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Prospective secondary teachers' noticing of students' thinking about the limit concept: pathways of development

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Abstract
Research has shown that there is a need to examine prospective teachers' development trajectories related to noticing expertise. An important content in the Spanish high school curriculum (16–18 years old) is the limit concept. Given the importance of this concept in the curriculum and the difficulties some prospective teachers have, developing their noticing of students' mathematical thinking of this concept in teacher education programs is crucial to achieve high school student mathematics achievement. This study examines how prospective secondary school mathematics teachers (PTs) notice students' mathematical thinking about the limit concept as they participated in a teaching module. PTs had to anticipate and interpret students' mathematical understanding and make instructional decisions to support students' conceptual progression using information about high school students' understanding of the limit concept. We examined PTs changes related to how they anticipated, interpreted and made instructional decisions during the teaching module. We identified a change in how PTs conceived the understanding of the dynamic limit concept: from all-or-nothing dichotomy to progression; and a change in the instructional decisions they made: from decisions focused on changing the type of discontinuity to conceptual decisions. These changes allow us to characterise development noticing pathways. Our findings also help to identify the teaching module characteristics that support the development of PTs noticing.

Keywords Prospective secondary teachers · Noticing · Limit · Development

1 Introduction

Research shows that a teacher's ability to elicit student thinking in mathematics instruction is positively associated with student mathematics achievement (Jacobs et al., 2010). Teachers must identify and interpret students' mathematical understanding and take instructional decisions based on these students' understanding. Therefore, it is crucial for any teacher to be able to notice students' mathematical thinking in classroom situations. In the words of Schoenfeld (2011) "what you see and don't see shapes what you do and don't do" (p. 228).

Research has significantly increased in recent years in the field of noticing (recent reviews in Dindyal et al., 2021; König et al., 2022; Weyers et al., 2023a, 2023b). Sherin et al. (2011) laid out five core questions regarding the field of noticing, including: Is teacher noticing trainable?, and what development trajectories related to noticing expertise exist for prospective and practising teachers?

Regarding the first question, research has shown that noticing can be developed in teacher training programmes (e.g., Dindyal et al., 2021; Fernández et al., 2018b) and has demonstrated that certain tools and contexts can support this development. For example, the use of representations of practice (Buchbinder & Kuntze, 2018) understood as a depiction of a classroom situation in different formats such as videos (van Es & Sherin, 2008), written work (Fernández et al., 2013; Sánchez-Matamoros et al., 2015) or animations and comics (Herbst & Kosko, 2014) are useful to promote prospective and in-service teachers' noticing. Another useful tool is the use of research syntheses in mathematics

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education linked to the students' understanding of a mathematical concept (theoretical lens, Fernández & Choy, 2020). They equip prospective teachers with a specific language to refer to a student's understanding, helping them to focus on relevant aspects of students' understandings (Fernández et al., 2018b; Moreno et al., 2021).

With regard to the second question: what development trajectories related to noticing expertise exist for prospective and practising teachers? Recently, related to this question, König et al. (2022) noted the research gap and the necessity to determine "factors influencing the change or growth of teacher attention over the course of interventions" (p. 14). We thus need to explore and characterise how prospective mathematics teachers and practising teachers develop noticing during instruction. Such studies could help to identify prospective teachers' development trajectories relating to noticing expertise.

Some previous studies have done some attempts to identify characteristics of this development. Fernández et al. (2013) characterised a four-level developmental sequence to describe how pre-service primary school teachers (6–12 years old) learn to attend to and interpret students' mathematical thinking when analysing students' written work on proportional reasoning tasks. Sánchez-Matamoros et al. (2015) provided noticing development descriptors that reflected how prospective secondary school teachers (12 -18 years old) recognised different levels of student understanding of the derivative concept. This characterisation of how noticing develops was linked to the prospective teachers' progressive understanding of the mathematical elements used by students to solve problems. Krupa et al. (2017) demonstrated the impact of a curricular module on prospective teacher noticing of student thinking related to the solving of linear equations. The findings indicated gains in the prospective teachers' ability to attend to and interpret student thinking, but no changes were found in their ability to decide how to respond based on student understanding. Van den Kieboom et al. (2017) found that prospective teachers mainly noticed the strategies used by their students to solve a task without focusing on student thinking about equal sign and equality. These results were obtained after prospective teachers had participated in an instructional intervention. Moreno et al. (2021) described prospective kindergarten teacher's (3–6 years old) use of a magnitude-length learning trajectory and its measurement to interpret students' mathematical understanding and make instructional decisions. This development "can be understood as a process of instrumentation that reveals how noticing skills develop" (p. 57).

Recently, in the specific content of the limit concept, Scheiner (2023) has investigated prospective teachers framing and noticing of students' mathematical thinking. In this study, aspects of students' thinking that the prospective teachers paid attention to has been identified, as well as the stances they took when interpreting students' thinking, and the instructional moves they proposed. This study provides information with regard to the changes in teachers' framing and noticing of students' thinking during a course.

An important content in the Spanish high school curriculum (16–18 years old) is the limit of a function. The concept of limit of a function acquires relevance in the curriculum not only because it is necessary to construct other concepts of calculus (Cornu, 1991), but also because it provides meaning to phenomena such as trends over time in population dynamics or the instantaneous velocity of a mobile. The latter is important since the current Spanish high school curriculum is competency-based and promote the understanding of conceptual and procedural knowledge needed for the resolution of STEM problems in real and intra-mathematical contexts.

However, some prospective mathematics teachers understand the concept of limit of a function as unreachable functions so the metric definition of limit is not well understood (Sulastri et al., 2021). In the university context, tables of values and graphs of functions are rarely used to solve limit problems, and prospective teachers preferred to solve them by direct substitution in the metric definition. However, both the numerical and graphical modes of representation are common in high school education, for understanding the concept of limit.

Given the importance of this concept in the Spanish curriculum and the difficulties some prospective teachers have, developing their noticing of students' mathematical thinking in this concept in teacher education programs in crucial to achieve high school student mathematics achievement. Therefore, our objective is examining the development of prospective mathematics secondary school teachers' (12–18 years old) noticing of students' mathematical thinking about the limit of a function as they participated in a teaching (instruction) module.

In the following section, we review the literature on high school student understanding of the limit concept, and we conceptualise noticing of student mathematics thinking about the limit concept in our study.

2 Theoretical background

2.1 Characteristics of high school student understanding of the limit of a function at a point

Several studies have shown that the concept of limit is a difficult notion for high school students (Fernández-Plaza & Simpson, 2016; Kidron, 2010). Cottrill et al. (1996) suggest that students' difficulties in understanding the metric conception of the limit

$$\lim_{x \to a} f(x) = L \Leftrightarrow \forall\, \varepsilon > 0, \exists\, \partial > 0 \text{ such that } \forall\, x,\ 0 < |x - a| < \partial \Rightarrow |f(x) - L| < \varepsilon$$

may result from a limited understanding of the dynamic conception: when the value in the domain x approaches to the real number a its image $f(x)$ approaches to L *(being L the limit of the function f in a point a)*. They consider that to reach the metric conception of the limit, it is necessary to construct the notion of domain and range approximation, as well as to learn to use the function to coordinate the approximations (dynamic conception; Oehrtman, 2009).

Representation modes (graphical, algebraic, and numerical) also play a key role in the understanding of the limit of a function (Pons, 2014). Monaghan (2001) indicates that the notion of limit presented in graphical mode is easier to understand than that presented in numerical mode. Moru (2009) shows that in graphical mode, students denied the existence of a limit where the function was not defined, did not interpret piecewise functions adequately, or needed the formula to find the limit value. Consequently, understanding the concept of limit in one mode of representation does not necessarily imply a complete understanding of the limit concept (Elia et al., 2009).

Therefore, to understand the dynamic limit concept of a function at a point (used in our study), coordinating domain and range approximation processes in the different representation modes (graphical, algebraic and numerical) is key. Valls et al. (2011) and Pons (2014) identified the mathematical elements needed for the understanding of the dynamic limit concept: (i) function; (ii) domain approximations (x approaches a) and range approximations ($f(x)$ approaches L); (iii), coordination of the domain and range approximations through function f (when x approaches a, $f(x)$ approaches L). For instance, in the problem 1 of Fig. 3, the coordination of the domain and range approximations around the point $x=2$ implies: when x approaches 2 from the left, the image approaches 4 and when x approaches 2 from the right, the image approaches 4. Then, when x approaches 2, the image approaches 4.

These authors characterised levels of understanding from the perspective of the coordination of the range and domain side approximations in the graphical, algebraic and numerical representation modes (Fig. 1). The characteristic that marks the transition from one level of understanding to an advanced one is the coordination of the domain and range approximations when the side approximations (approximations from the right and from the left) do not coincide.

In this study, we considered the mathematical elements involved in the understanding of the dynamic limit concept identified by Pons (2014) and Valls et al. (2011) as well as the levels of understanding in the design of the teaching module.

2.2 Noticing student mathematical thinking of the limit of a function at a point

Our conceptualisation of noticing is based on that of professional noticing of student mathematical thinking set out by Jacobs et al. (2010). The latter determined three interrelated skills: (i) attending to students' strategies; (ii) interpreting students' mathematical understandings; and (iii) deciding how to respond based on students' understanding. Research on teaching practice has also noted the importance of prospective teachers' knowledge of students' thinking in terms of anticipating students' answers (Fernández et al., 2018a;

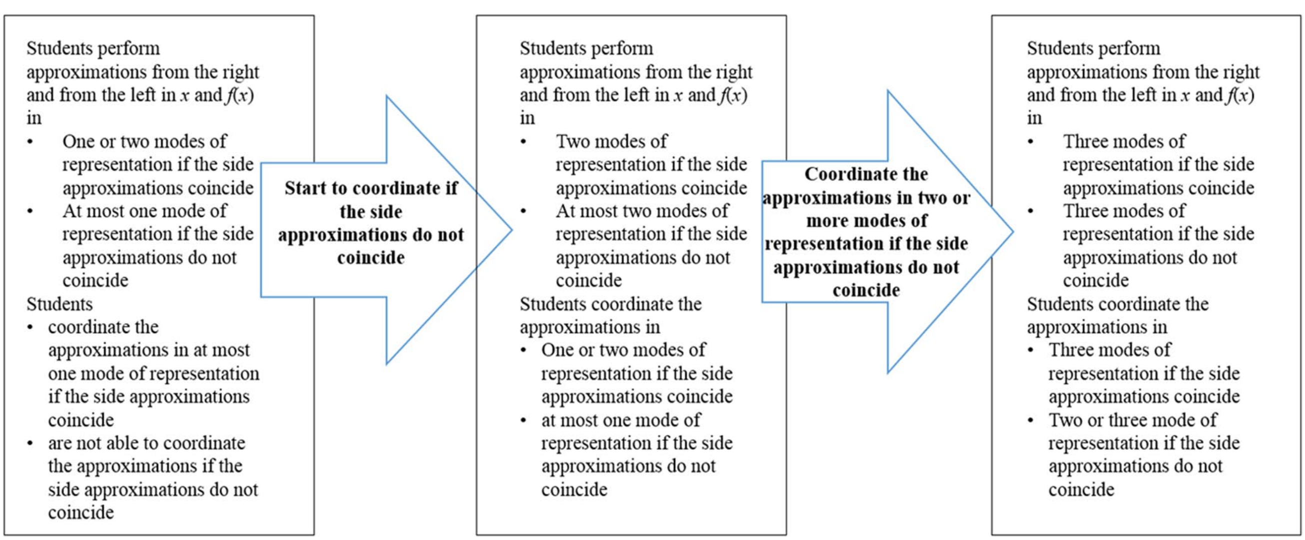


Fig. 1 Levels of understanding of the dynamic limit concept from Pons (2014)

Llinares et al., 2016). Anticipating and interpreting students' mathematical understanding are core-teaching tasks (Llinares et al., 2016; Stein et al., 2008). Anticipating includes predicting how students might interpret and solve mathematical activities (Stein et al., 2008).

When prospective teachers anticipate student answers to problems on the limit of a function, they need to consider the mathematical elements implied in the understanding of the concept and when they attend to students' answers they have to find evidence of student understanding of these elements: (i) function; (ii) domain and range side approximations; and (iii), coordination of the domain and range approximations through function *f*.

When prospective teachers anticipate or interpret student mathematical understanding, they need to link the mathematical elements they have considered in the anticipation or have attended in students' answers to the knowledge that corresponds to the student levels of understanding (Fig. 1). For instance, if the high school student has coordinated the approximations from the right and from the left in the domain and in the range in only the numerical mode of representation when the side approximations coincide, prospective teachers have to interpret that this high school student shows characteristics of level 1 of understanding.

Prospective teachers must propose instructional decisions that help students to transition towards a more advanced level of understanding considering their interpretation. For example, an instructional decision that could help the previous high school student is using different modes of representation (numerical and graphical) to start to coordinate the approximations in different representations modes and using functions where the sides approximations do not coincide in some points.

Our study is based on the following hypothesis: if prospective teachers focus on Key Developmental Understandings (KDUs) (Simon, 2006) of the dynamic limit concept of a function, they will generate hypotheses on how their students' mathematical understanding is developing. As a result, they might enhance their interpretations of their students' mathematical thinking and take instructional decisions based on their students' understandings.

A KDU corresponds to a student's conceptual change, i.e., a shift in the student's ability to think about and/or perceive certain mathematical relationships (Simon, 2006). A KDU in the understanding of the dynamic limit concept is the coordination of domain and range approximations across different representation modes (Pons, 2014; Valls et al., 2011). From this perspective, in order to anticipate or interpret students' mathematical understanding and to propose instructional decisions that could help to transition between levels, it is necessary that prospective teachers focus on this KDU (Llinares et al., 2016).

Thus, in line with the objective, we formulated the following research questions:

- How prospective teachers notice students' mathematical thinking about the limit of a function during their participation in a teaching module?
- What are the characteristics of the teaching module that support prospective teachers' noticing?

3 Method

3.1 Participants and context

The participants were 25 PTs enrolled in a University Master's Degree at the University of Alicante (Spain) to become a secondary mathematics teacher. These PTs can teach in secondary education (13–16 years old) and in high education (16–18 years old). The background of these PTs was diverse: mathematicians, physicists, engineers and architects. These PTs did not have experience as teachers and they had not participated in any other subject related to mathematics education, previously.

One of the Master's subjects in the mathematics speciality is *Learning Mathematics in Secondary School.* PTs are enrolled in this subject before their practices at secondary schools. An objective of this subject is to develop the prospective teachers' noticing skill of students' mathematical thinking. The subject is composed of several teaching (instruction) modules which address students' understanding of different mathematical concepts. The teaching modules "are composed of sessions in which theoretical documents are provided (based on research in Didactics of Mathematics) as well as professional tasks" (Fernández et al., 2018b, p.47). The study presented here focuses on the teaching module of the limit concept. This concept should be explained by PTs in high education (16–18 years old).

3.2 Teaching module: learning to notice student understanding of the limit of a function at a point

The main objective of this teaching module is that PTs learn to identify characteristics of high school student understanding of the limit of a function at a point and provide instructional decisions that will help high school students to progress in their understanding. The teaching module comprised four 2-h sessions (Fig. 2). PTs participated in groups of five (a total of five groups: G1, G2, G3, G4 and G5). The PTs themselves formed the groups without any specific criteria.

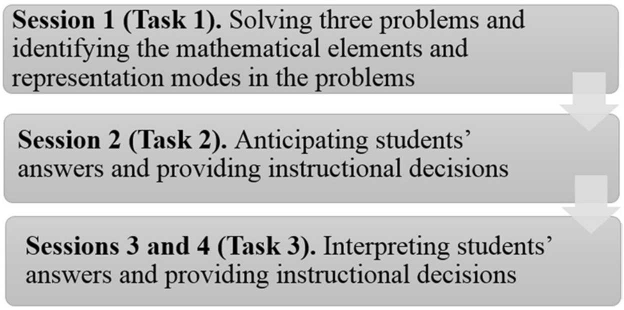


Fig. 2 Overview of the sessions in the teaching module

3.2.1 Session 1. Task 1: Solving three problems and identifying the mathematical elements and representation modes in the problems

PTs had to solve three problems related to the limit of a function at a point concept (adapted from a textbook) indicating the mathematical elements and representation modes used to solve them (Fig. 3). We provided PTs with a document (a theoretical lens) that included the definition of the limit based on the dynamic concept, as well as the relevant mathematical elements for its understanding (information in Section 2.1., Valls et al., 2011 and Pons, 2014).

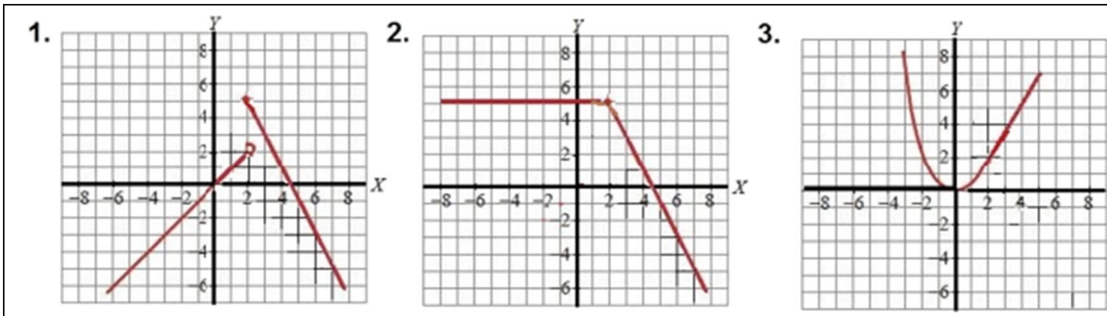
Problem 1 was presented in algebraic mode with non-coincident side limits at $x=1$ (section a) and coincident side limits at $x=2$ (section b). The mathematical elements involved in the resolution are: (i) a piecewise-defined function; (ii) the domain side approximations at $x=1$ and $x=2$, and range side approximations to determine the behaviour of $f(x)$ around $f(x)=3$ and $f(x)=4$; (iii) the coordination of the domain and range approximations around the points $x=1$ and $x=2$ (e.g., when x approaches 1 from the left, $f(x)=3$).

Problem 2 was presented in numerical mode. The sided limits coincide in the first table and do not coincide in the second. The mathematical elements involved in the resolution are: (i) the given functions; (ii), the domain and range approximations. In the first table, x_1 approaches 1 from the left and from the right (section a1) while in the range, $f(x_1)$ approaches 2 from the left and from the right (section a2). In the second table, x_2 approaches 1 from the right and from the left, and $g(x_2)$ approaches -1 from the left and 2 from the right (sections a1 and a3); and lastly, (iii), the coordination of domain and range approximations (e.g., when x_1 approaches 1 from the left, $f(x_1)=2$; sections b1 and b2).

Finally, problem 3 was presented in graphical mode. In the first graph, the side limits do not coincide at $x=2$. In the second and third graphs, the side limits coincide, so the function limit exists at $x=2$. The mathematical elements involved in the resolution are: (i) the given functions; (ii) the domain approximations (e.g., in graph 1, x approaches 2 from the left and from the right) and the range approximations (e.g., in graph 1, $f(x)$ approaches 2 from the left and 5 from the right); and (iii), the coordination of the domain and range approximations (e.g., in graph 1, when x approaches 2 from the left, $f(x)$ approaches 2, and when x approaches 2 from the right, $f(x)$ approaches 5).

3.2.2 Session 2. Task 2: Anticipating students' answers and providing instructional decisions

PTs had to anticipate high school student answers to the problems previously solved and that reflected different characteristics of students' understandings. They also had to propose new problems that would help high school students progress in their understanding. They had to answer the following questions:



a) The limit of the function in x= 2 is 2
b) The limit of the function in x= 2 is 5
c) There is no limit of the function in x= 2

Fig. 3 Session 1 problems (Colera et al., 2008)

- Indicate exactly what Maria (hypothetical student) would have to say and do in each problem to show that she achieved the learning objective (the understanding of the limit of a function at a point concept). Justify your answer based on the mathematical elements and representation modes.
- Indicate exactly what Pedro (hypothetical student) would have to say and do in each problem to show an understanding of certain elements of the limit of a function at a point concept while remaining unable to achieve the learning objective. Justify your answer based on the mathematical elements and representation modes.
- If you were the teacher of these students, how would you modify/extend the problems to confirm that Maria achieved the learning objective? How would you modify/extend the problems so that Pedro achieves the intended learning objective? Justify your answer.

3.2.3 Sessions 3 and 4. Task 3: Interpreting students' answers and providing instructional decisions

The PTs had to interpret the answers given by four high school students (Pablo, Rebecca, Luiggi and Jorge) to the same three problems (representation of practice) and to propose new problems (or modify them) to help students progress in their understanding. The answers of the four high school students presented different levels of understanding of the dynamic limit concept of a function at a point (Pons, 2014; Fig. 1 Section 2.1). Pablo (Fig. 4) and Luiggi are in the higher level (level 3) since they coordinate de domain and range approximations in three representation modes (algebraic, numerical and graphical). They only differ in making explicitly (Luiggi) or not (Pablo) the existence of limit. Rebecca (Fig. 5) is in the lower level (level 1) since she shows evidence of coordinating the domain and range approximations in the graphical mode (and when the side limits coincide). Jorge (Fig. 6) is in the medium level (level 2) since he shows evidence of coordinating the domain and range approximations in numerical and graphical modes (the latter only when side limits coincide).

In this task, the PTs had to answer the following questions:

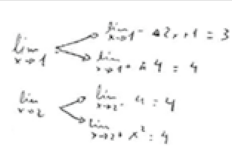
- In each problem, describe what mathematical elements of the dynamic limit concept were used by student X to solve them. Indicate whether student X encountered any difficulties and why.
- Based on the descriptions of how student X solved the three problems, is it possible to identify any characteristics of how student X understands the concept of the limit of a function at a point? Justify your answer considering the mathematical elements and representation modes.
- Considering student X's understanding of the limit of a function at a point, propose a new problem (or modify one of the given problems) to help high school students progress in their understanding. Justify your answer.

PTs were provided with a document (a theoretical lens) with the levels of understanding of the dynamic limit concept from Pons (2014) and presented in Fig. 1.

3.3 Analysis

The study data corresponded to the answers of the five groups of PTs to task 2 and task 3. Using an inductive analysis (Strauss & Corbin, 1994), categories were generated on how these groups of PTs anticipated high school students' answers, and how they interpreted the students' answers. Categories were also generated based on the different instructional decisions provided to help students progress in their understanding (for both the anticipation and interpretation tasks). Four researchers first read the answers and generated initial categories from a small sample of data. Then, new data were added and discussed to revise the initial category system. All data were analysed and coded by the four researchers until we reached a 100% of agreement in the final categories.

Fig. 4 Pablo's answers (Pablo's handwritten answers have been translated into English)

<table>
<tr><td><u>Problem 1</u></td><td><u>Problem 2</u></td></tr>
<tr><td></td><td>a1) x_1 approaches 1 from the left and from the right
x_2 approaches 1 from the left and from the right
a2) Approaches 2 from the right and 2 from the left
a3) Approaches 2 from the right and -1 from the left

b1) when x_1 approaches 1, $f(x_1)$ approaches 2
b2) from the left, x_2 approaches 1 and $g(x_2)$ approaches -1. From the right, x_2 approaches 1 and $g(x_2)$ approaches 2.</td></tr>
<tr><td colspan="2"><u>Problem 3</u>
a) Graph 3 because the limit in x= 2 exists and from the right and from the left is 2.
b) Graph 2 because the limit in x = 2 exists and from the right and from the left is 5.
c) Graph 1 because the limit in x =2 does not exit.</td></tr>
</table>

Fig. 5 Rebecca's answers (Rebecca's handwritten answers have been translated into English)

Problem 1	Problem 2
a) $\lim_{x \to 1} f(x) = \lim_{x \to 1} 2x+1 = \boxed{3}$ b) $\lim_{x \to 2} f(x) = \lim_{x \to 2} 4 = \boxed{4}$	a) 1. x_1 approaches 0 from the left and x_1 approaches 1 from the right x_2 approaches 0 from the left and x_2 approaches 1 from the right 2. $\lim_{x \to x_1^-} f(x_1) = \boxed{1}$ $\lim_{x \to x_1^+} f(x_1) = \boxed{2}$ 3. $\lim_{x \to x_2^-} g(x_2) = \boxed{0}$ $\lim_{x \to x_2^+} g(x_2) = \boxed{2}$ b) 1. $x_1^- = 0$ $\quad x_1^+ = 1$ $f(x_1)^- = 1$ $\quad f(x_1)^+ = 2$ 2. $x_2^- = 0$ $\quad x_2^+ = 1$ $f(x_2)^- = 0$ $\quad f(x_2)^+ = 2$ coinciden.
Problem 3 a) Graph 3 because the limit from the right and from the left is 2. b) Graph 2 because the limit from the right and from the left is 5. c) Graph 1 because the limit from the right and from the left do not coincide.	

Fig. 6 Jorge's answers (Jorge's handwritten answers have been translated into English)

Problem 1	Problem 2
a) 2 because the 1 is replaced in $2x+1$ since $x \leq 1$ b) 4 because 2 is replaced in 4 since $1<x\leq 2$	a1) x_1 approaches 1 from the left and from the right x_2 approaches 1 from the left and from the right a2) Approaches 2 from the right and from the left a3) Approaches 2 from the right and -1 from the left b1) when x_1 approaches 1, $f(x_1)$ approaches 2 b2) $g(x_2)$ approaches -1 from the left and it approaches 2 from the right when x_2 approaches 1.
Problem 3 a) Graph 3 because the limit coincides from the right and from the left. b) Graph 2 because the limit coincides from the right and from the left. c) Graph 1.	

Two categories emerged from the analysis of how PTs anticipated and interpreted students' answers. These categories depended on how PTs conceived the understanding of the dynamic limit concept of a function at a point (Table 1):

In relation to the instructional decisions provided by PTs, three categories emerged (Table 2):

Based on the categories that emerged in relation to how PTs conceived student understanding and the instruction decisions, we examined the changes that took place between the anticipation task and the interpretation task. Examples of these changes are shown in the results section.

4 Results

We identified a change in how PTs conceived the understanding of the dynamic limit concept of a function at a point (from all-or-nothing dichotomy to progression) as well as, a change in the instructional decisions they made

Table 1 Categories of how PTs conceived student understanding

Category	Descriptors
All-or-nothing dichotomy	PTs consider only two types of high school student understanding: • students who understand everything, i.e., students able to coordinate domain and range approximations in the different representation modes, • students who do not understand at all, so they fail to coordinate domain and range approximations in any mode of representation
Progression	PTs consider progressions in high school student understanding, as demonstrated by the fact of: • relating lack of understanding to the idea of not coordinating domain and range approximations in some representation modes

Table 2 Categories of instructional decisions given by PTs

Category	Descriptors
General decisions	No specific activities are proposed
Decisions focused on changing the type of discontinuity	Activities focused on changing the type of discontinuity
Conceptual decisions	Activities that consider the cognitive processes needed to transition between understanding levels in the same type of discontinuity (the coordination of approximations in the domain and range in different modes of representation and the use of the coordination in new situations)

(from decisions focused on changing the type of discontinuity to conceptual decisions). We also identified a group that did not change the conception (all-or nothing).

4.1 No change: All-or-nothing conception in the anticipating and interpreting tasks

One group (G3) did not change how they conceived the understanding of the dynamic limit concept of a function throughout the teaching module. This group conceived it as an all-or-nothing dichotomy: i.e., that high school students understand the concept when they are able to coordinate domain and range approximations in cases where the side limits coincide or not across the different representation modes. And students do not understand the concept if they only consider side approximations in cases where the function is defined at a point (therefore, they do not coordinate domain and range approximations in any mode of representation). Thus, the G3 anticipated (task 2) that Maria, in her answer, would coordinate the domain and range approximations in the three representation modes and that Pedro would only perform the side approximations when the function is defined at the point.

This PT group was characterised by their incorrect or rhetorical use of the mathematical elements of the dynamic limit concept of a function at a point to justify high school student understanding in the anticipated answer. Furthermore, they did not identify the coordination of domain and range approximations in the different representation modes as KDU. Thus, when justifying Maria's anticipated answer in algebraic mode (Fig. 7), this PT group confused

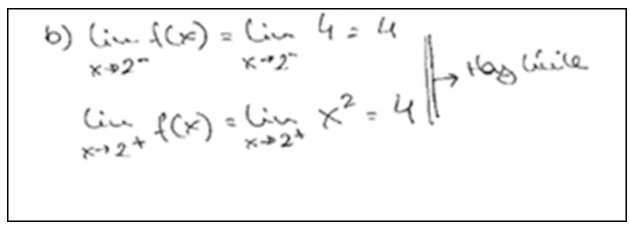


Fig. 7 G3's anticipated answer for Maria in algebraic mode [English translation: "The limit exits"]

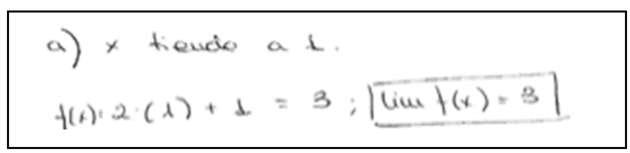


Fig. 8 G3's anticipated answer for Pedro in algebraic mode [English translation: "x approaches 1"]

the coordination of range and domain approximations with the existence or not of a limit. They stated that *"the fact of ordering the resolution on one side and on the other (referring to the side limits) indicates that she knows how to coordinate the approximations"*.

When justifying Pedro's answer in the algebraic mode, these PTs used the mathematical elements in a rhetorical way. Thus, they indicated, without relating the mathematical elements to Pedro's anticipated answer (Fig. 8), that *"Pedro shows that he has some (but not complete) knowledge of the function concept. He only calculates the limit from the left* *(he has no knowledge of the notion of approximation on both sides of the domain). He has no knowledge of the domain and range approximation processes".*

In the interpretation task, they were not able to recognise the characteristics of each high school students' understanding. This PT group continued using mathematical elements incorrectly or rhetorically when describing the high school students' answers without identifying the coordination of domain and range approximations in the different representation modes as a KDU. For example, in Pablo's answer to the numerical and graphical problems, the PTs did not correctly identify how the high school student used the domain and range approximations and the coordination of these approximations. They merely provided general comments on the correctness of the answers and used the mathematical elements rhetorically [they associated the side approximations in the domain and in the range with the side limits]:

> [G3, numerical mode]. The student understands domain and range approximations in isolation, separately. He encounters difficulties when both side approximations need to be coordinated.
> [G3, graphic mode]. Graphically, he performs the exercise correctly, and demonstrates that he understands the concept of limit visually, but he is not analytically capable of conducting the side approximations in the range.

This PT group also failed to identify the characteristics of students understanding. For example, for Pablo, they wrote:

> He understands the idea of function and limit graphically, but in the algebraic representation and in the tables, he understands side approximation in isolation, separately.

In addition, they advanced general decisions, without specifying a particular exercise or problem:

> [Decision for Pablo] We would use the same graphical exercise and explain it in the two other ways [referring to the algebraic and numerical modes], so that he can relate them.

4.2 Change in the conception of understanding "from all-or-nothing dichotomous to progression": KDU identification

Two PT groups (G2 and G5) conceived the understanding as an "all or nothing" dichotomy in the anticipation task. However, in the interpretation task, their conception of understanding changed, and they conceived a progression in the high school students' understanding.

These PT groups were characterised by being able to justify the answers they anticipated for Pedro and Maria using the mathematical elements. They were unable, however, to identify progression in the students' understanding as they did not identify the coordination of domain and range approximations in the different representation modes as a KDU.

For example, G2 anticipated that in her answers, Maria would coordinate the variable x approximations at the different intervals of definition and the function $f(x)$ values in the three representation modes. The coordination between the domain and range approximations is reflected in the algebraic and graphical modes in the recognition of the function branches at the different intervals and the establishment of the function limit according to whether the side limits coincide (Fig. 9). In numerical mode, this coordination is visible in the coordination of the domain and range approximations in sections b1 and b2.

These PTs provided the justification below:

> [Maria] She demonstrates that she grasps the concept of function as she used it correctly throughout the exercise. We would have considered that she had not understood the function concept if she had always chosen the same or wrong branch in the example. The notion of domain approximations corresponds to the fact that she selects the correct function branch. She demonstrates her understanding of range approximations when she substitutes the approximation of the independent variable in the limit. She demonstrates coordination when she establishes the limit value according to the branch. Finally, she shows an understanding of the limit concept and its existence when she checks that the range approximations coincide.

In Pedro's answer, the PTs indicated that the student should only be able to make side approximations where the function is defined at the point. Thus, in algebraic mode:

> [Pedro] Presents an erroneous understanding of domain approximation (both from the right and from the left) when he does not correctly select the function branch: he chose the one that corresponded to the value towards which the independent variable was tending.

In the interpretation task, G2 identified the mathematical elements in all four students' answers to the different problems. The following excerpts related to Pablo's answers:

> [Problem 1] Pablo correctly uses the concept of function and performs the side approximations. In addition, he succeeded at coordinating the approximations by establishing the relationship between the domain and range. However, he does not complete the exercise as he does not indicate whether the limit exists or not.
> [Problem 2] He interprets the data in the table correctly, which shows a correct use of the function con-

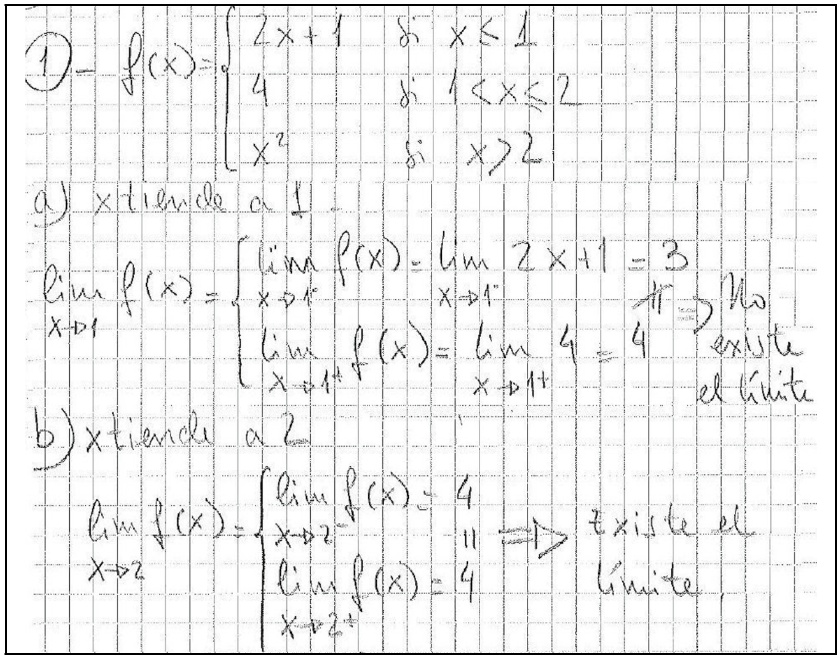


Fig. 9 G2’s anticipated answer for Maria in algebraic mode [English translation: a) “x approaches 1” “The limit doesn’t exit”; b) “x approaches 2”. “The limit exists"]

> cept. He correctly performs the side approximations and knows how to carry out the coordination […]. In section a1, he shows that he can correctly perform the domain approximation. In sections a2 and a3, he shows that he correctly performs the range approximation. In section b, we can see that he is coordinating.
> [Problem 3] He knows how to handle the graphical representations of the functions. He correctly performs the side approximations and correctly coordinates the domain and range approximations.

In addition, this group identified the characteristics of Pablo’s understanding of the limit of a function, detecting that his understanding corresponded to level 3:

> Pablo understands the limit concept because he correctly performs the approximations from the left and from the right at the point studied and is able to coordinate the approximations in the three representation modes. Therefore, we consider that Pablo is at level 3.

This group identified the coordination of the domain and range approximations in the different representation modes as KDU in the understanding of the dynamic limit concept. Indeed, the PTs considered that the student "is able to coordinate the approximations in the three representation modes".

Group G2 began by proposing general decisions in the anticipation task and group G5 by suggesting decisions focused on changing the type of discontinuity. After changing their interpretations of students’ understanding, both groups proposed instructional decisions focused on changing the type of discontinuity. For example, G2 proposed the function shown in Fig. 10 for Luiggi and Pablo in the interpretation task. This group changed the type of discontinuity. This decision is not focused on continuing using the coordination in new situations within the same type of discontinuity.

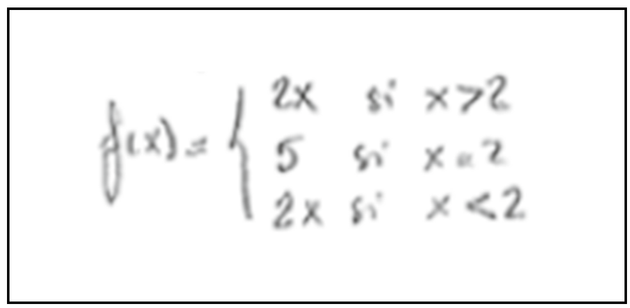


Fig. 10 G2’s instructional decision

4.3 Changes of instructional decisions “from decisions based on changing the type of discontinuity to conceptual decisions”: the use of the KDU in new situations

The two other PT groups (G1 and G4) began by conceiving understanding as a progression and continued with this conception throughout the teaching module. A characteristic of these PTs is that they not only identified the coordination of domain and range approximations in the different representation modes as KDU, but they also used this KDU to make decisions. Therefore, these PTs identified the mathematical elements and related them in order to interpret different levels of students' understanding in both the anticipation and interpretation task. They also made instructional decisions based on the use of coordinating approximations in a new situation.

For the numerical mode, G1 anticipated a correct resolution of both section a (domain and range approximations) and section b (coordination of domain and range approximations), arguing the following:

> In numerical mode, Pedro shows a better understanding of the idea of both domain and range approximations with the help of tables. He even coordinates both approximations when the side limits coincide or not (section b).

In algebraic mode, the PTs anticipated that he would have difficulties because he would not understand what is meant by domain and range approximation and the coordination of approximations, justifying the anticipated answer in Fig. 11 as follows:

> In algebraic mode, Pedro has difficulties because although he understands the notion of function, he does not understand what is meant by domain and range approximation and their coordination. He interprets solving a limit as finding the value of the function at that point (at x=1 and x=2).

In the interpretation task, they recognised features of each student's understanding in terms of whether they coordinated domain and range approximations or not in the different representation modes using the mathematical elements. Therefore, these PTs continued to identify the coordination of domain and range approximations in the different representation modes as KDU. For example, they identified the mathematical elements in Pablo's answers for each mode of representation:

> [Problem 1] He grasps the notion of function (piecewise). He calculates the side limits (domain and range approximations from the right and the left). He coordinates the approximations but fails to interpret that (a) has no limit and (b) does.
> [Problem 2] He realises that the two tables represent two functions (function idea). He notices that the first row refers to the approximations from the right and from the left in the domain and the second row to the approximations from the right and from the left in the range. In (b) he finds the relationship between a value and its image so he coordinates both approximations.
> [Problem 3] He notices that each graph corresponds to a function (function concept). He observes the point of interest in each graph and looks at its value (domain and range approximation). He coordinates both approximations, for the coincident and the non-coincident limit.

In addition, this PT group identified the characteristics of Pablo's understanding:

> Pablo has a high level of understanding as he makes both coincident and non-coincident side approximations in x and $f(x)$ in all three algebraic, graphical and numerical modes.

In the anticipation task, G1 proposed decisions based on changing the type of discontinuity (Fig. 12) for Maria. In this activity, G1 introduced a piecewise function with an infinite jump.

In the interpretation task, they proposed conceptual decisions to support cognitive processes. For example, they proposed that Pablo draw a function based on certain analytical conditions linked to the coordination of domain and

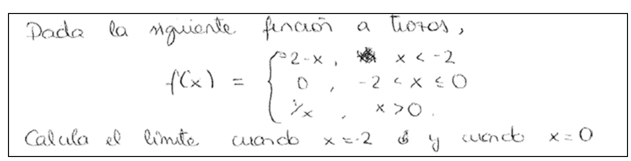
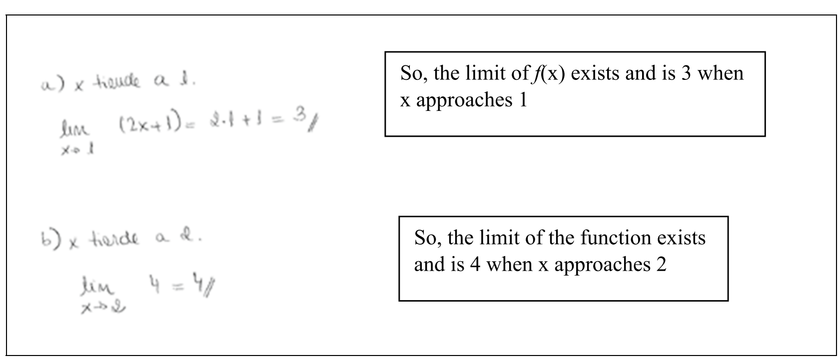


Fig. 12 Decision proposed by G1 for Maria (anticipation task). [English translation: "Given the following piecewise function" "Obtain the limit when x approaches -2 and when x approaches 0"]

Fig. 11 G1's anticipated answer to problem 1 for Pedro



range approximations using a function with the same type of discontinuity:

> Pablo has a high level of understanding, so we could set him tasks to reinforce his knowledge. For example, if the limit of a function is 3 at $x=5$, draw a function that satisfies this condition.

To accomplish this task, it is necessary to reverse the processes acquired in the construction of the dynamic concept, considering all the data needed to build the requested graphical representation.

5 Conclusions and discussion

The present study explored how PTs develop their noticing skill by characterising the changes manifested by the PTs as they participated in a teaching module. We discuss below the results of the study. They fall into two sections. First, the changes in PT noticing that allow us to characterise development pathways. Second, characteristics of our learning environment that support the development of PT noticing.

5.1 Changes in PT noticing: characterising development pathways

We found that throughout teaching module, four of the five PT groups manifested changes. This led us to identify two different development pathways that evidence progression in terms of the way PTs anticipate and interpret students' answers and decide using a theoretical lens (mathematical elements and levels of understanding).

Two groups (G2 and G5) changed from a dichotomous conception of all-or-nothing to a conception of progression. The main characteristic of this change is that these PTs went from not identifying the coordination of the domain and range approximations in the different representation modes as a KDU to identifying it as a KDU. This recognition allowed them to relate the lack of understanding to the idea of not coordinating the approximations in some representation modes, and therefore, to identifying characteristics of high school student understanding. They progressed from rhetorically or incorrectly using the mathematical elements in their descriptions to using the mathematical elements to describe and, finally to interpret students' mathematical understanding. They interpreted students' mathematical understanding when they were able to identify the coordination as a KDU (Fig. 13).

The PTs' recognition of the coordination of domain and range approximations as KDU reflects how PTs began to be aware of the relationship between mathematical knowledge (the mathematical elements involved in the dynamic limit concept) and knowledge of students' mathematical thinking (different levels of understanding). This is thus how they recognised that certain high school student behaviours could be significant or not from a mathematics learning perspective. Our results support previous studies that have shown that the recognition of KDUs in specific mathematical contents can

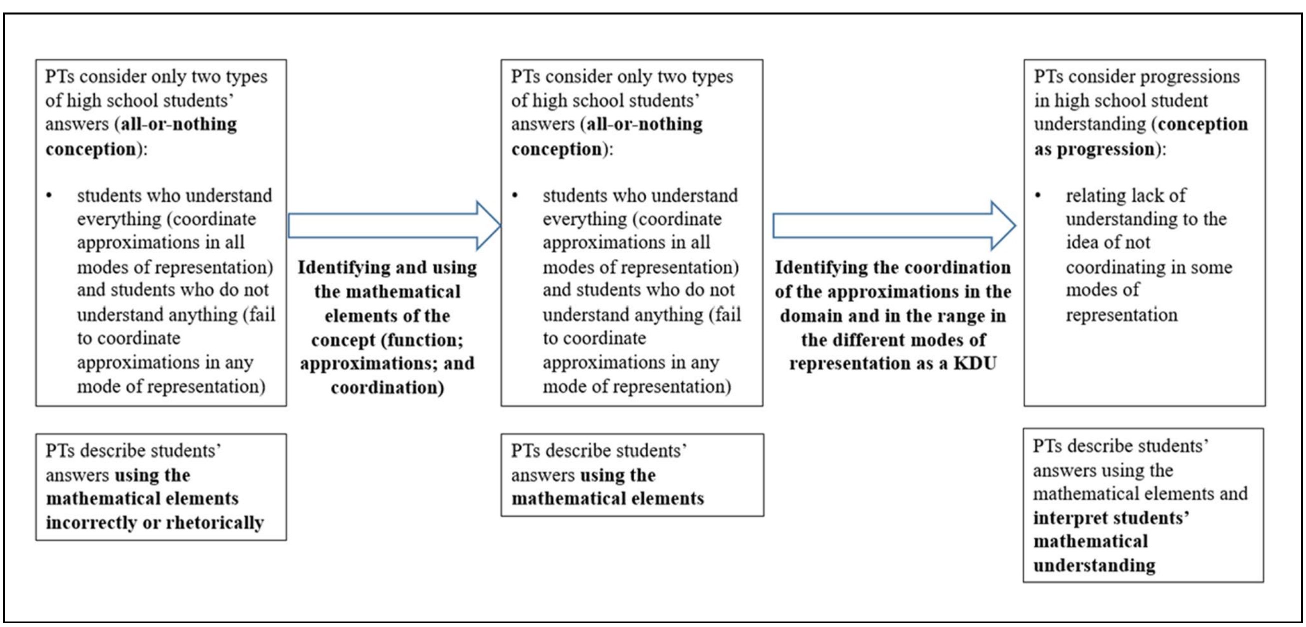


Fig. 13 Noticing development pathway: from using mathematical elements incorrectly or rhetorically to describe students answers to use them to describe, and finally, to interpret students' mathematical understanding

be considered as a benchmark in PT learning about student mathematical thinking (Llinares et al., 2016).

Two other groups (G1 and G4) consolidated their conception of understanding as a progression. Indeed, they applied the idea of coordinating the domain and range approximations in new situations. At the beginning, some of these PTs proposed decisions based on changing the type of discontinuity and ultimately advanced decisions based on cognitive processes considering the levels of understanding (Fig. 14). These decisions were given when the PTs were able to use the coordination (KDU) in new situations.

Finally, one group (G3) did not show any significant changes at any time in the teaching module. They conceived student understanding as dichotomous, using the mathematical elements rhetorically or incorrectly in their descriptions. Moreover, they provided general decisions. This group of PTs showed how the development of noticing can be influenced by other factors, such as the mathematical knowledge involved in the activities (Moreno et al., 2021), prior experiences or beliefs (Schoenfeld, 2011).

The identified pathways contribute to answering the following question, applied to the specific mathematical content domain of a function limit: *What development trajectories related to noticing expertise exist for prospective and practising teachers?* (Sherin, et al., 2011). These pathways start to show characteristics of the noticing development link to the ability of using the mathematical elements to describe students' answers and to interpret students' mathematical understanding. The transition between describing and interpreting was shown when PTs were able to identify the coordination of approximations in different modes of representations as a KDU. Moreover, PTs were able to provide conceptual decisions focused on students' understanding when they were able to use the KDU in new situations. However, future research could use more different tasks (anticipating, interpreting and deciding) along a course designed could support or extend our findings.

Furthermore, the findings have significant implications for teacher education since they can be used in teacher education programmes. These pathways can be used to assess PT noticing and teacher educators can employ them to help PTs develop their noticing competence.

5.2 Characteristics of our teaching module that support the development of PT noticing

The changes demonstrated that the teaching module designed supported the development of PT noticing of student mathematical understanding. Therefore, the designed tasks and the theoretical documents provided helped PTs to focus their attention on identifying important mathematical elements, on interpreting student mathematical understanding and on deciding how to respond based on students' understandings.

The task of interpreting consisted of representations of practice and some guiding questions. The representations of practice used reflected various characteristics of the understanding of the limit concept, and they proved to be useful instruments to activate PTs reflections and to provide them with opportunities to link the theoretical ideas with real contexts of mathematics teaching practice (Fernández et al., 2018b). The theoretical document that synthesised previous research on the mathematical elements of the limit concept provided PTs with a specific language (Fernández & Choy, 2020; Fernández et al., 2018b; Moreno et al., 2021) and supported PTs in making sense of the dynamic conception of límit (Cornu, 1991). It played a key role in scaffolding the noticing. The development of anticipating and interpreting skills in teaching modules led PTs the ability of recognizing students' multiple ways of thinking and it is fundamental knowledge for effective mathematics teaching.

The teaching module has allowed social and discursive practices to occur shaped by cultural attitudes towards the nature of mathematics and its teaching and learning (Schoenfeld, 2011). This has allowed for changes in some PTs framing (Scheiner, 2023) particularly, in those PT that have

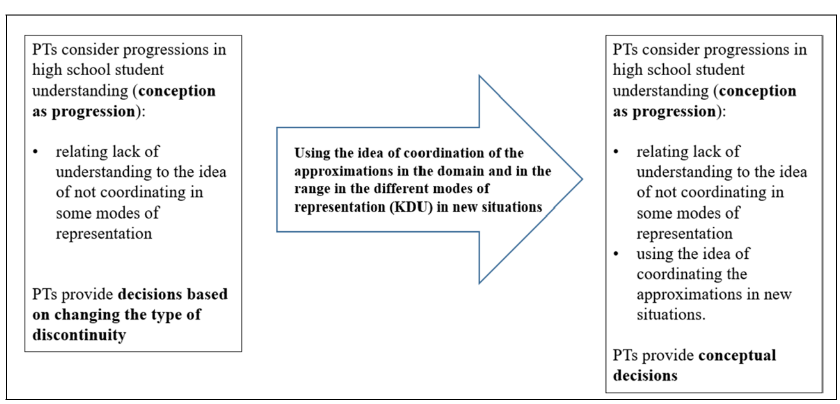


Fig. 14 Noticing development pathway: from recognising the KDU to using the KDU in new situations

developed noticing and conceived the teaching and learning of the limit concept from a new perspective.

Regarding PTs' decisions, some PTs seemed not to consider the levels of understanding provided, which inform us about the difficulty of the deciding skill development focused on students' understanding (Krupa et al., 2017). Including tasks in teaching modules aiming at analyzing mathematical activities considering the mathematical elements or cognitive processes that must be mobilized could facilitate making decisions on students' mathematical understanding.

In the light of our results, future research could focus on improving the teaching module designed, including other contextualized examples of anticipation and interpretation of students' answers such as video fragments, to assess its affectivity.

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Declarations

Conflict of interest There is no conflict of interest in the manuscript.

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