Multipliers on $\mathcal{S}_\omega(\mathbb{R}^N)$

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Abstract

The aim of this paper is to introduce and to study the space $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ of the multipliers of the space $\mathcal{S}_\omega(\mathbb{R}^N)$ of the ω -ultradifferentiable rapidly decreasing functions of Beurling type. We determine various properties of the space $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$. Moreover, we define and compare some lc-topologies of which $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ can be naturally endowed.

Keywords Multipliers · Weight functions · Ultradifferentiable rapidly decreasing function spaces of Beurling type

Mathematics Subject Classification Primary 46E10 · 46F05; Secondary 47B38

1 Introduction

Classes of ultradifferentiable functions have been intensively investigated since the 20ies of the last century. The subject indeed has a long tradition that goes back to the work of Gevrey [11]. Along the lines of [11], Komatsu introduced in [16] a way to define the ultradifferentiable functions. It consists in measuring their growth behaviour in terms of a weight sequence $\{M_p\}_{p \in \mathbb{N}_0}$, which satisfies certain conditions. Later Beurling [1] (see also Björck [2]) showed that one can also use weight functions ω to measure the smoothness of C^∞ -functions with compact support by the decay properties of their Fourier transform. This approach was modified by Braun, Meise, and Taylor [8], who showed that these classes can be defined by the decay behaviour of their derivatives by using the Young conjugate of the function $t \mapsto \omega(e^t)$. But, in

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general, the classes defined in one way *cannot* be defined in the other way (see, f.i., [6]).

The study of the space of multipliers and convolutors, $\mathcal{O}_M$ and $\mathcal{O}'_C$, of the space $\mathcal{S}(\mathbb{R}^N)$ of rapidly decreasing functions was started by Schwartz [27]. Since then the spaces $\mathcal{O}_M$ and $\mathcal{O}'_C$ attracted the attention of several authors, even recently (see [15, 19–22, 24] and the references therein). On the other hand, the space of multipliers and convolutors was introduced and studied in the setting of other classes of C^∞ -function spaces (see, f.i., [10, 14, 18, 25, 26, 28, 29] and the references therein).

In the last years the attention has focused on the study of the space $\mathcal{S}_\omega(\mathbb{R}^N)$ of the ultradifferentiable rapidly decreasing functions of Beurling type, as introduced by Björck [2] (see, [3–5], for instance, and the references therein). Inspired by this line of research and by the previous work, in this paper we introduce and study the space $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ of the slowly increasing functions of Beurling type in the setting of ultradifferentiable function space as introduced in [8]. In particular, we show that $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ is the space of the multipliers of the space $\mathcal{S}_\omega(\mathbb{R}^N)$ and of its dual $\mathcal{S}'_\omega(\mathbb{R}^N)$. We also define and compare some locally convex topologies (briefly, lc) of which $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ can be naturally endowed.

The paper is organized as follows. Section 2 collects some known definitions and properties about the weight functions in the sense of Braun, Meise and Taylor [8] and the space $\mathcal{S}_\omega(\mathbb{R}^N)$. Section 3 is devoted to the study of the space $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and of related topological properties. In Sect. 3 we also introduce the space $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ of the very slowly increasing functions of Beurling type. In particular, we show the link between these spaces and their topological properties. In a forthcoming paper we prove, f.i., that $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ is the space of convolutors of $\mathcal{S}'_\omega(\mathbb{R}^N)$ and the Fourier exchange between the spaces $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ and $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$. In Sect. 4 we show that $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ is the space of the multipliers of $\mathcal{S}_\omega(\mathbb{R}^N)$. Finally, in Sect. 5 we study and compare some lc-topologies of which $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ can be endowed in a natural way.

2 Weight functions and the space $\mathcal{S}_\omega(\mathbb{R}^N)$

We begin with the definition of non-quasianalytic weight function in the sense of Braun-Meise-Taylor [8] suitable for the Beurling case, i.e., we also consider the logarithm as a weight function.

Definition 2.1 A non-quasianalytic weight function is a continuous increasing function $\omega : [0, \infty) \rightarrow [0, \infty)$ satisfying the following properties:

- (α) there exists $K \geq 1$ such that $\omega(2t) \leq K(1 + \omega(t))$ for every $t \geq 0$;
- (β) $\int_1^\infty \frac{\omega(t)}{1+t^2} dt < \infty$;
- (γ) there exist $a \in \mathbb{R}$, $b > 0$ such that $\omega(t) \geq a + b \log(1 + t)$, for every $t \geq 0$;
- (δ) $\varphi_\omega(t) = \omega \circ \exp(t)$ is a convex function.

Given a non-quasianalytic weight function ω , we can extend it on $\mathbb{C}^N$ by setting $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{C}^N$, where $|\cdot|$ denotes the standard euclidean norm.

Remark 2.2 We now recall some known facts on weight functions that shall be useful in the following; the proofs can be found in the literature.

Let ω be a non-quasianalytic weight function. Then the following properties are satisfied.

(1) Condition (α) implies for every $t_1, t_2 \geq 0$ that

$$\omega(t_1 + t_2) \leq K(1 + \omega(t_1) + \omega(t_2)). \quad (2.1)$$

Observe that this condition is weaker than subadditivity (i.e., $\omega(t_1 + t_2) \leq \omega(t_1) + \omega(t_2)$). The weight functions satisfying (α) are not necessarily subadditive in general.

(2) Condition (α) implies that there exists $L \geq 1$ such that for every $t \geq 0$

$$\omega(et) \leq L(1 + \omega(t)). \quad (2.2)$$

(3) By condition (β) and the fact that ω is an increasing function, we have that $\omega(t) = o(t)$ as $t \rightarrow \infty$. This can be deduced by the fact that for every $t > 0$ we have

$$\frac{\omega(t)}{t} = \int_t^\infty \frac{\omega(s)}{s^2} ds \leq \int_t^\infty \frac{\omega(s)}{s^2} ds.$$

Let ω be a non-quasianalytic weight function. We define the Young conjugate φ_ω^* of φ_ω as the function $\varphi_\omega^* : [0, \infty) \rightarrow [0, \infty)$ defined by

$$\varphi_\omega^*(s) := \sup_{t \geq 0} \{st - \varphi_\omega(t)\}, \quad s \geq 0. \quad (2.3)$$

There is no loss of generality to assume that ω vanishes on $[0, 1]$. So, φ_ω^* has only non-negative values. By Fenchel-Moreau Theorem (see, f.i., [7]), we have that φ_ω^* is convex and increasing, $\varphi_\omega^*(0) = 0$ and $(\varphi_\omega^*)^* = \varphi_\omega$. Further useful properties of φ_ω^* are listed in the following lemma, see [8].

Lemma 2.3 *Let $\omega : [0, \infty) \rightarrow [0, \infty)$ be a non-quasianalytic weight function. Then the following properties are satisfied.*

- (1) $\lim_{t \rightarrow \infty} \frac{t}{\varphi_\omega^*(t)} = 0$.
- (2) $\frac{\varphi_\omega^*(t)}{t}$ is an increasing function in $(0, \infty)$.
- (3) For every $s, t \geq 0$ and $\lambda > 0$

$$2\lambda\varphi_\omega^*\left(\frac{s+t}{2\lambda}\right) \leq \lambda\varphi_\omega^*\left(\frac{s}{\lambda}\right) + \lambda\varphi_\omega^*\left(\frac{t}{\lambda}\right) \leq \lambda\varphi_\omega^*\left(\frac{s+t}{\lambda}\right). \quad (2.4)$$

- (4) For every $t \geq 0$ and $\lambda > 0$

$$\lambda L\varphi_\omega^*\left(\frac{t}{\lambda L}\right) + t \leq \lambda\varphi_\omega^*\left(\frac{t}{\lambda}\right) + \lambda L, \quad (2.5)$$

where $L \geq 1$ is the constant appearing in formula (2.2).

Remark 2.4 Lemma 2.3 allows to list some properties that we will frequently use in the sequel.

- (1) For every $m, M \in \mathbb{N}$ with $M \geq mL$, where L is the constant appearing in formula (2.2), and for every $\alpha \in \mathbb{N}_0^N$

$$2^{|\alpha|} \exp \left(M \varphi_\omega^* \left(\frac{|\alpha|}{M} \right) \right) \leq C \exp \left(m \varphi_\omega^* \left(\frac{|\alpha|}{m} \right) \right), \quad (2.6)$$

with $C := e^{mL}$.

- (2) For every $\alpha, \beta \in \mathbb{N}_0^N$ and $m \in \mathbb{N}$

$$2m \varphi_\omega^* \left(\frac{|\alpha + \beta|}{2m} \right) \leq m \varphi_\omega^* \left(\frac{|\alpha|}{m} \right) + m \varphi_\omega^* \left(\frac{|\beta|}{m} \right) \leq m \varphi_\omega^* \left(\frac{|\alpha + \beta|}{m} \right). \quad (2.7)$$

We now introduce the ultradifferentiable function spaces and their duals of Beurling type in the sense of Braun, Meise and Taylor [8].

Definition 2.5 Let ω be a non-quasianalytic weight.

- (a) For a compact subset K of $\mathbb{R}^N$ and $\lambda > 0$ define

$$\mathcal{E}_{\omega, \lambda}(K) := \left\{ f \in C^\infty(K) : p_{K, \lambda}(f) := \sup_{x \in K} \sup_{\alpha \in \mathbb{N}_0^N} |\partial^\alpha f(x)| \exp \left(-\lambda \varphi_\omega^* \left(\frac{|\alpha|}{\lambda} \right) \right) < \infty \right\}.$$

Then $(\mathcal{E}_{\omega, \lambda}(K), p_{K, \lambda})$ is a Banach space.

- (b) For an open subset Ω of $\mathbb{R}^N$ define

$$\mathcal{E}_\omega(\Omega) := \{ f \in C^\infty(\Omega) : p_{K, m}(f) < \infty \forall K \Subset \Omega, m \in \mathbb{N} \}$$

and endow it with its natural Fréchet space topology, i.e., with the lc-topology generated by the system of seminorms $\{p_{K, m}\}_{K \Subset \Omega, m \in \mathbb{N}}$. The elements of $\mathcal{E}_\omega(\Omega)$ are called ω -ultradifferentiable functions of Beurling type on Ω . The dual $\mathcal{E}'_\omega(\Omega)$ of $\mathcal{E}_\omega(\Omega)$ is endowed with its strong topology.

- (c) For a compact subset K of $\mathbb{R}^N$ define

$$\mathcal{D}_\omega(K) := \{ f \in \mathcal{E}_\omega(\mathbb{R}^N) : \text{supp } f \subseteq K \}$$

and endow it with the Fréchet space topology generated by the sequence $\{p_{K, m}\}_{m \in \mathbb{N}}$ of norms. For an open subset Ω of $\mathbb{R}^N$ define

$$\mathcal{D}_\omega(\Omega) := \text{ind }_{j \rightarrow} \mathcal{D}_\omega(K_j),$$

where $\{K_j\}_{j \in \mathbb{N}}$ is any fundamental sequence of compact subsets of Ω . The elements of $\mathcal{D}_\omega(\Omega)$ are called *test functions of Beurling type* on Ω . The dual $\mathcal{D}'_\omega(\Omega)$ of $\mathcal{D}_\omega(\Omega)$ is endowed with its strong topology. The elements of $\mathcal{D}'_\omega(\Omega)$ are called ω -ultradistributions of Beurling type on Ω .

Remark 2.6 Let ω be a non-quasianalytic weight function. Then the following properties are satisfied.

- (1) For every open subset Ω of $\mathbb{R}^N$ the space $\mathcal{E}_\omega(\Omega)$ is a nuclear Fréchet space, see [8, Proposition 4.9].
- (2) For every compact subset K of $\mathbb{R}^N$ we have $\mathcal{D}_\omega(K) \neq \{0\}$ and $\mathcal{D}_\omega(K)$ is a nuclear Fréchet space, see [8, Remark 3.2 (1) and Corollary 3.6(2)].
- (3) For every open subset Ω of $\mathbb{R}^N$ the inclusion $\mathcal{D}_\omega(\Omega) \hookrightarrow \mathcal{E}_\omega(\Omega)$ is continuous with dense range, see [8, Proposition 4.7 (1)].

We consider the following notation for the Fourier transform of a function $f \in L^1(\mathbb{R}^N)$:

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^N} f(x) e^{-ix\xi} dx, \quad \xi \in \mathbb{R}^N,$$

with standard extensions to more general spaces of functions or distributions. We introduce the space of weighted rapidly decreasing functions of Beurling type as defined in [2, Definition 1.8.1].

Definition 2.7 Let ω be a non-quasianalytic weight function. We denote by $\mathcal{S}_\omega(\mathbb{R}^N)$ the set of all functions $f \in L^1(\mathbb{R}^N)$ such that $f, \hat{f} \in C^\infty(\mathbb{R}^N)$ and for each $\lambda > 0$ and $\alpha \in \mathbb{N}_0^N$ we have

$$\|\exp(\lambda\omega)\partial^\alpha f\|_\infty < \infty \text{ and } \|\exp(\lambda\omega)\partial^\alpha \hat{f}\|_\infty < \infty. \quad (2.8)$$

The elements of $\mathcal{S}_\omega(\mathbb{R}^N)$ are called *ω -ultradifferentiable rapidly decreasing functions of Beurling type*.

The space $\mathcal{S}_\omega(\mathbb{R}^N)$ is a Fréchet space with respect to the lc-topology generated by the sequence of norms

$$\|\exp(n\omega)\partial^\alpha f\|_\infty + \|\exp(n\omega)\partial^\alpha \hat{f}\|_\infty, \quad f \in \mathcal{S}_\omega(\mathbb{R}^N), \quad n \in \mathbb{N}.$$

We denote by $\mathcal{S}'_\omega(\mathbb{R}^N)$ the dual of $\mathcal{S}_\omega(\mathbb{R}^N)$ endowed with its strong topology.

Remark 2.8 Let ω be a non-quasianalytic weight function. Then the following properties are satisfied.

- (1) The condition (γ) of Definition 2.1 implies that $\mathcal{S}_\omega(\mathbb{R}^N) \subseteq \mathcal{S}(\mathbb{R}^N)$ with continuous inclusion. Accordingly, we can rewrite the definition of $\mathcal{S}_\omega(\mathbb{R}^N)$ as the set of all the rapidly decreasing functions that satisfy the condition (2.8) in Definition 2.7.
- (2) The Fourier transform $\mathcal{F} : \mathcal{S}_\omega(\mathbb{R}^N) \rightarrow \mathcal{S}_\omega(\mathbb{R}^N)$ is a continuous isomorphism, that can be extended in the usual way to $\mathcal{S}'_\omega(\mathbb{R}^N)$, see [2, Proposition 1.8.2].
- (3) The space $\mathcal{S}_\omega(\mathbb{R}^N)$ is closed under convolution, under multiplication, translation and modulation, where the translation and modulation operators are defined by $\tau_y f(x) := f(x - y)$ and $M_t f(x) := e^{itx} f(x)$, respectively, where $t, x, y \in \mathbb{R}^N$, see [2, Propositions 1.8.3 and 1.8.5].

- (4) The inclusion $\mathcal{D}_\omega(\mathbb{R}^N) \hookrightarrow \mathcal{S}_\omega(\mathbb{R}^N) \hookrightarrow \mathcal{E}_\omega(\mathbb{R}^N)$ are continuous with dense range, see [8, Proposition 4.7.(1)] and [2, Propositions 1.8.6 and 1.8.7]. Therefore, $\mathcal{E}'_\omega(\mathbb{R}^N) \subset \mathcal{S}'_\omega(\mathbb{R}^N) \subset \mathcal{D}'_\omega(\mathbb{R}^N)$.
- (5) The space $\mathcal{S}_\omega(\mathbb{R}^N)$ is a nuclear Fréchet space, see [5, Theorem 3.3].

The space $\mathcal{S}_\omega(\mathbb{R}^N)$ is a Fréchet space with different equivalent systems of seminorms. Indeed, the following result holds (see [4, Theorem 4.8] and [3, Theorems 2.6]).

Proposition 2.9 *Let ω be a non-quasianalytic weight function and consider $f \in \mathcal{S}(\mathbb{R}^N)$. Then $f \in \mathcal{S}_\omega(\mathbb{R}^N)$ if and only if one of the following conditions is satisfied.*

- (1) (i) $\forall \lambda > 0, \alpha \in \mathbb{N}_0^N, 1 \leq p \leq \infty \exists C_{\alpha,\lambda,p} > 0$ such that $\|\exp(\lambda\omega)\partial^\alpha f\|_p \leq C_{\alpha,\lambda,p}$, and
 (ii) $\forall \lambda > 0, \alpha \in \mathbb{N}_0^N, 1 \leq p \leq \infty \exists C_{\alpha,\lambda,p} > 0$ such that $\|\exp(\lambda\omega)\partial^\alpha \hat{f}\|_p \leq C_{\alpha,\lambda,p}$.
- (2) (i) $\forall \lambda > 0, 1 \leq p \leq \infty \exists C_{\lambda,p} > 0$ such that $\|\exp(\lambda\omega)f\|_p \leq C_{\lambda,p}$, and
 (ii) $\forall \lambda > 0, 1 \leq p \leq \infty \exists C_{\lambda,p} > 0$ such that $\|\exp(\lambda\omega)\hat{f}\|_p \leq C_{\lambda,p}$.
- (3) $\forall \lambda, \mu > 0, 1 \leq p \leq \infty \exists C_{\lambda,\mu,p} > 0$ such that

$$\sup_{\alpha \in \mathbb{N}_0^N} \|\exp(\mu\omega)\partial^\alpha f\|_p \exp\left(-\lambda\varphi_\omega^*\left(\frac{|\alpha|}{\lambda}\right)\right) \leq C_{\lambda,\mu,p}.$$

Remark 2.10 We observe that the assumption $f \in \mathcal{S}(\mathbb{R}^N)$ in Proposition 2.9 can be replaced by the weaker assumption $f \in C^\infty(\mathbb{R}^N)$. Indeed, the condition (γ) in Definition 2.1 implies for every $x \in \mathbb{R}^N$ and $\alpha \in \mathbb{N}_0^N$ that

$$|x^\alpha| \leq \exp\left(-\frac{|\alpha|}{b}\right) \exp\left(\frac{|\alpha|}{b}\omega(x)\right),$$

where b is the constant appearing in condition (γ) . Therefore, if one (and hence all) of the equivalent conditions (1)–(3) of Proposition 2.9 is satisfied, then for every $\alpha, \beta \in \mathbb{N}_0^N$ and $1 \leq p \leq \infty$ we have

$$\|x^\alpha \partial^\beta f\|_p \leq \exp\left(-\frac{|\alpha|}{b}\right) \left\| \exp\left(\frac{|\alpha|}{b}\omega\right) \partial^\beta f \right\|_p < \infty.$$

Accordingly, $f \in \mathcal{S}(\mathbb{R}^N)$.

In the following, we will use this system of norms generating the Fréchet topology of $\mathcal{S}_\omega(\mathbb{R}^N)$:

$$q_{\lambda,\mu}(f) := \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in \mathbb{R}^N} \exp\left(-\lambda\varphi_\omega^*\left(\frac{|\alpha|}{\lambda}\right)\right) \exp(\mu\omega(x)) |\partial^\alpha f(x)|, \quad \lambda, \mu > 0, \quad f \in \mathcal{S}_\omega(\mathbb{R}^N).$$

In particular, the Fréchet topology of $\mathcal{S}_\omega(\mathbb{R}^N)$ is generated by the sequence of norms $\{q_{m,n}\}_{m,n \in \mathbb{N}}$.

3 The spaces $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$

We first introduce the spaces $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ and then we collect some basic properties about them.

Definition 3.1 Let ω be a non-quasianalytic weight function.

(a) For $m \in \mathbb{N}$ and $n \in \mathbb{Z}$ we define the space $\mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$ as the set of all functions $f \in C^\infty(\mathbb{R}^N)$ satisfying the following condition:

$$r_{m,n}(f) := \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in \mathbb{R}^N} |\partial^\alpha f(x)| \exp \left(-n\omega(x) - m\varphi_\omega^* \left(\frac{|\alpha|}{m} \right) \right) < \infty. \quad (3.1)$$

(b) We denote by $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ the set of all functions $f \in C^\infty(\mathbb{R}^N)$ such that for each $m \in \mathbb{N}$ there exist $C > 0$ and $n \in \mathbb{N}$ such that for every $\alpha \in \mathbb{N}_0^N$ and $x \in \mathbb{R}^N$ we have

$$|\partial^\alpha f(x)| \leq C \exp \left(n\omega(x) + m\varphi_\omega^* \left(\frac{|\alpha|}{m} \right) \right); \quad (3.2)$$

or equivalently,

$$\mathcal{O}_{M,\omega}(\mathbb{R}^N) := \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \mathcal{O}_{n,\omega}^m(\mathbb{R}^N). \quad (3.3)$$

The elements of $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ are called *slowly increasing functions of Beurling type*.

(c) We denote by $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ the set of all functions $f \in C^\infty(\mathbb{R}^N)$ for which there exists $n \in \mathbb{N}$ such that for every $m \in \mathbb{N}$ there exists $C > 0$ so that for every $\alpha \in \mathbb{N}_0^N$ and $x \in \mathbb{R}^N$ we have

$$|\partial^\alpha f(x)| \leq C \exp \left(n\omega(x) + m\varphi_\omega^* \left(\frac{|\alpha|}{m} \right) \right); \quad (3.4)$$

or equivalently,

$$\mathcal{O}_{C,\omega}(\mathbb{R}^N) := \bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \mathcal{O}_{n,\omega}^m(\mathbb{R}^N). \quad (3.5)$$

The elements of $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ are called *very slowly increasing functions of Beurling type*.

Remark 3.2 By Definition 3.1, we immediately obtain the following properties.

- (1) $\mathcal{D}_\omega(\mathbb{R}^N) \subseteq \mathcal{O}_{C,\omega}(\mathbb{R}^N)$ and $\mathcal{D}_\omega(\mathbb{R}^N) \subseteq \mathcal{O}_{M,\omega}(\mathbb{R}^N)$.
- (2) For every $\alpha \in \mathbb{N}_0^N$ the function $x^\alpha \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$. Indeed, fixed any $\alpha \in \mathbb{N}_0^N$, we have that $\partial^\beta x^\alpha = \prod_{i=1}^N \alpha_i(\alpha_i - 1) \dots (\alpha_i - \beta_i + 1) x_i^{\alpha_i - \beta_i}$ for $\beta \leq \alpha$, and that

$\partial^\beta x^\alpha = 0$ for $\beta > \alpha$. Since condition (γ) implies that $(1+t) \leq e^{-\frac{a}{b}} e^{\frac{1}{b}\omega(t)}$ for all $t \geq 0$ and ω is an increasing function on $[0, \infty)$, we have for every $\beta \leq \alpha$ and $x \in \mathbb{R}^N$ that

$$|\partial^\beta x^\alpha| \leq \prod_{i=1}^N \alpha_i! e^{-\frac{a}{b}(\alpha_i - \beta_i)} e^{\frac{1}{b}\omega(|x_i|)} \leq \alpha! e^{-\frac{a}{b}|\alpha|} e^{\frac{|\alpha|}{b}\omega(x)}.$$

Accordingly, $x^\alpha \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$.

- (3) The function $\log(1 + |x|^2) \in \mathcal{O}_{C,\omega}(\mathbb{R}^N)$ and the proof follows as in the case (2) above.

We now give an useful characterization of the elements of the spaces $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$.

Proposition 3.3 *Let ω be a non-quasianalytic weight function. Then the following properties are satisfied.*

- (1) *A function $f \in C^\infty(\mathbb{R}^N)$ is a slowly increasing function of Beurling type if and only if $f \in \mathcal{E}_\omega(\mathbb{R}^N)$ and for each $m \in \mathbb{N}$ there exist $C, R > 0$ and $n \in \mathbb{N}$ such that for every $\alpha \in \mathbb{N}_0^N$ and $x \in \mathbb{R}^N$ with $|x| \geq R$ we have*

$$|\partial^\alpha f(x)| \leq C \exp \left(n\omega(x) + m\varphi_\omega^* \left(\frac{|\alpha|}{m} \right) \right). \quad (3.6)$$

- (2) *A function $f \in C^\infty(\mathbb{R}^N)$ is a very slowly increasing function of Beurling type if and only if $f \in \mathcal{E}_\omega(\mathbb{R}^N)$ and there exists $n \in \mathbb{N}$ such that for every $m \in \mathbb{N}$ there exist $C, R > 0$ so that for every $\alpha \in \mathbb{N}_0^N$ and $x \in \mathbb{R}^N$ with $|x| \geq R$ we have*

$$|\partial^\alpha f(x)| \leq C \exp \left(n\omega(x) + m\varphi_\omega^* \left(\frac{|\alpha|}{m} \right) \right). \quad (3.7)$$

Proof (1) The necessity of the condition (3.6) is obvious. We need only to prove that $f \in \mathcal{E}_\omega(\mathbb{R}^N)$. So, fix a compact subset K of $\mathbb{R}^N$. Then for each $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for every $\alpha \in \mathbb{N}_0^N$ and $x \in K$ we have

$$|\partial^\alpha f(x)| \leq r_{m,n}(f) \exp \left(n\omega(x) + m\varphi_\omega^* \left(\frac{|\alpha|}{m} \right) \right) \leq D r_{m,n}(f) \exp \left(m\varphi_\omega^* \left(\frac{|\alpha|}{m} \right) \right),$$

where $D := \max_{y \in K} \{\exp(n\omega(y))\} < \infty$ is a constant depending only on n and K . Hence, it follows for every $m \in \mathbb{N}$ that

$$p_{K,m}(f) = \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in K} |\partial^\alpha f(x)| \exp \left(-m\varphi_\omega^* \left(\frac{|\alpha|}{m} \right) \right) \leq D r_{m,n}(f) < \infty.$$

Since K is arbitrary, this implies that $f \in \mathcal{E}_\omega(\mathbb{R}^N)$.

We now prove the sufficiency of the condition.

Fix $m \in \mathbb{N}$. By assumption there exist $C, R > 0$ and $n \in \mathbb{N}$ such that the inequality (3.6) is satisfied and $f \in \mathcal{E}_\omega(\mathbb{R}^N)$. Accordingly, we have

$$K_1 := \sup_{\alpha \in \mathbb{N}_0^N} \sup_{|x| \leq R} |\partial^\alpha f(x)| \exp \left(-m\varphi_\omega^* \left(\frac{|\alpha|}{m} \right) \right) < \infty.$$

If we set $K_2 := \min_{|x| \leq R} \exp(n\omega(x)) \geq 1$, then we obtain for every $|x| \leq R$ and $\alpha \in \mathbb{N}_0^N$ that

$$\begin{aligned} |\partial^\alpha f(x)| &\leq K_1 \exp \left(m\varphi_\omega^* \left(\frac{|\alpha|}{m} \right) \right) = \frac{K_1}{K_2} K_2 \exp \left(m\varphi_\omega^* \left(\frac{|\alpha|}{m} \right) \right) \\ &\leq \frac{K_1}{K_2} \exp(n\omega(x)) \exp \left(m\varphi_\omega^* \left(\frac{|\alpha|}{m} \right) \right). \end{aligned}$$

Therefore, setting $C' := \max \left\{ \frac{K_1}{K_2}, C \right\}$, the inequality (3.2) follows.

(2) follows by the same arguments for the proof of property (1) above. $\square$

We now establish some features concerning topological properties of the spaces introduced in Definition 3.1.

Proposition 3.4 *Let ω be a non-quasianalytic weight function. Then the following properties are satisfied.*

- (1) *Let $m \in \mathbb{N}$ and $n \in \mathbb{Z}$. Then $(\mathcal{O}_{n,\omega}^m(\mathbb{R}^N), r_{m,n})$ is a Banach space.*
- (2) *For every $n, n' \in \mathbb{Z}$ with $n \leq n'$ and $m \in \mathbb{N}$, the inclusion*

$$(\mathcal{O}_{n,\omega}^m(\mathbb{R}^N), r_{m,n}) \hookrightarrow (\mathcal{O}_{n',\omega}^m(\mathbb{R}^N), r_{m,n'}) \quad (3.8)$$

is well-defined and continuous.

- (3) *For every $n \in \mathbb{Z}$ and $m, m' \in \mathbb{N}$ with $m \leq m'$, the inclusion*

$$(\mathcal{O}_{n,\omega}^{m'}(\mathbb{R}^N), r_{m',n}) \hookrightarrow (\mathcal{O}_{n,\omega}^m(\mathbb{R}^N), r_{m,n}) \quad (3.9)$$

is well-defined and continuous.

Proof (1) Fix $m \in \mathbb{N}$ and $n \in \mathbb{Z}$. It suffices to show only the completeness. So, we fix a Cauchy sequence $\{f_j\}_{j \in \mathbb{N}}$ in $(\mathcal{O}_{n,\omega}^m(\mathbb{R}^N), r_{m,n})$ and observe that

$$|\partial^\alpha f_j(x) - \partial^\alpha f_{j'}(x)| \leq \exp \left(n\omega(x) + m\varphi_\omega^* \left(\frac{|\alpha|}{m} \right) \right) r_{m,n}(f_j - f_{j'}), \quad (3.10)$$

for all $j, j' \in \mathbb{N}$, $\alpha \in \mathbb{N}_0^N$ and $x \in \mathbb{R}^N$. Therefore, for any compact subset K of $\mathbb{R}^N$, we have

$$\sup_{\beta \leq \alpha} \sup_{x \in K} |\partial^\beta f_j(x) - \partial^\beta f_{j'}(x)| \leq C_{K,\alpha} r_{m,n}(f_j - f_{j'}),$$

for all $j, j' \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^N$, where $C_{K,\alpha} := \sup_{x \in K} \exp(n\omega(x)) \sup_{\beta \leq \alpha} \exp\left(m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) < \infty$. Since $r_{m,n}(f_j - f_{j'}) \rightarrow 0$ as $j, j' \rightarrow \infty$, and $C^\infty(\mathbb{R}^N)$ is a Fréchet space, it follows that $\{f_j\}_{j \in \mathbb{N}}$ is also a Cauchy sequence in $C^\infty(\mathbb{R}^N)$. Accordingly, there exists $f \in C^\infty(\mathbb{R}^N)$ such that $f_j \rightarrow f$ in $C^\infty(\mathbb{R}^N)$, as $j \rightarrow \infty$. In particular, $\partial^\alpha f_j \rightarrow \partial^\alpha f$ uniformly on compact subsets of $\mathbb{R}^N$ for every $\alpha \in \mathbb{N}_0^N$.

Now, fix $\epsilon > 0$. Since $\{f_j\}_{j \in \mathbb{N}}$ is a Cauchy sequence in $(\mathcal{O}_{n,\omega}^m(\mathbb{R}^N), r_{m,n})$, there exists $j_0 \in \mathbb{N}$ such that $r_{m,n}(f_j - f_{j'}) \leq \epsilon$ for all $j, j' \geq j_0$. It follows by (3.10) that for every $j, j' \geq j_0$, $x \in \mathbb{R}^N$ and $\alpha \in \mathbb{N}_0^N$ we have

$$|\partial^\alpha f_j(x) - \partial^\alpha f_{j'}(x)| \leq \epsilon \exp\left(n\omega(x) + m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right).$$

Letting $j' \rightarrow \infty$ in the inequality above, we obtain for every $j \geq j_0$, $x \in \mathbb{R}^N$ and $\alpha \in \mathbb{N}_0^N$ that

$$|\partial^\alpha f_j(x) - \partial^\alpha f(x)| \leq \epsilon \exp\left(n\omega(x) + m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right).$$

Accordingly, we have

$$\begin{aligned} r_{m,n}(f_j - f) &\leq \epsilon \quad \forall j \geq j_0, \\ r_{m,n}(f) &\leq r_{m,n}(f - f_{j_0}) + r_{m,n}(f_{j_0}) \leq \epsilon + r_{m,n}(f_{j_0}) < \infty. \end{aligned}$$

This means that $f \in \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$ and that $f_j \rightarrow f$ in $(\mathcal{O}_{n,\omega}^m(\mathbb{R}^N), r_{m,n})$ for $j \rightarrow \infty$, as $\epsilon > 0$ is arbitrary.

(2) Fix $n, n' \in \mathbb{Z}$ with $n \leq n'$ and $m \in \mathbb{N}$. Then for every $f \in \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$, we have

$$\begin{aligned} r_{m,n'}(f) &= \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in \mathbb{R}^N} |\partial^\alpha f(x)| \exp\left(-n'\omega(x) - m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) \\ &\leq \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in \mathbb{R}^N} |\partial^\alpha f(x)| \exp\left(-n\omega(x) - m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) = r_{m,n}(f). \end{aligned}$$

Therefore, the inclusion in (3.8) is well-defined and continuous.

(3) Fix $m, m' \in \mathbb{N}$ with $m \leq m'$ and $n \in \mathbb{Z}$. Since $\varphi_\omega^*(t)/t$ is an increasing function in $(0, \infty)$ (see Lemma 2.3(2)), for every $f \in \mathcal{O}_{n,\omega}^{m'}(\mathbb{R}^N)$ we have

$$\begin{aligned} r_{m,n}(f) &= \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in \mathbb{R}^N} |\partial^\alpha f(x)| \exp\left(-n\omega(x) - m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) \\ &\leq \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in \mathbb{R}^N} |\partial^\alpha f(x)| \exp\left(-n\omega(x) - m'\varphi_\omega^*\left(\frac{|\alpha|}{m'}\right)\right) = r_{m',n}(f). \end{aligned}$$

Therefore, the inclusion in (3.9) is well-defined and continuous. $\square$

Via Proposition 3.4(1)–(2) we deduce that the sequence $\{(\mathcal{O}_{n,\omega}^m(\mathbb{R}^N), r_{m,n})\}_{n \in \mathbb{N}}$ of Banach spaces forms for each $m \in \mathbb{N}$ an inductive spectrum. The space $\bigcup_{n=1}^\infty \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$, endowed with the inductive topology defined by $\{(\mathcal{O}_{n,\omega}^m(\mathbb{R}^N), r_{m,n})\}_{n \in \mathbb{N}}$, is then an (LB)-space for each $m \in \mathbb{N}$. On the other hand, via Proposition 3.4(1)–(3) we also deduce for every $n \in \mathbb{N}$ that the sequence $\{(\mathcal{O}_{n,\omega}^m(\mathbb{R}^N), r_{m,n})\}_{m \in \mathbb{N}}$ of Banach spaces forms a projective spectrum. So, for every $n \in \mathbb{N}$ the space $\bigcap_{m=1}^\infty \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$, endowed with the projective topology defined by $\{(\mathcal{O}_{n,\omega}^m(\mathbb{R}^N), r_{m,n})\}_{m \in \mathbb{N}}$, is a Fréchet space. In the following we always suppose that the spaces $\bigcup_{n=1}^\infty \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$ and $\bigcap_{m=1}^\infty \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$ are equipped respectively with the (LB)-topology and Fréchet topology defined above. In particular, the spaces $\bigcup_{n=1}^\infty \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$ and $\bigcap_{m=1}^\infty \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$ satisfy the following properties.

Proposition 3.5 *Let ω be a non-quasianalytic weight function and $m \in \mathbb{N}$. Then $\bigcup_{n=1}^\infty \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$ is a complete (LB)-space.*

Proof In order to show the completeness of the space $\bigcup_{n=1}^\infty \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$, we first prove that the inclusion $\bigcup_{n=1}^\infty \mathcal{O}_{n,\omega}^m(\mathbb{R}^N) \hookrightarrow C^\infty(\mathbb{R}^N)$ is continuous as follows.

Fix $n \in \mathbb{N}$. Then for every compact subset K of $\mathbb{R}^N$ and $\alpha \in \mathbb{N}_0^N$ we have

$$\begin{aligned} & \sup_{x \in K} \sup_{\beta \leq \alpha} |\partial^\beta f(x)| \\ &= \sup_{x \in K} \sup_{\beta \leq \alpha} |\partial^\beta f(x)| \exp\left(-n\omega(x) - m\varphi_\omega^*\left(\frac{|\beta|}{m}\right)\right) \exp\left(n\omega(x) + m\varphi_\omega^*\left(\frac{|\beta|}{m}\right)\right) \\ &\leq C_{K,\alpha} r_{m,n}(f) \end{aligned}$$

for each $f \in \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$, where $C_{K,\alpha} := \sup_{x \in K} \sup_{\beta \leq \alpha} \exp\left(n\omega(x) + m\varphi_\omega^*\left(\frac{|\beta|}{m}\right)\right) < \infty$ is a positive constant depending on K and α . This means that the inclusion $(\mathcal{O}_{n,\omega}^m(\mathbb{R}^N), r_{m,n}) \hookrightarrow C^\infty(\mathbb{R}^N)$ is continuous. Since $n \in \mathbb{N}$ is arbitrary and $\bigcup_{n=1}^\infty \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$ is an (LB)-space, the inclusion $\bigcup_{n=1}^\infty \mathcal{O}_{n,\omega}^m(\mathbb{R}^N) \hookrightarrow C^\infty(\mathbb{R}^N)$ is continuous too. Therefore, there exists a Hausdorff lc-topology τ on $\bigcup_{n=1}^\infty \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$ with the property that the closed unit ball of each $\mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$ is relatively τ -compact. But, the closed unit ball of each $\mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$ is also τ -compact. Indeed, let $n \in \mathbb{N}$ and let $\{f_j\}_{j \in \mathbb{N}} \subseteq \{g \in \mathcal{O}_{n,\omega}^m(\mathbb{R}^N) : r_{m,n}(g) \leq 1\}$ τ -convergent to some $f \in C^\infty(\mathbb{R}^N)$. Then for every $x \in \mathbb{R}^N$, $j \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^N$ we have

$$|\partial^\alpha f_j(x)| \leq \exp\left(n\omega(x) + m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right). \quad (3.11)$$

Since $f_j \rightarrow f$ in $C^\infty(\mathbb{R}^N)$ as $j \rightarrow \infty$ and hence $\partial^\alpha f_j \rightarrow \partial^\alpha f$ pointwise on $\mathbb{R}^N$ for each $\alpha \in \mathbb{N}_0^N$, it follows by letting $j \rightarrow \infty$ in (3.11) for every $x \in \mathbb{R}^N$ and $\alpha \in \mathbb{N}_0^N$ that

$$|\partial^\alpha f(x)| \leq \exp\left(n\omega(x) + m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right).$$

This implies that $r_{m,n}(f) \leq 1$. So, $\{g \in \mathcal{O}_{n,\omega}^m(\mathbb{R}^N) : r_{m,n}(g) \leq 1\}$ is τ -closed.

The result now follows by Mujica [23, Theorem 1]. $\square$

Proposition 3.6 *Let ω be a non-quasianalytic weight function. Then the following properties are satisfied.*

(1) *For every $n, n' \in \mathbb{Z}$ with $n \leq n'$, the inclusion*

$$\bigcap_{m=1}^{\infty} \mathcal{O}_{n,\omega}^m(\mathbb{R}^N) \hookrightarrow \bigcap_{m=1}^{\infty} \mathcal{O}_{n',\omega}^m(\mathbb{R}^N) \quad (3.12)$$

is well-defined and continuous.

(2) *For every $m, m' \in \mathbb{N}$ with $m \leq m'$, the inclusion*

$$\bigcup_{n=1}^{\infty} \mathcal{O}_{n,\omega}^{m'}(\mathbb{R}^N) \hookrightarrow \bigcup_{n=1}^{\infty} \mathcal{O}_{n,\omega}^m(\mathbb{R}^N) \quad (3.13)$$

is well-defined and continuous.

Proof (1) Fix $n, n' \in \mathbb{Z}$ with $n \leq n'$. Then, for every $h \in \mathbb{N}$ and $f \in \bigcap_{m=1}^{\infty} \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$, we have

$$\begin{aligned} r_{h,n'}(f) &= \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in \mathbb{R}^N} |\partial^\alpha f(x)| \exp \left(-n' \omega(x) - h \varphi_\omega^* \left(\frac{|\alpha|}{h} \right) \right) \\ &\leq \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in \mathbb{R}^N} |\partial^\alpha f(x)| \exp \left(-n \omega(x) - h \varphi_\omega^* \left(\frac{|\alpha|}{h} \right) \right) = r_{h,n}(f). \end{aligned}$$

This means that the inclusion in (3.12) is well-defined and continuous.

(2) Fix $m, m' \in \mathbb{N}$ with $m \leq m'$ and $h \in \mathbb{N}$. Then by Proposition 3.4(3) the inclusion $(\mathcal{O}_{h,\omega}^{m'}(\mathbb{R}^N), r_{m',h}) \hookrightarrow (\mathcal{O}_{h,\omega}^m(\mathbb{R}^N), r_{m,h})$ is continuous. On the other hand, the inclusion $(\mathcal{O}_{h,\omega}^m(\mathbb{R}^N), r_{m,h}) \hookrightarrow \bigcup_{n=1}^{\infty} \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$ is also continuous. Accordingly, the inclusion

$$(\mathcal{O}_{h,\omega}^{m'}(\mathbb{R}^N), r_{m',h}) \hookrightarrow \bigcup_{n=1}^{\infty} \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$$

is continuous. Since $h \in \mathbb{N}$ is arbitrary and $\bigcup_{h=1}^{\infty} \mathcal{O}_{h,\omega}^{m'}(\mathbb{R}^N)$ is an (LB)-space, it follows that the inclusion

$$\bigcup_{h=1}^{\infty} \mathcal{O}_{h,\omega}^{m'}(\mathbb{R}^N) \hookrightarrow \bigcup_{n=1}^{\infty} \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$$

is continuous, i.e., the inclusion in (3.12) is well-defined and continuous. $\square$

Proposition 3.6(1) yields that the sequence $\{\bigcap_{m=1}^\infty \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)\}_{n \in \mathbb{N}}$ of Fréchet spaces forms an inductive spectrum. So, we can endow the space $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ with the inductive topology defined by $\{\bigcap_{m=1}^\infty \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)\}_{n \in \mathbb{N}}$. Thereby, $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$, equipped with such an inductive topology, is an (LF)-space. Proposition 3.6(2) also yields that the sequence $\{\bigcup_{n=1}^\infty \mathcal{O}_{m,\omega}^n(\mathbb{R}^N)\}_{m \in \mathbb{N}}$ of (LB)-spaces is a projective spectrum. So, we can endow the space $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ with the projective topology defined by $\{\bigcup_{n=1}^\infty \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)\}_{m \in \mathbb{N}}$. Thereby, $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$, equipped with this projective topology, is a projective limit of (LB)-spaces. By Proposition 3.5, it is then a complete lc-space. In the following we always assume that the spaces $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ and $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ are endowed with the lc-topologies defined above.

Remark 3.7 We observe that, for fixed $m \in \mathbb{N}$ and $n \in \mathbb{N}$, we have that

$$q_{m,n}(f) = \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in \mathbb{R}^N} \exp\left(-m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) \exp(n\omega(x)) |\partial^\alpha f(x)| = r_{m,-n}(f).$$

for every $f \in \mathcal{S}_\omega(\mathbb{R}^N)$. Since $\{q_{m,n}\}_{m,n \in \mathbb{N}}$ is a fundamental sequence of norms generating the Fréchet topology of $\mathcal{S}_\omega(\mathbb{R}^N)$, it follows that $\mathcal{S}_\omega(\mathbb{R}^N) = \bigcap_{n=1}^\infty \bigcap_{m=1}^\infty \mathcal{O}_{-n,\omega}^m(\mathbb{R}^N)$.

Theorem 3.8 *Let ω be a non-quasianalytic weight function. Then the following properties are satisfied.*

(1) *The inclusion*

$$\mathcal{O}_{C,\omega}(\mathbb{R}^N) \hookrightarrow \mathcal{O}_{M,\omega}(\mathbb{R}^N) \quad (3.14)$$

is well-defined and continuous.

(2) *The inclusions*

$$\mathcal{S}_\omega(\mathbb{R}^N) \hookrightarrow \mathcal{O}_{C,\omega}(\mathbb{R}^N) \hookrightarrow \mathcal{E}_\omega(\mathbb{R}^N) \quad (3.15)$$

are well-defined and continuous.

Proof (1) Fix $f \in \mathcal{O}_{C,\omega}(\mathbb{R}^N)$ and $m \in \mathbb{N}$. Then there exists $n \in \mathbb{N}$ such that $r_{m,n}(f) < \infty$ and so, $f \in \mathcal{O}_{n,\omega}^m(\mathbb{R}^N) \subset \bigcup_{h=1}^\infty \mathcal{O}_{h,\omega}^m(\mathbb{R}^N)$. Since $m \in \mathbb{N}$ is arbitrary, we can conclude that $f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$. We have so shown that the inclusion is well-defined. We now prove that such an inclusion is continuous as follows.

We first observe that for every $m' \in \mathbb{N}$ and $n, n' \in \mathbb{N}$ with $n \leq n'$, the inclusion

$$\bigcap_{m=1}^\infty \mathcal{O}_{n,\omega}^m(\mathbb{R}^N) \hookrightarrow \mathcal{O}_{n',\omega}^{m'}(\mathbb{R}^N)$$

is well-defined and continuous. Indeed, for every $f \in \bigcap_{m=1}^\infty \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$ we have

$$r_{m',n'}(f) = \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in \mathbb{R}^N} |\partial^\alpha f(x)| \exp\left(-n'\omega(x) - m'\varphi_\omega^*\left(\frac{|\alpha|}{m'}\right)\right)$$

$$\leq \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in \mathbb{R}^N} |\partial^\alpha f(x)| \exp \left(-n\omega(x) - m' \varphi_\omega^* \left(\frac{|\alpha|}{m'} \right) \right) = r_{m',n}(f).$$

Since $r_{m',n} \in \{r_{m,n}\}_{m \in \mathbb{N}}$ and $\{r_{m,n}\}_{m \in \mathbb{N}}$ is a sequence of norms defining the lc-topology of $\bigcap_{m=1}^\infty \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$, it follows that the inclusion $\bigcap_{m=1}^\infty \mathcal{O}_{n,\omega}^m(\mathbb{R}^N) \hookrightarrow \mathcal{O}_{n',\omega}^{m'}(\mathbb{R}^N)$ is continuous. Taking into account that $\bigcup_{h=1}^\infty \mathcal{O}_{h,\omega}^{m'}(\mathbb{R}^N)$ is an (LB)-space and so, $\mathcal{O}_{n',\omega}^{m'}(\mathbb{R}^N) \hookrightarrow \bigcup_{h=1}^\infty \mathcal{O}_{h,\omega}^{m'}(\mathbb{R}^N)$ continuously, we deduce for every $m' \in \mathbb{N}$ and $n \in \mathbb{N}$ that the inclusion

$$\bigcap_{m=1}^\infty \mathcal{O}_{n,\omega}^m(\mathbb{R}^N) \hookrightarrow \bigcup_{h=1}^\infty \mathcal{O}_{h,\omega}^{m'}(\mathbb{R}^N)$$

is well-defined and continuous. Since $\mathcal{O}_{M,\omega}(\mathbb{R}^N) = \bigcap_{m'=1}^\infty \bigcup_{h=1}^\infty \mathcal{O}_{h,\omega}^{m'}(\mathbb{R}^N)$ is the projective limit of the sequence $\left\{ \bigcup_{h=1}^\infty \mathcal{O}_{h,\omega}^{m'}(\mathbb{R}^N) \right\}_{m' \in \mathbb{N}}$ of (LB)-spaces, it follows for every $n \in \mathbb{N}$ that the inclusion

$$\bigcap_{m=1}^\infty \mathcal{O}_{n,\omega}^m(\mathbb{R}^N) \hookrightarrow \mathcal{O}_{M,\omega}(\mathbb{R}^N)$$

is well-defined and continuous. Finally, since $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ is the inductive limit of the sequence $\left\{ \bigcap_{m=1}^\infty \mathcal{O}_{n,\omega}^m(\mathbb{R}^N) \right\}_{n \in \mathbb{N}}$ of Fréchet space, it follows that the inclusion

$$\mathcal{O}_{C,\omega}(\mathbb{R}^N) \hookrightarrow \mathcal{O}_{M,\omega}(\mathbb{R}^N)$$

is continuous.

(2) We first show that the inclusion $\mathcal{O}_{C,\omega}(\mathbb{R}^N) \hookrightarrow \mathcal{E}_\omega(\mathbb{R}^N)$ is continuous. So, fix $f \in \mathcal{O}_{C,\omega}(\mathbb{R}^N)$ and $n \in \mathbb{N}$. Then $f \in \bigcap_{h=1}^\infty \mathcal{O}_{n,\omega}^h(\mathbb{R}^N)$. This implies for every $m \in \mathbb{N}$ and K compact subset of $\mathbb{R}^N$ that

$$\begin{aligned} p_{K,m}(f) &= \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in K} |\partial^\alpha f(x)| \exp \left(-m \varphi_\omega^* \left(\frac{|\alpha|}{m} \right) \right) \\ &= \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in K} |\partial^\alpha f(x)| \exp \left(-m \varphi_\omega^* \left(\frac{|\alpha|}{m} \right) \right) \exp(n\omega(x) - n\omega(x)) \\ &\leq D \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in K} |\partial^\alpha f(x)| \exp \left(-m \varphi_\omega^* \left(\frac{|\alpha|}{m} \right) \right) \exp(-n\omega(x)) \leq D r_{m,n}(f) < \infty, \end{aligned}$$

where $D := \max_{y \in K} \{\exp(n\omega(y))\} < \infty$ is a constant depending only on n and K . Since $\{r_{m,n}\}_{m \in \mathbb{N}}$ is a sequence of norms generating the lc-topology of the Fréchet space $\bigcap_{h=1}^\infty \mathcal{O}_{n,\omega}^h(\mathbb{R}^N)$ and $\{p_{K,m}\}_{K \in \mathbb{R}^N}$ is a sequence of seminorms generating the lc-topology of the Fréchet space $\mathcal{E}_\omega(\mathbb{R}^N)$, it follows that the inclusion

$$\bigcap_{h=1}^{\infty} \mathcal{O}_{n,\omega}^h(\mathbb{R}^N) \hookrightarrow \mathcal{E}_\omega(\mathbb{R}^N)$$

is continuous. Since $n \in \mathbb{N}$ is arbitrary and $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ is the inductive limit of the Fréchet spaces $\{\bigcap_{h=1}^{\infty} \mathcal{O}_{n,\omega}^h(\mathbb{R}^N)\}_{n \in \mathbb{N}}$, it follows that the inclusion on the right of (3.15) is continuous.

We now show that the inclusion $\mathcal{S}_\omega(\mathbb{R}^N) \hookrightarrow \mathcal{O}_{C,\omega}(\mathbb{R}^N)$ is continuous. So, fix $f \in \mathcal{S}_\omega(\mathbb{R}^N)$. Then for every $m, n \in \mathbb{N}$ we have that $q_{m,n}(f) < \infty$, thereby implying that

$$\begin{aligned} r_{m,n}(f) &= \alpha \in \mathbb{N}_0^N \sup_{x \in \mathbb{R}^N} \exp\left(-m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) \exp(-n\omega(x)) |\partial^\alpha f(x)| \\ &< q_{m,n}(f) < \infty. \end{aligned} \quad (3.16)$$

Hence, $f \in \mathcal{O}_{C,\omega}(\mathbb{R}^N)$. This shows that the inclusion is well-defined. To prove the continuity of the inclusion, we observe that for every $n, n' \in \mathbb{N}$ and $m \in \mathbb{N}$ the inclusion

$$\bigcap_{m=1}^{\infty} \mathcal{O}_{-n,\omega}^m(\mathbb{R}^N) \hookrightarrow \bigcap_{m=1}^{\infty} \mathcal{O}_{n',\omega}^m(\mathbb{R}^N)$$

is continuous. Since $\bigcap_{m=1}^{\infty} \mathcal{O}_{n',\omega}^m(\mathbb{R}^N) \hookrightarrow \mathcal{O}_{C,\omega}(\mathbb{R}^N)$ continuously for all $n' \in \mathbb{N}$, it follows for every $m \in \mathbb{N}$ that the inclusion

$$\bigcap_{m=1}^{\infty} \mathcal{O}_{-n,\omega}^m(\mathbb{R}^N) \hookrightarrow \mathcal{O}_{C,\omega}(\mathbb{R}^N)$$

is continuous. Accordingly, as $\mathcal{S}_\omega(\mathbb{R}^N)$ is the projective limit of the Fréchet spaces $\{\bigcap_{m=1}^{\infty} \mathcal{O}_{-n,\omega}^m(\mathbb{R}^N)\}_{n \in \mathbb{N}}$, we can conclude that the inclusion on the left of (3.14) is continuous. $\square$

Finally, $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ have the following important property.

Theorem 3.9 *Let ω be a non-quasianalytic weight function. Then the following inclusions*

$$\mathcal{D}_\omega(\mathbb{R}^N) \subseteq \mathcal{S}_\omega(\mathbb{R}^N) \subseteq \mathcal{O}_{C,\omega}(\mathbb{R}^N) \subseteq \mathcal{O}_{M,\omega}(\mathbb{R}^N) \subseteq \mathcal{E}_\omega(\mathbb{R}^N)$$

are dense.

Proof The space $\mathcal{D}_\omega(\mathbb{R}^N)$ is a dense subspace of $\mathcal{S}_\omega(\mathbb{R}^N)$ and of $\mathcal{E}_\omega(\mathbb{R}^N)$, see [8, Proposition 4.7.(1)] and [2, Propositions 1.8.6 and 1.8.7]. So, both the spaces $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ and $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ are dense subspaces of $\mathcal{E}_\omega(\mathbb{R}^N)$.

We now prove that $\mathcal{S}_\omega(\mathbb{R}^N)$ is a dense subspace of $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ as follows.

Since $\mathcal{D}_\omega(\mathbb{R}^N) \subseteq \mathcal{S}_\omega(\mathbb{R}^N)$, it suffices to show only that $\mathcal{D}_\omega(\mathbb{R}^N)$ is a dense subspace of $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$. So, fix $f \in \mathcal{O}_{C,\omega}(\mathbb{R}^N)$ and $\phi \in \mathcal{D}_\omega(\mathbb{R}^N)$ such that $\phi \equiv 1$ on $\overline{B}_1(0)$ and $0 \leq \phi \leq 1$. Then for every $\epsilon > 0$ the function $\phi_\epsilon(x) := \phi(\epsilon x)f(x)$, for $x \in \mathbb{R}^N$, belongs to $\mathcal{D}_\omega(\mathbb{R}^N)$ because $\mathcal{O}_{C,\omega}(\mathbb{R}^N) \subseteq \mathcal{E}_\omega(\mathbb{R}^N)$. Since $f \in \mathcal{O}_{C,\omega}(\mathbb{R}^N)$, we have that $f \in \bigcap_{m=1}^\infty \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$ for some $n \in \mathbb{N}$. This implies that $f \in \bigcap_{m=1}^\infty \mathcal{O}_{h,\omega}^m(\mathbb{R}^N)$ for every $h \in \mathbb{N}$ with $h \geq n$. We claim that $r_{m,n+1}(\phi_\epsilon - f) \rightarrow 0$ as $\epsilon \rightarrow 0^+$ for every $m \in \mathbb{N}$. To show the claim, we proceed as follows.

Fix $m \in \mathbb{N}$ and let $M \in \mathbb{N}$ such that $M \geq Lm$, where $L \geq 1$ is the constant appearing in formula (2.2). Then for every $\alpha \in \mathbb{N}_0^N$, $x \in \mathbb{R}^N$ and $\epsilon > 0$ we have

$$\begin{aligned}
 |\partial^\alpha \phi_\epsilon(x) - \partial^\alpha f(x)| &\leq |\partial^\alpha f(x)(\phi(\epsilon x) - 1)| + \sum_{\beta < \alpha} \binom{\alpha}{\beta} |\partial^\beta f(x)| \epsilon^{\alpha-\beta} |\partial^{\alpha-\beta} \phi(\epsilon x)| \\
 &\leq |\partial^\alpha f(x)(\phi(\epsilon x) - 1)| + \sum_{\beta < \alpha} \binom{\alpha}{\beta} \epsilon^{\alpha-\beta} r_{M,n+1}(f) \exp((n+1)\omega(x)) \\
 &\quad \times \exp\left(M\varphi_\omega^*\left(\frac{|\beta|}{M}\right)\right) r_{M,0}(\phi) \exp\left(M\varphi_\omega^*\left(\frac{|\alpha-\beta|}{M}\right)\right) \\
 &\leq |\partial^\alpha f(x)(\phi(\epsilon x) - 1)| + \sum_{\beta < \alpha} \binom{\alpha}{\beta} \epsilon^{\alpha-\beta} r_{M,n+1}(f) \exp((n+1)\omega(x)) \\
 &\quad \times r_{M,0}(\phi) \exp\left(M\varphi_\omega^*\left(\frac{|\alpha|}{M}\right)\right) \\
 &\leq |\partial^\alpha f(x)(\phi(\epsilon x) - 1)| \\
 &\quad + r_{M,n+1}(f) r_{M,0}(\phi) \exp((n+1)\omega(x)) \exp\left(M\varphi_\omega^*\left(\frac{|\alpha|}{M}\right)\right) \epsilon^{2|\alpha|},
 \end{aligned} \tag{3.17}$$

after having observed that $\sum_{\beta < \alpha} \binom{\alpha}{\beta} \epsilon^{\alpha-\beta} \leq \epsilon^{2|\alpha|}$ for every $\alpha \in \mathbb{N}_0^N$ and using formula (2.7). Since $M \geq mL$, applying the inequality (2.6) it follows via (3.17) for every $\alpha \in \mathbb{N}_0^N$, $x \in \mathbb{R}^N$ and $\epsilon > 0$ that

$$\begin{aligned}
 |\partial^\alpha \phi_\epsilon(x) - \partial^\alpha f(x)| &\leq |\partial^\alpha f(x)(\phi(\epsilon x) - 1)| \\
 &\quad + \epsilon C r_{M,n+1}(f) r_{M,0}(\phi) \exp((n+1)\omega(x)) \exp\left(m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right).
 \end{aligned}$$

Therefore, we have for every $\epsilon > 0$ that

$$\begin{aligned}
 r_{m,n+1}(\phi_\epsilon - f) &= \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in \mathbb{R}^N} \exp\left(-m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) \exp(-(n+1)\omega(x)) |\partial^\alpha \phi_\epsilon(x) - \partial^\alpha f(x)| \\
 &\leq \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in \mathbb{R}^N} \exp\left(-m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) \exp(-(n+1)\omega(x)) |\partial^\alpha f(x)(\phi(\epsilon x) - 1)| \\
 &\quad + \epsilon C r_{M,n+1}(f) r_{M,0}(\phi).
 \end{aligned}$$

Since $f \in \bigcap_{m=1}^\infty \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$ and $\phi \in \mathcal{D}_\omega(\mathbb{R}^N)$, $r_{M,n+1}(f)r_{M,0}(\phi) < \infty$ and so, we have $\epsilon Cr_{M,n+1}(f)r_{M,0}(\phi) \rightarrow 0$ as $\epsilon \rightarrow 0^+$. In order to conclude the proof, it then remains to prove that

$$\sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in \mathbb{R}^N} \exp\left(-m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) \exp(-(n+1)\omega(x)) |\partial^\alpha f(x)(\phi(\epsilon x) - 1)| \rightarrow 0, \text{ as } \epsilon \rightarrow 0^+.$$

But, $\phi(\epsilon x) - 1 = 0$ whenever $|x| \leq \frac{1}{\epsilon}$. Accordingly, we have

$$\begin{aligned} & \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in \mathbb{R}^N} \exp\left(-m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) \exp(-(n+1)\omega(x)) |\partial^\alpha f(x)(\phi(\epsilon x) - 1)| \\ &= \sup_{\alpha \in \mathbb{N}_0^N} \sup_{|x| > \frac{1}{\epsilon}} \exp\left(-m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) \exp(-(n+1)\omega(x)) |\partial^\alpha f(x)(\phi(\epsilon x) - 1)| \\ &\leq r_{m,n}(f) \sup_{|x| > \frac{1}{\epsilon}} \exp(-\omega(x)). \end{aligned}$$

Since $\sup_{|x| > \frac{1}{\epsilon}} \exp(-\omega(x)) \rightarrow 0$ as $\epsilon \rightarrow 0^+$, the claim is proved.

From the arbitrariness of $m \in \mathbb{N}$, we can conclude that $\phi_\epsilon \rightarrow f$ in $\bigcap_{m=1}^\infty \mathcal{O}_{n+1,\omega}^m(\mathbb{R}^N)$ as $\epsilon \rightarrow 0^+$ and hence in $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$, taking into account that $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ is the inductive limit of the Fréchet spaces $\{\bigcap_{m=1}^\infty \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)\}_{n \in \mathbb{N}}$.

We now prove that $\mathcal{S}_\omega(\mathbb{R}^N)$ is a dense subspace of $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$, thereby obtaining by Theorem 3.8 that $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ is also a dense subspace of $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$. Since $\mathcal{D}_\omega(\mathbb{R}^N) \subseteq \mathcal{S}_\omega(\mathbb{R}^N)$, it suffices to show only that $\mathcal{D}_\omega(\mathbb{R}^N)$ is a dense subspace of $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$. So, fix $f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and $\phi \in \mathcal{D}_\omega(\mathbb{R}^N)$ such that $\phi \equiv 1$ on $\overline{B_1(0)}$ and $0 \leq \phi \leq 1$. Then for every $\epsilon > 0$ the function $\phi_\epsilon(x) := \phi(\epsilon x)f(x)$, for $x \in \mathbb{R}^N$, belongs to $\mathcal{D}_\omega(\mathbb{R}^N)$ because $\mathcal{O}_{M,\omega}(\mathbb{R}^N) \subseteq \mathcal{E}_\omega(\mathbb{R}^N)$. To show that $\phi_\epsilon \rightarrow f$ in $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$, we proceed as follows.

Fix $m \in \mathbb{N}$ and let $M \in \mathbb{N}$ such that $M \geq mL$, where $L \geq 1$ is the constant appearing in formula (2.2). Since $f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$, there exist $n(m), n(M) \in \mathbb{N}$ such that $f \in \mathcal{O}_{n(m),\omega}^m(\mathbb{R}^N)$ and $f \in \mathcal{O}_{n(M),\omega}^M(\mathbb{R}^N)$. Clearly, this implies that $f \in \mathcal{O}_{h,\omega}^m(\mathbb{R}^N) \cap \mathcal{O}_{h,\omega}^M(\mathbb{R}^N)$ for every $h \geq n := \max\{n(M), n(m)\}$. Therefore, we can proceed as above to show that for every $\epsilon > 0$ we have

$$\begin{aligned} r_{m,n+1}(\phi_\epsilon - f) &\leq \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in \mathbb{R}^N} \exp\left(-m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) \exp(-(n+1)\omega(x)) \\ &\quad \times |\partial^\alpha f(x)(\phi(\epsilon x) - 1)| + \epsilon Cr_{M,n+1}(f)r_{M,0}(\phi). \end{aligned}$$

Since $f \in \mathcal{O}_{n+1,\omega}^M(\mathbb{R}^N)$ and $\phi \in \mathcal{D}_\omega(\mathbb{R}^N)$, we have $r_{M,n+1}(f)r_{M,0}(\phi) < \infty$ and hence, $\epsilon Cr_{M,n+1}(f)r_{M,0}(\phi) \rightarrow 0$ as $\epsilon \rightarrow 0^+$. In order to conclude the proof, it then remains to prove that

$$\sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in \mathbb{R}^N} \exp\left(-m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) \exp(-(n+1)\omega(x)) |\partial^\alpha f(x)(\phi(\epsilon x) - 1)| \rightarrow 0, \text{ as } \epsilon \rightarrow 0^+.$$

As before, we have $\phi(\epsilon x) - 1 = 0$ whenever $|x| \leq \frac{1}{\epsilon}$. Therefore, for every $\epsilon > 0$ we have

$$\begin{aligned} & \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in \mathbb{R}^N} \exp\left(-m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) \exp(-(n+1)\omega(x)) |\partial^\alpha f(x)(\phi(\epsilon x) - 1)| \\ & \leq r_{m,n}(f) \sup_{|x| > \frac{1}{\epsilon}} \exp(-\omega(x)), \end{aligned}$$

where $\sup_{|x| > \frac{1}{\epsilon}} \exp(-\omega(x)) \rightarrow 0$ as $\epsilon \rightarrow 0^+$ and $r_{m,n}(f) < \infty$. This means that $\phi_\epsilon \rightarrow f$ in the Banach space $\mathcal{O}_{n+1,\omega}^m(\mathbb{R}^N)$ as $\epsilon \rightarrow 0^+$, and hence in the (LB)-space $\bigcup_{h=1}^\infty \mathcal{O}_{h,\omega}^m(\mathbb{R}^N)$. Since $m \in \mathbb{N}$ is arbitrary and $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ is the projective limit of the (LB)-spaces $\left\{ \bigcup_{h=1}^\infty \mathcal{O}_{h,\omega}^m(\mathbb{R}^N) \right\}_{m \in \mathbb{N}}$, we can conclude that $\phi_\epsilon \rightarrow f$ in the space $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ as $\epsilon \rightarrow 0^+$. This completes the proof. $\square$

4 $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ is the space of multipliers of the spaces $\mathcal{S}_\omega(\mathbb{R}^N)$ and $\mathcal{S}'_\omega(\mathbb{R}^N)$

The main aim of this section is to prove that $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ is the space of multipliers of both the spaces $\mathcal{S}_\omega(\mathbb{R}^N)$ and $\mathcal{S}'_\omega(\mathbb{R}^N)$. In order to do this, we first show some preliminary facts.

Lemma 4.1 *Let ω be a non-quasianalytic weight function and $f \in C^\infty(\mathbb{R}^N)$. If $fg \in \mathcal{S}_\omega(\mathbb{R}^N)$ for every $g \in \mathcal{S}_\omega(\mathbb{R}^N)$, then $f \in \mathcal{E}_\omega(\mathbb{R}^N)$.*

Proof Fix a compact subset K of $\mathbb{R}^N$ and $g \in \mathcal{D}_\omega(\mathbb{R}^N) \subseteq \mathcal{S}_\omega(\mathbb{R}^N)$ such that $g \equiv 1$ on K and $0 \leq g \leq 1$. Then $fg \in \mathcal{S}_\omega(\mathbb{R}^N)$ and so, for every $m \in \mathbb{N}$ we have

$$q_{m,1}(fg) = \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in \mathbb{R}^N} |\partial^\alpha (fg)(x)| \exp\left(-m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) \exp(\omega(x)) < \infty.$$

Since $g \equiv 1$ on K , it follows that

$$\begin{aligned} p_{K,m}(f) &= \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in K} |\partial^\alpha f(x)| \exp\left(-m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) \\ &\leq \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in \mathbb{R}^N} |\partial^\alpha (fg)(x)| \exp\left(-m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) \exp(\omega(x)) \\ &= q_{m,1}(fg) < \infty. \end{aligned}$$

Since K is arbitrary, we can conclude that $f \in \mathcal{E}_\omega(\mathbb{R}^N)$. $\square$

Lemma 4.2 *Let ω be a non-quasianalytic weight function. Let $\rho \in \mathcal{D}_\omega(\mathbb{R}^N)$ so that $\text{supp } \rho \subseteq B_1(0)$ and $\{x_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^N$ be a sequence such that $|x_j| \geq |x_{j-1}| + 2$ for every $j \geq 2$ and $|x_1| > 1$. If we set*

$$g(x) := \sum_{j \in \mathbb{N}} \rho(x - x_j) \exp(-j\omega(x_j)), \quad x \in \mathbb{R}^N, \quad (4.1)$$

then $g \in \mathcal{S}_\omega(\mathbb{R}^N)$.

Proof Since the functions of the sequence $\{\rho(\cdot - x_j)\}_{j \in \mathbb{N}}$ have disjoint supports, the series on the right of (4.1) converges in $C^\infty(\mathbb{R}^N)$ and so, the function g on the left of (4.1) belongs to $C^\infty(\mathbb{R}^N)$. On the other hand, the property (γ) of ω implies that $\exp(\omega(x)) \geq \exp(a)(1 + |x|)^b$ for every $x \in \mathbb{R}^N$. Therefore, $g \in \mathcal{S}(\mathbb{R}^N)$, see [13, Proposition 5, Chap. 4 §11]. In order to conclude the proof, we proceed as follows.

Fix $\lambda, \mu > 0$ and $x \in \mathbb{R}^N$. Then either $x \in B_1(x_l)$ for some $j \in \mathbb{N}$ or $x \notin B_1(x_l)$ for all $l \in \mathbb{N}$. Accordingly, we have for every $\alpha \in \mathbb{N}_0^N$ that either

$$\partial^\alpha g(x) = \exp(-j\omega(x_j)) \partial^\alpha \rho(x - x_j),$$

or $\partial^\alpha g(x) = 0$. Since $\rho \in \mathcal{D}_\omega(\mathbb{R}^N)$, it follows for every $\alpha \in \mathbb{N}_0^N$ that

$$\begin{aligned} \exp(\mu\omega(x)) |\partial^\alpha g(x)| &\leq \\ &\leq \exp(\mu\omega(x)) \exp(-j\omega(x_j)) \exp\left(-\mu K\omega(x - x_j) + \lambda\varphi_\omega^*\left(\frac{|\alpha|}{\lambda}\right)\right) q_{\lambda, K\mu}(\rho), \end{aligned} \quad (4.2)$$

where K is the constant appearing in the property (α) of ω . But, we have

$$\omega(x) = \omega((x - x_j) + x_j) \leq K(\omega(x - x_j) + \omega(x_j) + 1).$$

So, by (4.2) we obtain for every $\alpha \in \mathbb{N}_0^N$ that

$$\begin{aligned} \exp(\mu\omega(x)) |\partial^\alpha g(x)| &\leq e^K \exp(K\mu\omega(x - x_j) + K\mu\omega(x_j)) \\ &\quad \times \exp(-j\omega(x_j)) \exp\left(-\mu K\omega(x - x_j) + \lambda\varphi_\omega^*\left(\frac{|\alpha|}{\lambda}\right)\right) q_{\lambda, K\mu}(\rho) \\ &= e^K \exp((K\mu - j)\omega(x_j)) \exp\left(\lambda\varphi_\omega^*\left(\frac{|\alpha|}{\lambda}\right)\right) q_{\lambda, K\mu}(\rho) \end{aligned}$$

and hence,

$$\exp\left(-\lambda\varphi_\omega^*\left(\frac{|\alpha|}{\lambda}\right)\right) \exp(\mu\omega(x)) |\partial^\alpha g(x)| \leq e^K \exp((K\mu - j)\omega(x_j)) q_{\lambda, K\mu}(\rho).$$

Since $\exp((K\mu - j)\omega(x_j)) \leq 1$ whenever $j \geq K\mu$ and x is arbitrary, it follows that

$$q_{\lambda, \mu}(g) \leq e^K \max_{j < K\mu} \exp((K\mu - j)\omega(x_j)) q_{\lambda, K\mu}(\rho) < \infty.$$

But, λ and μ are also arbitrary. So, we can conclude that $g \in \mathcal{S}_\omega(\mathbb{R}^N)$. $\square$

Remark 4.3 Let ω be a non-quasianalytic weight function. Let $\rho \in \mathcal{D}_\omega(\mathbb{R}^N)$ so that $\text{supp } \rho \subseteq B_1(0)$ and $\{x_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^N$ be a sequence such that $|x_j| \geq |x_{j-1}| + 2$ for every $j \geq 2$ and $|x_1| > 1$. If $h \in \mathcal{S}_\omega(\mathbb{R}^N)$ and $g(x) := \sum_{j \in \mathbb{N}} h(x_j) \rho(x - x_j)$ for each $x \in \mathbb{R}^N$, then $g \in \mathcal{S}_\omega(\mathbb{R}^N)$. The proof follows by arguing as in the proof of Lemma 4.2, after having observed that $\sup_{x \in \mathbb{R}^N} |h(x)| \exp(j\omega(x)) < \infty$ for each $j \in \mathbb{N}$.

We can now state and prove that $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ is the space of multipliers of $\mathcal{S}_\omega(\mathbb{R}^N)$.

Theorem 4.4 *Let ω be a non-quasianalytic weight function and $f \in C^\infty(\mathbb{R}^N)$. Then the following properties are equivalent.*

- (1) $f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$.
- (2) For every $g \in \mathcal{S}_\omega(\mathbb{R}^N)$ we have $fg \in \mathcal{S}_\omega(\mathbb{R}^N)$.

Moreover, if $f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$, then the linear operator $M_f : \mathcal{S}_\omega(\mathbb{R}^N) \rightarrow \mathcal{S}_\omega(\mathbb{R}^N)$ defined by $M_f(g) := fg$, for $g \in \mathcal{S}_\omega(\mathbb{R}^N)$, is continuous.

Proof (1) $\Rightarrow$ (2). Fix $g \in \mathcal{S}_\omega(\mathbb{R}^N)$. Then $fg \in C^\infty(\mathbb{R}^N)$. Moreover, for every $\alpha \in \mathbb{N}_0^N$ and $x \in \mathbb{R}^N$ we have

$$|\partial^\alpha(fg)(x)| \leq \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} |\partial^\gamma f(x)| |\partial^{\alpha-\gamma} g(x)|. \quad (4.3)$$

Fixed $m \in \mathbb{N}$, let $M \in \mathbb{N}$ so that $M \geq Lm$, where $L \geq 1$ is the constant appearing in formula (2.2). Since $f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$, there exist $n \in \mathbb{N}$ and $C > 0$ such that for every $\gamma \in \mathbb{N}_0^N$ and $x \in \mathbb{R}^N$ we have

$$|\partial^\gamma f(x)| \leq C \exp\left(n\omega(x) + M\varphi_\omega^*\left(\frac{|\gamma|}{M}\right)\right). \quad (4.4)$$

On the other hand, $g \in \mathcal{S}_\omega(\mathbb{R}^N)$. So, setting $M' := \max\{M, n+m\}$ and recalling that $\varphi_\omega^*(t)/t$ is an increasing function in $(0, \infty)$, we have for every $\delta \in \mathbb{N}_0^N$ and $x \in \mathbb{R}^N$ that

$$\begin{aligned} |\partial^\delta g(x)| &\leq q_{M',M'}(g) \exp\left(M'\varphi_\omega^*\left(\frac{|\delta|}{M'}\right) - M'\omega(x)\right) \\ &\leq q_{M',M'}(g) \exp\left(M\varphi_\omega^*\left(\frac{|\delta|}{M}\right) - M'\omega(x)\right). \end{aligned} \quad (4.5)$$

By combining (4.3), (4.4) and (4.5), we obtain for every $\alpha \in \mathbb{N}_0^N$ and $x \in \mathbb{R}^N$ that

$$\begin{aligned} \exp(m\omega(x)) |\partial^\alpha(fg)(x)| &\leq \\ &\leq \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} C \exp((n+m)\omega(x)) \exp\left(M\varphi_\omega^*\left(\frac{|\gamma|}{M}\right)\right) |\partial^{\alpha-\gamma} g(x)| \end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \exp((n+m)\omega(x)) \exp\left(M\varphi_\omega^*\left(\frac{|\gamma|}{M}\right)\right) q_{M',M'}(g) \\
&\quad \times \exp\left(M\varphi_\omega^*\left(\frac{|\alpha-\gamma|}{M}\right) - M'\omega(x)\right) \\
&\leq C q_{M',M'}(g) \exp((n+m-M')\omega(x)) \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \exp\left(M\varphi_\omega^*\left(\frac{|\gamma|}{M}\right)\right) \times \\
&\quad \times \exp\left(M\varphi_\omega^*\left(\frac{|\alpha-\gamma|}{M}\right)\right).
\end{aligned}$$

Using inequality (2.7) and taking in mind that $\sup_{x \in \mathbb{R}^N} \exp((n+m-M')\omega(x)) < \infty$, it follows for every $\alpha \in \mathbb{N}_0^N$ and $x \in \mathbb{R}^N$ that

$$\exp(m\omega(x)) |\partial^\alpha (fg)(x)| \leq C q_{M',M'}(g) 2^{|\alpha|} \exp\left(M\varphi_\omega^*\left(\frac{|\alpha|}{M}\right)\right). \quad (4.6)$$

Since $M \geq mL$, applying formula (2.6) we obtain via (4.6) that for every $\alpha \in \mathbb{N}_0^N$ and $x \in \mathbb{R}^N$

$$\exp(m\omega(x)) |\partial^\alpha (fg)(x)| \leq C q_{M',M'}(g) \exp\left(m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right)$$

and so

$$q_{m,m}(fg) \leq C \exp(mL) q_{M',M'}(g) < \infty. \quad (4.7)$$

The thesis follows from the arbitrariness of $m \in \mathbb{N}$.

(2) $\Rightarrow$ (1). We first observe that by Lemma 4.1 we have $f \in \mathcal{E}_\omega(\mathbb{R}^N)$. We now suppose that $f \notin \mathcal{O}_{M,\omega}(\mathbb{R}^N)$. Then by Proposition 3.3(1) there exists $m \in \mathbb{N}$ such that for each $C, R > 0$ and $n \in \mathbb{N}$ there exist $x \in \mathbb{R}^N$ with $|x| > R$ and $\alpha \in \mathbb{N}_0^N$ such that

$$|\partial^\alpha f(x)| \geq C \exp\left(n\omega(x) + m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right).$$

So, we can choose two sequences $\{\alpha_j\}_{j \in \mathbb{N}} \subset \mathbb{N}_0^N$ and $\{x_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^N$ with $|x_{j+1}| > |x_j| + 2$ for all $j \in \mathbb{N}$ and $|x_1| > 1$ such that

$$|\partial^{\alpha_j} f(x_j)| \geq \exp\left(j\omega(x_j) + m\varphi_\omega^*\left(\frac{|\alpha_j|}{m}\right)\right). \quad (4.8)$$

Let $\rho \in \mathcal{D}_\omega(\mathbb{R}^N) \subset \mathcal{S}_\omega(\mathbb{R}^N)$ such that $\text{supp } \rho \subseteq B_1(0)$ and $\rho \equiv 1$ on $B_r(0)$ for some $0 < r < 1$. For every $x \in \mathbb{R}^N$ let

$$g(x) := \sum_{j \in \mathbb{N}} \rho(x - x_j) \exp(-j\omega(x_j)).$$

Then by Lemma 4.2 we have $g \in \mathcal{S}_\omega(\mathbb{R}^N)$ and so, $q_{m,m}(fg) < \infty$. In particular, we have that

$$\sup_{j \in \mathbb{N}} |\partial^{\alpha_j}(fg)(x_j)| \exp\left(m\omega(x_j) - m\varphi_\omega^*\left(\frac{|\alpha_j|}{m}\right)\right) \leq q_{m,m}(fg) < \infty. \quad (4.9)$$

On the other hand, for every $j \in \mathbb{N}$ we have $g \equiv \exp(-j\omega(x_j))$ in $\overline{B}_r(x_j)$. Accordingly, for every $j \in \mathbb{N}$, $x \in \overline{B}_r(x_j)$ and $\alpha \in \mathbb{N}_0^N$ we have

$$\partial^\alpha(fg)(x) = \exp(-j\omega(x_j))\partial^\alpha f(x).$$

Therefore, by (4.8) and (4.9) it follows that

$$\begin{aligned} q_{m,m}(fg) &\geq \sup_{j \in \mathbb{N}} |\partial^{\alpha_j} f(x_j)| \exp\left(m\omega(x_j) - m\varphi_\omega^*\left(\frac{|\alpha_j|}{m}\right)\right) \exp(-j\omega(x_j)) \\ &\geq \sup_{j \in \mathbb{N}} \exp(m\omega(x_j)), \end{aligned}$$

thereby implying that $\sup_{j \in \mathbb{N}} \exp(m\omega(x_j)) < \infty$. But, $\lim_{j \rightarrow \infty} \exp(m\omega(x_j)) = +\infty$ and hence, $\sup_{j \in \mathbb{N}} \exp(m\omega(x_j)) = \infty$. This is a contradiction.

Fix any $f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$. Then the operator $M_f: \mathcal{S}_\omega(\mathbb{R}^N) \rightarrow \mathcal{S}_\omega(\mathbb{R}^N)$ is well defined by the proof above. Actually, M_f is also continuous as follows directly from (4.7). $\square$

Remark 4.5 (a) Let $f \in \mathcal{S}_\omega(\mathbb{R}^N)$. Then $fg \in \mathcal{S}_\omega(\mathbb{R}^N)$ for every $g \in \mathcal{S}_\omega(\mathbb{R}^N)$. So, by Theorem 4.4 it follows that $f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$.

(b) Let $f \in \mathcal{E}_\omega(\mathbb{R}^N)$. Then $fg \in \mathcal{S}_\omega(\mathbb{R}^N)$ for every $g \in \mathcal{S}_\omega(\mathbb{R}^N)$ if and only if $(\partial^\alpha f)g \in \mathcal{S}_\omega(\mathbb{R}^N)$ for every $g \in \mathcal{S}_\omega(\mathbb{R}^N)$ and $\alpha \in \mathbb{N}_0^N$.

Indeed, the condition on the right clearly implies the condition on the left. Conversely, the assumption $fg \in \mathcal{S}_\omega(\mathbb{R}^N)$ for every $g \in \mathcal{S}_\omega(\mathbb{R}^N)$ implies for every $j = 1, \dots, N$ that

$$(\partial_j f)g = \partial_j(fg) - f(\partial_j g) \in \mathcal{S}_\omega(\mathbb{R}^N).$$

So, proceeding by induction the result follows.

Consequently, by Theorem 4.4 we can conclude that for fixed $f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and $\alpha \in \mathbb{N}_0^N$ the function $\partial^\alpha f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$ too, and the linear operator $M_{\partial^\alpha f}: \mathcal{S}_\omega(\mathbb{R}^N) \rightarrow \mathcal{S}_\omega(\mathbb{R}^N)$ is continuous.

Finally, we show that $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ is also the space of multipliers of $\mathcal{S}'_\omega(\mathbb{R}^N)$.

Theorem 4.6 *Let ω be a non-quasianalytic weight function and $f \in \mathcal{E}_\omega(\mathbb{R}^N)$. Then the following properties are equivalent.*

- (1) $f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$.
- (2) For every $T \in \mathcal{S}'_\omega(\mathbb{R}^N)$ we have $fT \in \mathcal{S}'_\omega(\mathbb{R}^N)$.

Moreover, if $f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$, then the linear operator $\mathcal{M}_f: \mathcal{S}'_\omega(\mathbb{R}^N) \rightarrow \mathcal{S}'_\omega(\mathbb{R}^N)$ defined by $\mathcal{M}_f(T) := fT$, for $T \in \mathcal{S}'_\omega(\mathbb{R}^N)$, is continuous.

Proof (1) $\Rightarrow$ (2). Fix $T \in \mathcal{S}'_\omega(\mathbb{R}^N)$. By Theorem 4.4 the linear operator $M_f: \mathcal{S}_\omega(\mathbb{R}^N) \rightarrow \mathcal{S}_\omega(\mathbb{R}^N)$ given by $M_f(g) = fg$, for $g \in \mathcal{S}_\omega(\mathbb{R}^N)$, is continuous and so, the linear functional $T \circ M_f: \mathcal{S}_\omega(\mathbb{R}^N) \rightarrow \mathbb{C}$ is also continuous. Accordingly, $T \circ M_f \in \mathcal{S}'_\omega(\mathbb{R}^N)$. But, for every $g \in \mathcal{S}_\omega(\mathbb{R}^N)$ we have

$$(T \circ M_f)(g) = \langle T, M_f(g) \rangle = \langle T, fg \rangle = \langle fT, g \rangle.$$

Therefore, $fT = T \circ M_f \in \mathcal{S}'_\omega(\mathbb{R}^N)$. This completes the proof.

(2) $\Rightarrow$ (1). Suppose for every $T \in \mathcal{S}'_\omega(\mathbb{R}^N)$ that $fT \in \mathcal{S}'_\omega(\mathbb{R}^N)$. Then for a fixed $g \in \mathcal{S}_\omega(\mathbb{R}^N)$ the linear functional $L: \mathcal{S}'_\omega(\mathbb{R}^N) \rightarrow \mathbb{C}$ defined by $L(T) := \langle fT, g \rangle$, for $T \in \mathcal{S}'_\omega(\mathbb{R}^N)$, is continuous and so, $L \in \mathcal{S}''_\omega(\mathbb{R}^N)$. Since $\mathcal{S}_\omega(\mathbb{R}^N)$ is reflexive, there exists $h \in \mathcal{S}_\omega(\mathbb{R}^N)$ such that $L(T) = \langle T, h \rangle$ for every $T \in \mathcal{S}'_\omega(\mathbb{R}^N)$, i.e., $\langle fT, g \rangle = \langle T, h \rangle$ for every $T \in \mathcal{S}'_\omega(\mathbb{R}^N)$. Accordingly, for every $T \in \mathcal{D}_\omega(\mathbb{R}^N) \subseteq \mathcal{S}'_\omega(\mathbb{R}^N)$ we have $\langle fT, g \rangle = \langle T, h \rangle$. Since for every $T \in \mathcal{D}_\omega(\mathbb{R}^N)$ we have $\langle fT, g \rangle = \langle T, fg \rangle$ and $\mathcal{D}_\omega(\mathbb{R}^N)$ is a dense subspace of $\mathcal{S}_\omega(\mathbb{R}^N)$, it follows that $fg = h \in \mathcal{S}_\omega(\mathbb{R}^N)$. So, as $g \in \mathcal{S}_\omega(\mathbb{R}^N)$ is arbitrary, by Theorem 4.4 we can conclude that $f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$.

Fix any $f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$. Then for every $g \in \mathcal{S}_\omega(\mathbb{R}^N)$ and $T \in \mathcal{S}'_\omega(\mathbb{R}^N)$ we have

$$\langle fT, g \rangle = \langle T, fg \rangle.$$

This means that the linear operator $\mathcal{M}_f: \mathcal{S}'_\omega(\mathbb{R}^N) \rightarrow \mathcal{S}'_\omega(\mathbb{R}^N)$ is the transpose of the continuous linear operator $M_f: \mathcal{S}_\omega(\mathbb{R}^N) \rightarrow \mathcal{S}_\omega(\mathbb{R}^N)$. Therefore, the linear operator $\mathcal{M}_f: \mathcal{S}'_\omega(\mathbb{R}^N) \rightarrow \mathcal{S}'_\omega(\mathbb{R}^N)$ is necessarily continuous. $\square$

5 Other topologies on $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$

In this section we show that the space $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ can be naturally endowed with other lc-topologies. We also compare these lc-topologies with each other and with the projective lc-topology defined on $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ by the spectrum $\{\cup_{n=1}^\infty \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)\}_{m \in \mathbb{N}}$.

We begin by giving another useful characterization of the space $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$.

Theorem 5.1 *Let ω be a non-quasianalytic weight function and $f \in C^\infty(\mathbb{R}^N)$. Then the following properties are equivalent.*

- (1) $f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$.
- (2) For every $g \in \mathcal{S}_\omega(\mathbb{R}^N)$ and $m \in \mathbb{N}$ we have

$$q_{m,g}(f) := \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in \mathbb{R}^N} |g(x)| |\partial^\alpha f(x)| \exp\left(-m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) < \infty. \quad (5.1)$$

Proof (1) $\Rightarrow$ (2). Since $f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$, we have for every $m \in \mathbb{N}$ that there exist $C > 0$ and $n \in \mathbb{N}$ such that for every $\alpha \in \mathbb{N}_0^N$ and $x \in \mathbb{R}^N$ the following inequality is

satisfied

$$|\partial^\alpha f(x)| \leq C \exp \left(n\omega(x) + m\varphi_\omega^* \left(\frac{|\alpha|}{m} \right) \right).$$

Therefore, for any fixed $g \in \mathcal{S}_\omega(\mathbb{R}^N)$, it follows for every $m \in \mathbb{N}$, $\alpha \in \mathbb{N}_0^N$ and $x \in \mathbb{R}^N$ that

$$|g(x)| |\partial^\alpha f(x)| \exp \left(-m\varphi_\omega^* \left(\frac{|\alpha|}{m} \right) \right) \leq C \exp(n\omega(x)) |g(x)|,$$

with $n \in \mathbb{N}$ depending only of m . This implies that

$$q_{m,g}(f) \leq C \sup_{x \in \mathbb{R}^N} \exp(n\omega(x)) |g(x)| < \infty.$$

Since $g \in \mathcal{S}_\omega(\mathbb{R}^N)$ is arbitrary, the thesis follows.

(2) $\Rightarrow$ (1). We first show that $f \in \mathcal{E}_\omega(\mathbb{R}^N)$. So, fixed any compact subset K of $\mathbb{R}^N$, let $g \in \mathcal{D}_\omega(\mathbb{R}^N)$ such that $g \equiv 1$ on K . Then it follows for every $m \in \mathbb{N}$ that

$$\begin{aligned} p_{K,m}(f) &= \sup_{x \in K} \sup_{\alpha \in \mathbb{N}_0^N} |\partial^\alpha f(x)| \exp \left(-m\varphi_\omega^* \left(\frac{|\alpha|}{m} \right) \right) \\ &\leq \sup_{x \in K} \sup_{\alpha \in \mathbb{N}_0^N} |g(x)| |\partial^\alpha f(x)| \exp \left(-m\varphi_\omega^* \left(\frac{|\alpha|}{m} \right) \right) \\ &\leq q_{m,g}(f) < \infty. \end{aligned} \tag{5.2}$$

Since K is an arbitrary compact subset of $\mathbb{R}^N$, we can conclude that $f \in \mathcal{E}_\omega(\mathbb{R}^N)$.

Suppose that $f \notin \mathcal{O}_{M,\omega}(\mathbb{R}^N)$. Since $f \in \mathcal{E}_\omega(\mathbb{R}^N)$, it follows from Proposition 3.3(1) that there exists $m \in \mathbb{N}$ such that for each $C, R > 0$ and $n \in \mathbb{N}$ there exist $x \in \mathbb{R}^N$ with $|x| > R$ and $\alpha \in \mathbb{N}_0^N$ such that

$$|\partial^\alpha f(x)| \geq C \exp \left(n\omega(x) + m\varphi_\omega^* \left(\frac{|\alpha|}{m} \right) \right).$$

So, we can choose two sequences $\{\alpha_j\}_{j \in \mathbb{N}} \subset \mathbb{N}_0^N$ and $\{x_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^N$ with $|x_{j+1}| > |x_j| + 2$ for all $j \in \mathbb{N}$ and $|x_1| > 1$ such that

$$|\partial^{\alpha_j} f(x_j)| \geq \exp \left(j\omega(x_j) + m\varphi_\omega^* \left(\frac{|\alpha_j|}{m} \right) \right). \tag{5.3}$$

Next, let $\rho \in \mathcal{D}_\omega(\mathbb{R}^N) \subset \mathcal{S}_\omega(\mathbb{R}^N)$ such that $\text{supp } \rho \subseteq B_1(0)$ and $\rho \equiv 1$ on $B_r(0)$ for some $0 < r < 1$. For every $x \in \mathbb{R}^N$ let

$$g(x) := \sum_{j \in \mathbb{N}} \rho(x - x_j) \exp(-j\omega(x_j)).$$

Then by Lemma 4.2 we have $g \in \mathcal{S}_\omega(\mathbb{R}^N)$ and hence, it necessarily holds that $q_{m,g}(f) < \infty$. In particular, we have that

$$\sup_{j \in \mathbb{N}} |g(x_j)| |\partial^{\alpha_j} f(x_j)| \exp \left(m\omega(x_j) - m\varphi_\omega^* \left(\frac{|\alpha_j|}{m} \right) \right) \leq q_{m,g}(f) < \infty. \quad (5.4)$$

But, $g(x_j) = \exp(-j\omega(x_j))$ for every $j \in \mathbb{N}$. So, it follows by (5.3) and (5.4) that

$$\begin{aligned} q_{m,g}(f) &\geq \sup_{j \in \mathbb{N}} \exp(-j\omega(x_j)) |\partial^{\alpha_j} f(x_j)| \exp \left(m\omega(x_j) - m\varphi_\omega^* \left(\frac{|\alpha_j|}{m} \right) \right) \\ &\geq \sup_{j \in \mathbb{N}} \exp(m\omega(x_j)), \end{aligned}$$

thereby implying that $\sup_{j \in \mathbb{N}} \exp(m\omega(x_j)) < \infty$. This is a contradiction because $\lim_{j \rightarrow \infty} \exp(m\omega(x_j)) = \infty$. Hence, f necessarily belongs to $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$. $\square$

Theorem 5.1 implies that the set $\{q_{m,g}\}_{m \in \mathbb{N}, g \in \mathcal{S}_\omega(\mathbb{R}^N)}$ forms a fundamental system of norms on $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$. Denote by τ the Hausdorff lc-topology on $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ generated by $\{q_{m,g}\}_{m \in \mathbb{N}, g \in \mathcal{S}_\omega(\mathbb{R}^N)}$.

We now collect some properties of the Hausdorff lc-space $(\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau)$.

Theorem 5.2 *Let ω be a non-quasianalytic weight function. Then the following properties are satisfied.*

(1) *The inclusion*

$$(\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau) \hookrightarrow \mathcal{E}_\omega(\mathbb{R}^N) \quad (5.5)$$

is continuous with dense range.

(2) *$(\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau)$ is a complete lc-space.*

Proof (1) The continuity of the inclusion $(\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau) \hookrightarrow \mathcal{E}_\omega(\mathbb{R}^N)$ follows by repeating the arguments at the beginning of the proof of Theorem 5.1 (2) $\Rightarrow$ (1), i.e., of (5.2). On the other hand, the facts that the inclusion $\mathcal{D}_\omega(\mathbb{R}^N) \hookrightarrow \mathcal{E}_\omega(\mathbb{R}^N)$ has dense range (see Remark 2.6(3)) and $\mathcal{D}_\omega(\mathbb{R}^N) \subseteq \mathcal{O}_{M,\omega}(\mathbb{R}^N)$ (see Remark 3.2(1)) clearly imply that the inclusion $(\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau) \hookrightarrow \mathcal{E}_\omega(\mathbb{R}^N)$ has dense range too.

(2) Let $\{f_i\}_{i \in I}$ be a Cauchy net in $(\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau)$. Since the inclusion $(\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau) \hookrightarrow C^\infty(\mathbb{R}^N)$ is continuous as it is easy to prove, it follows that $\{f_i\}_{i \in I}$ is also a Cauchy net in $C^\infty(\mathbb{R}^N)$. But, $C^\infty(\mathbb{R}^N)$ is a Fréchet space and hence, a complete lc-space. So, there exists $f \in C^\infty(\mathbb{R}^N)$ such that $f_i \rightarrow f$ in $C^\infty(\mathbb{R}^N)$. We claim that $f_i \rightarrow f$ in $(\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau)$. To see this, we fix $g \in \mathcal{S}_\omega(\mathbb{R}^N)$, $m \in \mathbb{N}$ and $\epsilon > 0$. Since $\{f_i\}_{i \in I}$ is a Cauchy net in $(\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau)$, there exists $i_0 \in I$ such that for every $i, i' \geq i_0$ we have

$$q_{m,g}(f_i - f_{i'}) = \sup_{x \in \mathbb{R}^N} \sup_{\alpha \in \mathbb{N}_0^n} |g(x)| |\partial^\alpha (f_i - f_{i'})(x)| \exp \left(-m\varphi_\omega^* \left(\frac{|\alpha|}{m} \right) \right) < \epsilon,$$

i.e., for every $i, i' \geq i_0, \alpha \in \mathbb{N}_0^N$ and $x \in \mathbb{R}^N$ we have

$$|g(x)| |\partial^\alpha (f_i - f_{i'})(x)| < \epsilon \exp \left(m \varphi_\omega^* \left(\frac{|\alpha|}{m} \right) \right).$$

Since $f_i \rightarrow f$ in $C^\infty(\mathbb{R}^N)$ implies that $\partial^\alpha f_i \rightarrow \partial^\alpha f$ pointwise in $\mathbb{R}^N$ for every $\alpha \in \mathbb{N}_0^N$, by letting i' to infinity it follows for every $i \geq i_0, \alpha \in \mathbb{N}_0^N$ and $x \in \mathbb{R}^N$ that

$$|g(x)| |\partial^\alpha (f_i - f)(x)| < \epsilon \exp \left(m \varphi_\omega^* \left(\frac{|\alpha|}{m} \right) \right)$$

and so,

$$\begin{aligned} |g(x)| |\partial^\alpha f(x)| &\leq |g(x)| |\partial^\alpha (f - f_{i_0})(x)| + |g(x)| |\partial^\alpha f_{i_0}(x)| \\ &< \epsilon \exp \left(m \varphi_\omega^* \left(\frac{|\alpha|}{m} \right) \right) + |g(x)| |\partial^\alpha f_{i_0}(x)|. \end{aligned}$$

Accordingly, we have for every $i \geq i_0$ that

$$q_{m,g}(f_i - f) \leq \epsilon \quad \text{and} \quad q_{m,g}(f) \leq \epsilon + q_{m,g}(f_{i_0}).$$

Since $g \in \mathcal{S}_\omega(\mathbb{R}^N)$ and $m \in \mathbb{N}$ are arbitrary, this shows via Theorem 5.1 that $f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and that $f_i \rightarrow f$ in $(\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau)$. $\square$

We recall that the sequence $\{\cup_{n=1}^\infty \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)\}_{m \in \mathbb{N}}$ of (LB)-spaces forms a projective spectrum and $\mathcal{O}_{M,\omega}(\mathbb{R}^N) = \cap_{m=1}^\infty \cup_{n=1}^\infty \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$. We denote by t the projective topology on the space $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ defined by $\{\cup_{n=1}^\infty \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)\}_{m \in \mathbb{N}}$. The next aim is to compare the topology τ with the topology t . To this end, we introduce the following spaces.

Definition 5.3 Let ω be a non-quasianalytic weight function. For $m \in \mathbb{N}$ we define the space

$$\mathcal{O}_{s_\omega}^m(\mathbb{R}^N) := \left\{ f \in C^\infty(\mathbb{R}^N) : \forall g \in \mathcal{S}_\omega(\mathbb{R}^N) \quad q_{m,g}(f) < \infty \right\} \quad (5.6)$$

and endow it with the lc-topology τ_m generated by the system of norms $\{q_{m,g}\}_{g \in \mathcal{S}_\omega(\mathbb{R}^N)}$.

It is straightforward to verify that the following topological equality holds

$$(\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau) = \cap_{m=1}^\infty (\mathcal{O}_{s_\omega}^m(\mathbb{R}^N), \tau_m), \quad (5.7)$$

when the space on the right hand side is endowed with the corresponding projective limit topology. Moreover, the following results hold.

Proposition 5.4 Let ω be a non-quasianalytic weight function and $m \in \mathbb{N}$. Then the following properties are satisfied.

- (1) The inclusion $\cup_{n=1}^{\infty} \mathcal{O}_{n,\omega}^m(\mathbb{R}^N) \hookrightarrow (\mathcal{O}_{s_\omega}^m(\mathbb{R}^N), \tau_m)$ is well-defined and continuous.
- (2) $\cup_{n=1}^{\infty} \mathcal{O}_{n,\omega}^m(\mathbb{R}^N) = \mathcal{O}_{s_\omega}^m(\mathbb{R}^N)$ algebraically. Moreover, the spaces $\cup_{n=1}^{\infty} \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$ and $(\mathcal{O}_{s_\omega}^m(\mathbb{R}^N), \tau_m)$ have the same bounded sets.
- (3) $(\mathcal{O}_{s_\omega}^m(\mathbb{R}^N), \tau_m)$ is a complete lc-space.

Proof (1) Fix $g \in \mathcal{S}_\omega(\mathbb{R}^N)$. Then for each $n \in \mathbb{N}$ there exists $c_n > 0$ such that for every $x \in \mathbb{R}^N$ we have

$$|g(x)| \leq c_n \exp(-n\omega(x)). \quad (5.8)$$

This implies for every $n \in \mathbb{N}$ that the inclusion

$$\mathcal{O}_{n,\omega}^m(\mathbb{R}^N) \hookrightarrow \mathcal{O}_{s_\omega}^m(\mathbb{R}^N)$$

is well-defined and continuous. Indeed, for a fixed $n \in \mathbb{N}$, we obtain via (5.8) that

$$\begin{aligned} q_{m,g}(f) &= \sup_{x \in \mathbb{R}^N} \sup_{\alpha \in \mathbb{N}_0^N} |g(x)| |\partial^\alpha f(x)| \exp\left(-m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) \\ &\leq c_n \sup_{x \in \mathbb{R}^N} \sup_{\alpha \in \mathbb{N}_0^N} |\partial^\alpha f(x)| \exp\left(-n\omega(x) - m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) = c_n r_{m,n}(f). \end{aligned}$$

Since $g \in \mathcal{S}_\omega(\mathbb{R}^N)$ is arbitrary, the continuity of the inclusion $\mathcal{O}_{n,\omega}^m(\mathbb{R}^N) \hookrightarrow \mathcal{O}_{s_\omega}^m(\mathbb{R}^N)$ follows.

Since $\cup_{n=1}^{\infty} \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$ is an (LB)-space, we deduce that the inclusion $\cup_{n=1}^{\infty} \mathcal{O}_{n,\omega}^m(\mathbb{R}^N) \hookrightarrow (\mathcal{O}_{s_\omega}^m(\mathbb{R}^N), \tau_m)$ is continuous.

(2) By (1) above it suffices to show only that every bounded subset of $(\mathcal{O}_{s_\omega}^m(\mathbb{R}^N), \tau_m)$ is also a bounded subset of $\cup_{n=1}^{\infty} \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$. To this end, we fix a bounded subset B of $(\mathcal{O}_{s_\omega}^m(\mathbb{R}^N), \tau_m)$. We would show that there exists $n_0 \in \mathbb{N}$ such that $\sup_{f \in B} r_{m,n_0}(f) < \infty$. If this is not the case, then $\sup_{f \in B} r_{m,n}(f) = \infty$ for every $n \in \mathbb{N}$. To get a contradiction, we proceed as follows.

For each $j \in \mathbb{N}$ let $K_j := \overline{B}_j(0) \times \{\alpha \in \mathbb{N}_0^N : |\alpha| \leq j\}$. Then $\cup_{j \in \mathbb{N}} K_j = \mathbb{R}^N \times \mathbb{N}_0^N$. On the other hand, taking in account that the function $g_0(x) := \exp(-|x|^2)$, for $x \in \mathbb{R}^N$, belongs to $\mathcal{S}_\omega(\mathbb{R}^N)$ we have for every $f \in B$ and $n, j \in \mathbb{N}$ that

$$\begin{aligned} &\sup_{(x,\alpha) \in K_j} |\partial^\alpha f(x)| \exp\left(-n\omega(x) - m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) \\ &\leq \sup_{(x,\alpha) \in K_j} \exp(|x|^2 - n\omega(x)) |\partial^\alpha f(x)| g_0(x) \exp\left(-m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) \leq k_j q_{m,g_0}(f), \end{aligned}$$

where $k_j := \sup_{(x,\alpha) \in K_j} \exp(|x|^2 - n\omega(x)) < \infty$, and hence

$$\sup_{f \in B} \sup_{(x,\alpha) \in K_j} |\partial^\alpha f(x)| \exp\left(-n\omega(x) - m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) < \infty.$$

Since $\sup_{f \in B} r_{m,n}(f) = \infty$, an argument by induction then yields that there exist a sequence $\{f_n\}_{n \in \mathbb{N}} \subseteq B$ and a strictly increasing sequence $\{j_n\}_{n \in \mathbb{N}}$ of positive integers such that for every $n \in \mathbb{N}$ there exists some point $(x_n, \alpha_n) \in \overset{\circ}{K}_{j_{n+1}} \setminus K_{j_{n-1}}$ for which

$$|\partial^{\alpha_n} f_n(x_n)| \exp\left(-n\omega(x_n) - m\varphi_\omega^*\left(\frac{|\alpha_n|}{m}\right)\right) > n. \quad (5.9)$$

Now, we choose a function $\rho \in \mathcal{D}_\omega(\mathbb{R}^N)$ such that $\text{supp } \rho \subseteq B_1(0)$ and $\rho(0) = 1$. We define

$$g(x) := \sum_{k=1}^{\infty} \exp(-k\omega(x)) \rho(x - x_k), \quad x \in \mathbb{R}^N. \quad (5.10)$$

Then by Lemma 4.2 we have $g \in \mathcal{S}_\omega(\mathbb{R}^N)$. In particular, $g(x_n) = \exp(-n\omega(x_n))$ for every $n \in \mathbb{N}$. Hence, by (5.9) it follows for every $n \in \mathbb{N}$ that

$$\begin{aligned} q_{m,g}(f_n) &\geq g(x_n) |\partial^{\alpha_n} f_n(x_n)| \exp\left(-m\varphi_\omega^*\left(\frac{|\alpha_n|}{m}\right)\right) \\ &\geq \exp(-n\omega(x_n)) |\partial^{\alpha_n} f_n(x_n)| \exp\left(-m\varphi_\omega^*\left(\frac{|\alpha_n|}{m}\right)\right) > n. \end{aligned}$$

This shows that $\sup_{f \in B} q_{m,g}(f) = \infty$, which is a contradiction as B is a bounded subset of $(\mathcal{O}_{s_\omega}^m(\mathbb{R}^N), \tau)$ and hence $\sup_{f \in B} q_{m,g}(f) < \infty$.

(3) follows as in the proof of Theorem 5.2(2). $\square$

Let $X = \text{ind}_{n \rightarrow} X_n$ be an (LB)-space with canonical inclusions $j_n: X_n \rightarrow X$ for each $n \in \mathbb{N}$. Recall that X is called *regular* if every bounded subset of X is contained and bounded in a step X_m for some $m \in \mathbb{N}$. Every complete (LB)-space is regular, [17, (5) p.225]. Accordingly, Proposition 3.5 implies that the space $\bigcup_{n=1}^{\infty} \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$ is a regular (LB)-space for each $m \in \mathbb{N}$. On the other hand, Proposition 5.4 yields another proof of the regularity of the (LB)-spaces $\bigcup_{n=1}^{\infty} \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$.

Proposition 5.5 *Let ω be a non-quasianalytic weight function and $m \in \mathbb{N}$. Then $\bigcup_{n=1}^{\infty} \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$ is a regular (LB)-space. Moreover, $\bigcup_{n=1}^{\infty} \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$ is the bornological space associated with the space $(\mathcal{O}_{s_\omega}^m(\mathbb{R}^N), \tau_m)$.*

Proof The result follows from Proposition 5.4(2). Indeed, in the proof of Proposition 5.4(2) it has been established also that every bounded subset of $(\mathcal{O}_{s_\omega}^m(\mathbb{R}^N), \tau_m)$ is contained and bounded in the Banach space $\mathcal{O}_{n,\omega}^m(\mathbb{R}^N)$ for some $n \in \mathbb{N}$. $\square$

Further immediate consequences of Proposition 5.4 are the following results.

Proposition 5.6 *Let ω be a non-quasianalytic weight function. Then the inclusion*

$$(\mathcal{O}_{M,\omega}(\mathbb{R}^N), t) \hookrightarrow (\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau) \quad (5.11)$$

is continuous, i.e., the topology τ is coarser than the topology t . Moreover, the spaces $(\mathcal{O}_{M,\omega}(\mathbb{R}^N), t)$ and $(\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau)$ have the same bounded subsets.

Proof Since

$$\begin{aligned} (\mathcal{O}_{M,\omega}(\mathbb{R}^N), t) &= \text{proj}_{m \leftarrow \bigcup_{n=1}^{\infty} \mathcal{O}_{n,\omega}^m(\mathbb{R}^N)} \\ \text{and } (\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau) &= \text{proj}_{m \leftarrow (\mathcal{O}_{s_\omega}^m(\mathbb{R}^N), \tau_m)}, \end{aligned}$$

the result follows immediately from Proposition 5.4(1)-(2). $\square$

Proposition 5.7 *Let ω be a non-quasianalytic weight function. Then the inclusion*

$$\mathcal{S}_\omega(\mathbb{R}^N) \hookrightarrow (\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau) \quad (5.12)$$

is continuous with dense range.

Proof The result immediately follows from Proposition 5.6, Theorems 3.8 and 3.9. $\square$

Let X be a Hausdorff lc-space and Γ_X be a system of continuous seminorms generating the topology of X . Then the strong operator topology τ_s in the space $\mathcal{L}(X)$ of all continuous linear operators from X into itself is determined by the family of seminorms $q_x(S) := q(Sx)$ ($S \in \mathcal{L}(X)$) for each $x \in X$ and $q \in \Gamma_X$. In such a case we write $\mathcal{L}_s(X)$. Denoted by $\mathcal{B}(X)$ the collection of all bounded subsets of X , the topology τ_b of uniform convergence on bounded sets is defined in $\mathcal{L}(X)$ by the seminorms $q_B(S) := \sup_{x \in B} q(Sx)$ ($S \in \mathcal{L}(X)$) for each $B \in \mathcal{B}(X)$ and $q \in \Gamma_X$. In such a case we write $\mathcal{L}_b(X)$.

By Theorem 4.4 the space $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ can be identified with the space $\mathcal{M}(\mathcal{S}_\omega(\mathbb{R}^N))$ of all multipliers on $\mathcal{S}_\omega(\mathbb{R}^N)$ via the map $M: \mathcal{O}_{M,\omega}(\mathbb{R}^N) \rightarrow \mathcal{M}(\mathcal{S}_\omega(\mathbb{R}^N))$ defined by $M(f) := M_f$ for each $f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$. Since $\mathcal{M}(\mathcal{S}_\omega(\mathbb{R}^N))$ is a subspace of $\mathcal{L}(\mathcal{S}_\omega(\mathbb{R}^N))$, the space $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ (via the map M) can be then endowed with either the topology τ_b induced by $\mathcal{L}_b(\mathcal{S}_\omega(\mathbb{R}^N))$ or the topology τ_s induced by $\mathcal{L}_s(\mathcal{S}_\omega(\mathbb{R}^N))$. In the next result we compare the three topologies τ_b , τ_s and τ .

To this end, we first show the following variant of Lemma 4.2.

Lemma 5.8 *Let ω be a non-quasianalytic weight function. Let $h: \mathbb{R}^N \rightarrow \mathbb{R}$ be a non-negative function satisfying the condition*

$$\forall \lambda > 0 \quad \lim_{|x| \rightarrow \infty} \exp(\lambda \omega(x)) h(x) = 0. \quad (5.13)$$

Then there exists $g \in \mathcal{S}_\omega(\mathbb{R}^N)$ such that

$$\forall x \in \mathbb{R}^N \quad h(x) \leq g(x). \quad (5.14)$$

Proof Let $\rho \in \mathcal{D}_\omega(\mathbb{R}^N)$ such that $\rho \geq 0$, $\rho \equiv 1$ on $B_1(0)$ and $\text{supp } \rho \subseteq B_1(0)$. Let $\{x_j\}_{j \in \mathbb{N}} \subseteq \mathbb{R}^N$ be a sequence satisfying the following properties: $\lim_{j \rightarrow \infty} |x_j| = \infty$; there exists $H \in \mathbb{N}$ such that $\{|j \in \mathbb{N}: x \in B_2(x_j)\}| \leq H$ for every $x \in \mathbb{R}^N$; for any $x \in \mathbb{R}^N$ there exists $j \in \mathbb{N}$ such that $x \in B_1(x_j)$.

Let

$$a_j := \sup_{x \in \overline{B}_2(x_j)} h(x), \quad j \in \mathbb{N}, \quad (5.15)$$

and

$$g(x) := \sum_{j \in \mathbb{N}} a_j \rho(x - x_j), \quad x \in \mathbb{R}^N, \quad (5.16)$$

where the series on the right of (5.16) is a finite sum for every $x \in \mathbb{R}^N$.

For any fixed $x \in \mathbb{R}^N$, let $j \in \mathbb{N}$ such that $x \in B_1(x_j)$. Then $g(x) \geq a_j \geq h(x)$. This shows that $h(x) \leq g(x)$ for all $x \in \mathbb{R}^N$. So, it remains to establish that $g \in \mathcal{S}_\omega(\mathbb{R}^N)$.

Since the property (γ) of ω implies that $\exp(\omega(x)) \geq \exp(a)(1 + |x|)^b$ for every $x \in \mathbb{R}^N$, by [9, Lemma 3.6, p.127] we can conclude that $g \in \mathcal{S}(\mathbb{R}^N)$. In order to conclude the proof, we proceed as follows.

Fix $\lambda, \mu > 0$ and $x \in \mathbb{R}^N$. Then we have for every $\alpha \in \mathbb{N}_0^N$ that

$$\partial^\alpha g(x) = \sum_{x \in \overline{B}_2(x_j)} a_j \partial^\alpha \rho(x - x_j),$$

where the set $J(x) := \{j \in \mathbb{N} : x \in \overline{B}_2(x_j)\}$ has cardinality less or equal to that of H , with H independent of x . Since $\rho \in \mathcal{D}_\omega(\mathbb{R}^N)$, it follows for every $\alpha \in \mathbb{N}_0^N$ that

$$\begin{aligned} \exp(\mu\omega(x)) |\partial^\alpha g(x)| &\leq \\ &\leq H \sup_{j \in J(x)} a_j \exp(\mu\omega(x)) \exp\left(-\mu K\omega(x - x_j) + \lambda\varphi_\omega^*\left(\frac{|\alpha|}{\lambda}\right)\right) q_{\lambda, \mu K}(\rho), \end{aligned} \quad (5.17)$$

where K is the constant appearing in the property (α) of ω . On the other hand, by (5.13) there exists $C > 0$ such that $h(x) \leq C \exp(-\mu K^2 \omega(x))$ for all $x \in \mathbb{R}^N$ and so, we have for every $j \in \mathbb{N}$ that

$$a_j = \sup_{x \in \overline{B}_2(x_j)} h(x) = \sup_{y \in \overline{B}_2(0)} h(y + x_j) \leq C \sup_{y \in \overline{B}_2(0)} \exp(-\mu K^2 \omega(y + x_j)). \quad (5.18)$$

But, the following inequalities are satisfied

$$\omega(x) = \omega((x - x_j) + x_j) \leq K(\omega(x - x_j) + \omega(x_j) + 1)$$

and

$$\omega(x_j) = \omega((y + x_j) - y) \leq K(\omega(y + x_j) + \omega(y) + 1).$$

Accordingly, by (5.17) and (5.18) it follows for every $\alpha \in \mathbb{N}_0^N$ that

$$\begin{aligned} \exp(\mu\omega(x))|\partial^\alpha g(x)| &\leq HC \sup_{j \in J(x)} \sup_{y \in \overline{B}_2(0)} \exp(K^2\mu(\omega(y) + 1)) \exp(-K\mu\omega(x_j)) \\ &\quad \times e^{\mu K} \exp(\mu K\omega(x - x_j)) \exp(\mu K\omega(x_j)) \exp(-\mu K\omega(x - x_j)) \\ &\quad + \lambda \varphi_\omega^* \left(\frac{|\alpha|}{\lambda} \right) q_{\lambda, \mu K}(\rho) \\ &= HC e^{\mu K} \sup_{y \in \overline{B}_2(0)} \exp(K^2\mu(\omega(y) + 1)) \exp \left(\lambda \varphi_\omega^* \left(\frac{|\alpha|}{\lambda} \right) \right) q_{\lambda, \mu K}(\rho) \end{aligned}$$

and hence,

$$\exp \left(-\lambda \varphi_\omega^* \left(\frac{|\alpha|}{\lambda} \right) \right) \exp(\mu\omega(x))|\partial^\alpha g(x)| \leq HC e^{\mu K} \sup_{y \in \overline{B}_2(0)} \exp(K^2\mu(\omega(y) + 1)) q_{\lambda, \mu K}(\rho).$$

Since $D := \sup_{y \in \overline{B}_2(0)} \exp(K^2\mu(\omega(y) + 1)) < \infty$ is a constant independent of x and x is arbitrary, we can conclude that

$$q_{\lambda, \mu}(g) \leq HC D e^{\mu K} q_{\lambda, \mu K}(\rho) < \infty.$$

But, λ and μ are also arbitrary. So, this implies that $g \in \mathcal{S}_\omega(\mathbb{R}^N)$. $\square$

Theorem 5.9 *Let ω be a non-quasianalytic weight function. Then the inclusions*

$$(\mathcal{O}_{M, \omega}(\mathbb{R}^N), \tau) \hookrightarrow (\mathcal{O}_{M, \omega}(\mathbb{R}^N), \tau_b) \hookrightarrow (\mathcal{O}_{M, \omega}(\mathbb{R}^N), \tau_s) \quad (5.19)$$

are continuous. Moreover, the spaces $(\mathcal{O}_{M, \omega}(\mathbb{R}^N), \tau)$, $(\mathcal{O}_{M, \omega}(\mathbb{R}^N), \tau_b)$ and $(\mathcal{O}_{M, \omega}(\mathbb{R}^N), \tau_s)$ have the same bounded subsets.

Proof Since $\tau_s \subseteq \tau_b$, it suffices to show that $\tau_b \subseteq \tau$. To this end, let W be a 0-neighbourhood of $\mathcal{L}_b(\mathcal{S}_\omega(\mathbb{R}^N))$. Then there exist a 0-neighbourhood V of $\mathcal{S}_\omega(\mathbb{R}^N)$ and a bounded subset B of $\mathcal{S}_\omega(\mathbb{R}^N)$ such that

$$\{T \in \mathcal{L}(\mathcal{S}_\omega(\mathbb{R}^N)) : T(B) \subseteq V\} \subseteq W.$$

We can suppose that $V = \{h \in \mathcal{S}_\omega(\mathbb{R}^N) : q_{m, n}(f) \leq \epsilon\}$ for some $m, n \in \mathbb{N}$ and $\epsilon > 0$. To conclude the proof, we have to show that there exists a 0-neighbourhood U of $(\mathcal{O}_{M, \omega}(\mathbb{R}^N), \tau)$ such that $fg \in V$ for all $f \in U$ and $g \in B$. To this end, let $M \in \mathbb{N}$ such that $M \geq Lm$ and $\epsilon' > 0$ such that $\epsilon' < \epsilon e^{-mL}$, where $L \geq 1$ is the constant appearing in formula (2.5). Then the set U of all functions $f \in \mathcal{O}_{M, \omega}(\mathbb{R}^N)$ satisfying the condition

$$\sup_{x \in \mathbb{R}^N} \sup_{\gamma, \delta \in \mathbb{N}_0^N} \exp \left(n\omega(x) - M\varphi_\omega^* \left(\frac{|\gamma + \delta|}{M} \right) \right) |\partial^\gamma f(x)| |\partial^\delta g(x)| \leq \epsilon', \quad g \in B, \quad (5.20)$$

is a 0-neighbourhood of $(\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau)$ for which $fg \in V$ for all $f \in U$ and $g \in B$. Indeed, for fixed $f \in U$ and $g \in B$, we have for every $\alpha \in \mathbb{N}_0^N$ and $x \in \mathbb{R}^N$ that

$$\begin{aligned} |\partial^\alpha(fg)(x)| &\leq \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} |\partial^\gamma f(x)| |\partial^{\alpha-\gamma} g(x)| \leq \epsilon' \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \exp\left(-n\omega(x) + M\varphi_\omega^*\left(\frac{|\alpha|}{M}\right)\right) \\ &\leq \epsilon' 2^{|\alpha|} \exp(-n\omega(x)) \exp\left(M\varphi_\omega^*\left(\frac{|\alpha|}{M}\right)\right) \\ &\leq \epsilon' e^{mL} \exp(-n\omega(x)) \exp\left(m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) \\ &\leq \epsilon \exp\left(-n\omega(x) + m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right). \end{aligned}$$

after having used inequality (2.6). Therefore, $q_{m,n}(fg) \leq \epsilon$ and so $fg \in V$.

It remains to show that U is a 0-neighbourhood of $(\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau)$. To see this, we define

$$h(x) := \exp(n\omega(x)) \sup_{g \in B} \sup_{\delta \in \mathbb{N}_0^N} \exp\left(-M\varphi_\omega^*\left(\frac{|\delta|}{M}\right)\right) |\partial^\delta g(x)|, \quad x \in \mathbb{R}^N.$$

Since B a bounded subset of $\mathcal{S}_\omega(\mathbb{R}^N)$, we have for every $\lambda \geq 0$ and $x \in \mathbb{R}^N$ that

$$\begin{aligned} \exp(\lambda\omega(x))h(x) &\leq \exp((\lambda+n)\omega(x)) \sup_{g \in B} \sup_{\delta \in \mathbb{N}_0^N} \exp\left(-M\varphi_\omega^*\left(\frac{|\delta|}{M}\right)\right) |\partial^\delta g(x)| \\ &\leq \sup_{g \in B} q_{M,\lambda+n}(g) < \infty. \end{aligned}$$

Accordingly, h is a well-defined non-negative function on $\mathbb{R}^N$ satisfying condition (5.13). So, by Lemma 5.8 there exists $g_1 \in \mathcal{S}_\omega(\mathbb{R}^N)$ such that $h(x) \leq g_1(x)$ for all $x \in \mathbb{R}^N$. Then $U' := \{f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N) : q_{M,g_1}(f) \leq \epsilon'\}$ is a 0-neighbourhood of $(\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau)$. Moreover, for fixed $f \in U'$ and $g \in B$, we have for every $\gamma, \delta \in \mathbb{N}_0^N$ that

$$\begin{aligned} |\partial^\gamma f(x)| |\partial^\delta g(x)| &\leq |\partial^\gamma f(x)| g_1(x) \exp(-n\omega(x)) \exp\left(M\varphi_\omega^*\left(\frac{|\delta|}{M}\right)\right) \\ &\leq \epsilon' \exp\left(M\varphi_\omega^*\left(\frac{|\gamma|}{M}\right)\right) \exp(-n\omega(x)) \exp\left(M\varphi_\omega^*\left(\frac{|\delta|}{M}\right)\right) \\ &= \epsilon' \exp(-n\omega(x)) \exp\left(M\varphi_\omega^*\left(\frac{|\gamma|}{M}\right) + M\varphi_\omega^*\left(\frac{|\delta|}{M}\right)\right) \\ &\leq \epsilon' \exp(-n\omega(x)) \exp\left(M\varphi_\omega^*\left(\frac{|\gamma+\delta|}{M}\right)\right). \end{aligned}$$

Since $f \in U'$ and $g \in B$ are arbitrary, this implies that $U' \subseteq U$. So, U is a 0-neighbourhood of $(\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau)$.

In order to show that the spaces $(\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau)$, $(\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau_b)$ and $(\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau_s)$ have the same bounded sets, it suffices to prove only that every bounded subset of $(\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau_s)$ is also a bounded subset of $(\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau)$. To this end, we fix a bounded subset B of $(\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau_s)$ and suppose that B is not a bounded subset of $(\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau)$. Then there exist $g \in \mathcal{S}_\omega(\mathbb{R}^N)$ and $m \in \mathbb{N}$ such that

$$\sup_{f \in B} q_{m,g}(f) = \infty. \quad (5.21)$$

To get a contradiction, we first observe that the inclusion $(\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau_s) \hookrightarrow \mathcal{E}_\omega(\mathbb{R}^N)$ is also continuous. Indeed, fixed any compact subset K of $\mathbb{R}^N$, let $h \in \mathcal{D}_\omega(\mathbb{R}^N)$ such that $h \equiv 1$ on K . Then $\partial^\alpha(fh) = \partial^\alpha f$ on K for each $f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and $\alpha \in \mathbb{N}_0^N$. This implies for every $f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and $l \in \mathbb{N}$ that

$$\begin{aligned} p_{K,m}(f) &= \sup_{x \in K} \sup_{\alpha \in \mathbb{N}_0^N} |\partial^\alpha f(x)| \exp \left(-l\varphi_\omega^* \left(\frac{|\alpha|}{l} \right) \right) \\ &= \sup_{x \in K} \sup_{\alpha \in \mathbb{N}_0^N} |\partial^\alpha (fh)(x)| \exp \left(\omega(x) - l\varphi_\omega^* \left(\frac{|\alpha|}{l} \right) \right) \exp(-\omega(x)) \\ &\leq C_K q_{l,1}(fh), \end{aligned} \quad (5.22)$$

where $C_K := \sup_{x \in K} \exp(-\omega(x)) < \infty$ is a positive constant depending only on K . Accordingly, the inclusion $(\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau_s) \hookrightarrow \mathcal{E}_\omega(\mathbb{R}^N)$ is continuous.

Let $K_j := \overline{B}_j(0) \times \{\alpha \in \mathbb{N}_0^N : |\alpha| \leq j\}$ for each $j \in \mathbb{N}$ (so, $\cup_{j \in \mathbb{N}} K_j = \mathbb{R}^N \times \mathbb{N}_0^N$). The continuity of the inclusion $(\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau_s) \hookrightarrow \mathcal{E}_\omega(\mathbb{R}^N)$ implies for every $f \in B$ and $n, j \in \mathbb{N}$ that

$$\begin{aligned} \sup_{(x,\alpha) \in K_j} |g(x)| |\partial^\alpha f(x)| \exp \left(-m\varphi_\omega^* \left(\frac{|\alpha|}{m} \right) \right) &\leq \\ &\leq k_j \sup_{(x,\alpha) \in K_j} |\partial^\alpha f(x)| \exp \left(-m\varphi_\omega^* \left(\frac{|\alpha|}{m} \right) \right) \leq k_j p_{\overline{B}_j(0),m}(f), \end{aligned}$$

where $k_j := \sup_{x \in \overline{B}_j(0)} |g(x)| < \infty$, and hence

$$\sup_{f \in B} \sup_{(x,\alpha) \in K_j} |g(x)| |\partial^\alpha f(x)| \exp \left(-m\varphi_\omega^* \left(\frac{|\alpha|}{m} \right) \right) < \infty.$$

Taking in mind (5.21), we can then argue by induction to find a sequence $\{f_n\}_{n \in \mathbb{N}} \subseteq B$ and a strictly increasing sequence $\{j_n\}_{n \in \mathbb{N}}$ of positive integers such that for every $n \in \mathbb{N}$ there exists some point $(x_n, \alpha_n) \in \overset{\circ}{K}_{j_{n+1}} \setminus K_{j_n}$ for which

$$|g(x_n)| |\partial^{\alpha_n} f_n(x_n)| \exp \left(-m\varphi_\omega^* \left(\frac{|\alpha|}{m} \right) \right) > n. \quad (5.23)$$

Now, we choose a function $\rho \in \mathcal{D}_\omega(\mathbb{R}^N)$ such that $\text{supp } \rho \subseteq B_1(0)$ and $\rho \equiv 1$ on $B_r(0)$ for some $0 < r < 1$. We define

$$g_0(x) := \sum_{k=1}^{\infty} g(x_k) \rho(x - x_k), \quad x \in \mathbb{R}^N. \quad (5.24)$$

Then by Remark 4.3 we have $g_0 \in \mathcal{S}_\omega(\mathbb{R}^N)$. In particular, $g_0 \equiv g(x_n)$ in $\overline{B}_r(x_n)$ for each $n \in \mathbb{N}$. Accordingly, for every $n \in \mathbb{N}$, $x \in \overline{B}_r(x_n)$ and $\alpha \in \mathbb{N}_0^N$ we have

$$\partial^\alpha (fg_0)(x) = \partial^\alpha f(x)g(x_n).$$

Hence, by (5.23) it follows for every $n \in \mathbb{N}$ that

$$\begin{aligned} q_{m,m}(f_n g_0) &\geq |g(x_n)| |\partial^{\alpha_n} f_n(x_n)| \exp \left(m\omega(x) - m\varphi_\omega^* \left(\frac{|\alpha_n|}{m} \right) \right) \\ &\geq |g(x_n)| |\partial^{\alpha_n} f_n(x_n)| \exp \left(-m\varphi_\omega^* \left(\frac{|\alpha_n|}{m} \right) \right) > n. \end{aligned}$$

This shows that $\sup_{f \in B} q_{m,m}(fg_0) = \infty$, which is a contradiction because B is a bounded subset of $(\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau_s)$. Hence $\sup_{f \in B} q_{m,g}(f) < \infty$. $\square$

Finally, we have

Proposition 5.10 *Let ω be a non-quasianalytic weight function. Then $(\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau_b)$ and its strong dual are nuclear lc-spaces. Moreover, $(\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau_b)$ is complete.*

Proof Since $\mathcal{S}_\omega(\mathbb{R}^N)$ is a nuclear Fréchet space by [5, Theorem 3.3], the space $\mathcal{L}_b(\mathcal{S}_\omega(\mathbb{R}^N))$ and its strong dual space are nuclear lc-spaces, [12, Corollaire 3, Chap. II, §2, p.48]. Therefore, by [12, Théorème 9, Chap. II, §2, p.47] $(\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau_b)$ and its strong dual are also nuclear lc-spaces.

It is straightforward to show the completeness of $(\mathcal{O}_{M,\omega}(\mathbb{R}^N), \tau_b)$. $\square$

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