On the boundedness of the Weyl transform for homogeneous distributions on $\mathbb{R}^2$

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Abstract

Starting from earlier work by M.W. Wong, we give necessary and sufficient criteria, respectively, for the boundedness of the Weyl transform W_σ as an operator on $L^2(\mathbb{R})$ if σ is a homogeneous distribution on $\mathbb{R}^2$.

Keywords Weyl transform · Pseudo-differential operators · Homogeneous distributions

Mathematics Subject Classification 47G30 · 46F10

1 Introduction and notation

In our first paper on Weyl transforms, see [8], we endeavored to generalize the theorems and examples in [11] concerning the boundedness and compactness, respectively, of Weyl transforms of radially symmetric *functions* to the case of radially symmetric *distributions*. (Incidentally, let me indicate three printing errors in [8]: On p. 769, the space $\mathcal{D}'_{L^1}(\mathbb{R}^2)$ should be replaced by $\mathcal{D}'_{L^1}(\mathbb{R}^1)$ and on p. 775, the expression $e^{i \exp(2x)}$ was twice contorted to $e^i \exp(2x)$; finally, in line 13 on p. 785, the factor s_0^z got lost in the expression $(2\pi)^{-1} s_0^z e^{ih(s_0)} \sqrt{2\pi i/h''(s_0)}$.)

In [8], we did not touch on Theorem 27.3 on tensor products in polar coordinates in [11, p. 128] for a variety of reasons: First, this theorem concerns a much more general situation (of which radially symmetric functions are just a tiny special case), second, the setting is much more intricate and arduous than for radially symmetric functions, third, we did not immediately succeed in finding a generalization to distributions with meaningful and manageable examples, and fourth, there is no hope (as far as we can see) for a single *necessary and sufficient* criterion for boundedness, in contrast to the radial case.

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This paper tries to work out [11, Thm. 27.3, p. 128] from a distribution-theoretic viewpoint. We give a distributional formulation of it in Proposition 1, and we specialize Proposition 1 to the case of *homogeneous* distributions σ on $\mathbb{R}^2$ in Proposition 4. Therein we state a sufficient condition for the boundedness of the Weyl transform W_σ on $L^2(\mathbb{R})$ in terms of the Fourier coefficients of the characteristic F of σ . In the preparation for the general case, we deal with the case of homogeneity 0 in Proposition 2, where the matrix coefficients of the Weyl transforms can be expressed more explicitly than in the general case. In Proposition 3, we give necessary conditions on the degree λ of homogeneity and on the characteristic F of $\sigma = F \cdot r^\lambda$ that must be satisfied in order that W_σ can be bounded.

Let us introduce some notation. Besides the familiar Banach spaces $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, we use the Banach spaces of bounded and of compact linear operators on $L^2(\mathbb{R}^n)$, respectively, which we denote by $\mathcal{L}(L^2(\mathbb{R}^n))$ and by $\text{Com}(L^2(\mathbb{R}^n))$, respectively.

We employ the standard notation for the distribution spaces $\mathcal{D}'$, $\mathcal{S}'$, $\mathcal{E}'$, the dual spaces of the spaces $\mathcal{D}$, $\mathcal{S}$, $\mathcal{E}$ of “test functions”, of “rapidly decreasing functions” and of C^∞ functions, respectively, see [4,9]. In order to display the active variable in a distribution, say $x \in \mathbb{R}^n$, we use notation as $T(x)$ or $T \in \mathcal{D}'(\mathbb{R}_x^n)$. For the evaluation of a distribution $T \in \mathcal{E}'$ on a test function $\phi \in \mathcal{E}$, we use angle brackets, i.e., $\langle \phi, T \rangle$ or, more precisely ${}_E \langle \phi, T \rangle_{E'}$.

The spaces $\mathcal{D}_{L^p}$, $\mathcal{D}'_{L^p}$, $1 \leq p \leq \infty$, were introduced in [9, Ch. VI, § 8, p. 199], and we need, just for $p = 1$ and $p = \infty$, so-called “weighted” $\mathcal{D}_{L^p}$ and $\mathcal{D}'_{L^p}$ -spaces, i.e.,

$$\mathcal{D}_{L^p, \mu} := \mathcal{D}_{L^p} \cdot (1 + |x|^2)^{-\mu/2}, \quad \mathcal{D}'_{L^p, \mu} := \mathcal{D}'_{L^p} \cdot (1 + |x|^2)^{-\mu/2}, \\ 1 \leq p \leq \infty, \quad \mu \in \mathbb{R},$$

see [6].

$\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{Z}$ denote the sets of positive, of non-negative, and of all integers, respectively. For sequences with the index sets $\mathbb{N}_0$ and $\mathbb{Z}$, respectively, we use the Banach spaces l^p , $1 \leq p \leq \infty$, and the Fréchet space s of fast decreasing sequences with its dual s' of slowly increasing sequences.

$\mathbb{S}^1$ denotes the unit circle in $\mathbb{R}^2$, which we identify with $\mathbb{C}$, and we write ω for the generic variable on $\mathbb{S}^1 \subset \mathbb{C}$ and $d\omega$ for the measure of length on $\mathbb{S}^1$. With these notations, $(2\pi)^{-1/2} \omega^k$, $k \in \mathbb{Z}$, is an orthonormal basis in $L^2(\mathbb{S}^1)$ and $f(\omega) = \sum_{k \in \mathbb{Z}} c_k \omega^k$ is the usual Fourier series expansion for $f \in L^2(\mathbb{S}^1)$ if $c_k = \frac{1}{2\pi} \int_{\mathbb{S}^1} f(\omega) \omega^{-k} d\omega$, $k \in \mathbb{Z}$. Furthermore, $L^1(\mathbb{S}^1)$ is embedded into $\mathcal{D}'(\mathbb{S}^1)$ by $\langle \phi, f \rangle = \int \phi(\omega) f(\omega) d\omega$.

The Heaviside function is denoted by Y , see [9, p. 36], and we write $\delta_\tau(t) \in \mathcal{D}'(\mathbb{R}_t^1)$, $\tau \in \mathbb{R}$, for the delta distribution with support in τ , which is the derivative of $Y(t - \tau)$, i.e., $\langle \phi, \delta_\tau \rangle = \phi(\tau)$ for $\phi \in \mathcal{D}(\mathbb{R}^1)$. In contrast, $\delta \in \mathcal{D}'(\mathbb{R}^n)$ without any subscript stands for the delta distribution at the origin. For $z \in \mathbb{C}$ with $\text{Re } z > -1$, we write t_+^z for the locally integrable function $Y(t)t^z$ and we obtain, by analytic continuation, the analytic distribution-valued function

$$t_+^z : \mathbb{C} \setminus (-\mathbb{N}) \longrightarrow \mathcal{S}'(\mathbb{R}_t^1) : z \longmapsto t_+^z,$$

see [7, Ex. 1.4.8, p. 49].

The pull-back $h^*T = T \circ h \in \mathcal{D}'(\Omega)$ of a distribution T in one variable t with respect to a submersive $\mathcal{C}^\infty$ function $h : \Omega \rightarrow \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ open, is defined as in [4, Equ. (7.2.4/5), p. 82] or in [7, Def. 1.2.12, p. 19], i.e.,

$$\langle \phi, h^*T \rangle = \left\langle \frac{d}{dt} \left(\int_{\Omega} Y(t - h(x)) \phi(x) dx \right), T \right\rangle, \quad \phi \in \mathcal{D}(\Omega).$$

We use the Fourier transform $\mathcal{F}$ in the form

$$(\mathcal{F}\phi)(\xi) := \int_{\mathbb{R}^n} e^{-i\xi x} \phi(x) dx, \quad \phi \in \mathcal{S}(\mathbb{R}^n),$$

this being extended to $\mathcal{S}'$ by continuity. For the partial Fourier transforms of a distribution $T \in \mathcal{S}'(\mathbb{R}_{xy}^{m+n})$ with respect to $x \in \mathbb{R}^m$ or $y \in \mathbb{R}^n$, respectively, we use the notation $\mathcal{F}_x T$ and $\mathcal{F}_y T$, respectively.

The Hermite polynomials H_j and Hermite functions h_j are defined as usually:

$$H_j(x) = (-1)^j e^{x^2} \cdot (e^{-x^2})^{(j)}, \quad h_j(x) = e^{-x^2/2} H_j(x), \quad j \in \mathbb{N}_0, \quad x \in \mathbb{R},$$

see [5, Section 8.95], [11, Sections 18, 19]. Similarly, the Laguerre polynomials $L_j^\alpha(x)$ of order α are defined by

$$L_j^\alpha(x) := j!^{-1} x^{-\alpha} e^x \cdot (e^{-x} x^{\alpha+j})^{(j)}, \quad x > 0, \quad \alpha > -1, \quad j \in \mathbb{N}_0,$$

see [5, Section 8.97], [11, Section 20, p. 95].

2 Review of definitions and classical results

The Weyl transformation W can be described as a partial Fourier transform composed with a linear transformation in $\mathbb{R}_{xy}^{2n}$, to wit

$$W : \mathcal{S}'(\mathbb{R}^{2n}) \longrightarrow \mathcal{S}'(\mathbb{R}^{2n}) : \sigma(x, y) \longmapsto W_\sigma(x, y).$$

where

$$W_\sigma(x, y) = (2\pi)^{-n} (\mathcal{F}_y \sigma) \left(\frac{x+y}{2}, y-x \right),$$

see [3, p. 79], [11, Equ. (4.2), p. 19], [8, Sect. 2]. Hence if $\sigma \in \mathcal{S}'(\mathbb{R}^{2n})$, then its Weyl transform W_σ is the pseudo-differential operator represented by the continuous linear mapping $W_\sigma : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ with the kernel $W_\sigma(x, y) \in \mathcal{S}'(\mathbb{R}^{2n})$, i.e.,

$$(W_\sigma \phi)(x) = \langle \phi(y), W_\sigma(x, y) \rangle \in \mathcal{S}'(\mathbb{R}_x^n), \quad \phi \in \mathcal{S}(\mathbb{R}^n).$$

The Weyl transform W can be expressed by the “Wigner transform,” which is a continuous sesquilinear mapping

$$w : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^{2n}).$$

The mappings w and W are connected by

$$\langle \bar{\psi}, W_\sigma \phi \rangle = \langle w(\phi, \psi), \sigma \rangle, \quad \phi, \psi \in \mathcal{S}(\mathbb{R}^n). \quad (2.1)$$

This implies

$$\begin{aligned} \langle \bar{\psi}, W_\sigma \phi \rangle &= (2\pi)^{-n} \overline{\langle \psi(x) \phi(y), (\mathcal{F}_y \sigma)(\frac{x+y}{2}, y-x) \rangle} \\ &= (2\pi)^{-n} \overline{\langle \psi(x - \frac{y}{2}) \phi(x + \frac{y}{2}), \mathcal{F}_y \sigma \rangle} \\ &= (2\pi)^{-n} \left\langle \mathcal{F}_y \left(\overline{\psi(x - \frac{y}{2})} \phi(x + \frac{y}{2}) \right), \sigma \right\rangle = \langle w(\phi, \psi), \sigma \rangle \end{aligned}$$

and hence

$$(w(\phi, \psi))(x, y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-iy\eta} \phi(x + \frac{\eta}{2}) \overline{\psi(x - \frac{\eta}{2})} d\eta.$$

Note that we use for w a normalization constant different from that in [11, Equ. (3.12), p. 15], and that we also changed, for clarity, the notation for the Wigner transform from $W(\phi, \psi)$ to $w(\phi, \psi)$.

In this study, we shall focus on symbols σ which are homogeneous distributions in two dimensions, and we shall derive necessary and sufficient criteria, respectively, for the boundedness of the corresponding pseudo-differential W_σ on $L^2(\mathbb{R}^1)$, i.e., criteria on whether W_σ can be extended to a bounded mapping $L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$, or, in other words, whether the norm $\|W_\sigma \phi\|_2$ can be bounded by a constant multiple of $\|\phi\|_2$ for $\phi \in \mathcal{S}(\mathbb{R}^1)$.

Let us now recapitulate some basic facts about homogeneous distributions σ on $\mathbb{R}^2$, see [6,7] for more details. If $\sigma \in \mathcal{D}'(\mathbb{R}^2 \setminus \{0\})$ is homogeneous of degree $\lambda \in \mathbb{C}$, then the structure theorem (see [6, Sect. 2.5]) implies that σ can be extended to $\mathbb{R}^2$ and can be represented in the form $\sigma = F \cdot r^\lambda$ where the so-called “characteristic” F belongs to $\mathcal{D}'(\mathbb{S}^1)$ and

$$\langle \phi, F \cdot r^\lambda \rangle = \mathcal{S}(\mathbb{R}^1) \left\langle \mathcal{D}(\mathbb{S}^1) \langle \phi(t\omega), F(\omega) \rangle_{\mathcal{D}'(\mathbb{S}^1)}, t_+^{\lambda+1} \right\rangle_{\mathcal{S}'(\mathbb{R}^1)}, \quad \phi \in \mathcal{S}(\mathbb{R}^2).$$

If we identify $\mathbb{S}^1 \subset \mathbb{R}^2 \simeq \mathbb{C}$, then

$$\mathcal{D}'(\mathbb{S}^1) \xrightarrow{\sim} s'(\mathbb{Z}) : F \longmapsto (c_k)_{k \in \mathbb{Z}}$$

where $c_k = \frac{1}{2\pi} \langle \omega^{-k}, F \rangle \in \mathbb{C}$ for $k \in \mathbb{Z}$ and $\sum_{k \in \mathbb{Z}} c_k \omega^k$ converges to $F(\omega)$ in $\mathcal{D}'(\mathbb{S}^1)$. The Hilbert spaces

$$\mathcal{H}^{(\mu)}(\mathbb{S}^1) = \left\{ F(\omega) = \sum_{k \in \mathbb{Z}} c_k \omega^k \in \mathcal{D}'(\mathbb{S}^1); (c_k(1 + |k|)^\mu)_{k \in \mathbb{Z}} \in l^2(\mathbb{Z}) \right\}, \quad \mu \in \mathbb{R},$$

are the usual Sobolev spaces on $\mathbb{S}^1$.

The distribution-valued function $z \mapsto t_+^z$ is meromorphic with simple poles at $z = -k$, $k \in \mathbb{N}$. We have $\text{Res}_{z=-k} t_+^z = (-1)^{k-1} (k-1)! \delta^{(k-1)}$ and we set

$$t_+^{-k} := \text{Pf}_{z=-k} t_+^z = \lim_{z \rightarrow -k} \left(t^z - (z+k)^{-1} \text{Res}_{\lambda=-k} t^\lambda \right), \quad k \in \mathbb{N}.$$

For $\lambda \in \mathbb{C} \setminus (-2 - \mathbb{N}_0)$, this implies that $F \cdot r^\lambda$ is homogeneous not only in $\mathbb{R}^2 \setminus \{0\}$, but also on $\mathbb{R}^2$, whereas for $\lambda = -2 - j$, $j \in \mathbb{N}_0$, $F \cdot r^\lambda$ is homogeneous on $\mathbb{R}^2$ if and only if the Fourier coefficients $c_k = (2\pi)^{-1} \langle \omega^{-k}, F \rangle$ vanish for $|k| \leq j$, $k \in \mathbb{Z}$, see [6, Thm. 2.5.1, p. 58].

Furthermore, if $\mathcal{S}'_\lambda(\mathbb{R}^2)$, $\lambda \in \mathbb{C}$, denotes the closed subspace of $\mathcal{S}'(\mathbb{R}^2)$ of distributions which are homogeneous of degree λ in $\mathbb{R}^2$, then $\mathcal{F} : \mathcal{S}'_\lambda(\mathbb{R}^2) \xrightarrow{\sim} \mathcal{S}'_{-2-\lambda}(\mathbb{R}^2)$ is an isomorphism of locally convex spaces. More explicitly, if $\lambda \in \mathbb{C} \setminus \mathbb{Z}$, then $\mathcal{F}(F \cdot r^\lambda) = G \cdot r^{-2-\lambda}$ where $F = \sum_{k \in \mathbb{Z}} c_k \omega^k$, $G = \sum_{k \in \mathbb{Z}} d_k \omega^k$ and

$$d_k = 2^{2+\lambda} \pi i^k \frac{\Gamma(1 + \frac{\lambda-k}{2})}{\Gamma(-\frac{\lambda+k}{2})} c_k, \quad k \in \mathbb{Z}. \quad (2.2)$$

Formula (2.2) is a special case of the Bochner–Hecke formula, see [10, Ch. IV, Thm. 4.1, p. 160]. For a direct verification, note that, with $z = x + iy = r\omega$, $r = |z|$, and using [7, Equ. (1.6.18), p. 99], we have

$$\begin{aligned} \mathcal{F}(\omega^k \cdot r^\lambda) &= \mathcal{F}(z^k \cdot r^{\lambda-k}) = i^{|k|} (\partial_x + \text{sign}(k) i \partial_y)^{|k|} \mathcal{F}(r^{\lambda-|k|}) \\ &= 2^{2+\lambda} \pi i^{|k|} \Gamma\left(1 + \frac{\lambda-|k|}{2}\right) \left(\tfrac{1}{2}(\partial_x + \text{sign}(k) i \partial_y)\right)^{|k|} \left(\frac{r^{|k|-2-\lambda}}{\Gamma(\frac{|k|-\lambda}{2})}\right) \\ &= 2^{2+\lambda} \pi i^{|k|} \frac{\Gamma\left(1 + \frac{\lambda-|k|}{2}\right)}{\Gamma\left(-\frac{\lambda+|k|}{2}\right)} \omega^k \cdot r^{-2-\lambda}, \quad k \in \mathbb{Z}. \end{aligned}$$

We finally observe that the factor in front of c_k in formula (2.2) does not depend on the sign of k on account of the complement formula for the gamma function.

As we shall see in Proposition 3 below, a homogeneous symbol $\sigma \in \mathcal{S}'(\mathbb{R}^2)$ can have a bounded Weyl transform W_σ only under the condition that the degree λ of homogeneity fulfills $-2 \leq \text{Re } \lambda \leq 0$. For homogeneous and radially symmetric symbols $\sigma = r^\lambda \in \mathcal{S}'(\mathbb{R}^2)$, we have shown in [8, Sect. 5, Ex. 1] that W_σ is bounded on $L^2(\mathbb{R}^1)$ if and only if $-2 \leq \text{Re } \lambda \leq 0$, $\lambda \neq -2$. (Note that $r^{-2} = \text{Pf}_{\lambda=-2} r^\lambda$ is not homogeneous on $\mathbb{R}^2$. We remark, incidentally, that W_σ is unbounded for $\sigma = r^{-2} \in \mathcal{S}'(\mathbb{R}^2)$, see the remark to Proposition 3.)

Let us describe next what can be inferred from classical results. According to [11, Thms. 11.1, 11.3, pp. 55, 57], compare also [8, Prop. 1, p. 773], W_σ is bounded and even compact as an operator on $L^2(\mathbb{R}^n)$ if σ or $\mathcal{F}\sigma$ belong to $\cup_{p \in [1, 2]} L^p(\mathbb{R}^{2n})$. In particular,

if $\sigma = F(\omega) \cdot r^\lambda \in \mathcal{S}'(\mathbb{R}^2)$ with $F \in L^2(\mathbb{S}^1)$ and $-2 < \operatorname{Re} \lambda < -1$, then $\sigma \in L^1(\mathbb{R}^2) + L^2(\mathbb{R}^2)$ and hence $W_\sigma \in \operatorname{Com}(L^2(\mathbb{R}^1))$. Similarly, if $-1 < \mu = \operatorname{Re} \lambda < 0$, and $F \in \mathcal{H}^{(\mu+1)}(\mathbb{S}^1)$, i.e., $F(\omega) = \sum_{k \in \mathbb{Z}} c_k \omega^k$ such that $(c_k |k|^{\mu+1})_{k \in \mathbb{Z}} \in l^2(\mathbb{Z})$, then we employ formula (2.2) and conclude once again that $W_\sigma \in \operatorname{Com}(L^2(\mathbb{R}^1))$ from the asymptotic behaviour

$$\frac{\Gamma(1 + \frac{\lambda - |k|}{2})}{\Gamma(-\frac{\lambda + |k|}{2})} = (-1)^k \frac{\Gamma(1 + \frac{|k| + \lambda}{2})}{\Gamma(\frac{|k| - \lambda}{2})} \sim (-1)^k \left(\frac{|k|}{2}\right)^{\lambda+1}, \quad |k| \rightarrow \infty,$$

see [5, Equ. 8.328.2]. We shall deduce more general results in Sect. 6 below.

3 The matrix representation of W_σ

The normalized Hermite functions

$$e_j = h_j / \|h_j\|_2 = (2^j j! \sqrt{\pi})^{-1/2} h_j \in \mathcal{S}(\mathbb{R}), \quad j \in \mathbb{N}_0, \quad (3.1)$$

are an orthonormal basis in $L^2(\mathbb{R})$, see, e.g., [11, Thm. 19.3, p. 94]. This yields an isomorphism of Hilbert spaces:

$$L^2(\mathbb{R}) \xrightarrow{\sim} l^2(\mathbb{N}_0) : f \mapsto \left(\int_{\mathbb{R}} f(x) e_j(x) dx \right)_{j \in \mathbb{N}_0}.$$

As was observed already by Schwartz, cf. [9, Ch. VII, Sect. 7, Ex. 7, pp. 260, 262], the sequence e_j , $j \in \mathbb{N}_0$, also serves to identify the spaces $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ with the sequence spaces $s(\mathbb{N}_0)$ and $s'(\mathbb{N}_0)$, respectively. Hence we obtain a commutative diagram:

$$\begin{array}{ccccc} \mathcal{S}(\mathbb{R}) & \hookrightarrow & L^2(\mathbb{R}) & \hookrightarrow & \mathcal{S}'(\mathbb{R}) \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ s(\mathbb{N}_0) & \hookrightarrow & l^2(\mathbb{N}_0) & \hookrightarrow & s'(\mathbb{N}_0) \end{array}$$

Here the vertical isomorphisms in the diagram are all given by $T \mapsto (\langle e_j, T \rangle)_{j \in \mathbb{N}_0}$.

By the same token, using $\mathcal{S}'(\mathbb{R}^2) \xrightarrow{\sim} s'(\mathbb{N}_0^2)$, the mapping $W_\sigma : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}'(\mathbb{R})$ is represented by the matrix $B_\sigma = (b_{jk})_{j,k \in \mathbb{N}_0} \in s'(\mathbb{N}_0^2)$ where $W_\sigma e_k = \sum_{j \in \mathbb{N}_0} e_j b_{jk}$ for $k \in \mathbb{N}_0$ and the sums converge in $\mathcal{S}'(\mathbb{R})$. The matrix coefficients b_{jk} of W_σ are given by

$$b_{jk} = \langle e_j, W_\sigma e_k \rangle = \langle w(e_k, e_j), \sigma \rangle, \quad j, k \in \mathbb{N}_0, \quad (3.2)$$

on account of formula (2.1). Therefore, the operator W_σ is bounded on $L^2(\mathbb{R})$ if and only if

$$\exists C > 0 : \forall \phi \in s(\mathbb{N}_0) : \left\| \left(\sum_{k=0}^{\infty} b_{jk} \phi_k \right)_{j \in \mathbb{N}_0} \right\|_{l^2} \leq C \|\phi\|_{l^2}$$

holds.

Let us specialize now to distributions $\sigma \in \mathcal{S}'(\mathbb{R}^2)$ that are tensor products in polar coordinates. For the radial dependency of σ , we take $R \in \mathcal{S}'([0, \infty))$, which is the dual space of $\mathcal{S}([0, \infty)) = \{\phi|_{[0, \infty)} : \phi \in \mathcal{S}(\mathbb{R})\}$ equipped with the seminorms

$$\|\phi\|_{k,l} = \|(1+x)^k \phi^{(l)}(x)\|_{L^\infty([0, \infty))}, \quad k, l \in \mathbb{N}_0.$$

As in [8, p. 776], we observe that

$$\mathcal{S}'([0, \infty)) \xrightarrow{\sim} \{S \in \mathcal{S}'(\mathbb{R}); \text{supp } S \subset [0, \infty)\} : S \longmapsto (\phi \mapsto \langle \phi|_{[0, \infty)}, S \rangle)$$

is an isomorphism of locally convex spaces.

If, furthermore, $F \in \mathcal{D}'(\mathbb{S}^1)$, then we define $F \cdot R \in \mathcal{S}'(\mathbb{R}^2)$ by the following:

$$\langle \phi, F \cdot R \rangle = \mathcal{S}([0, \infty)) \langle r \cdot \langle \phi(r\omega), F(\omega) \rangle, R(r) \rangle_{\mathcal{S}'([0, \infty))}, \quad \phi \in \mathcal{S}(\mathbb{R}^2). \quad (3.3)$$

Note that this is well-defined since the function

$$\mathbb{R} \longrightarrow \mathbb{C} : r \longmapsto r \cdot \langle \phi(r\omega), F(\omega) \rangle$$

belongs to $\mathcal{S}(\mathbb{R}^1)$, but that the distribution $R \in \mathcal{S}'([0, \infty))$ is not uniquely determined up to a factor by the distribution $\sigma = F \cdot R$ since, e.g., $F \cdot \delta = 0$.

In the following Proposition 1, we state formulas for the matrix coefficients of W_σ with respect to the normalized Hermite functions e_j, e_k if $\sigma = F \cdot R \in \mathcal{S}'(\mathbb{R}^2)$ is a tensor product in polar coordinates. This proposition is but a slight extension of Theorem 27.3 in [11, p. 128] from temperate functions to temperate distributions.

Proposition 1 *Let e_j be the normalized Hermite functions as in formula (3.1) and L_k^α be the Laguerre polynomials as in Sect. 1. Let $F(\omega) = \sum_{k \in \mathbb{Z}} c_k \omega^k \in \mathcal{D}'(\mathbb{S}^1)$ and $R \in \mathcal{S}'([0, \infty))$ and let $\sigma = F \cdot R \in \mathcal{S}'(\mathbb{R}^2)$ be defined as in formula (3.3). Let $W_\sigma e_k = \sum_{j \in \mathbb{N}_0} e_j b_{jk}$, $k \in \mathbb{N}_0$, with $(b_{jk})_{j,k \in \mathbb{N}_0} \in s'(\mathbb{N}_0^2)$ being the matrix representation of W_σ and define the symmetric matrix $(a_{jk})_{j,k \in \mathbb{N}_0} \in s'(\mathbb{N}_0^2)$ by*

$$a_{jk} = 2(-1)^k \sqrt{\frac{k!}{j!}} 2^{(j-k)/2} \mathcal{S}([0, \infty)) \langle r^{j-k+1} L_k^{j-k}(2r^2) e^{-r^2}, R(r) \rangle_{\mathcal{S}'([0, \infty))}, \quad j \geq k, \quad (3.4)$$

and $a_{jk} = a_{kj}$ for $j \leq k$, $j, k \in \mathbb{N}_0$.

Then the following holds:

- (a) For $j, k \in \mathbb{N}_0$ we have $b_{jk} = a_{jk} c_{k-j}$;
- (b) if $\mu \in \mathbb{R}$ and

$$\exists C > 0 : \forall j, k \in \mathbb{N}_0 : |a_{jk}| \leq C \cdot (1 + |j - k|)^\mu \quad (3.5)$$

and $((1 + |k|)^\mu c_k)_{k \in \mathbb{Z}} \in l^1(\mathbb{Z})$, then W_σ is bounded, i.e., $W_\sigma \in \mathcal{L}(L^2(\mathbb{R}))$.

Proof (a) The formula for b_{jk} in the proposition directly follows from the calculation of the Wigner transforms $w(e_j, e_k)$, $j, k \in \mathbb{N}_0$. This calculation goes back to C. Itzykson, W. Miller, and N. Ya. Vilenkin, see the historical comment in [3, p. 65]. Taking account of the different notations in the literature, we have, for $j, k \in \mathbb{N}_0$ with $j \geq k$,

$$(w(e_j, e_k))(r\omega) = \frac{2(-1)^k}{2\pi} \sqrt{\frac{k!}{j!}} 2^{(j-k)/2} r^{j-k} \omega^{k-j} L_k^{j-k}(2r^2) e^{-r^2}, \quad (3.6)$$

where $r > 0$ and $\omega \in \mathbb{S}^1$, see [11, Thm. 24.1, p. 113], [3, Thm. 1.104, p. 64]. Furthermore we have, quite trivially, $w(e_k, e_j) = \overline{w(e_j, e_k)}$ for $j, k \in \mathbb{N}_0$.

Hence employing formulas (3.2), (3.3), (3.4) and (3.6), we obtain

$$b_{jk} = \langle w(e_k, e_j), \sigma \rangle = \langle r \cdot \overline{w(e_j, e_k)(r\omega)}, F(\omega), R(r) \rangle = a_{jk} c_{k-j}, \quad j, k \in \mathbb{N}_0.$$

(b) In the second part, we will closely follow the proof of Theorem 27.3 in [11, p. 129]. Assuming condition (3.5) and $((1+|k|)^\mu c_k)_{k \in \mathbb{Z}} \in l^1(\mathbb{Z})$, we set $d_k = C(1+|k|)^\mu |c_k|$, $k \in \mathbb{Z}$. Then the sequence $(d_k)_{k \in \mathbb{Z}}$ belongs to $l^1(\mathbb{Z})$ and

$$|b_{jk}| = |a_{jk}| \cdot |c_{k-j}| \leq C \cdot (1+|k-j|)^\mu |c_{k-j}| = d_{k-j}, \quad j, k \in \mathbb{N}_0.$$

Hence, if $\phi \in \mathcal{S}(\mathbb{N}_0)$ and if we set $\phi_k = 0$ for $k < 0$, then the matrix $B_\sigma = (b_{jk})_{j,k \in \mathbb{N}_0}$ of W_σ fulfills

$$\begin{aligned} \|B_\sigma \phi\|_{l^2(\mathbb{N}_0)} &= \left\| \left(\sum_{k=0}^{\infty} b_{jk} \phi_k \right)_{j \in \mathbb{N}_0} \right\|_{l^2(\mathbb{N}_0)} \leq \left\| \left(\sum_{k \in \mathbb{Z}} d_{k-j} |\phi_k| \right)_{j \in \mathbb{N}_0} \right\|_{l^2(\mathbb{N}_0)} \\ &= \left\| \left(\sum_{l \in \mathbb{Z}} d_l |\phi_{j+l}| \right)_{j \in \mathbb{N}_0} \right\|_{l^2(\mathbb{N}_0)} \leq \|\phi\|_{l^2(\mathbb{N}_0)} \sum_{l \in \mathbb{Z}} d_l, \end{aligned}$$

i.e., B_σ and thus also W_σ are bounded. This completes the proof. $\square$

Remarks (1) Let us first discuss the special case of $\sigma \in \mathcal{S}'(\mathbb{R}^2)$ being radially symmetric. We can set then $F = 1$ and hence σ is given by $R(r) \in \mathcal{S}'([0, \infty))$. As we have mentioned above, R is not uniquely determined by σ . However, as explained in [8, p. 776], there is a unique $S \in \mathcal{S}'([0, \infty))$ such that σ is represented by the formula

$$\langle \phi, \sigma \rangle = \mathcal{S}([0, \infty)) \left\langle \frac{1}{2} \int_{\mathbb{S}^1} \phi(\sqrt{t} \omega) d\omega, S(t) \right\rangle_{\mathcal{S}'([0, \infty))}, \quad \phi \in \mathcal{S}(\mathbb{R}^2). \quad (3.7)$$

This means that $S \in \mathcal{S}'([0, \infty))$ satisfies

$$\langle 2r\psi(r^2), R(r) \rangle = \langle \psi, S \rangle, \quad \psi \in \mathcal{S}([0, \infty)).$$

If we set $F = 1$ in Proposition 1, we obtain $c_k = \delta_{k0}$ and hence the matrix B_σ is diagonal with the eigenvalues

$$\lambda_j = a_{jj} = 2(-1)^j \langle r L_j^0(2r^2) e^{-r^2}, R(r) \rangle = (-1)^j \langle L_j^0(2t) e^{-t}, S(t) \rangle, \quad j \in \mathbb{N}_0. \quad (3.8)$$

Formula (3.8) coincides with [8, Prop. 3, Equ. (3.5), p. 777].

However note that condition (3.5) in Proposition 1 above is overly restrictive in the case of radially symmetric distributions since, clearly, the diagonal matrix B_σ is bounded on $l^2(\mathbb{N}_0)$ if and only if the sequence of eigenvalues $(a_{jj})_{j \in \mathbb{N}_0}$ is bounded. Hence no restriction on a_{jk} , $j \neq k$, is necessary. More generally, if F is a trigonometric polynomial, i.e., if $F(\omega) = \sum_{k \in M} c_k \omega^k$ where M is a finite subset of $\mathbb{Z}$ with $c_k \neq 0$ for $k \in M$ and if $\sigma = F \cdot R$, then W_σ is bounded on $L^2(\mathbb{R})$ if and only if the set $\{a_{jk}; k - j \in M\}$ is bounded in $\mathbb{C}$.

(2) In [8, Prop. 4, Equ. (4.2), p. 779], a representation of the eigenvalues $\lambda_j = a_{jj}$, $j \in \mathbb{N}_0$, by integrals with rational kernels was derived for $\sigma \in \mathcal{S}'(\mathbb{R}^2)$ radially symmetric and such that $\sigma(x, y) = (\mathcal{F}T)(x^2 + y^2)$ with $T \in \mathcal{D}'_{L^1, -1}(\mathbb{R})$. This formula, i.e.,

$$\lambda_j = \mathcal{D}_{L^\infty, 1} \left\langle \frac{(1 - it)^j}{(1 + it)^{j+1}}, T(t) \right\rangle_{\mathcal{D}'_{L^1, -1}}, \quad j \in \mathbb{N}_0,$$

was useful in some examples in calculating the asymptotic behavior of λ_j for $j \rightarrow \infty$, see [8, Sect. 5].

Since we will make no use here of the corresponding “rational” representation of a_{jk} , let us just state the formula without proof: If $j, k \in \mathbb{N}_0$, $j \geq k$, $T \in \mathcal{D}'_{L^1, -1}(\mathbb{R})$ and $R(r) = (\mathcal{F}T)(r^2) \cdot r^{j-k}$ holds in $\mathcal{S}'([0, \infty))$, then

$$a_{jk} = \sqrt{\frac{j!}{k!}} 2^{(j-k)/2} \mathcal{D}_{L^\infty, 1} \left\langle \frac{(1 - it)^k}{(1 + it)^{j+1}}, T(t) \right\rangle_{\mathcal{D}'_{L^1, -1}}. \quad (3.9)$$

As in the diagonal case, formula (3.9) can be derived from formula (3.4) by Fourier transformation, compare the procedure in [11, Sects. 25, 26].

4 The case of homogeneous symbols of degree 0

We specialize now our investigation to homogeneous symbols $\sigma = F \cdot r^\lambda$, $\lambda \in \mathbb{C}$, $F(\omega) = \sum_{k \in \mathbb{Z}} c_k \omega^k \in \mathcal{D}'(\mathbb{S}^1)$. We assume that σ is homogeneous on $\mathbb{R}^2$ and hence that $c_k = 0$ for $|k| \leq j$ if $\lambda = -2 - j$, $j \in \mathbb{N}_0$, cf. Sect. 2.

In order to calculate $a_{jk}(\lambda)$ for $R(r) = r_+^\lambda \in \mathcal{S}'([0, \infty))$ according to formula (3.4), we first assume that $\lambda \in \mathbb{C} \setminus \mathbb{Z}$ with $\operatorname{Re} \lambda > -2$. Then [1, Equ. 11.44, p. 110]

and [5, Equ. 9.131.1] yield the following for $j, k \in \mathbb{N}_0$ with $j \geq k$:

$$\begin{aligned} a_{jk}(\lambda) &= 2(-1)^k \sqrt{\frac{k!}{j!}} 2^{(j-k)/2} \int_0^\infty r^{\lambda+j-k+1} L_k^{j-k}(2r^2) e^{-r^2} dr \\ &= \frac{(-1)^k}{2^{\lambda/2+1}} \sqrt{\frac{k!}{j!}} \int_0^\infty t^{(\lambda+j-k)/2} L_k^{j-k}(t) e^{-t/2} dt \\ &= 2^{(j-k)/2} \frac{\Gamma(\frac{\lambda+j+k}{2} + 1)}{\sqrt{j!k!}} \cdot {}_2F_1\left(-k, \frac{j-k-\lambda}{2}; -\frac{j+k+\lambda}{2}; -1\right) \quad (4.1) \end{aligned}$$

$$= 2^{(j+k)/2} \frac{\Gamma(\frac{\lambda+j+k}{2} + 1)}{\sqrt{j!k!}} \cdot {}_2F_1\left(-j, -k; -\frac{j+k+\lambda}{2}; \frac{1}{2}\right) \quad (4.2)$$

By analytic continuation, formulas (4.1), (4.2) hold for $\lambda \in \mathbb{C} \setminus \mathbb{Z}$ and, due to the symmetry of (4.2) in j, k , for all $j, k \in \mathbb{N}_0$. But note that the hypergeometric function ${}_2F_1(a, b; c; z)$ is defined for $c \in -\mathbb{N}_0$ only if $[a \in -\mathbb{N}_0 \text{ and } a > c]$ or $[b \in -\mathbb{N}_0 \text{ and } b > c]$. Because of that and because of the gamma factor in (4.1), (4.2), we have to exercise some care if $\lambda \in \mathbb{Z}$.

Let us now specialize further and consider the case $\lambda = 0$, i.e., the case of homogeneous symbols $\sigma = F(\omega)$ of degree 0.

Proposition 2 *Let $F(\omega) = \sum_{k \in \mathbb{Z}} c_k \omega^k \in \mathcal{D}'(\mathbb{S}^1)$ and $\sigma = F \cdot 1 \in S'(\mathbb{R}^2)$ be homogeneous of degree 0. Then the following holds:*

(a) *If $\alpha_j = \sqrt{\Gamma(\frac{j}{2} + 1) / \Gamma(\frac{j+1}{2})}$ for $j \in \mathbb{N}_0$ and $a_{jk} = a_{jk}(0)$ is as in Proposition 1 for $R(r) = 1$, then*

$$a_{jk}(0) = \left\{ \begin{array}{ll} \frac{\alpha_{\max(j,k)}}{\alpha_{\min(j,k)}} : \min(j, k) \text{ is even,} \\ \frac{\alpha_{\min(j,k)}}{\alpha_{\max(j,k)}} : \min(j, k) \text{ is odd} \end{array} \right\}, \quad j, k \in \mathbb{N}_0. \quad (4.3)$$

(b) *If $(\sqrt[4]{|k|} c_k)_{k \in \mathbb{Z}} \in l^1(\mathbb{Z})$, then $W_\sigma \in \mathcal{L}(L^2(\mathbb{R}))$.*

Proof (a) If $j = k \in \mathbb{N}_0$, then formula (4.1) yields $a_{jj}(0) = 1$. If $j \neq k$, then formula (4.2) is valid. Let us first assume $j > k$ and k even. For $\operatorname{Re} z < -k$, [2, Equ. 7.3.7.4] yields

$${}_2F_1\left(z, -k; \frac{z-k}{2}; \frac{1}{2}\right) = \sqrt{\pi} \frac{\Gamma(\frac{z-k}{2})}{\Gamma(\frac{z}{2})\Gamma(\frac{1-k}{2})} = \frac{\Gamma(1 - \frac{z}{2})\Gamma(\frac{k+1}{2})}{\sqrt{\pi} \Gamma(\frac{k-z}{2} + 1)}$$

and hence

$$a_{jk}(0) = 2^{(j+k)/2} \frac{\Gamma(\frac{j}{2} + 1)\Gamma(\frac{k+1}{2})}{\sqrt{\pi} \Gamma(j+1)\Gamma(k+1)} = \frac{\alpha_j}{\alpha_k}$$

on account of the complement formula and of the duplication formula for the gamma function.

Similarly, if k is odd and $\operatorname{Re} z < -k$, then [2, Equ. 7.3.7.4] furnishes

$${}_2F_1\left(z, -k; \frac{z-k}{2}; \frac{1}{2}\right) = \sqrt{\pi} \frac{\Gamma(\frac{z-k}{2})}{\Gamma(\frac{z+1}{2})\Gamma(-\frac{k}{2})} = \frac{\Gamma(\frac{1-z}{2})\Gamma(\frac{k}{2}+1)}{\sqrt{\pi} \Gamma(\frac{k-z}{2}+1)}$$

and hence $a_{jk}(0) = \alpha_k/\alpha_j$ for $j > k$. This proves formula (4.3).

(b) Let us next estimate $a_{jk}(0)$. Using Euler's psi function $\psi(x) = (\log \Gamma(x))'$ for $x > 0$ we obtain

$$\log \alpha_j = \frac{1}{2} \int_{(j+1)/2}^{j/2+1} \psi(x) dx. \quad (4.4)$$

Equation (4.4) implies, incidentally, that α_j is monotonically increasing since the same holds for the function ψ due to the logarithmic convexity of the gamma function, see, e.g., [5, Equ. 8.363.3].

Since $C_1 + \log x \leq \psi(x) \leq C_2 + \log x$ holds for all $x \geq \frac{1}{2}$ and appropriate constants $C_1, C_2 \in \mathbb{R}$, we conclude that

$$\exists C > 0 : \forall j \in \mathbb{N}_0 : C^{-1} \sqrt[4]{j+1} \leq \alpha_j \leq C \sqrt[4]{j+1}. \quad (4.5)$$

This implies

$$\begin{aligned} \exists C > 0 : \forall j, k \in \mathbb{N}_0 \text{ with } j \geq k : 0 \leq a_{jk}(0) &\leq \frac{\alpha_j}{\alpha_k} \\ &\leq C \sqrt[4]{\frac{j+1}{k+1}} = C \left(1 + \frac{j-k}{k+1}\right)^{1/4}. \end{aligned} \quad (4.6)$$

Therefore, condition (3.5) in Proposition 1 holds for $\mu = \frac{1}{4}$ and this implies that W_σ is bounded on $L^2(\mathbb{R})$ if the condition $(\sqrt[4]{|k|} c_k)_{k \in \mathbb{Z}} \in l^1(\mathbb{Z})$ is assumed. The proof is complete. $\square$

Remark The matrix $(a_{jk}(\lambda))_{j,k \in \mathbb{N}_0}$ can be calculated similarly for each integral λ . E.g., if $\lambda = -1$, then [2, Equ. 7.3.7.5] implies that $a_{jk}(-1) = (\alpha_j \alpha_k)^{-1}$ if $\min(j, k)$ is even and $a_{jk}(-1)$ vanishes if $\min(j, k)$ is odd. Hence

$$\exists C > 0 : \forall j, k \in \mathbb{N}_0 : 0 \leq a_{jk}(-1) \leq C(1 + |j - k|)^{-1/4}$$

and $W_\sigma \in \mathcal{L}(L^2(\mathbb{R}))$ for $\sigma = F \cdot r^{-1}$ if $\sum_{k \in \mathbb{Z}} |c_k|(1 + |k|)^{-1/4} < \infty$.

5 Necessary conditions

We shall prove here that $\sigma \in \mathcal{S}'(\mathbb{R}^2)$ homogeneous of degree λ can have a bounded Weyl transform only if the real part of λ lies in the interval $[-2, 0]$ and if the characteristic F of σ belongs to the Sobolev space $\mathcal{H}^{(1/4+\mu/2)}(\mathbb{S}^1)$. In deriving this result, we need yet another representation of $a_{jk}(\lambda)$ by the hypergeometric function.

If we use the second equation in [1, Equ. 11.44, p. 110] or, alternatively, apply (employing a limit procedure) [5, Equ. 9.131.2] to formula (4.1), then we obtain the following for $\lambda \in \mathbb{C} \setminus \mathbb{Z}$ and $j, k \in \mathbb{N}_0$ with $j \geq k$:

$$a_{jk}(\lambda) = 2^{(j-k)/2} \sqrt{\frac{j!}{k!}} \frac{\Gamma(\frac{\lambda+j-k}{2} + 1)}{(j-k)!} \cdot {}_2F_1\left(-k, \frac{j-k-\lambda}{2}; j-k+1; 2\right). \quad (5.1)$$

Let us now digress a little bit and derive the “complement formula” for the matrix coefficients $a_{jk}(\lambda)$ from formula (5.1). Upon applying [5, Equ. 9.131.1] to formula (5.1) we obtain, still for $j \geq k$,

$$a_{jk}(\lambda) = 2^{(j-k)/2} \sqrt{\frac{j!}{k!}} \frac{\Gamma(\frac{\lambda+j-k}{2} + 1)}{(j-k)!} \cdot (-1)^k {}_2F_1\left(-k, \frac{j-k+\lambda+2}{2}; j-k+1; 2\right),$$

wherefrom we conclude that

$$a_{jk}(\lambda) = (-1)^k \frac{\Gamma(\frac{\lambda+j-k}{2} + 1)}{\Gamma(\frac{-\lambda+j-k}{2})} a_{jk}(-2-\lambda), \quad j, k \in \mathbb{N}_0, \lambda \in \mathbb{C} \setminus \mathbb{Z}. \quad (5.2)$$

Note that the factor in (5.2) in front of $a_{jk}(-2-\lambda)$ is symmetric in j and k .

We observe that the special case of $j = k$ in (5.2) was already given in [8, p. 782] and that formula (5.2) implies $a_{jk}(-1) = 0$ if the minimum of j and k is odd, compare the remark following Proposition 2. Also note that formula (5.2) is a consequence of formula (2.2) and of the fact that W_σ and $W_{\mathcal{F}\sigma}$ are closely connected.

Proposition 3 *Let $\sigma \in \mathcal{S}'(\mathbb{R}^2)$ be homogeneous on $\mathbb{R}^2$ of degree $\lambda \in \mathbb{C}$ and such that W_σ is bounded on $L^2(\mathbb{R})$. Then $-2 \leq \operatorname{Re} \lambda \leq 0$ and [either σ is a multiple of δ or $\sigma = F \cdot r^\lambda$ with $F \in \mathcal{H}^{(1/4+\mu/2)}(\mathbb{S}^1)$ for $\mu = \operatorname{Re} \lambda$].*

Proof (a) Let us first settle the case of homogeneous σ with support contained in $\{0\}$. Then $\lambda = -2 - j$, $j \in \mathbb{N}_0$, and σ is a linear combination of $\partial_x^k \partial_y^l \delta$ for $k, l \in \mathbb{N}_0$ with $k + l = j$. This implies that W_σ is a linear combination of the operators A_{kl} , $k + l = j$, where $(A_{kl}\phi)(x) = x^k \phi^{(l)}(-x)$ for $\phi \in \mathcal{S}(\mathbb{R})$. Therefore, W_σ is bounded in this case if and only if $j = 0$. Indeed, for $\phi \in \mathcal{S}(\mathbb{R})$, we have

$$\lim_{N \rightarrow \infty} N^{-k} \|A_{kl}(\phi(x + N))\|_2 = \|\phi^{(l)}\|_2$$

and the boundedness of A_{kl} implies $k = 0$, and

$$\|A_{0l}(\phi(Nx))\|_2 = N^{l-1/2} \|\phi^{(l)}\|_2, \quad N > 0,$$

and the boundedness of A_{0l} implies $l = 0$ and hence σ is a multiple of δ .

(b) Let now $\sigma = F \cdot r^\lambda$ with $F(\omega) = \sum_{k \in \mathbb{Z}} c_k \omega^k \in \mathcal{D}'(\mathbb{S}^1)$ and assume that $c_m \neq 0$ for some $m \in \mathbb{Z}$. In particular, in the case of $\lambda = -2 - j$, $j \in \mathbb{N}_0$, the homogeneity of σ on $\mathbb{R}^2$ implies that $|m| > j$.

Since $W_\sigma \in \mathcal{L}(L^2(\mathbb{R}))$, its matrix coefficients $b_{jk} = \langle e_j, W_\sigma e_k \rangle$ are bounded in modulus by $\|W_\sigma\|_{\mathcal{L}(L^2(\mathbb{R}))}$. Hence also the complex sequence $a_{k,k+m}(\lambda) = b_{k,k+m}/c_m$, $k \in \mathbb{N}_0$, $k \geq -m$, must be bounded.

Let us next determine the asymptotic behavior of $a_{k+m,k}(\lambda)$ for $k \rightarrow \infty$ and $m \in \mathbb{N}_0$ fixed, and for $\lambda \in \mathbb{C} \setminus (-2 - \mathbb{N}_0)$ or such that $\lambda = -2 - j$, $j \in \mathbb{N}_0$, and $m > j$.

Upon inserting the standard integral representation for the hypergeometric function (see [5, Equ. 9.111]) into formula (5.1) and using analytic continuation, we obtain

$$a_{k+m,k}(\lambda) = 2^{m/2} \sqrt{\frac{(k+m)!}{k!}} \left\langle t_+^{(m-\lambda)/2-1} \cdot (1-t)_+^{(m+\lambda)/2}, (1-2t)^k \right\rangle, \quad k, m \in \mathbb{N}_0. \quad (5.3)$$

Note that $t_+^{(m-\lambda)/2-1} / \Gamma(\frac{m-\lambda}{2})$ is an entire distribution-valued function of λ which can be multiplied with $(1-t)_+^{(m+\lambda)/2}$ since the singular supports of the two distributions are disjoint. Also note that $(1-t)_+^{(m+\lambda)/2}$ is analytic at the value of λ which we consider.

We then obtain the asymptotic behavior of $a_{k+m,k}(\lambda)$ for $k \rightarrow \infty$ by approximating $(1-2t)^k$ in formula (5.3) by $(1-t)^{2k}$ near zero and by $(-t^2)^k$ near one, respectively. This implies

$$\begin{aligned} a_{k+m,k}(\lambda) &\sim 2^{m/2} \sqrt{\frac{(k+m)!}{k!}} \left[\frac{\Gamma(\frac{m+\lambda}{2} + 2k + 1)}{\Gamma(m + 2k + 1)} + (-1)^k \frac{\Gamma(\frac{m-\lambda}{2} + 2k)\Gamma(\frac{m+\lambda}{2} + 1)}{\Gamma(\frac{m-\lambda}{2})\Gamma(m + 2k + 1)} \right] \\ &\sim (2k)^{m/2} \left[(2k)^{(\lambda-m)/2} + (-1)^k \frac{\Gamma(\frac{m+\lambda}{2} + 1)}{\Gamma(\frac{m-\lambda}{2})} (2k)^{-(m+\lambda+2)/2} \right]. \end{aligned}$$

Hence, eventually,

$$a_{k+m,k}(\lambda) \sim \begin{cases} (2k)^{\lambda/2} & : \operatorname{Re} \lambda > -1, \\ (2k)^{\lambda/2} + (-1)^k \frac{\Gamma(\frac{m+\lambda}{2} + 1)}{\Gamma(\frac{m-\lambda}{2})} (2k)^{-\lambda/2-1} & : \operatorname{Re} \lambda = -1, \\ (-1)^k \frac{\Gamma(\frac{m+\lambda}{2} + 1)}{\Gamma(\frac{m-\lambda}{2})} (2k)^{-\lambda/2-1} & : \operatorname{Re} \lambda < -1, \end{cases} \quad (5.4)$$

for $k \rightarrow \infty$ and $m \in \mathbb{N}_0$ with $m > |2 + \lambda|$ if $\lambda \in -2 - \mathbb{N}_0$. Here the precise meaning of $v_k \sim w_k$ is $\lim_{k \rightarrow \infty} k^{-c}(v_k - w_k) = 0$ for $c = \frac{1}{2} \max(\operatorname{Re} \lambda, -2 - \operatorname{Re} \lambda)$. We remark that the special case of $m = 0$ was shown already in [8, p. 783].

According to the above, if $c_m \neq 0$ for some $m \in \mathbb{N}_0$, then the boundedness of $a_{k+m,k}(\lambda)$ implies $\operatorname{Re} \lambda \in [-2, 0]$ due to formula (5.4). If instead $c_m \neq 0$ for $m < 0$, then we use the boundedness of $a_{k,k+m}(\lambda) = a_{l+|m|,l}(\lambda)$ for $l = k + m$ with $k \geq |m|$. (c) Let us finally show that $F \in \mathcal{H}^{(1/4+\mu/2)}(\mathbb{S}^1)$ for $\mu = \operatorname{Re} \lambda$. First we observe that

$$a_{j0}(\lambda) = 2^{j/2} \frac{\Gamma(\frac{\lambda+j}{2} + 1)}{\sqrt{j!}} = \sqrt[4]{\pi} \sqrt{\frac{\Gamma(\frac{\lambda+j}{2} + 1)^2}{\Gamma(\frac{j+1}{2})\Gamma(\frac{j}{2} + 1)}} \sim \sqrt[4]{\pi} \left(\frac{j}{2}\right)^{1/4+\lambda/2}, \quad j \rightarrow \infty,$$

according to [5, Equ. 8.328.2]. If e_j , $j \in \mathbb{N}_0$, denotes, as before, the orthonormal basis of normalized Hermite functions, see formula (3.1), we infer therefore from

$$W_\sigma e_0 = \sum_{j=0}^{\infty} e_j b_{j0} = \sum_{j=0}^{\infty} e_j a_{j0} c_{-j}$$

and from $W_\sigma \in \mathcal{L}(L^2(\mathbb{R}))$ that $(a_{j0} c_{-j})_{j \in \mathbb{N}_0} \in l^2(\mathbb{N}_0)$ and hence that also the sequence $(c_{-j} (1+j)^{1/4+\mu/2})_{j \in \mathbb{N}_0}$ belongs to $l^2(\mathbb{N}_0)$.

Similarly, if $\phi \in s(\mathbb{N}_0)$ with $\|\phi\|_{l^2(\mathbb{N}_0)} \leq 1$, then

$$\|W_\sigma\|_{\mathcal{L}(L^2(\mathbb{R}))} \geq \left| \left\langle W_\sigma \left(\sum_{k \in \mathbb{N}_0} \phi_k e_k \right), e_0 \right\rangle \right| = \left| \sum_{k=0}^{\infty} a_{0k} c_k \phi_k \right|$$

and thus $(c_k (1+k)^{1/4+\mu/2})_{k \in \mathbb{N}_0} \in l^2(\mathbb{N}_0)$. Altogether we conclude that F belongs to the Sobolev space $\mathcal{H}^{(1/4+\mu/2)}(\mathbb{S}^1)$ and this completes the proof. $\square$

Remark Let us yet comment on the situation for symbols σ that are homogeneous on $\mathbb{R}^2 \setminus \{0\}$, but only associated homogeneous on $\mathbb{R}^2$, i.e., $\sigma = F \cdot r^{-2-j}$, $j \in \mathbb{N}_0$, and $F(\omega) = \sum_{k \in \mathbb{Z}} c_k \omega^k$ with $c_k \neq 0$ for some $k \in \mathbb{Z}$ with $|k| \leq j$. In this case, W_σ cannot be bounded on $L^2(\mathbb{R})$. We have omitted this case in Proposition 3 since the asymptotic expansion in formula (5.4) becomes slightly more complicated. As to be expected, logarithms appear. Just to get an idea, let us analyze the case of homogeneity $\lambda = -2$.

According to the complement formula (5.2), we obtain

$$a_{jk}(-2) = (-1)^{\min(j,k)} \cdot \frac{2a_{jk}(0)}{|j-k|}, \quad j \neq k \in \mathbb{N}_0.$$

In particular, this implies

$$\exists C > 0 : \forall j \neq k \in \mathbb{N}_0 : |a_{jk}(-2)| \leq C(1 + |j-k|)^{-3/4}$$

and hence $W_\sigma \in \mathcal{L}(L^2(\mathbb{R}))$ if $\sigma = F \cdot r^{-2}$, $F = \sum_{k \in \mathbb{Z}} c_k \omega^k$, σ is homogeneous on $\mathbb{R}^2$, i.e., $c_0 = 0$, and $\sum_{k \in \mathbb{Z}} |c_k| (1 + |k|)^{-3/4} < \infty$.

On the other hand, $a_{jj}(-2)$ grows logarithmically for $j \rightarrow \infty$ and thus W_σ is not bounded if $\sigma = r^{-2} = \text{Pf}_{\lambda=-2} r^\lambda$. In fact, setting $m = 0$ and $k = j \in \mathbb{N}$ in formula (5.3) yields

$$\begin{aligned} a_{jj}(-2) &= \text{Pf}_{\lambda=-2} a_{jj}(\lambda) = \text{Pf}_{z=0} a_{jj}(-2-2z) \\ &= \left\langle \text{Pf}_{z=0} \left(\frac{t_+^z}{\Gamma(1+z)} \cdot (1-t)_+^{-1-z} \right), (1-2t)^j \right\rangle \\ &= \left\langle \frac{d}{dz} \left(\frac{t_+^z}{\Gamma(1+z)} \right) \Big|_{z=0} \cdot \text{Res}_{z=0} (1-t)_+^{-1-z} + Y(t)(1-t)_+^{-1}, (1-2t)^j \right\rangle \end{aligned}$$

$$\begin{aligned}
&= \langle (\psi(1) - \log t) \cdot \delta(1-t) + Y(t)(1-t)_+^{-1}, (1-2t)^j \rangle \\
&= \psi(1)(-1)^j + \int_0^1 \frac{(1-2t)^j - (-1)^j}{1-t} dt \\
&= (-1)^j \left[\psi(1) + 2j \int_0^1 (1-2s)^{j-1} \log s \, ds \right] \\
&= (-1)^j \left[\psi(1) + 2j \int_0^1 (1-s)^{2j-2} \log s \, ds + o(1) \right] \\
&= 2(-1)^{j+1} [\psi(2j) - \psi(1)] + o(1).
\end{aligned}$$

Hence $a_{jj}(-2) = 2(-1)^{j+1} [\log(2j) - \psi(1)] + o(1)$ for $j \rightarrow \infty$. (Here the terms $o(1)$ denote series converging to 0).

6 A sufficient condition ensuring the boundedness of W_σ

When comparing the three examples $\lambda = 0$ (in Proposition 2), $\lambda = -1$ (in the remark following Proposition 2) and $\lambda = -2$ (in the remark following Proposition 3), we see that we have to stipulate an l^1 condition on the Fourier coefficients c_k of the characteristic F of the symbol $\sigma = F \cdot r^\lambda$ in order to guarantee that W_σ is bounded. The following proposition incorporates these special cases into a general statement.

Proposition 4 *Let $F(\omega) = \sum_{k \in \mathbb{Z}} c_k \omega^k \in \mathcal{D}'(\mathbb{S}^1)$ and $\lambda \in \mathbb{C}$ such that $\mu = \operatorname{Re} \lambda \in [-2, 0]$ and assume that $\sigma = F \cdot r^\lambda$ is homogeneous on $\mathbb{R}^2$. (This just means that c_0 vanishes if $\lambda = -2$.) Then the following holds:*

If $\sum_{k \in \mathbb{Z}} |c_k| (1 + |k|)^{1/4 + \mu/2} < \infty$, then W_σ is a bounded operator on $L^2(\mathbb{R})$.

Proof (a) We have dealt with the special case of $\lambda = -2$ in the remark following Proposition 3. We can therefore assume that $\lambda \neq -2$. According to condition (3.5) in Proposition 1, we then have to verify the following estimate:

$$\exists C > 0 : \forall j, k \in \mathbb{N}_0 : |a_{jk}(\lambda)| \leq C(1 + |j - k|)^{1/4 + \mu/2}. \quad (6.1)$$

Note that the asymptotic expansion in formula (5.4) already shows that $a_{jj}(\lambda)$, $j \in \mathbb{N}_0$, is a bounded sequence. Therefore it suffices to prove the estimate in (6.1) for $j > k$.

(b) Let us define now

$$\tilde{a}_{jk}(\lambda) := \int_0^\infty \frac{(1-s)^k}{(1+s)^{j+1}} s^{(j-k-\lambda)/2-1} ds, \quad j, k \in \mathbb{N}_0, \quad j > k. \quad (6.2)$$

We point out that the integral in (6.2) is absolutely convergent for $\mu = \operatorname{Re} \lambda \in [-2, 0]$.

Due to [5, Eqs. 3.197.1 and 9.131.1] the integral in (6.2) yields the following:

$$\tilde{a}_{jk}(\lambda) = \frac{\Gamma(\frac{j-k-\lambda}{2}) \Gamma(\frac{\lambda+j-k}{2} + 1)}{(j-k)!} \cdot {}_2F_1(-k, \frac{j-k-\lambda}{2}; j-k+1, 2). \quad (6.3)$$

Combining formulas (5.1) and (6.3), we obtain

$$a_{jk}(\lambda) = 2^{(j-k)/2} \sqrt{\frac{j!}{k!}} \cdot \frac{\tilde{a}_{jk}(\lambda)}{\Gamma(\frac{j-k-\lambda}{2})}, \quad j, k \in \mathbb{N}_0, \quad j > k, \quad \mu = \operatorname{Re} \lambda \in [-2, 0]. \quad (6.4)$$

(c) Let us estimate $\tilde{a}_{jk}(\lambda)$ for $j > k$, $\mu = \operatorname{Re} \lambda \in [-2, 0]$. By dividing the range of integration we obtain

$$\begin{aligned} \tilde{a}_{jk}(\lambda) &= \int_0^1 \frac{(1-s)^k}{(1+s)^{j+1}} s^{(j-k-\lambda)/2-1} ds + \int_0^1 \frac{(1-s^{-1})^k}{(1+s^{-1})^{j+1}} s^{-(j-k-\lambda)/2-1} ds \\ &= \int_0^1 \frac{(1-s)^k}{(1+s)^{j+1}} s^{(j-k)/2-1} [s^{-\lambda/2} + (-1)^k s^{\lambda/2+1}] ds \end{aligned}$$

and hence

$$|\tilde{a}_{jk}(\lambda)| \leq 2 \int_0^1 \frac{(1-s)^k}{(1+s)^{j+1}} s^{(j-k)/2-1} ds.$$

The standard integral representation [5, Equ. 9.111] of the hypergeometric function therefore yields

$$|\tilde{a}_{jk}(\lambda)| \leq 2 \frac{\Gamma(\frac{j-k}{2}) k!}{\Gamma(\frac{j+k}{2} + 1)} \cdot {}_2F_1(j+1, \frac{j-k}{2}; \frac{j+k}{2} + 1; -1).$$

Upon inserting [2, Equ. 7.3.6.1] we obtain the following estimate of $|\tilde{a}_{jk}(\lambda)|$ by an upper bound independent of λ and valid for $j > k$, $\mu = \operatorname{Re} \lambda \in [-2, 0]$:

$$|\tilde{a}_{jk}(\lambda)| \leq 2^{-j} \Gamma\left(\frac{j-k}{2}\right) k! \sqrt{\pi} \left[\frac{1}{\Gamma(\frac{j}{2} + 1) \Gamma(\frac{k+1}{2})} + \frac{1}{\Gamma(\frac{j+1}{2}) \Gamma(\frac{k}{2} + 1)} \right]. \quad (6.5)$$

(d) Combining formulas (6.4) and (6.5) and using the duplication formula for the gamma function yields, still for $j > k$, $\mu = \operatorname{Re} \lambda \in [-2, 0]$,

$$|a_{jk}(\lambda)| \leq \frac{\Gamma(\frac{j-k}{2})}{|\Gamma(\frac{j-k-\lambda}{2})|} \left(\frac{\alpha_j}{\alpha_k} + \frac{\alpha_k}{\alpha_j} \right).$$

Here $\alpha_j = \sqrt{\Gamma(\frac{j}{2} + 1) / \Gamma(\frac{j+1}{2})}$ is defined as in Proposition 2. The limit [5, Equ. 8.328.2] implies

$$\exists C > 0 : \forall j, k \in \mathbb{N}_0 \text{ with } j \geq k : \frac{\Gamma(\frac{j-k}{2})}{|\Gamma(\frac{j-k-\lambda}{2})|} \leq C(1+j-k)^{\mu/2}$$

with $\mu = \operatorname{Re} \lambda$. Hence the estimate of α_j / α_k in formula (4.6) and the monotonicity of α_j imply

$$\exists C > 0 : \forall j, k \in \mathbb{N}_0 \text{ with } j > k : |a_{jk}(\lambda)| \leq C(1 + j - k)^{1/4 + \mu/2}.$$

Therefore, the estimate (6.1) holds in generality and this completes the proof. $\square$

Remark Let us finally compare the assumption

$$F \in L^2(\mathbb{S}^1), \text{ i.e., } (c_k)_{k \in \mathbb{Z}} \in l^2(\mathbb{Z}) \quad (\text{A})$$

with the hypothesis made in Proposition 4, namely

$$((1 + |k|)^{1/4 + \mu/2} c_k)_{k \in \mathbb{Z}} \in l^1(\mathbb{Z}). \quad (\text{B})$$

At the end of Sect. 2, we have seen that in the case of $-2 < \mu = \operatorname{Re} \lambda < -1$, assumption (A) implies the boundedness of W_σ on $L^2(\mathbb{R})$ for $\sigma = F \cdot r^\lambda$. Instead, assumption (B) implies that W_σ is bounded whenever $\mu \in [-2, 0]$ according to Proposition 4.

If $-2 < \mu < -\frac{3}{2}$, then $((1 + |k|)^{1/4 + \mu/2} c_k)_{k \in \mathbb{Z}} \in l^2(\mathbb{Z})$ and hence (A) implies (B). So condition (B) is the less restrictive and more general one. In contrast, when $-\frac{3}{2} \leq \mu < -1$, then neither of the two assumptions (A) and (B) implies the other one. In fact, e.g., for $\mu = -\frac{3}{2}$ and $c_k = (1 + |k|)^{-1/2} / \log(1 + |k|)$, condition (A) holds whereas (B) fails. Conversely, if $\mu = -\frac{3}{2}$ and $c_k = 2^{j/2} j^{-2}$ for $k = 2^j$, $j \in \mathbb{N}$, and $c_k = 0$ for $k \notin \{2^j; j \in \mathbb{N}\}$, then

$$\sum_{k \in \mathbb{Z}} (1 + |k|)^{-1/2} |c_k| = \sum_{j=1}^{\infty} (1 + 2^j)^{-1/2} 2^{j/2} j^{-2} \leq \sum_{j=1}^{\infty} j^{-2} < \infty$$

but $(c_k)_{k \in \mathbb{Z}} \notin l^2(\mathbb{Z})$. Hence (B) holds and (A) fails.

As was pointed out already in the introduction, in contrast to the case of radially symmetric distributions, we cannot hope for a condition that is both necessary and sufficient for the boundedness of W_σ if σ is a tensor product in polar coordinates. This is a consequence of the fact that the Hermite functions in this case do not yield eigenfunctions of W_σ . So the approach in Proposition 1 has its limitations from the word go.

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