be a finite or infinite interval of the real line $\mathbb{R}$ and $\alpha > 0$. Also let $\psi(x)$ be an increasing and positive monotone function on $(a, b]$, having a continuous derivative $\psi'(x)$ on $\mathbf{J}$. The left-sided and right-sided fractional integrals of a function f with respect to another function ψ , are defined by

$$\mathcal{I}_{a+}^{\alpha;\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} f(t) dt. \quad (8)$$

and

$$\mathcal{I}_{b-}^{\alpha;\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t) (\psi(t) - \psi(x))^{\alpha-1} f(t) dt. \quad (9)$$

Definition 10 [13,37] Let $n - 1 < \alpha < n$, with $n \in \mathbb{N}$, $\mathbf{J}$ is an interval such that $-\infty \leq a < b \leq +\infty$ and $f, \psi \in \mathcal{C}^n(\mathbf{J}, \mathbb{R})$ are two functions such that ψ is increasing and $\psi(x) \neq 0$, for all $x \in \mathbf{J}$. The ψ -Hilfer fractional derivative left-sided and right-sided, denoted by ${}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi}(\cdot)$ of a function f of order α and type $0 \leq \beta \leq 1$, is defined by

$${}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(x) = \mathcal{I}_{a+}^{\beta(n-\alpha);\psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \mathcal{I}_{a+}^{(1-\beta)(n-\alpha);\psi} f(x) \quad (10)$$

and

$${}^H\mathbb{D}_{b-}^{\alpha,\beta;\psi} f(x) = \mathcal{I}_{b-}^{\beta(n-\alpha);\psi} \left(-\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n \mathcal{I}_{b-}^{(1-\beta)(n-\alpha);\psi} f(x) \quad (11)$$

where $\mathcal{I}_{a+}^{\alpha;\psi}(\cdot)$ and $\mathcal{I}_{b-}^{\alpha;\psi}(\cdot)$ by defined in Eq. (8) and Eq. (9), respectively.

Definition 11 [14] Let a generator $g : \mathbf{J} \rightarrow [0, \infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function. Also let ψ be an increasing and positive function on $(a, b]$, having a continuous derivative $\psi'(x)$. The left-sided and the right-sided ψ -Riemann-Liouville pseudo-fractional integrals of order $\alpha > 0$ of a measurable function $f : \mathbf{J} \rightarrow \mathbf{J}$ with respect to function ψ on $\mathbf{J}$ are defined by:

$$\begin{aligned} \mathbb{I}_{\oplus, \odot, a+}^{\alpha; \psi} f(x) &= g^{-1} \left(\mathcal{I}_{a+}^{\alpha; \psi} g(f(x)) \right) \\ &= \int_{[a, x]}^{\oplus} \left[g^{-1} \left(\frac{\psi'(t) (\psi(x) - \psi(t))^{\alpha-1}}{\Gamma(\alpha)} \right) \odot f(t) \right] \odot dt \quad (12) \end{aligned}$$

and

$$\begin{aligned} \mathbb{I}_{\oplus, \odot, b-}^{\alpha; \psi} f(x) &= g^{-1} \left(\mathcal{I}_{b-}^{\alpha; \psi} g(f(x)) \right) \\ &= \int_{[x, b]}^{\oplus} \left[g^{-1} \left(\frac{\psi'(t) (\psi(t) - \psi(x))^{\alpha-1}}{\Gamma(\alpha)} \right) \odot f(t) \right] \odot dt \quad (13) \end{aligned}$$

where $\mathcal{I}_{a+}^{\alpha; \psi}(\cdot)$ and $\mathcal{I}_{b-}^{\alpha; \psi}(\cdot)$ are given by Eq. (8) and Eq. (9), respectively.

Definition 12 [14] Let a generator $g : \mathbf{J} \rightarrow [0, \infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function. Also let $\psi \in C^n(\mathbf{J}, \mathbb{R})$, a function such that ψ be an increasing and positive function on $(a, b]$ having a continuous derivative ψ' and $\psi'(x) \neq 0$ for all $x \in \mathbf{J}$. The left-sided and right-sided ψ -Hilfer pseudo-fractional derivative of order $n - 1 < \alpha < n$ and type $0 \leq \beta \leq 1$, of a measurable function $f : \mathbf{J} \rightarrow \mathbf{J}$ is defined by

$$\begin{aligned} \mathbb{H}_{\oplus, \odot, a+}^{\alpha, \beta; \psi} f(x) &= g^{-1} \left({}^H\mathbb{D}_{a+}^{\alpha, \beta; \psi} g(f(x)) \right) \\ &= \mathbb{I}_{\oplus, \odot, a+}^{\beta(n-\alpha); \psi} g^{-1} \left(\left(\frac{D}{\psi'(x)} \right)^n \right) \odot \mathbb{I}_{\oplus, \odot, a+}^{1-\gamma; \psi} f(x) \quad (14) \end{aligned}$$

and

$$\begin{aligned} \mathbb{H}_{\oplus, \odot, b-}^{\alpha, \beta; \psi} f(x) &= g^{-1} \left({}^H\mathbb{D}_{b-}^{\alpha, \beta; \psi} g(f(x)) \right) \\ &= \mathbb{I}_{\oplus, \odot, b-}^{\beta(n-\alpha); \psi} g^{-1} \left(\left(-\frac{D}{\psi'(x)} \right)^n \right) \odot \mathbb{I}_{\oplus, \odot, b-}^{1-\gamma; \psi} f(x) \quad (15) \end{aligned}$$

where ${}^H\mathbb{D}_{a+}^{\alpha, \beta; \psi}(\cdot)$ and ${}^H\mathbb{D}_{b-}^{\alpha, \beta; \psi}(\cdot)$ are ψ -Hilfer fractional derivative are given by Eq. (10) and Eq. (11).

Note that

$$\begin{aligned} \mathbb{H}_{\oplus, \odot, a+}^{\alpha, \beta; \psi} f(x) &= g^{-1} \left(\mathcal{I}_{a+}^{\gamma-\alpha; \psi} {}^{RL}\mathbb{D}_{a+}^{\gamma; \psi} g(f(x)) \right) \\ &= \mathbb{I}_{\oplus, \odot, a+}^{\gamma-\alpha; \psi} {}^{RL}\mathbb{D}_{\oplus, \odot, a+}^{\gamma; \psi} f(x) \quad (16) \end{aligned}$$

and

$$\begin{aligned}\mathbb{H}_{\oplus, \odot, b-}^{\alpha, \beta; \psi} f(x) &= g^{-1} \left(\mathcal{I}_{b-}^{\gamma-\alpha; \psi} {}^{RL} \mathbb{D}_{b-}^{\gamma; \psi} g(f(x)) \right) \\ &= \mathbb{I}_{\oplus, \odot, b-}^{\gamma-\alpha; \psi} {}^{RL} \mathbb{D}_{\oplus, \odot, b-}^{\gamma; \psi} f(x)\end{aligned}\quad (17)$$

where $\gamma = \alpha + \beta(n - \alpha)$.

If $\alpha > 0$ and $\mathbf{A}$ is $n \times n$ matrix, then

$$\mathbb{H}_{\oplus, \odot, a+}^{\alpha, \beta; \psi} \mathbb{E}_{\alpha, \beta}(\mathbf{A}(\psi(x) - \psi(a))^\alpha) = \mathbf{A} \mathbb{E}_{\alpha, \beta}(\mathbf{A}(\psi(x) - \psi(a))^\alpha). \quad (18)$$

Theorem 3 [14] Let $f : \mathbf{J} \rightarrow \mathbf{J}$ be a measurable functions. If $n \in \mathbb{N}$, then we have

1. $\mathbb{H}_{\oplus, \odot, a+}^{0, \beta; \psi} f(x) = f(x);$
2. $\mathbb{H}_{\oplus, \odot, a+}^{1, \beta; \psi} f(x) = g^{-1} \left(\left(\frac{D}{\psi'(x)} \right) g(f(x)) \right);$
3. $\mathbb{H}_{\oplus, \odot, a+}^{n, \beta; \psi} f(x) = \left(\frac{D}{\psi'(x)} \right)^{(n)\oplus} f(x);$
4. $\mathbb{H}_{\oplus, \odot, a+}^{\alpha, \beta; \psi} (f_1(x) \oplus f_2(x)) = \mathbb{H}_{\oplus, \odot, a+}^{\alpha, \beta; \psi} f_1(x) \oplus \mathbb{H}_{\oplus, \odot, a+}^{\alpha, \beta; \psi} f_2(x);$
5. $\mathbb{H}_{\oplus, \odot, a+}^{\alpha, \beta; \psi} (\lambda \odot f(x)) = \lambda \odot \mathbb{H}_{\oplus, \odot, a+}^{\alpha, \beta; \psi} f(x);$
6. $\mathbb{H}_{\oplus, \odot, a+}^{\alpha, \beta; \psi} \mathbb{I}_{\oplus, \odot, a+}^{\alpha; \psi} f(x) = f(x);$
7. $\mathbb{I}_{\oplus, \odot, a+}^{\alpha; \psi} \mathbb{H}_{\oplus, \odot, a+}^{\alpha, \beta; \psi} f(x) = f(x) \ominus \left[\bigoplus_{k=1}^n g^{-1}(C_k) \odot g^{-1}(\psi(x) - \psi(a))^{\gamma-k} \right]$ with

$$\gamma = \alpha + \beta(n - \alpha) \text{ and } C_k = \frac{(g \circ f)^{[n-k]} \mathbb{I}_{\oplus, \odot, a+}^{n-\gamma; \psi} g(f(a))}{\Gamma(\gamma - k + 1)}.$$

Definition 13 [17] Let $f, \psi : [0, \infty) \rightarrow \mathbb{R}$ be real valued functions such that ψ be a non negative increasing function with $\psi(0) = 0$. Then, the Laplace transform of f with respect to ψ is defined by

$$\mathcal{L}_\psi(f(t)) = F(s) = \int_0^\infty e^{-s\psi(t)} \psi'(t) f(t) dt$$

for all $s \in \mathbb{C}$ such that this integral converges. Here, $\mathcal{L}_\psi(\cdot)$ denotes the Laplace transform with respect to ψ , which we call a generalized Laplace transform.

Definition 14 [17] Assume that the function f is defined for $f \geq 0$. Then, the Laplace transform of f , denoted by $\mathcal{L}(f)$, is defined by the improper integral

$$\mathcal{L}(f(t)) \equiv F(s) = \int_0^\infty e^{-st} f(t) dt \quad (19)$$

provided that the integral in Eq. (19) exists, i.e., that the integral is convergent. The corresponding inverse Laplace transform is given by

$$\mathcal{L}^{-1}(f(t)) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s) ds \quad (20)$$

with $s \in \mathbb{C}$ such that $\operatorname{Re}(s) = c$.

Let $m-1 < \alpha < m$, $m \in \mathbb{N}$, $-\infty \leq a < b \leq \infty$, $0 \leq \beta \leq 1$, and $f, \psi \in C^m(\mathbf{J}, \mathbb{R})$ be functions such that ψ is increasing and $\psi'(x) \neq 0$ for all $x \in \mathbf{J}$. Then, the ψ -Hilfer fractional derivative of order α and type β is given by [13,37]

$${}^H\mathbb{D}_{a+}^{\alpha, \beta, \psi} f(x) = I_{a+}^{\beta(m-\alpha), \psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^m I_{a+}^{(1-\beta)(m-\alpha), \psi} f(x). \quad (21)$$

Taking $\beta \rightarrow 1$ in Eq. (21), we obtain the Caputo fractional derivative given by [13]

$${}^C\mathbb{D}_{a+}^{\alpha, \psi} f(x) = I_{a+}^{m-\alpha, \psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^m f(x). \quad (22)$$

Also, taking $\beta \rightarrow 0$ in Eq. (21), we get the Riemann-Liouville fractional derivative given by

$${}^{RL}\mathbb{D}_{a+}^{\alpha, \psi} f(x) = \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^m I_{a+}^{m-\alpha, \psi} f(x). \quad (23)$$

These generalized fractional operators can be written as the conjugation of the standard fractional operators with the operation of composition with ψ or ψ^{-1} [18]

$$\begin{aligned} I_{a+}^{\alpha, \psi} &= Q_\psi \circ I_{t+}^{\alpha, \psi(a)} \circ (Q_\psi)^{-1}, \\ {}^{RL}\mathbb{D}_{a+}^{\alpha, \psi} &= Q_\psi \circ {}^{RL}\mathbb{D}_{t+}^{\alpha, \psi(a)} \circ (Q_\psi)^{-1}, \\ {}^C\mathbb{D}_{a+}^{\alpha, \psi} &= Q_\psi \circ {}^C\mathbb{D}_{t+}^{\alpha, \psi(a)} \circ (Q_\psi)^{-1}, \end{aligned} \quad (24)$$

and

$${}^H\mathbb{D}_{a+}^{\alpha, \beta, \psi} = Q_\psi \circ {}^H\mathbb{D}_{t+}^{\alpha, \beta, \psi(a)} \circ (Q_\psi)^{-1}, \quad (25)$$

where the functional operator Q_ψ is defined by

$$(Q_\psi f)(x) = f(\psi(x)). \quad (26)$$

Theorem 4 [18] *The generalized Laplace transform may be written as a combination of the classical Laplace transform with the operation of composition with ψ or ψ^{-1} , as follows*

$$\mathcal{L}_\psi = \mathcal{L} \circ Q_\psi^{-1} \quad (27)$$

where the fundamental operator Q_ψ is defined in Eq. (26).

Corollary 1 [17] *The inverse generalized Laplace transform may be written as a combination of the inverse classical Laplace transform with the operation of composition with ψ or ψ^{-1} , as follows*

$$\mathcal{L}_{\psi}^{-1} = Q_{\psi} \circ \mathcal{L}^{-1} \quad (28)$$

or, in other words

$$\mathcal{L}_{\psi}^{-1}(F(s)) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{s\psi(t)} F(s) ds. \quad (29)$$

Corollary 2 [17] *If $f(t)$ is a function whose classical Laplace transform is $F(s)$, the generalized Laplace transform of the function $f \circ \psi = f(\psi(t))$ is also $F(s)$,*

$$\mathcal{L}(f(t)) = F(s) \implies \mathcal{L}_{\psi}(f(\psi(t))) = F(s).$$

Below some particular cases, presented as examples [17,18].

Example 1 Let $\mu \in \mathbb{C}$ be such that $\operatorname{Re}(\mu) > -1$, then

$$\mathcal{L}_{\psi}((\psi(t))^{\mu}) = \frac{\Gamma(\mu + 1)}{s^{\mu+1}}$$

for $\operatorname{Re}(s) > 0$.

Example 2 Let $\lambda \in \mathbb{R}$, then

$$\mathcal{L}_{\psi}(e^{\lambda\psi(t)}) = \frac{1}{s - \lambda}$$

for $\operatorname{Re}(s) > \lambda$.

Example 3 Let $\mu \in \mathbb{C}$ be such that $\operatorname{Re}(\mu) > 0$ and $\mathbb{E}_{\mu}(\cdot)$ be the one-parameter Mittag-Leffler function, then

$$\mathcal{L}_{\psi}(\mathbb{E}_{\mu}(\lambda\psi(t))^{\mu}) = \frac{s^{\mu-1}}{s^{\mu} - \lambda}$$

for $|\lambda/s^{\mu}| < 1$.

Example 4 Let $\mu \in \mathbb{C}$ be such that $\operatorname{Re}(\mu) > 0$ and $\mathbb{E}_{\mu,\mu}(\cdot)$ be the two-parameter Mittag-Leffler function, then

$$\mathcal{L}_{\psi}(\mathbb{E}_{\mu,\mu}(\lambda\psi(t))^{\mu}) = \frac{1}{s^{\mu} - \lambda}$$

for $|\lambda/s^{\mu}| < 1$.

Example 5 Assume that $\mu \in \mathbb{C}$ be such that $\operatorname{Re}(\mu) > 0$ and $|\lambda/s^\mu| < 1$. If $\mathbb{E}_{\mu,v}^\gamma(\cdot)$ denotes the three-parameter Mittag-Leffler function, we evaluate the Laplace transform of the so-called Prabhakar function, then we have

$$\mathcal{L}_\psi((\psi(t))^{v-1} \mathbb{E}_{\mu,v}^\gamma(\lambda \psi(t)^\mu)) = \frac{s^{\mu\gamma-v}}{(s^\mu - \lambda)^\gamma}.$$

Theorem 5 [17,18] Let $\mu > 0$ and let f be a function of ψ -exponential order, piecewise continuous over each finite interval $[0, T]$. Then,

$$\mathcal{L}_\psi \left(I_{a+}^{\mu,\psi} f(t) \right) = s^{-\mu} \mathcal{L}_\psi(f(t)).$$

Theorem 6 [17,18] Assume $m-1 < \mu < 1$, $0 \leq v \leq 1$ and f a function such that $f(t)$, $D^{j,\psi} I_{0+}^{(1-v)(m-\mu),\psi} f(t) \in C[0, \infty)$ are of ψ -exponential order for $j = 0, 1, 2, \dots, m-1$, while ${}^H\mathbb{D}_{0+}^{\mu,v,\psi} f(t)$ is piecewise continuous on $[0, \infty)$. Then,

$$\mathcal{L}_\psi \left({}^H\mathbb{D}_{0+}^{\mu,v,\psi} f(t) \right) = s^\mu \mathcal{L}_\psi(f(t)) - \sum_{i=0}^{m-1} s^{m(1-v)+\mu v-i-1} \left(I_{0+}^{(1-v)(m-\mu)-v} f(0) \right). \quad (30)$$

Definition 15 [17] Let f and h be of ψ -exponential order, piecewise continuous functions over each finite interval $[0, T]$. Then, the ψ -convolution of f and h , denoted by $f *_\psi h$ is given by

$$f *_\psi h = \int_0^t f \left(\psi^{-1}(\psi(t) - \psi(\tau)) \right) \psi'(\tau) h(\tau) d\tau. \quad (31)$$

Theorem 7 [18] Let f, p, h be of ψ -exponential order, piecewise continuous functions over each finite interval $[0, T]$, and let a and b be constants. Then,

1. $f *_\psi p = p *_\psi f$.
2. $(f *_\psi p) *_\psi h = f *_\psi (p *_\psi h)$.
3. $f *_\psi (ap + bh) = af *_\psi p + bf *_\psi h$.

Theorem 8 [18] Assume that f and p are piecewise continuous functions on $[0, T]$ and of ψ -exponential order $c > 0$. Then,

$$\mathcal{L}_\psi(f *_\psi p) = \mathcal{L}_\psi(f) \mathcal{L}_\psi(p) \quad (32)$$

Definition 16 [12,24,25,27] Let the pseudo-operations $\oplus$ and $\odot$ be defined through a monotone and continuous function $g : \mathbf{J} \rightarrow [0, \infty]$:

(1) The g -integral of a measurable function $f : [c, d] \rightarrow [a, b]$ is given by

$$\int_{[c,d]}^\oplus f \odot dt = g^{-1} \left(\int_c^d g(f(t)) dt \right),$$

(2) The g -Laplace transform of a function f is defined by

$$\mathcal{L}^\oplus(f(t)) = g^{-1}(\mathcal{L}(g(f(t)))) .$$

3 Generalized g -Laplace transform

In this section, we will present a generalization for the g -Laplace transform (g -calculus) with respect to another function ψ and its respective inverse. In this perspective we discuss some properties, in particular, we obtain the g -Laplace transform of the ψ -Hilfer fractional derivative and the ψ -Riemann-Liouville fractional integral.

Definition 17 Let a generator $g : \mathbf{J} \rightarrow [0, \infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function. Let $f, \psi : [0, \infty) \rightarrow \mathbb{R}$ be real valued functions such that ψ be a non-negative increasing function such that $\psi(0) = 0$. Then, the g -Laplace transform of f with respect to another function ψ is defined by

$$\mathcal{L}_\psi^\oplus(f(t)) = g^{-1}(\mathcal{L}_\psi(g(f(t))))$$

where $\mathcal{L}_\psi(\cdot)$ is defined in Definition 13.

Theorem 9 Let a generator $g : \mathbf{J} \rightarrow [0, \infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function. Let $f, \psi : [0, \infty) \rightarrow \mathbb{R}$ be real valued functions such that ψ be increasing function. The inverse generalized Laplace transform may be written as a combination of the inverse classical Laplace transform with the operation of composition with ψ and ψ^{-1} as follows

$$\mathcal{L}_\psi^\oplus = \mathcal{L}^\oplus \circ Q_\psi^{-1}$$

where the functional operator $(Q_\psi^{-1}f)(t) = f(\psi^{-1}(t))$.

Proof Let $f : t \rightarrow f(t)$. On the other hand,

$$Q_\psi^{-1}f(t) : t \rightarrow (Q_\psi^{-1}f)(t) = f(\psi^{-1}(t)).$$

So, using the definition of the generalized g -Laplace transform, we have

$$\begin{aligned} \mathcal{L}^\oplus \circ Q_\psi^{-1}(f) : t &\rightarrow \mathcal{L}_\psi^\oplus \circ (Q_\psi^{-1}f)(t) \\ &= \mathcal{L}^\oplus (Q_\psi^{-1}f)(t) = g^{-1}(\mathcal{L}g(Q_\psi^{-1}f)(t)) \\ &= g^{-1} \left(\int_0^\infty e^{-st} g(Q_\psi^{-1}f)(t) dt \right) \\ &= g^{-1} \left(\int_0^\infty e^{-st} g[f(\psi^{-1}(t))](t) dt \right) \\ &= g^{-1} \left(\int_0^\infty e^{-s\psi(u)} g(f(u))\psi'(u) du \right) \end{aligned}$$

thus, the result follows. $\square$

At this point, we are motivated to present the inverse generalized g -Laplace transform, based on the following corollary.

Corollary 3 *The inverse generalized g -Laplace transform with respect to another function, ψ , may be written as a combination of the inverse classical g -Laplace transform with the operation of composition with ψ or ψ^{-1} as follows*

$$\mathcal{L}_{\psi}^{\oplus -1} = Q_{\psi} \circ \mathcal{L}^{\oplus -1}$$

or, in other words

$$\begin{aligned} \mathcal{L}_{\psi}^{\oplus -1} \{F(s)\} &= g^{-1} \left\{ \mathcal{L}_{\psi}^{-1} g(F(s)) \right\} \\ &= g^{-1} \left\{ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{s\psi(t)} g(F(s)) ds \right\} \end{aligned}$$

As a result we can write

$$\mathcal{L}_{\psi}^{\oplus} \{\lambda \odot f(t) \oplus h(t)\} = \lambda \odot \mathcal{L}_{\psi}^{\oplus} \{f(t)\} \oplus \mathcal{L}_{\psi}^{\oplus} \{g(t)\}$$

whose proof follows from the definition.

Definition 18 [38] Given a pseudo-linear space V over $(\mathbf{J}, \oplus, \odot)$, a generalized norm (g -norm) is a mapping $\|\cdot\|_g : V \rightarrow \mathbf{J}_+$ such that

1. $\|v\|_g \geq_g 0$, $\forall v \in V$ and $\|v\|_g = 0$ if and only if $v = 0$
2. $\|c \odot v\|_g = |c|_g \odot \|v\|_g$, $\forall c \in [0, b]$ and $v \in V$
3. $\|v \oplus w\|_g \leq_g \|v\|_g \oplus \|w\|_g$, $\forall v, w \in V$.

A function $f : [0, \infty) \rightarrow \mathbb{R}^n$ is said to be of ψ -exponential order $c > 0$, if there exist positive constants M, c, T , such that $\|f\|_g \leq M e^{c\psi(t)}$, for $t \geq T$.

Theorem 10 *If $f : [0, \infty) \rightarrow \mathbb{R}$ is a piecewise continuous function and is of ψ -exponential order $c > 0$, where ψ is a non-negative increasing function with $\psi(0) = 0$, then the generalized g -Laplace transform of f exists for $s > c$.*

Proof For the ψ -exponentially bounded function f , we have

$$\begin{aligned} \left| \mathcal{L}_{\psi}^{\oplus} \{f(t)\} \right| &= \left| g^{-1} (\mathcal{L}_{\psi} g(f(t))) \right| \\ &\leq g^{-1} \left(\frac{M}{s - c} \right) \end{aligned}$$

where $\left| \mathcal{L}_{\psi} \{f(t)\} \right| \leq \frac{M}{s - c}$ and in the last step we used the assumption that $s > c$. Therefore, the integral is convergent. $\square$

Theorem 11 Let $\alpha > 0$ and f be a piecewise continuous function on each interval $[0, t]$ and of ψ -exponential order. Then

$$\mathcal{L}_\psi^\oplus \left\{ \mathbb{I}_{\oplus, \odot, 0+}^{\alpha, \psi} f(t) \right\} = g^{-1}(s^{-\alpha}) \odot \mathcal{L}_\psi^\oplus f(t)$$

Proof In fact, from Definition 17 and remembering that

$$\mathcal{L}_\psi \left(I_{0+}^{\alpha, \psi} f(t) \right) = s^{-\alpha} \mathcal{L}_\psi(f(t))$$

we have

$$\begin{aligned} \mathcal{L}_\psi^\oplus \left\{ \mathbb{I}_{\oplus, \odot, 0+}^{\alpha, \psi} f(t) \right\} &= g^{-1} \left(\mathcal{L}_\psi g \left(\mathbb{I}_{\oplus, \odot, 0+}^{\alpha, \psi} f(t) \right) \right) \\ &= g^{-1} \left[\mathcal{L}_\psi g \left(g^{-1} \left(I_{0+}^{\alpha, \psi} g(f(t)) \right) \right) \right] \\ &= g^{-1} \left[\mathcal{L}_\psi \left(I_{0+}^{\alpha, \psi} g(f(t)) \right) \right] \\ &= g^{-1} \left[s^{-\alpha} \mathcal{L}_\psi g(f(t)) \right] \\ &= g^{-1}(s^{-\alpha}) \odot g^{-1} \left(\mathcal{L}_\psi g(f(t)) \right) \\ &= g^{-1}(s^{-\alpha}) \odot \mathcal{L}_\psi^\oplus f(t) \end{aligned}$$

which conclude the proof. $\square$

Theorem 12 Let g be the same as in Definition 17, $m-1 < \alpha < m$, $0 \leq \beta \leq 1$, and $s \in \mathbb{R}$. Let f be a function such that $f(t)$, $D_{0+}^{j, \psi} I_{0+}^{(1-\beta)(n-\alpha), \psi} f(t) \in C[0, \infty)$ are of ψ -exponential order for $j = 0, 1, 2, \dots, m-1$, while ${}^H\mathbb{D}_{\oplus, \odot, 0+}^{\alpha, \beta, \psi} f(t)$ is piecewise continuous on $[0, \infty)$. Then, the generalized g -Laplace transform of the ψ -Hilfer fractional derivative of order α and type β , with $0 \leq \beta \leq 1$ is given by

$$\begin{aligned} \mathcal{L}_\psi^\oplus \left[{}^H\mathbb{D}_{\oplus, \odot, 0+}^{\alpha, \beta, \psi} f(t) \right] &= \left(g^{-1}(s^\alpha) \odot \mathcal{L}_\psi^\oplus f(t) \right) \\ &\ominus \left[\bigoplus_{i=1}^{m-1} g^{-1} \left(s^{m(1-\beta)+\alpha\beta-i-1} \right) \odot \mathbb{I}_{\oplus, \odot, 0+}^{(1-\beta)(m-\alpha)-i, \psi} f(0) \right]. \end{aligned}$$

Proof In fact, from Definition 17 and remembering that

$$\mathcal{L}_\psi \left(I_{0+}^{\alpha, \psi} f(t) \right) = s^{-\alpha} \mathcal{L}_\psi(f(t))$$

we have

$$\begin{aligned} \mathcal{L}_\psi^\oplus \left\{ \mathbb{I}_{\oplus, \odot, 0+}^{\alpha, \psi} f(t) \right\} &= g^{-1} \left(\mathcal{L}_\psi g \left(\mathbb{I}_{\oplus, \odot, 0+}^{\alpha, \psi} f(t) \right) \right) \\ &= g^{-1} \left[\mathcal{L}_\psi g \left(g^{-1} \left(I_{0+}^{\alpha, \psi} g(f(t)) \right) \right) \right] \\ &= g^{-1} \left[\mathcal{L}_\psi \left(I_{0+}^{\alpha, \psi} g(f(t)) \right) \right] \\ &= g^{-1} \left[s^{-\alpha} \mathcal{L}_\psi g(f(t)) \right] \\ &= g^{-1}(s^{-\alpha}) \odot g^{-1} \left(\mathcal{L}_\psi g(f(t)) \right) \\ &= g^{-1}(s^{-\alpha}) \odot \mathcal{L}_\psi^\oplus f(t) \end{aligned}$$

which conclude the proof. $\square$

4 ψ -convolution

Definition 19 Let a generator $g : \mathbf{J} \rightarrow [0, \infty]$ of the pseudo-addition, $\oplus$, and the pseudo-multiplication, $\odot$, be an increasing function. Let f and h be of ψ -exponential order, piecewise continuous functions over each finite interval $[0, T]$. Then, the ψ -convolution of f and h is a function, $f \otimes_{\psi} h$, defined by

$$\begin{aligned} (f \otimes_{\psi} h)(t) &= \int_{[0,t]}^{\oplus} f\left(\psi^{-1}(\psi(t) - \psi(x))\right) \odot g^{-1}(\psi'(x)) \odot h(x) \odot dx \\ &= g^{-1}\left(\int_0^t g\left(\psi^{-1}(\psi(t) - \psi(x))\right) \psi'(x) g(h(x)) dx\right). \end{aligned} \quad (33)$$

Remark 1 Some particular cases of the ψ -convolution (Definition 19), given by from choosing of $g(\cdot)$ and $\psi(\cdot)$:

1. Note that, taking $g(x) = x$ in Definition 19, we have the classical convolution with respect to another function, (ψ) , given by the following relation

$$(f *_{\psi} h)(t) = \int_0^t f\left(\psi^{-1}(\psi(t) - \psi(x))\right) \psi'(x) h(x) dx.$$

2. (2) Taking $\psi(x) = x$ we get the definition of convolution in the sense of g -calculus, given by

$$(f \otimes h)(t) = \int_{[0,t]}^{\oplus} f(t - x) \odot h(x) \odot dx.$$

3. Finally, taking $g(x) = x = \psi(x)$ Definition 19, we obtain the classical convolution given by

$$(f * h)(t) = \int_0^t f(t - x) h(x) dx.$$

4. Making the same choice for Theorem 13 and Theorem 14, we obtain their respective particular cases.

Theorem 13 Let a generator $g : \mathbf{J} \rightarrow [0, \infty]$ of the pseudo-addition, $\oplus$, and the pseudo-multiplication, $\odot$ be an increasing function. Let f and h be ψ -exponential order, piecewise continuous functions over each finite interval $[0, T]$. Then, the following relation

$$f \otimes_{\psi} h = Q_{\psi}\left(\left(Q_{\psi}^{-1} f\right) \otimes \left(Q_{\psi}^{-1} h\right)\right)$$

or in other words

$$f \otimes_{\psi} h = \left((f \circ \psi^{-1}) \otimes (g \circ \psi^{-1}) \right) \circ \psi$$

Proof Using the classical g -condition and substituting $x = \psi(u)$ we get

$$\begin{aligned} f \otimes_{\psi} h(t) &= \int_{[0,t]}^{\oplus} f(t-x) \odot h(x) \odot dx \\ &= g^{-1} \left(\int_0^t g(f(t-x))g(h(x)) dx \right) \\ &= g^{-1} \left(\int_0^{\psi^{-1}(t)} g(f(t-\psi(u)))g(h(\psi(u)))\psi'(u) du \right) \end{aligned}$$

Applying Q_{π} we have

$$\begin{aligned} (f \otimes_{\psi} h) \circ \psi(t) &= \left(\int_0^t g(f(\psi(t)-\psi(u)))g(h(\psi(u)))\psi'(u) du \right) \\ &= g^{-1} \left(\int_0^t g[(f \circ \psi)(\psi^{-1}(\psi(t)-\psi(u)))]g[(h \circ \psi)(u)]\psi'(u) du \right) \\ &= g(\circ \psi) \otimes_{\psi} (g \circ \psi) \end{aligned}$$

which conclude the proof. $\square$

If we interpret convolution as a binary operation acting on two functions, and use alternative notation $\otimes(f, g)$ and $\otimes_{\psi}(f, g)$ instead of $f \otimes g$ and $f \otimes_{\psi} g$, then the results of Theorem 13 can be written, like Eq. (24) for ψ -fractional derivatives and integrals, as a conjugation of operators

$$\otimes_{\psi} = Q_{\psi} \circ \otimes \circ (Q_{\psi}^{-1}, Q_{\psi}^{-1})$$

namely

$$\otimes_{\psi}(f, g) = Q_{\psi} \left(\otimes \left(Q_{\psi}^{-1} f, Q_{\psi}^{-1} g \right) \right).$$

Theorem 14 Let a generator $g : \mathbf{J} \rightarrow [0, \infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function. Let f and h be piecewise continuous functions on $[0, T]$ and of ψ -exponential order $c > 0$. Then,

$$\mathcal{L}_{\psi}^{\oplus} \{f \otimes_{\psi} h\} = \mathcal{L}_{\psi}^{\oplus} \{f\} \mathcal{L}_{\psi}^{\oplus} \{h\}.$$

Proof In fact, note that by Theorem 13 and Theorem 9, we have

$$\begin{aligned} (\mathcal{L}_{\psi}^{\oplus}) \circ (\otimes_{\psi}) &= \left(\mathcal{L}^{\oplus} \circ Q_{\psi}^{-1} \right) \circ \left(Q_{\psi} \circ \otimes \circ (Q_{\psi}^{-1}, Q_{\psi}^{-1}) \right) \\ &= (\mathcal{L}^{\oplus} \circ \otimes) \circ (Q_{\psi}^{-1}, Q_{\psi}^{-1}) \end{aligned}$$

therefore

$$\begin{aligned} (\mathcal{L}_\psi^\oplus)(f, h) &= (\mathcal{L}^\oplus \circ \otimes) \circ (Q_\psi^{-1} f, Q_\psi^{-1} h) \\ &= \mathcal{L}^\oplus(Q_\psi^{-1} f) \mathcal{L}^\oplus(Q_\psi^{-1} h) \\ &= \mathcal{L}_\psi^\oplus\{f\} \mathcal{L}_\psi^\oplus\{h\} \end{aligned}$$

which conclude the proof. $\square$

The proof of Theorem 15 follows directly from Theorem 13.

Theorem 15 Let a generator $g : \mathbf{J} \rightarrow [0, \infty]$ of the pseudo-addition $\oplus$ and the pseudo multiplication $\odot$ be an increasing function. Let ψ , h and p be of ψ -exponential order, piecewise continuous function over each finite interval $[0, T]$, and let a and b constants. Then,

1. $f \otimes_\psi h = h \otimes_\psi f$.
2. $(f \otimes_\psi h) \otimes_\psi p = f \otimes_\psi (h \otimes_\psi p)$.
3. $f \otimes_\psi (ah \oplus bp) = af \otimes_\psi h \oplus bf \otimes_\psi p$.

5 Existence and uniqueness

In this section, we will discuss the existence and uniqueness of the solution of the non-linear pseudo-fractional differential equation in terms of the Mittag-Leffler function of one and two parameters.

Lemma 1 For $0 < \alpha \leq 1$, the solution of Eq. (4) is

$$\begin{aligned} x(t) &= x_0 \odot g^{-1} \left((\psi(t) - \psi(t_0))^{\gamma-1} \right) \\ &\odot \int_{[t_0, t]}^\oplus g^{-1} (\psi'(s)) \odot \mathbb{E}_{\alpha, \gamma} (g(\mathbf{A}) (\psi(t) - \psi(s))^\alpha) \odot ds \\ &\oplus \int_{[t_0, t]}^\oplus g^{-1} \left(\psi'(s) (\psi(t) - \psi(s))^{\alpha-1} \right) \odot \mathbb{E}_{\alpha, \alpha} (g(\mathbf{A}) (\psi(t) - \psi(s))^\alpha) \\ &\odot \odot f(s, x(s)) \odot ds. \end{aligned}$$

Proof Applying the operator $\mathbb{I}_{\oplus, \odot, t_0+}^{1-\gamma; \psi}(\cdot)$ on both sides of the Eq. (4) and using the Theorem 3, one has $x(t) \ominus [g^{-1}(C_1) \odot g^{-1}(\psi(t) - \psi(t_0))^{\gamma-k}] = \mathbb{I}_{\oplus, \odot, t_0+}^{1-\gamma; \psi}(\mathbf{A}x(t) \oplus f(t, x(t)))$.

This implies

$$\begin{aligned} x(t) &= g^{-1}(C_1) \odot g^{-1}(\psi(t) - \psi(t_0))^{\gamma-k} \oplus \int_{[t_0, t]}^\oplus g^{-1} \\ &\left(\frac{\psi'(s) (\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} \right) \odot (\mathbf{A}x(s) \oplus f(s, x(s))) \odot ds \end{aligned} \quad (34)$$

where $C_1 = \frac{I_{t_0+}^{1-\gamma;\psi} g(x(t))}{\Gamma(\gamma)}$.

Note that, the Eq. (34) can be rewritten in the form

$$x(t) = \frac{\mathbb{I}_{\oplus, \odot, t_0+}^{1-\gamma;\psi} x(t_0)}{\Gamma(\gamma)} \odot g^{-1}(\psi(t) - \psi(t_0))^{\gamma-k} \oplus \int_{[t_0, t]}^{\oplus} g^{-1} \left(\frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} \right) \odot (\mathbf{A}x(s) \oplus f(s, x(s))) \odot ds. \quad (35)$$

Now, through successive approximations, let's get an expression for Eq. (35).

Let for this set

$$x_0(t) = \frac{x_0}{\Gamma(\gamma)} \odot g^{-1}(\psi(t) - \psi(t_0))^{\gamma-1} \quad (36)$$

and

$$x_m(t) = \frac{x_0}{\Gamma(\gamma)} \odot g^{-1}(\psi(t) - \psi(t_0))^{\gamma-1} \oplus \int_{[t_0, t]}^{\oplus} g^{-1} \left(\frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} \right) \odot (\mathbf{A}x_{m-1}(s) \oplus f(s, x(s))) \odot ds. \quad (37)$$

Using the Eqs. (36) and (37), we find

$$\begin{aligned} x_1(t) &= x_0(t) \oplus \int_{[t_0, t]}^{\oplus} g^{-1} \left(\frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} \right) \odot (\mathbf{A}x_0(s) \oplus f(s, x(s))) \odot ds \\ &= \frac{x_0}{\Gamma(\gamma)} \odot g^{-1}(\psi(t) - \psi(t_0))^{\gamma-1} \oplus \int_{[t_0, t]}^{\oplus} g^{-1} \left(\frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} \right) \\ &\quad \odot \left(\mathbf{A} \frac{x_0}{\Gamma(\gamma)} \odot g^{-1}(\psi(s) - \psi(t_0))^{\gamma-1} \oplus f(s, x(s)) \right) \odot ds \\ &= \frac{x_0}{\Gamma(\gamma)} \odot g^{-1}(\psi(t) - \psi(t_0))^{\gamma-1} \oplus \\ &\quad \mathbf{A} \frac{x_0}{\Gamma(\gamma)} \odot \int_{[t_0, t]}^{\oplus} g^{-1} \left(\frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} \right) \odot g^{-1}(\psi(s) - \psi(t_0))^{\gamma-1} \odot ds \\ &\quad \oplus \int_{[t_0, t]}^{\oplus} g^{-1} \left(\frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} \right) \odot f(s, x(s)) \odot ds \\ &= \frac{x_0}{\Gamma(\gamma)} \odot g^{-1}(\psi(t) - \psi(t_0))^{\gamma-1} \oplus \mathbf{A} \frac{x_0}{\Gamma(\gamma)} \odot \mathbb{I}_{\oplus, \odot, t_0+}^{\alpha;\psi} \left(g^{-1}(\psi(t) - \psi(t_0))^{\gamma-1} \right) \\ &\quad \oplus \mathbb{I}_{\oplus, \odot, t_0+}^{\alpha;\psi} f(t, x(t)). \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
 x_2(t) &= \frac{x_0}{\Gamma(\gamma)} \odot g^{-1}(\psi(t) - \psi(t_0))^{\gamma-1} \oplus \int_{[t_0, t]}^{\oplus} g^{-1} \left(\frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} \right) \\
 &\quad \odot (\mathbf{A}x_1(s) \oplus f(s, x(s))) \odot ds \\
 &= \frac{x_0}{\Gamma(\gamma)} \odot g^{-1}(\psi(t) - \psi(t_0))^{\gamma-1} \oplus \int_{[t_0, t]}^{\oplus} g^{-1} \left(\frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} \right) \\
 &\quad \odot \mathbf{A} \frac{x_0}{\Gamma(\gamma)} \odot g^{-1}(\psi(s) - \psi(t_0))^{\gamma-1} \odot ds \\
 &\quad \oplus \int_{[t_0, t]}^{\oplus} g^{-1} \left(\frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} \right) \odot \mathbf{A}^2 \frac{x_0}{\Gamma(\gamma)} \odot \mathbb{I}_{\oplus, \odot, t_0+}^{\alpha; \psi} g^{-1}((\psi(t) - \psi(s))^{\gamma-1}) \odot ds \\
 &\quad \oplus \int_{[t_0, t]}^{\oplus} g^{-1} \left(\frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} \right) \odot \mathbf{A} \odot \mathbb{I}_{\oplus, \odot, t_0+}^{\alpha; \psi} \odot f(s, x(s)) \odot ds \\
 &\quad \oplus \int_{[t_0, t]}^{\oplus} g^{-1} \left(\frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} \right) \odot f(s) \odot ds \\
 &= x_0 \odot \int_{[t_0, t]}^{\oplus} \sum_{k=0}^2 \mathbf{A}^k \odot g^{-1} \left(\frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha k-1}}{\Gamma(\alpha k)} \right) \frac{g^{-1}((\psi(s) - \psi(t_0))^{\gamma-1})}{\Gamma(\gamma)} \odot ds \\
 &\quad \oplus \int_{[t_0, t]}^{\oplus} \sum_{k=1}^2 \mathbf{A}^k \odot g^{-1} \left(\frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha k-1}}{\Gamma(\alpha k)} \right) \odot f(s, x(s)) \odot ds. \tag{38}
 \end{aligned}$$

Continuing this process, we derive the following relation

$$\begin{aligned}
 x_m(t) &= x_0 \odot \int_{[t_0, t]}^{\oplus} \sum_{k=0}^m \mathbf{A}^k \odot g^{-1} \left(\frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha k-1}}{\Gamma(\alpha k)} \right) \frac{g^{-1}((\psi(s) - \psi(t_0))^{\gamma-1})}{\Gamma(\gamma)} \odot ds \\
 &\quad \oplus \int_{[t_0, t]}^{\oplus} \sum_{k=0}^m \mathbf{A}^k \odot g^{-1} \left(\frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha k+\alpha-1}}{\Gamma(\alpha k + \alpha)} \right) \odot f(s, x(s)) \odot ds. \tag{39}
 \end{aligned}$$

Taking limit as $m \rightarrow \infty$ on both sides of the Eq. (39), we have

$$\begin{aligned}
 \lim_{m \rightarrow \infty} x_m(t) &= x_0 \odot \int_{[t_0, t]}^{\oplus} \sum_{k=0}^{\infty} \mathbf{A}^k \odot g^{-1} \left(\frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha k-1}}{\Gamma(\alpha k + \gamma)} \right) g^{-1}((\psi(s) - \psi(t_0))^{\gamma-1}) \odot ds \\
 &\quad \oplus \int_{[t_0, t]}^{\oplus} \sum_{k=0}^{\infty} \mathbf{A}^k \odot g^{-1} \left(\frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha k+\alpha-1}}{\Gamma(\alpha k + \alpha)} \right) \odot f(s, x(s)) \odot ds \\
 &= x_0 \odot g^{-1}((\psi(t) - \psi(t_0))^{\gamma-1}) \odot \int_{[t_0, t]}^{\oplus} g^{-1}(\psi'(s)) \odot \mathbb{E}_{\alpha, \gamma}(g(\mathbf{A})(\psi(t) - \psi(s))^{\alpha}) \odot ds \\
 &\quad \oplus \int_{[t_0, t]}^{\oplus} g^{-1}(\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}) \odot \mathbb{E}_{\alpha, \alpha}(g(\mathbf{A})(\psi(t) - \psi(s))^{\alpha}) \odot f(s, x(s)) \odot ds.
 \end{aligned}$$

Therefore, we conclude that

$$\begin{aligned} x(t) &= x_0 \odot g^{-1} \left((\psi(t) - \psi(t_0))^{\gamma-1} \right) \odot \int_{[t_0, t]}^{\oplus} g^{-1}(\psi'(s)) \odot \mathbb{E}_{\alpha, \gamma} (g(\mathbf{A})(\psi(t) - \psi(s))^\alpha) \odot ds \\ &\quad \oplus \int_{[t_0, t]}^{\oplus} g^{-1}(\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}) \odot \mathbb{E}_{\alpha, \alpha} (g(\mathbf{A})(\psi(t) - \psi(s))^\alpha) \odot f(s, x(s)) \odot ds. \end{aligned} \quad (40)$$

□

The norm an n -vector function $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$ in $\mathbb{R}^n$ is defined by

$$\|x(t)\| = \left(\sum_{k=1}^n |x_k(t)|^2 \right)^{\frac{1}{2}}.$$

On the other hand, we have $\|x(t)\|_g = g^{-1} \|g(x(t))\|$. Let

$$X = \left\{ x, x \in \mathbb{C}, \mathbb{H}_{\oplus, \odot, t_0^+}^{\alpha, \beta, \psi} x \in \mathbb{C} \right\}.$$

Then X is a Banach space endowed with the norm

$$\|x\|_{X, g} = \|x\|_g.$$

In view of Lemma 1, we convert the initial value problem Eq. (4)

$$Fx = \left\{ x, x \in \mathbb{C}, \mathbb{H}_{\oplus, \odot, t_0^+}^{\alpha, \beta, \psi} x \in \mathbb{C} \right\}$$

where $F : X \rightarrow X$ is defined by

$$\begin{aligned} Fx(t) &= x_0 \odot g^{-1} \left((\psi(t) - \psi(t_0))^{\gamma-1} \right) \odot \int_{[t_0, t]}^{\oplus} g^{-1}(\psi'(s)) \odot \mathbb{E}_{\alpha, \gamma} \\ &\quad (g(\mathbf{A})(\psi(t) - \psi(s))^\alpha) \odot ds \oplus \\ &\quad \int_{[t_0, t]}^{\oplus} g^{-1} \left(\psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \right) \odot \mathbb{E}_{\alpha, \alpha} (g(\mathbf{A})(\psi(t) - \psi(s))^\alpha) \\ &\quad \odot f(s, x(s)) \odot ds \end{aligned} \quad (41)$$

Theorem 16 Let $f : [t_0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous function satisfying the Lipschitz condition

$$\|f(t, x_1(t)) \ominus f(t, x_2(t))\|_g \leq_g L \odot \{ \|x_1(t) \ominus x_2(t)\|_g \}, t \in J = [t_0, t_1], L > 0.$$

Then, the initial value problem Eq. (4) has a unique solution whenever

$$M \odot L \odot g^{-1} \left(\frac{(\psi(t) - \psi(t_0))^\alpha}{\alpha} \right) < 1$$

Proof The proof consists of two steps.

Step 1 We show that $F(B_r) \subset B_r$, i.e., we show that $F(x) \in B_r$, where B_r is a closed ball of radius $r > 0$ in X , that is, $\|f(t, 0)\|_g$ and

$$\|\mathbb{E}_{\alpha,i} (g(\mathbf{A})(\psi(t) - \psi(t_0))^\alpha)\|_g \leq_g M, \quad i = \alpha, \gamma$$

and

$$r > \max 2 \left\{ \|x_0\|_g \odot g^{-1}((\psi(T) - \psi(t_0))^{\gamma-1}) \odot M \oplus M \odot L \odot g^{-1} \left(\frac{(\psi(t) - \psi(t_0))^\alpha}{\alpha} \right) \right\}.$$

For $x \in B_r$, we have

$$\begin{aligned} \|Fx(t)\| &\leq \|x_0\| \\ &\odot g^{-1}((\psi(t) - \psi(t_0))^{\gamma-1}) \odot \int_{[t_0,t]}^{\oplus} g^{-1}(\psi'(s)) \odot \\ &\quad \|\mathbb{E}_{\alpha,\gamma}(g(\mathbf{A})(\psi(t) - \psi(s))^\alpha)\| \odot ds \\ &\oplus \int_{[t_0,t]}^{\oplus} g^{-1}(\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}) \\ &\quad \odot \|\mathbb{E}_{\alpha,\alpha}(g(\mathbf{A})(\psi(t) - \psi(s))^\alpha)\| \odot \|f(s, x(s))\| \odot ds \\ &\leq \|x_0\| \odot g^{-1}((\psi(T) - \psi(t_0))^{\gamma-1}) \odot M \oplus M \odot L \odot g^{-1} \left(\frac{(\psi(t) - \psi(t_0))^\alpha}{\alpha} \right) \end{aligned} \quad (42)$$

Thus, $\|Fx\|_g \leq r$, then $Fx \in B_r$, whenever $x \in B_r$.

Step 2 We next show that F is contraction mapping on X . For $x, y \in X$ and for each $t \in J$ we obtain

$$\begin{aligned} &\|(Fx)(t) \ominus (Fy)(t)\|_g \\ &= \left\| \int_{[t_0,t]}^{\oplus} g^{-1}(\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}) \odot \mathbb{E}_{\alpha,\alpha}(g(\mathbf{A})(\psi(t) - \psi(s))^\alpha) \odot f(s, x(s)) \odot ds \ominus \right. \\ &\quad \left. \int_{[t_0,t]}^{\oplus} g^{-1}(\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}) \odot \mathbb{E}_{\alpha,\alpha}(g(\mathbf{A})(\psi(t) - \psi(s))^\alpha) \odot f(s, y(s)) \odot ds \right\|_g \\ &\leq \int_{[t_0,t]}^{\oplus} \left\| g^{-1}(\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}) \right\|_g \odot \|\mathbb{E}_{\alpha,\alpha}(g(\mathbf{A})(\psi(t) - \psi(s))^\alpha)\|_g \odot \end{aligned}$$

$$\begin{aligned}
& \|f(s, x(s)) \ominus f(s, y(s))\|_g \odot ds \\
& \leq_g M \odot \int_{[t_0, t]}^{\oplus} g^{-1} \left(\psi'(s) (\psi(t) - \psi(s))^{\alpha-1} \right) \odot L \odot \|x(s) \ominus y(s)\|_g \odot ds \\
& \leq_g M \odot L \odot \|x \ominus y\|_g \odot \int_{[t_0, t]}^{\oplus} g^{-1} \left(\psi'(s) (\psi(t) - \psi(s))^{\alpha-1} \right) \odot ds \\
& = M \odot L \odot \|x \ominus y\|_g \odot g^{-1} \left(\int_{[t_0, t]}^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} ds \right) \\
& = M \odot L \odot \|x \ominus y\|_g \odot g^{-1} \left(\frac{(\psi(t) - \psi(t_0))^{\alpha}}{\alpha} \right) \\
& = M \odot L \odot g^{-1} \left(\frac{(\psi(t) - \psi(t_0))^{\alpha}}{\alpha} \right) \odot \|x \ominus y\|_g. \tag{43}
\end{aligned}$$

We conclude that F is a contraction since

$$M \odot L \odot g^{-1} \left(\frac{(\psi(t) - \psi(t_0))^{\alpha}}{\alpha} \right) < 1 \tag{44}$$

and the statement of the theorem follows by the classical Banach fixed point theorem. $\square$

Theorem 17 Let $f : [t_0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous function such that there exist a positive constant L , such that

$$\|f(t, x(t))\|_g \leq_g L \odot \|x(t)\|_g, \quad t \in J = [t_0, t_1]. \tag{45}$$

Then, the initial value Eq. (4) has a solution on J .

Proof We shall use the Schauder fixed point theorem to prove that F defined by Eq. (41) has a fixed point. The proof will be given in three steps.

Step 1 F is continuous.

Let n_n be a sequence in $\mathbb{R}^n$ such that $\lim_{n \rightarrow \infty} x_n = x$ in X . Keeping in mind the previous bounds of Mittag-Leffler functions, we have

$$\begin{aligned}
& \|(Fx_n)(t) \ominus (Fx)(t)\|_g \\
& \leq_g \int_{[t_0, t]}^{\oplus} g^{-1} \left(\psi'(s) (\psi(t) - \psi(s))^{\alpha-1} \right) \odot \|\mathbb{E}_{\alpha, \gamma}(g(\mathbf{A}))(\psi(t) - \psi(s))^{\alpha}\|_g \\
& \odot \|f(s, x_n(s)) \ominus f(s, x(s))\|_g \odot ds \\
& \leq_g M \odot \int_{[t_0, t]}^{\oplus} g^{-1} \left(\psi'(s) (\psi(t) - \psi(s))^{\alpha-1} \right) \odot \|f(s, x_n(s)) \ominus f(s, x(s))\|_g \odot ds.
\end{aligned}$$

Since f is continuous and bounded, g is continuous then by dominated convergent theorem, we deduce that the integrate tends to zero as n approaches to infinity. Thus,

$$\lim_{n \rightarrow \infty} \|(Fx_n)(t) \ominus (Fx)(t)\|_g = 0 \tag{46}$$

which implies that F is continuous.

Step 2 F maps bounded set.

We show that F maps bounded set. Indeed, it is enough to show that for any $r > 0$, there exists a positive constant $L > 0$, such that for each

$$x \in B_r = \{x \in X : \|x\|_g \leq r\}$$

we have

$$\begin{aligned} \|Fx(t)\|_g &\leq_g \|x_0\|_g \odot g^{-1}((\psi(t) - \psi(t_0))^{\gamma-1}) \odot \int_{[t_0, t]}^{\oplus} g^{-1}(\psi'(s)) \\ &\quad \odot \left\| \mathbb{E}_{\alpha, \gamma}(g(\mathbf{A})(\psi(t) - \psi(s))^\alpha) \right\|_g \odot ds \\ &\quad \oplus \int_{[t_0, t]}^{\oplus} g^{-1}(\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}) \odot \left\| \mathbb{E}_{\alpha, \alpha}(g(\mathbf{A})(\psi(t) - \psi(s))^\alpha) \right\|_g \\ &\quad \odot \|f(s, x(s))\|_g \odot ds \\ &\leq_g \|x_0\|_g \odot g^{-1}((\psi(t) - \psi(t_0))^{\gamma-1}) \odot M \odot g^{-1} \left(\int_{t_0}^t \psi'(s) ds \right) \\ &\quad \oplus M \odot L \odot \int_{[t_0, t]}^{\oplus} g^{-1}(\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}) \odot \|x(s)\|_g \odot ds \\ &\leq_g \|x_0\|_g \odot g^{-1}((\psi(t) - \psi(t_0))^{\gamma-1}) \odot M \odot g^{-1}(\psi(t) - \psi(t_0)) \\ &\quad \oplus M \odot L \odot \|x\|_g \odot g^{-1} \left(\frac{(\psi(t) - \psi(t_0))^\alpha}{\alpha} \right) \\ &= L \end{aligned} \tag{47}$$

Therefore, $\|Fx\|_g \leq_g L$.

Step 3 F maps bounded set into equicontinuous set.

We show that F maps bounded set into equicontinuous set of X . Let $t_1, t_2 \in J$ with $t_1 < t_2$ and B_r be a bounded set of X as in Step 2. Let $x \in B_r$, then

$$\begin{aligned} &\|(Fx)(t_2) \ominus (Fx)(t_1)\|_g \\ &= \left\| x_0 \odot g^{-1}((\psi(t_2) - \psi(t_0))^{\gamma-1}) \odot \int_{[t_0, t_2]}^{\oplus} g^{-1}(\psi'(s)) \odot \mathbb{E}_{\alpha, \gamma}(g(\mathbf{A})(\psi(t_2) - \psi(s))^\alpha) \odot ds \right. \\ &\quad \oplus \int_{[t_0, t_2]}^{\oplus} g^{-1}(\psi'(s)(\psi(t_2) - \psi(s))^{\alpha-1}) \odot \mathbb{E}_{\alpha, \alpha}(g(\mathbf{A})(\psi(t_2) - \psi(s))^\alpha) \odot f(s, x(s)) \odot ds \\ &\quad \ominus x_0 \odot g^{-1}((\psi(t_1) - \psi(t_0))^{\gamma-1}) \odot \int_{[t_0, t_1]}^{\oplus} g^{-1}(\psi'(s)) \odot \mathbb{E}_{\alpha, \gamma}(g(\mathbf{A})(\psi(t_1) - \psi(s))^\alpha) \odot ds \\ &\quad \left. \ominus \int_{[t_0, t_1]}^{\oplus} g^{-1}(\psi'(s)(\psi(t_1) - \psi(s))^{\alpha-1}) \odot \mathbb{E}_{\alpha, \alpha}(g(\mathbf{A})(\psi(t_1) - \psi(s))^\alpha) \odot f(s, x(s)) \odot ds \right\|_g \\ &= \left\| x_0 \odot g^{-1}((\psi(t_2) - \psi(t_0))^{\gamma-1}) \odot \left\{ \int_{[t_0, t_1]}^{\oplus} g^{-1}(\psi'(s)) \odot \mathbb{E}_{\alpha, \gamma}(g(\mathbf{A})(\psi(t_2) - \psi(s))^\alpha) \odot ds \right. \right. \\ &\quad \left. \oplus \int_{[t_1, t_2]}^{\oplus} g^{-1}(\psi'(s)) \odot \mathbb{E}_{\alpha, \gamma}(g(\mathbf{A})(\psi(t_2) - \psi(s))^\alpha) \odot ds \right\} \\ &\quad \left. \oplus \int_{[t_0, t_1]}^{\oplus} g^{-1}(\psi'(s)(\psi(t_2) - \psi(s))^{\alpha-1}) \odot \mathbb{E}_{\alpha, \alpha}(g(\mathbf{A})(\psi(t_2) - \psi(s))^\alpha) \odot f(s, x(s)) \odot ds \right\|_g \end{aligned}$$

$$\begin{aligned}
& \oplus \int_{[t_1, t_2]}^{\oplus} g^{-1}(\psi'(s)(\psi(t_2) - \psi(s))^{\alpha-1}) \odot \mathbb{E}_{\alpha, \alpha}(g(\mathbf{A})(\psi(t_2) - \psi(s))^{\alpha}) \odot f(s, x(s)) \odot ds \\
& \ominus x_0 \odot g^{-1}((\psi(t_1) - \psi(t_0))^{\gamma-1}) \odot \int_{[t_0, t_1]}^{\oplus} g^{-1}(\psi'(s)) \odot \mathbb{E}_{\alpha, \gamma}(g(\mathbf{A})(\psi(t_1) - \psi(s))^{\alpha}) \odot ds \\
& \ominus \int_{[t_0, t_1]}^{\oplus} g^{-1}(\psi'(s)(\psi(t_1) - \psi(s))^{\alpha-1}) \odot \mathbb{E}_{\alpha, \alpha}(g(\mathbf{A})(\psi(t_1) - \psi(s))^{\alpha}) \odot f(s, x(s)) \odot ds \Big\|_g \\
& \leq \|x_0\|_g \odot g^{-1}((\psi(t_2) - \psi(t_0))^{\gamma-1}) \odot ((\psi(t_1) - \psi(t_0))^{\gamma-1}) \\
& \odot \int_{[t_0, t_1]}^{\oplus} g^{-1}(\psi'(s)) \odot \|\mathbb{E}_{\alpha, \gamma}(g(\mathbf{A})(\psi(t_2) - \psi(s))^{\alpha}) \ominus \mathbb{E}_{\alpha, \gamma}(g(\mathbf{A})(\psi(t_1) - \psi(s))^{\alpha})\|_g \odot ds \\
& \oplus \int_{[t_1, t_2]}^{\oplus} g^{-1}(\psi'(s)) \odot \|\mathbb{E}_{\alpha, \gamma}(g(\mathbf{A})(\psi(t_2) - \psi(s))^{\alpha})\|_g \odot ds \\
& \oplus \int_{[t_0, t_1]}^{\oplus} \|g^{-1}(\psi'(s)(\psi(t_2) - \psi(s))^{\alpha-1}) \odot \mathbb{E}_{\alpha, \alpha}(g(\mathbf{A})(\psi(t_2) - \psi(s))^{\alpha}) \ominus \\
& g^{-1}(\psi'(s)(\psi(t_1) - \psi(s))^{\alpha-1}) \odot \mathbb{E}_{\alpha, \alpha}(g(\mathbf{A})(\psi(t_1) - \psi(s))^{\alpha})\|_g \odot \|f(s, x(s))\|_g \odot ds \\
& \oplus \int_{[t_1, t_2]}^{\oplus} g^{-1}(\psi'(s)(\psi(t_2) - \psi(s))^{\alpha-1}) \odot \|\mathbb{E}_{\alpha, \alpha}(g(\mathbf{A})(\psi(t_2) - \psi(s))^{\alpha})\|_g \odot \|f(s, x(s))\|_g \odot ds \\
& \leq \|x_0\|_g \odot g^{-1}(\psi(t_2) - \psi(t_0))^{\gamma-1} \odot g^{-1}((\psi(t_1) - \psi(t_0))^{\gamma-1}) \odot \int_{[t_0, t_1]}^{\oplus} g^{-1}(\psi'(s)) \odot \\
& \sum_{k=0}^{\infty} \frac{\|g(\mathbf{A})^k\|_g \|(\psi(t_2) - \psi(s))^{\alpha k} \ominus (\psi(t_1) - \psi(s))^{\alpha k}\|_g}{\Gamma(\alpha k + \gamma)} \odot ds \oplus M \odot \int_{[t_1, t_2]}^{\oplus} g^{-1}(\psi'(s)) \odot ds \\
& \oplus \int_{[t_0, t_1]}^{\oplus} \left\{ \|g^{-1}(\psi'(s)(\psi(t_2) - \psi(s))^{\alpha-1})\|_g \odot \sum_{k=0}^{\infty} \|g(\mathbf{A})^k\|_g \frac{\|(\psi(t_2) - \psi(s))^{\alpha k}\|_g}{\Gamma(\alpha k + \alpha)} \ominus \right. \\
& \left. \|g^{-1}(\psi'(s)(\psi(t_1) - \psi(s))^{\alpha-1})\|_g \odot \sum_{k=0}^{\infty} \|g(\mathbf{A})^k\|_g \frac{\|(\psi(t_1) - \psi(s))^{\alpha k}\|_g}{\Gamma(\alpha k + \alpha)} \right\} \odot L_1 \|x(s)\|_g \odot ds \\
& \oplus M \odot \int_{[t_1, t_2]}^{\oplus} g^{-1}(\psi'(s)(\psi(t_2) - \psi(s))^{\alpha-1}) \odot L_1 \|x(s)\|_g \odot ds \\
& \leq \|x_0\|_g \odot g^{-1}((\psi(t_2) - \psi(t_0))^{\gamma-1}) \odot g^{-1}((\psi(t_1) - \psi(t_0))^{\gamma-1}) \odot \\
& M \odot \int_{[t_0, t_1]}^{\oplus} g^{-1}(\psi'^{-1}(\psi(t_1) - \psi(t_2))) \oplus L_1 \odot \|x\|_g \odot \\
& \int_{[t_0, t_1]}^{\oplus} \left\{ \|g^{-1}(\psi'(s)(\psi(t_2) - \psi(s))^{\alpha-1})\|_g \odot \sum_{k=0}^{\infty} \frac{\|(\psi(t_2) - \psi(s))^{\alpha k}\|_g}{\Gamma(\alpha k + \alpha)} \ominus \right. \\
& \left. \|g^{-1}(\psi'(s)(\psi(t_1) - \psi(s))^{\alpha-1})\|_g \odot \sum_{k=0}^{\infty} \frac{\|(\psi(t_1) - \psi(s))^{\alpha k}\|_g}{\Gamma(\alpha k + \alpha)} \right\} \odot ds \\
& \oplus M \odot L_1 \odot \|x\|_g \odot g^{-1}\left(\frac{(\psi(t_2) - \psi(t_1))^{\alpha}}{\alpha}\right) \\
& \leq \|x_0\|_g \odot g^{-1}((\psi(t_2) - \psi(t_0))^{\gamma-1}) \odot g^{-1}((\psi(t_1) - \psi(t_0))^{\gamma-1}) \\
& \odot M \odot g^{-1}(\psi(t_1) - \psi(t_0)) \oplus M \odot g^{-1}(\psi(t) - \psi(t_0))
\end{aligned}$$

$$\begin{aligned}
& \oplus L_1 \odot \|x\|_g \odot \int_{[t_0, t_1]}^{\oplus} \left\{ \left\| g^{-1}(\psi'(s)(\psi(t_2) - \psi(s))^{\alpha-1}) \right\|_g \odot \sum_{k=0}^{\infty} \frac{\|(\psi(t_2) - \psi(s))^{\alpha k}\|_g}{\Gamma(\alpha k + \alpha)} \right. \\
& \left. \ominus \left\| g^{-1}(\psi'(s)(\psi(t_1) - \psi(s))^{\alpha-1}) \right\|_g \odot \sum_{k=0}^{\infty} \frac{\|(\psi(t_1) - \psi(s))^{\alpha k}\|_g}{\Gamma(\alpha k + \alpha)} \right\} \odot ds \\
& \oplus M \odot L_1 \odot \|x\|_g \odot g^{-1} \left(\frac{(\psi(t) - \psi(t_0))^{\alpha}}{\alpha} \right) \\
& \leq \|x_0\|_g \odot g^{-1} \left((\psi(t) - \psi(t_0))^{\alpha-1} \right) \odot M \oplus M \odot g^{-1}(\psi(t) - \psi(t_0)) \\
& \oplus M \odot L_1 \|x\|_g \odot g^{-1} \left(\frac{(\psi(t) - \psi(t_0))^{\alpha}}{\alpha} \right).
\end{aligned}$$

Since the Mittag-Leffler function is uniformly bounded, hence we interchange the summation and integration. As $t_1 \rightarrow t_2$, we deduce $\|Fx(t_2) - Fx(t_1)\|_g \rightarrow 0$.

As a consequence of steps 1 to 3 together with the Arzelà-Ascoli theorem, we can conclude that $F : X \rightarrow X$ is continuous and completely continuous. Therefore F is completely continuous. By Schauder fixed point theorem the IPVs, Eq. (4) has a solution. $\square$

6 Concluding remark and open problem

From the generalized g -Laplace will be possible to investigate some open questions in the theory of pseudo-fractional differential equations, in particular, the discussion of mild solutions involving sectorial and quasi-sectorial operators in Banach spaces and their consequences. Besides, traditional problems, such as obtaining analytical solution of linear pseudo-differential equations that model natural phenomena, will be allowed possible discussions via generalized g -Laplace transform. On the other hand, the discussion of the existence and uniqueness of the mild solutions of the pseudo-fractional differential equation in the sense of the pseudo-fractional derivative ψ -Hilfer, On the other hand, the discussion of the existence and uniqueness of the mild solution of the pseudo-fractional differential equation in the sense of the pseudo-fractional derivative ψ -Hilfer, will allow a new way to discuss new and important results.

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