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Learning more about derivative: leveraging online resources for varied realizations

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Abstract
Recent literature underlines the increasing use of online platforms in learning undergraduate mathematics, where students refer to these as supplementary resources to develop their mathematical understanding. Through an intrinsic case study, we focus on a highly viewed YouTube learning resource for learning derivative. The selected case is from *3Blue1Brown*, a YouTube channel whose founder has received an award from the American Mathematical Society. The video has garnered more than 3.3 million views in the past couple of years. Reflecting on the relevant literature, a realization tree for derivative is developed and then used as an analytical tool to analyze this resource to explore what realizations have been used in it to facilitate students' understanding of derivative. The findings indicate that the analyzed YouTube resource discusses various realizations of derivative, including all its five main realizations, and effectively utilizes new digital technology for discussing these realizations. Such an exceptional resource for learning mathematics leads us to suggest that mathematics lecturers raise their awareness about such online free resources and incorporate them into their teaching packages when appropriate to facilitate student learning.

Keywords Derivative · Commognition · realization tree · YouTube · Online resources

1 Introduction

Representations play a significant role in learning mathematics, such as helping students make sense of mathematical objects and tasks, communicating mathematical thinking to oneself and others, analyzing mathematical problems, and it is indeed instrumental in solving mathematical modelling tasks (Haghjoo et al., 2023). Duval (2018) also notes the importance of representations in learning mathematics: "The power of mathematical thinking is due to the use of heterogeneous registers of producing a wide range of semiotic representations: numerical systems, algebraic expressions, graphs, words, statements, geometrical shapes, diagrams, tables" (p. 724). Effective mathematics teaching is also characterized by a strong focus on using and connecting different mathematical representations (National Council of Teachers of Mathematics [NCTM], 2014).

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YouTube is one of the online resources many undergraduate students use to seek help when learning mathematics (e.g., Aguilar & Esparza Puga, 2020; Esparza Puga & Aguilar, 2023). Its utility has been reported for several purposes, such as developing a better understanding of the mathematical concepts discussed in the lectures (Aguilar & Esparza Puga, 2020). Therefore, investigating what opportunities YouTube resources could provide for teaching and learning mathematics is important to explore their possible integration into teaching and learning mathematics to address students' academic needs and preferences (Esparza Puga & Aguilar, 2023).

Some YouTube learning resources happen to be extremely popular nowadays, much more than others, and there is a need to better understand how the content creator of these extremely popular videos convey mathematics. In this study, through an intrinsic case study, we focus on analyzing a YouTube resource for learning derivatives, with a particular emphasis on utilizing multiple representations. The derivative is chosen as it is a central topic in calculus with a wide range of applications in many disciplines, such as economics and biology. It is also included in service calculus courses offered to STEM (Science, Technology, Engineering and Mathematics) major students. We consider the

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following research question to explore the potential learning opportunities of YouTube resources for teaching and learning derivative:

> What learning opportunities are available in YouTube resources for the derivative, specifically when aiming to enhance students' comprehension of various representations (realizations) of the derivative?

This research adds to the existing literature on mathematics education in several ways. Firstly, there has not been extensive research on analyzing online video resources for learning calculus. Secondly, to address the research question, we use an analytical framework from the commognition theory (Sfard, 2008) called the *realization tree* (RT) (see Section 2). The RT has been used in past research for different purposes, such as analyzing calculus textbooks (Haghjoo et al., 2023). However, this tool has not been used to analyze online learning resources, including YouTube videos. Consequently, this study potentially promotes its use in mathematics education research. Thirdly, the RT developed for the derivative could also be used to analyze other learning resources, such as textbooks and face-to-face teaching. It can be also used to design tasks and activities for teaching and learning derivative.

2 Commognition and the realization tree

Researchers adopting a commognitive perspective take a discursive approach to the teaching and learning of mathematics, emphasizing the key role of communication in all human activities, including teaching and learning. This stands in contrast to a more traditional cognitive approach characterized by viewing communication "as a mere window to something else – mental schemes, conceptions, inner representations" (Sfard, 2017, p. 41). In this context, mathematical thinking could be defined as communicating with oneself or others in a specific way, known as mathematical (Sfard, 2017). In the following, we begin by outlining how mathematics is perceived from a commognitive perspective. Then, rituals and explorations, along with their roles in learning mathematics, are discussed as the two main routines of mathematical discourse. We continue this section by discussing the deritualization process and the teacher's role in such a process. Afterwards, we turn our attention to realizations and their importance in learning mathematics, followed by a discussion of the realization tree and its significance in the teaching and learning of mathematics.

From a commognitive perspective, mathematics is a "historically established discourse" (Sfard, 2020, p. 95) with four distinctive interrelated characteristics. It has unique *keywords* (e.g., derivative), *visual mediators* (e.g.,$\frac{dy}{dx}$), *routines,* i.e., actions that are regularly used in the discourse (e.g., finding the derivative of a polynomial function), and *endorsed narratives* (e.g., the Rolle's theorem) (Sfard, 2008). *Rituals* and *explorations* are two main discursive routines introduced in the commognition theory. Rituals are process-oriented routines (Lavie et al., 2019); the goal is to perform a specific task and the motivation is often social and the discursant concerns with "how do I proceed? or how can I enact a specific procedure?" (Nachlieli & Tabach, 2019, p. 255). On the other hand, explorations are product-oriented routines, where the goal is "constructing and endorsing a new narrative about mathematical objects" (Lavie et al., 2019, p. 166). Here, a person who "learns exploratively" concerns with questions like "what do I want to achieve?" (Nachlieli & Tabach, 2019, p. 255). For example, a ritual is finding derivative of a function by following a given procedure without thinking of what the result represents. On the other hand, an exploration focusses on constructing a new narrative, e.g., for $n \in \mathbb{R}$, the derivative of $f(x) = x^n$ is $f'(x) = nx^{n-1}$, and endorsing it—determining whether it is true or not.

Mathematical routines to be beneficial "must evolve into full-fledged explorations" (Sfard, 2008, p. 167); nonetheless, earlier studies indicated that transitioning from ritual to exploration is a gradual and slow process, and many students do not make this transition in school (Sfard, 2017). *Deritualization* is the term used in commognition theory for such a transition, and several properties are highlighted for this process (e.g., *objectification* (See Lavie et al., 2019; Sfard, 2008)). As an example, objectification refer to the situation where learners do not need to rely on their past relevant experience with *concrete* objects when facing a task situation. They are able to tell a narrative on mathematical objects by utilizing some of the procedures related to the properties of the mathematical objects in question (Lavie et al., 2019).

Sfard (2017) pointed out that mathematics teachers can support the deritualization process by "demonstrating the type of explorative discourse they would like their students to develop" and "explicitly encourage the desired kind discourse by appropriate pedagogical moves" (p. 44). Among other things, teachers can focus on objectification: "Unlike ritualized discourse, which is mainly – and sometimes exclusively – about unobjectified signifiers, explorative discourse aims at stories about abstract mathematical objects (p. 45). In more detail, in ritualized discourse, the focus is on manipulating mathematical signifiers and performing actions on symbols such as solving and transposing (Sfard, 2017). For example, when teaching derivative, a ritual discourse could be when the teacher emphasizes techniques and procedures, i.e., routines for calculating derivative, including finding the derivative using the definition. This might involve focusing on finding $f(x+h)$ by substituting $x+h$ into $f(x)$ and using the technique students learn in the limits topic to solve the following limit $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$. In explorative discourse, on the other hand, the focus would be understanding what the derivative is, with less emphasis on its computational aspect, as can be seen in the case chosen for this study.

Realization is a term introduced in the commognition theory instead of representation to emphasize that mathematical objects are the product of human discourse (Sfard, 2008). They are discursive constructs that emerge for the sake of communication about the world we live in, not as entities perceived to exist independently of humans (Sfard, 2017). Furthermore, Sfard (2008) proposed a visual mediator called the *realization tree* which is a connected graph that shows different realizations of a given signifier (e.g., a mathematical object) with the realizations of these realizations and so forth. From a semiotic perspective, "a mathematical object can be defined as a mathematical signifier together with its realization tree" and a signifier could have many different realizations (Sfard, 2017, p. 43). Focusing on different realizations of a mathematical object is important as *saming* different realizations of a mathematical object is an important step towards *objectification* (Wallach et al., 2022). Sfard (2017) highlighted further that the primary goal of learning mathematics in school should be to assist students in establishing connections between various realizations of mathematical objects. However, she acknowledged the inherent complexity and challenges involved in this process. She noted that this aspect of learning mathematics is both challenging and often overlooked in research: "This is probably the most challenging, and at the same time, the least obvious and rarely studied aspect of learning mathematics" (p. 43). Also, past commognitive research pointed out the importance of discussing different realizations of mathematical objects in teaching to facilitate explorative participation in mathematical discourse (e.g., Weingarden & Heyd-Metzuyanim, 2023). Constructing thorough RTs for mathematical objects that are well grounded in the relevant literature could help mathematics educators and practitioners critically evaluate how to enhance teaching and learning mathematics by focusing on different realizations of the mathematical object in question in teaching and learning situations (Haghjoo et al., 2023).

In the next section, we discuss the relevant literature on teaching and learning derivative, as it was instrumental in refining the existing realization tree for the derivative (i.e., Haghjoo et al., 2023). Additionally, this literature could be useful for readers to gain a better understanding of why certain approaches have been utilized in the selected YouTube learning resource.

3 The teaching and learning of derivative

The derivative is one of the core topics of calculus in addition to its role in other subjects, such as chemistry and physics (Radmehr et al., 2023; Roundy et al., 2015). The teaching and learning of the derivative have garnered notable attention from mathematics education researchers (Haghjoo et al., 2023; Park, 2013, 2015; Thompson, 1994; Zandieh, 2000). In the following, we discuss pertinent literature related to challenges reported in relation to learning derivative, followed by proposed approaches for teaching this mathematical object.

3.1 Challenges with learning derivative

Previous literature (Biza, 2021; Orton, 1983; Kertil et al., 2023; Ryberg, 2018; Thompson, 1994; Zandieh & Knapp, 2006) has reported a number of challenges with learning the four out of the five main realizations of derivative. While there has been little to no investigation of students' understanding of the physical realization, more focus has been placed on the graphical, symbolic, and verbal realizations of the derivative. Below, we summarize the student challenges reported in the previous studies by connecting them to these four main realizations:

- understanding the distinction between average rate of change over an interval and instantaneous rate of a change at a point (i.e., related to verbal realizations),
- transition from secants to the notion of tangent(s) (i.e., related to graphical realizations),
- finding the average rate of change at a certain point (i.e., related to numerical realizations),
- lack of algebraic fluency, limit, and using and interpreting the related visual mediators (e.g., $\frac{dy}{dx}$), (i.e., related to symbolic realizations).

We note that the challenges regarding symbolic realizations could be due to the complexity of the definition, and different visual mediators of derivative (Biza, 2021; Thompson, 1994), as a result, students could have different level of participation in mathematical discourse about derivative. For instance, Zandieh and Knapp (2006) underlined that students have challenges in realizing derivative as a limit of rate of change. Furthermore, Park (2013) reported that many students did not realize derivative "at a point as a number and derivative function as a function" (p. 624). And it is not easy to connect and move between realizing derivative at the limit and the function levels (Park, 2015).

Another major challenge reported in previous studies is graphical realizations, particularly linking the graphs of a given function with its derivative and tangent lines (García-García & Dolores-Flores, 2021; Ryberg, 2018). For example, Ryberg (2018) reported that students could face challenges in realizing the differences between the graph of a function and its derivative graph. Furthermore, discerning/linking the slope of the graph and its derivative function's graph is challenging for many students.

From the point of moving between symbolic and graphical realizations, García-García and Dolores-Flores (2021) noted that when transitioning from the graph of the given function $f(x)$ to its derivative function $f'(x)$, many students often resort to searching for a symbolic realization of $f(x)$ to calculate its derivative, rather than exploring the provided graphical realizations. This is due to students are proficient in the routines for finding derivatives using symbolic realizations (Orton, 1983).

3.2 Teaching derivative and the role of realizations

In reviewing the literature, we notice that there is a particular interest in the use of different realizations for teaching derivative (Hong & Lee, 2022; Ryberg, 2018). Second, researchers underlined the careful use of symbolic realizations and transitioning to graphical realizations (Park, 2015, 2016). In other words, accordingly, there seems to be a need for the coordination and synergy among the use of graphical and symbolic realizations to create a link between different aspects of derivative; as a function, as a number at a point or as limit of the rate of the change (de Almeida & da Silva, 2018; Park, 2015, 2016; Ryberg, 2018). In addition, Ryberg (2018) suggested incorporating a diverse range of graphical realizations in the initial start of teaching derivative. Using the formal definition of derivative with a particular reference to the rate of change and its different graphical realizations is the main tool for having an explorative participation in mathematical discourse about derivative (Zengin, 2018). Therefore, student knowledge about limits and the limiting process, particularly in the context of the rate of change, is considered as a core element for having an explorative participation in this discourse (Hähkiöniemi, 2006). In the following, we discuss a few studies that investigate the teaching of derivative.

Park (2015) explored three instructors' classroom discourse on derivative. Instructors started from secants to the tangent line of a curve at a certain point to show the symbolic realizations of derivative, but without exploring the dialectics between the symbolic and graphical realizations. Later, instructors moved back and forth between the limit and the function levels to facilitate student participation in the discourse about derivative. In terms of teaching derivative graphs, Hong and Lee (2022) focused on the instructors' preferences. Instructors referred to different realizations, however, this was up to the instructors' personal orientations, resources, and the limited knowledge of the audience. As noted by Hong and Lee (2022), this led to a conflict and critically influenced instructors to rely on and gave a particular emphasis on graphical realizations in the classroom. Overall, in both studies, instructors referred to a number of different but interrelated realizations (Park, 2015), a common belief is that without providing multiple realizations, students' knowledge and participation in the discourse will be limited (Hong & Lee, 2022).

Regarding instructors' personal orientations and the available resources, as addressed by Hong and Lee (2022), textbooks could play a particular role that affect instructors' participation in discourse in classrooms and potentially influences student learning of derivative (Haghjoo et al., 2023). de Almeida and da Silva (2018) focused on semiotic systems regarding the notion of derivative in a textbook. They concluded that the textbook includes a variety of visual mediators and realizations (e.g., iconic, indexical, or symbolic), and learning derivative is a recursive process mediated by these visual mediators and realizations, with teachers playing a significant role in helping students navigate between them. Park (2016) also discussed the role of visual mediators and realizations further by noting that the use of similar notations as visual mediators for derivative at a point and derivative as a function could lead to epistemological obstacles. Overall, it appears that calculus instructors need to go beyond what is typically offered in calculus textbooks. There is a need to think critically about how different realizations of derivative should be discussed and brought forth in teaching. Haghjoo et al.'s (2023) study supports this claim. After analysing 14 calculus textbooks on derivative at a point, they noted that none of the analyzed textbooks included all five main realizations of derivative.

Recently, Radmehr et al. (2023), inspired by Haghjoo et al.'s RT (2023), designed a teaching activity in the context of chemistry for learning derivative. They suggested starting with physical realizations, providing the necessary data to engage students in the numerical realizations as also suggested by others (e.g., Diaz Eaton et al., 2019; Roundy et al., 2015) to teach mathematics "as a laboratory discipline" (Diaz Eaton et al., 2019, p. 807). This approach, among other benefits, could help students feel "more agency to readily engage with the conversation on models and modeling" (Diaz Eaton et al., 2019, p. 807). Radmehr et al. (2023) proposed that graphical, symbolic, and verbal realizations could then be the focus of teaching. They also pointed out among several calculus textbooks (e.g., Hass et al., 2018), the physical realizations were not emphasized when introducing derivative. For instance, in Hass et al. (2018), derivative is first introduced through graphical and symbolic realizations.

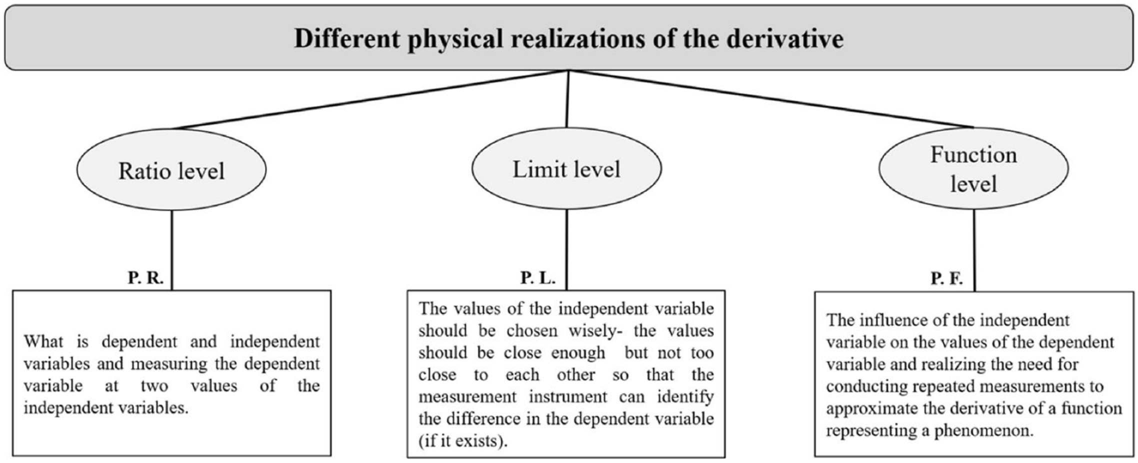


Fig. 1 Different physical realizations of derivative

4 Methodology

The present study is an intrinsic case study. This qualitative approach is characterized by selecting a unique case that needs to be described and detailed (Creswell & Poth, 2018). We selected a highly viewed YouTube resource for learning derivatives from a YouTube channel named *3Blue1Brown*. As of February 19, 2024, when we were doing the final revisions on this paper, it has 5.92 million subscribers, and its content has been viewed more than 468 million times. This channel is developed by Grant Sanderson, a Stanford graduate who received an award from the American Mathematical Society (AMS) for his contribution to teaching and learning of mathematics. The AMS (2022) acknowledged the channel as a "watchable and engaging YouTube channel … about discovery and creativity in mathematics … Through 3Blue1Brown videos and animations, Sanderson presents mathematics both as practically valuable and as an art form, rich with inviting stories and arresting images". The chosen YouTube video is part of a series of videos on calculus titled *the essence of calculus*.[1]This series comprises 12 learning resources, and the selected YouTube video, titled, *the paradox of the derivative*, is the second video in this series. It has garnered 3.3 million views and can be accessed at https://youtu.be/9vKqVkMQHKk?si=3rgpjTfEL5wDgT82. It is the most viewed YouTube learning resource on derivatives in English within the duration of up to 20 min, specifically focusing on teaching derivative. There is one YouTube video with more views in English (i.e., 3.5 million views) that is over six hours long, but its nature differs. In that video, the content creator solves 100 derivative tasks using derivative rules and some with the definition of the derivative, but the focus of the video is not on explaining what the derivative is. Furthermore, the YouTube channel is less popular compared to *3Blue1Brown*, with only 1.24 million subscribers, and its content has only been viewed over 11 million times.

4.1 Data analysis

We first developed the RT as discussed below in Section 4.2. While revising our RT, we watched several videos on derivative including this case, multiple times. Afterward, the video file was deductively analyzed using the refined RT. The first author conducted the initial analysis, and the second author reviewed his work. We then discussed areas of disagreement, which lead to the results discussed below. The video was viewed at least 10 times by each author to ensure that all realizations are addressed in the results section, including both what Grant says verbally and what appears on screen.

4.2 Different realizations of the derivative

The derivative is closely related to *function* and *limit*, as well as the notion of *rate of change* (Weigand, 2014). There are also two main ideas associated with this mathematical object: The derivative at a point (*local* aspect) and the derivative as a function (*global* aspect) (de Almeida & da Silva, 2018). The learning of the derivative is a recursive process, and visual mediators play a key role in such a process (de Almeida & da Silva, 2018). Weigand (2014) elaborated further to have explorative participation in the discourse about derivative, "it is necessary— besides understanding limit processes—to have adequate conceptions of the *rate of change* and to understand—in relation to limit processes—the transformation from the *average* rate of change to the *local* rate of change" (p. 604). Five main realizations have been identified in the literature for the derivative: *numerical*,

[1] https://youtu.be/WUvTyaaNkzM?si=XROwSXBqOWNvnDWb

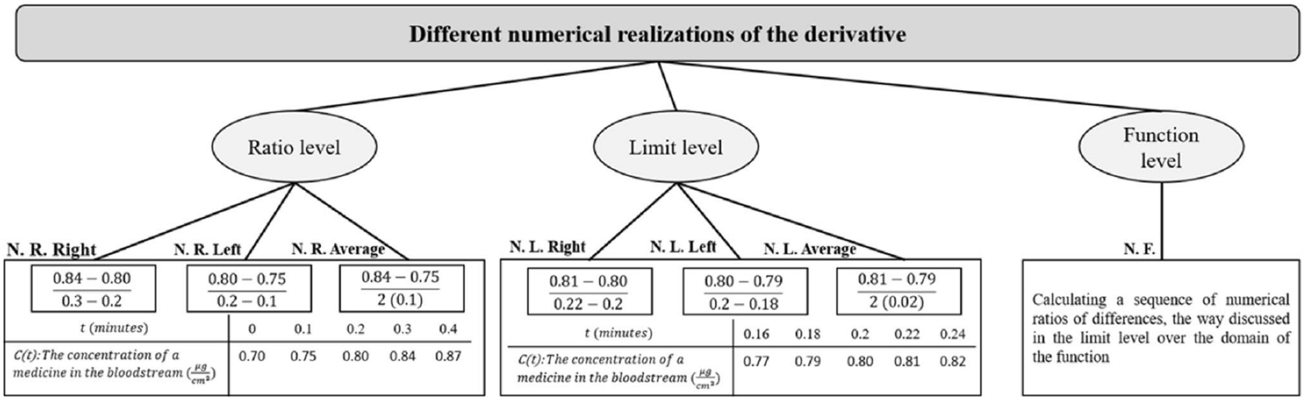


Fig. 2 Different numerical realizations of derivative

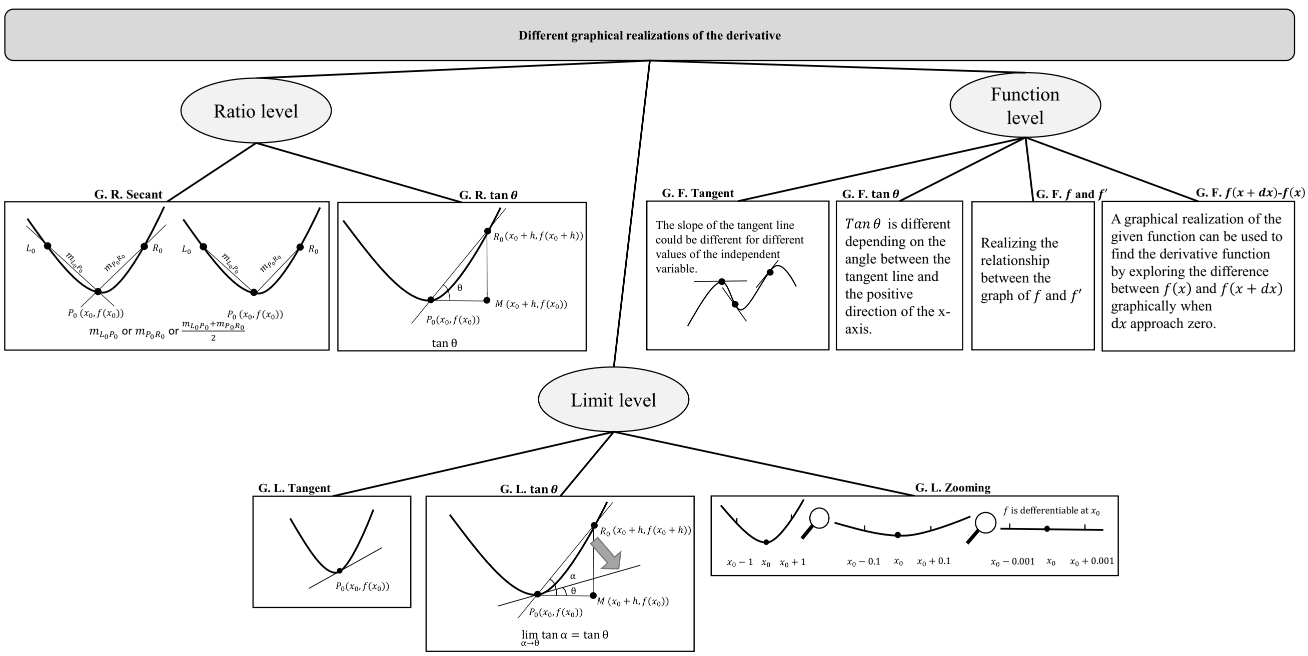


Fig. 3 Different graphical realizations of the derivative

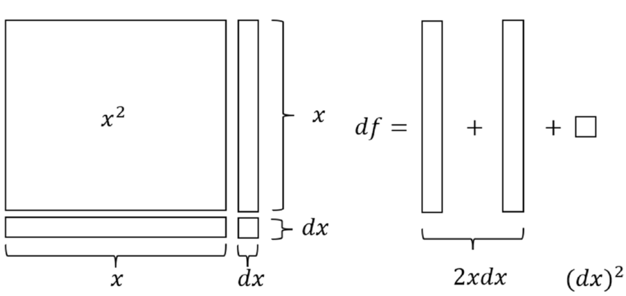


Fig. 4 A graphical realization of $f(x) = x^2$ and how to realize df to derive the derivative function

symbolic, *graphical*, *verbal*, and *physical.* These realizations are discussed at *ratio*, *limit*, and *function* levels (Roundy et al., 2015; Zandieh, 2000). Haghjoo et al. (2023) unpacked these five main realizations in an RT with 17 roots that discusses the derivative mainly in the ratio and limit levels. Reflecting on the literature and our teaching and research experience we developed an RT for the derivative with 35 roots that discuss the derivative at the ratio, limit, and function levels as described in the following. In the RT that we developed, the ellipses/nodes are not considered mathematical objects, which is why they have not been counted as roots in our RT. These nodes are used to cluster the realizations. Such an approach has not been used by Sfard (2008)

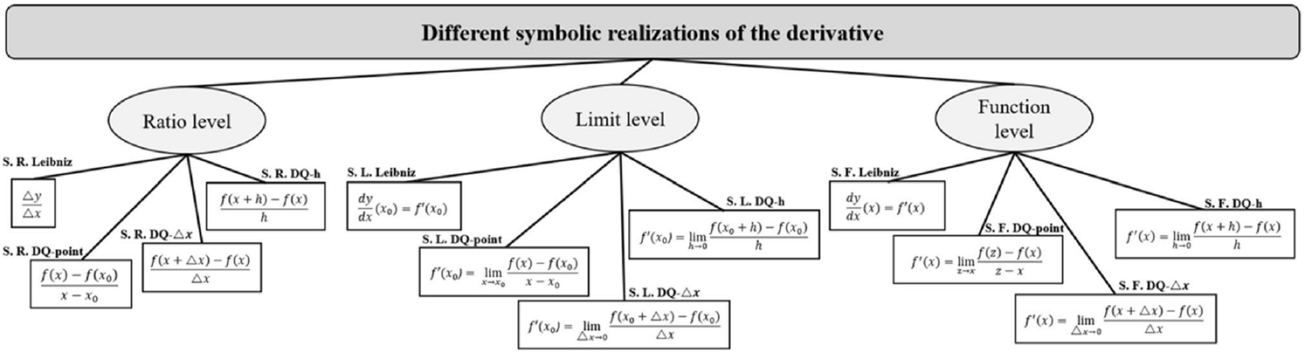


Fig. 5 Different symbolic realizations of derivative

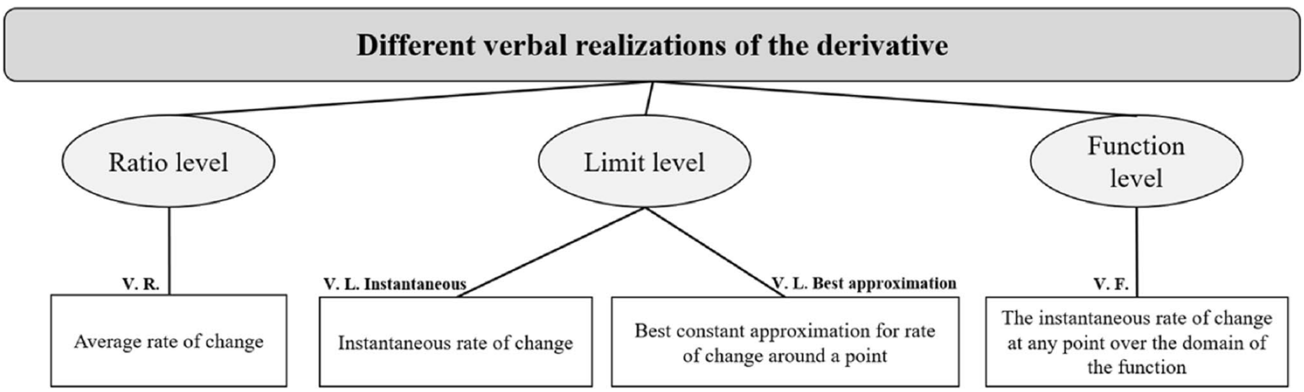


Fig. 6 Different verbal realizations of derivative

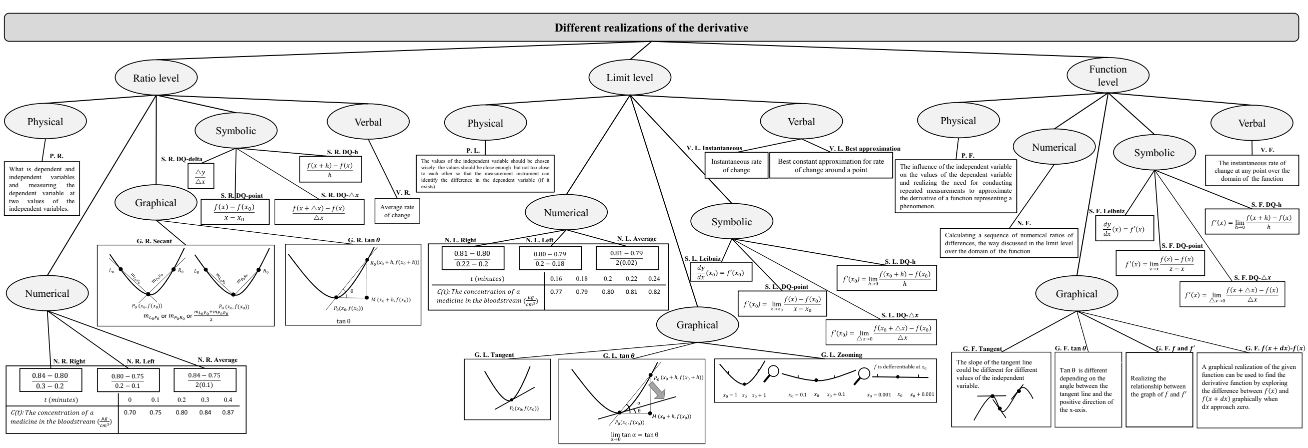


Fig. 7 A full realization tree for derivative

or others who use RT in their research (e.g., Weingarden & Heyd-Metzuyanim, 2023), but we found it useful to utilize here for the derivative considering its versatile realizations. In all figures, we utilized the following labeling approach. The first letter denotes the main type of realization (e.g., P for physical), and the second letter indicates the level (e.g., R for Ratio). If there are multiple realizations at that level, a brief term is also provided for readability. This allows readers to navigate the results section without needing to refer back to these figures constantly.

4.2.1 Physical realizations

This realization of derivative is related to the measurement process prior to approximating derivative using numerical approaches (see Fig. 1) (Roundy et al., 2015). Here at the ratio level, it is important that students realize what the dependent and independent variables are and measure the dependent variable in at least two values of the independent variable. At the limit level, it is important that students realize that the values of the independent variable should be chosen wisely. That is, the values should be close enough so that "this ratio to 'be' the derivative in the thick sense used by physicists and engineers" (Roundy et al., 2015, p. 923), meaning not too close to each other so that the measurement instrument can identify the difference in the dependent variable (if it exists) based on its sensitivity (Haghjoo et al., 2023). At the function level, it is related to realizing the influence of the independent variable on the values of the dependent variable and realizing the need for conducting repeated measurements to approximate the derivative of a function representing a phenomenon (Roundy et al., 2015).

4.2.2 Numerical realizations

This realization is closely related to physical realizations and expressed as $\frac{y_2-y_1}{x_2-x_1}$ where the values in this expression are realized to be numerical values (see Fig. 2) obtained from measuring dependent and independent variables (Roundy et al., 2015). This realization also could be used when the values of a function are given, typically in a table, instead of providing a graphical or symbolic realization of the function. At the ratio level, there are not many restrictions on the chosen values of the independent variables as discussed above, but at the limit level, Δx is very small but does not approach zero as in the formal definition. In other words, in this realization, the derivative has some "thickness" as opposed to the formal definition that requires an infinitesimal limit (Roundy et al., 2015, p. 923). At the function level, it is "a sequence of numerical ratios of difference" (Roundy et al., 2015, p. 923) that is used to estimate the derivative over the domain that is defined. These differences could be calculated by approaching x_0 from both the left and the hand sides of the point (i.e., $f'(x_0) \approx \frac{f(x_0+h)-f(x_0)}{h}$ and $f'(x_0) \approx \frac{f(x_0)-f(x_0-h)}{h}$) (Haghjoo et al., 2023). Furthermore, to have a better approximation, the average of these could be considered for the numerical approximation of the derivative (i.e., $f'(x_0) \approx \frac{f(x_0+h)-f(x_0-h)}{2h}$) as suggested by Hughes-Hallet et al. (2017, see p. 102).

4.2.3 Graphical realizations

The slope is the graphical realization of derivative (see Fig. 3) (Roundy et al., 2015; Zandieh, 2000). At the ratio level, it is the slope of a secant line joining two points on the graph of a function (Roundy et al., 2015). It could also be realized as the slope of a line segment that its endpoints are on the graph of a function. This slope could be calculated by considering one of the endpoints on the left or the right side of the point in the question or by considering the average slope of these two situations (Haghjoo et al., 2023). At the limit level, the graphical realization is the slope of the tangent line at the given/chosen point on the curve (Roundy et al., 2015). At the function level, the graphical realization of derivative is realizing that the slope of the tangent line could be different for different values of the independent variable (Roundy et al., 2015). $tan\theta$, where $\theta$is the angle between the secant line/line segment and the positive direction of the x-axis is also another realization of derivative at the ratio level. At the limit level, θ becomes the angle between the tangent line and the positive direction of the x -axis. Finally at the function level, this realization is about $tan\theta$ could be different depending on the angle between the tangent line and the positive direction of the x-axis across the domain of the function. Another graphical realization of derivative is that when zooming in the graph of a function at a point, if it is differentiable at that point, it looks like a straight line, regardless of the shape of the actual function (Haghjoo et al., 2023). This is typically done with the assistance of technology such as GeoGebra. The two final graphical realizations we consider are at the function level. The first is about realizing the relationship between the graph of f and f', for instance on the intervals where f is decreasing, the graph of f' is below the x-axis. The second is about considering a graphical realization of the given function (e.g., imagining $f(x) = x^2$ as the area of a square with a side length of x (Fig. 4) and exploring the difference between values of $f(x)$ and $f(x + dx)$ graphically when dx approaches zero to calculate derivative of the given function. This realization is well discussed in a YouTube video on derivative.[2]

[2] https://youtu.be/S0_qX4VJhMQ?si=sI0MsUohAAf_DubQ

4.2.4 Symbolic realizations

At the ratio level, the symbolic realization of derivative could be shown by a difference quotient (DQ), such as $\frac{f(x+h)-f(x)}{h}$, $\frac{f(x+\Delta x)-f(x)}{\Delta x}$, and $\frac{f(x)-f(x_0)}{x-x_0}$ where x_0 and $x \in D_f$ and h or Δx represents a small increment from x(see Fig. 5) (Roundy et al., 2015; Zandieh, 2000). At the limit level, the symbolic realization of the derivative is the formal definition of the derivative as the limit of DQ that could be expressed by $f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$, $f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$, and $f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$ (Roundy et al., 2015; Zandieh, 2000). At the function level, one needs to realize that the limiting routine must be repeated for all values in the domain of the derivative function (Zandieh, 2000). Zandieh (2000) highlighted this by choosing different visual mediators when expressing the symbolic realization of derivative at the function level, changing x_0 to x (i.e., $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$). The symbolic realization of derivative at the function level could also be expressed by $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ (Roundy et al., 2015) and $f'(x) = \lim_{z \to x} \frac{f(z)-f(x)}{z-x}$ (Hass et al., 2018). Leibniz's notation could be also considered as another set of symbolic realizations of the derivative (See for example, Hughes-Hallet et al., 2017). At the ratio level, it could be realized as $\frac{\Delta y}{\Delta x}$, and at the limit and the function levels could be realized as $\frac{dy}{dx}(x_0) = f'(x_0)$ and $\frac{dy}{dx}(x) = f'(x)$, respectively.

4.2.5 Verbal realizations

For derivative at the ratio level, the known formal verbal realization is the *average rate of change* and at the limit level is the *instantaneous rate of change* (see Fig. 6) (Roundy et al., 2015; Zandieh, 2000). Within less formal verbal realizations that could be considered for the derivative, we only include the *best constant approximation for rate of change around a point* at the limit level as discussed in the results section. At the function level, it is related to realizing that the instantaneous rate of change should be considered for all inputs over the derivative function domain (Roundy et al., 2015). Moreover, the average and instantaneous rate of change have numerous applications in the real world. Recently, Haghjoo et al. (2023) identified 26 different verbal realizations of the derivative over eight subjects by exploring several calculus textbooks, such as reaction rate in chemistry or growth rate in biology. However, verbal realizations of the derivative in other disciplines are not the focus of this RT (Fig. 7).

5 Results

The goal of this learning resource is set as (a) explaining what derivative is and (b) discussing possible misunderstandings that students might have when learning this topic: "there's some subtlety to this topic, and a lot of potential for paradoxes, if you're not careful, … the secondary goal is you have an appreciation for what those paradoxes are and how to avoid them". In analyzing the case, overall, we have identified 18 (Fig. 8a and b) out of the 35 roots we identified earlier. In the following, we discuss the main highlights of the video, including how some of these realizations have been addressed in more detail.

5.1 Introduction to the paradox of 'instantaneous rate of change'

Grant discusses the paradox with *V. L. Instantaneous* in many instances. The first one appears very early in the video (Fig. 9):

> It's common for people to say that the derivative measures "instantaneous rate of change" … that phrase is actually an oxymoron: Change is something that happens between separate points in time, and when you blind yourself to all but just a single instant, there is not really any room for change …

He utilizes a contextual example to address this paradox: a car that travels from point A to B, covering a distance of 100 *m* over a 10 *s* interval while experiencing varying speeds. He illustrates this motion by plotting the distance travelled on a vertical axis and representing time on a horizontal axis. Grant then introduces the distance function and explains how the graph can be interpreted, particularly emphasizing how the slope of the distance function reflects the car's velocity (*G. R. Secant*) (Fig. 10).

5.2 The relationship between the graph of *f* and *f'*

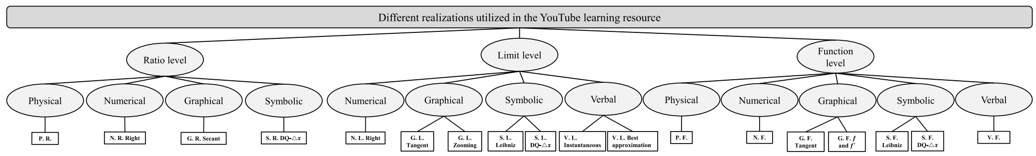
Afterwards, he focuses on the relationship between the graph of f and f' (*G.F. f and f'*) by adding the graph of the car's velocity in $\frac{m}{s}$ as a function of time on the screen[3] and discusses what this graph means:

> At early times, the velocity is very small. Up to the middle of the journey, the car builds up to some maximum velocity, covering a relatively large distance in each second [Figure 11]. Then it slows back down towards the speed of zero.

Grant then continues by highlighting that these two graphs are "definitely related to each other" and that changing the distance function, impacts the corresponding velocity function. He illustrates this by showing the graphs of several distance functions and their corresponding velocity functions (e.g., Fig. 12).

[3] The two graphs do not share the same y-axis; however, there is no comment about this matter in the video.

a



b

Time Point	Ratio Level	Limit Level	Function Level
0:34		V. L. Instantaneous	
2:12	G. R. Secant		
2:24			G. F. f and f'
2:32	G. R. Secant		
2:46			G. F. f and f'
3:29	P. R.		
3:45	N. R. Right		
3:56	P. R.		
4:01			G. F. f and f'
4:55		N. L. Right	
5:28		S. L. Leibniz	
5:35		G. L. Zooming	
5:41	G. R. Secant		
6:01			G. F. Tangent
6:01			V. F.
6:12			S. F. Leibniz
6:20			G. F. f and f'
6:30	P. R.		
6:35			P. F.
6:59			N. F.
7:02	S. R. DQ-Δx		
7:05			G. F. f and f'
7:30	G. R. Secant		
7:35		S. L. Leibniz	
7:45			G. F. f and f'
7:51			S. F. Leibniz
7:54	S. R. DQ-Δx		
7:55	P. R.		
8:01		N. L. Right	
8:04			S. F. Leibniz
8:04			S. F. DQ-Δx
8:14	G. R. Secant		
8:22		G. L. Tangent	
8:35			G. F. Tangent
8:50		G. L. Tangent	
8:53	G. R. Secant		

(continued from the previous column)

Time Point	Ratio Level	Limit Level	Function Level
8:54	G. R. Secant		
9:02		G. L. Tangent	
9:09			G. F. Tangent
9:36		V. L. Instantaneous	
9:39		V. L. Best Approximation	
9:54	G. R. Secant		
9:56		S. L. Leibniz	
10:12			S. F. Leibniz
10:12		N. L. Right	
10:21			S. F. DQ-Δx
10:45	G. R. Secant		
11:05		S. L. Leibniz	
11:27		S. L. DQ-Δx	
12:29	G. R. Secant		
12:32		G. L. Tangent	
13:02			V. F.
13:03			S. F. Leibniz
13:03			S. F. DQ-Δx
13:05			G. F. Tangent
13:06			S. F. Leibniz
13:55	G. R. Secant		
13:57		S. L. Leibniz	
14:04			S. F. Leibniz
14:04			S. F. DQ-Δx
14:19		V. L. Instantaneous	
14:47			S. F. Leibniz
14:55			G. F. Tangent
14:59		G. L. Tangent	
15:01		V. L. Instantaneous	
15:38		V. L. Best Approximation	
15:41		G. L. Tangent	
16:16			S. F. Leibniz
16:17		V.L. Best Approximation	
16:19		G. L. Tangent	
16:27		V.L. Instantaneous	
16:35		V.L. Best Approximation	

Fig. 8 **a** The realizations used in the case. **b** The realizations used in the case with time distribution

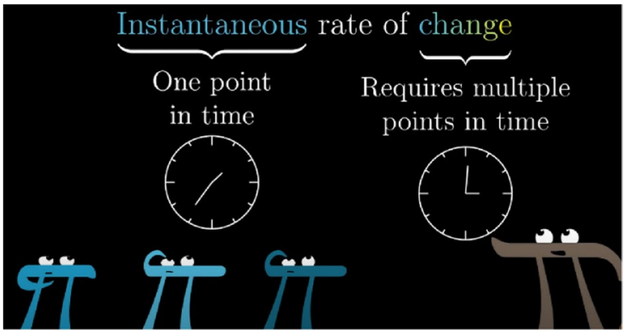


Fig. 9 A screenshot when Grant introduces the paradox of a verbal realization of the derivative (0:46) (Permission for all screenshots granted by the copyright holder, Grant Sanderson.)

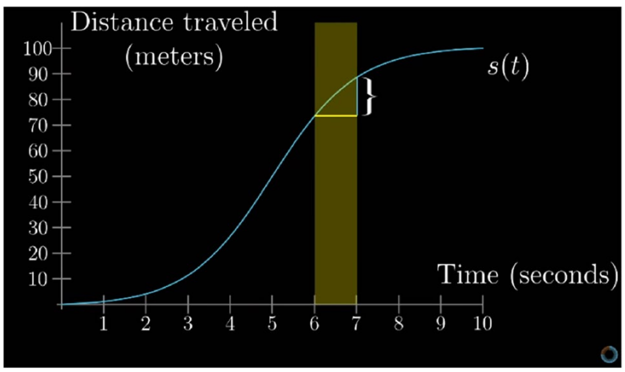


Fig. 10 A screenshot when Grant introduces the distance function and the slope of the function at each second (2:14)

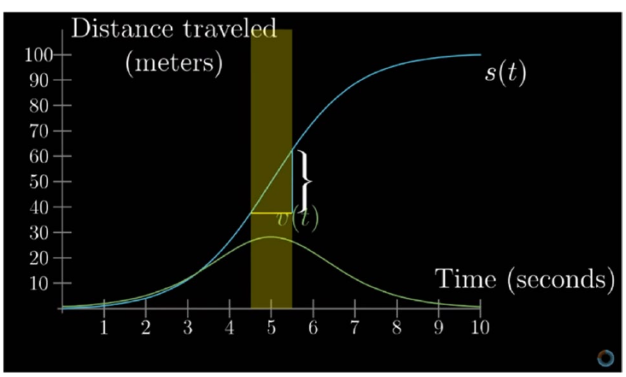


Fig. 11 A screenshot when Grant discusses the car's velocity function (2:37)

5.3 Associating the paradox with numerical realizations

Later in the video, Grant brings up the paradox he discussed earlier and points to *P. R.*:

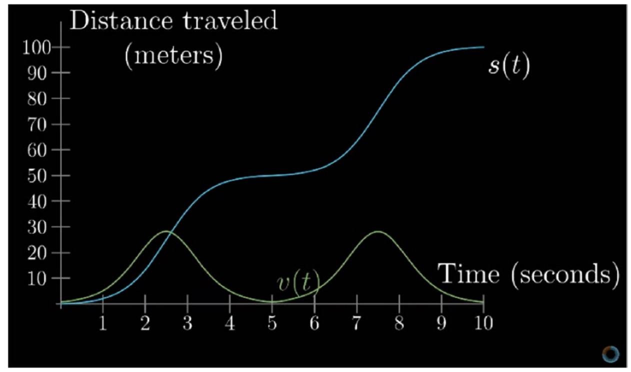


Fig. 12 A screenshot when Grant illustrates how distance and velocity functions are related (2:57)

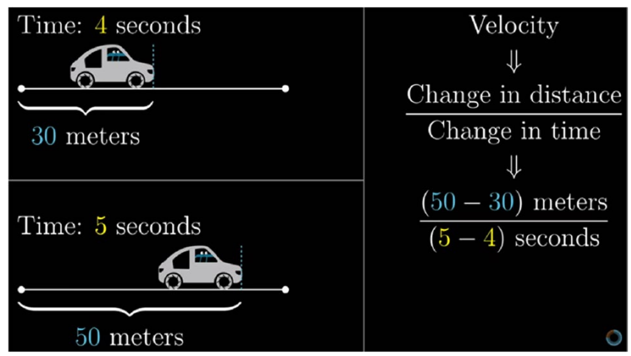


Fig. 13 A screenshot when Grant addresses N. R. Right (3:45)

> … velocity at a single moment makes no sense. If I show you a picture of a car, a snapshot in an instant, and ask you how fast it's going, you'd have no way of telling me. What you need are two separate points in time to compare …

On the screen (Fig. 13), he continues by addressing *N. R. Right* and then defining velocity as "the distance traveled per unit time".

Afterwards, he returns back to the argument about the paradox by pointing that:

> how is it that we're looking at a function for velocity that only takes in a single value of *t*, a single snapshot in time. It's weird, isn't it? We want to associate individual point in time with a velocity, but actually computing velocity requires comparing two separate points in time [*P. R.* while revisiting *G. F. f and f′* on the screen] …

Grant continues the discussion by first talking about the real-world example and then moves to the mathematical world. He starts by discussing in simple terms how cars' speedometers might approximate velocity (*N. L. Right*).

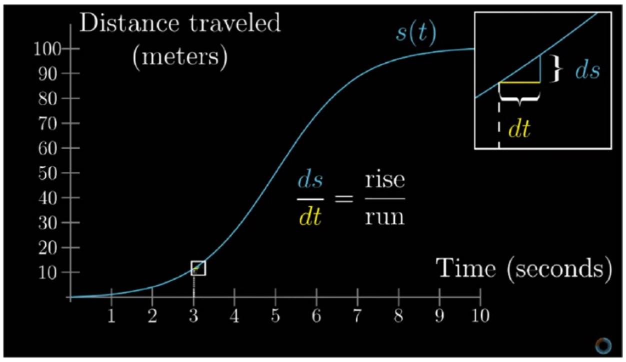


Fig. 14 A screenshot when Grant illustrates $\frac{ds}{dt}$ on the graph of a distance function after zooming in on a point (6:01)

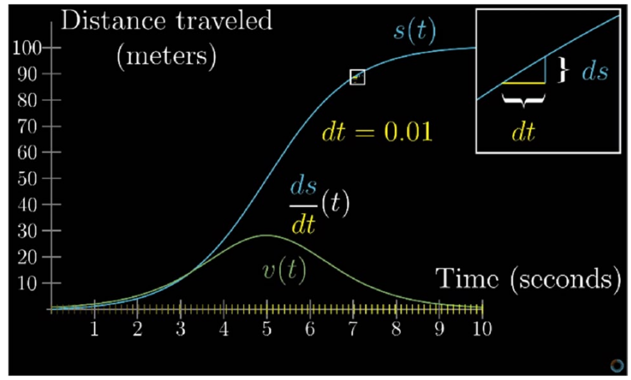


Fig. 15 A screenshot when Grant addresses *P. F.* and *N. F.* (6:35)

Grant highlights that "a physical car just sidesteps the paradox and does not actually compute speed at a single point in time. It computes speed during a very small amounts of time… [such] as 0.01 *s*".

5.4 Integrating digital technologies to discuss different realizations

Grant utilizes digital technology effectively across this learning resource to address different realizations of the derivative. The following episode could be one of the many good examples. Here Grant zooms in on the graph of a function (*G. L. Zooming*) (Fig. 14) and again focuses on *G. R. Secant* discussing what could be considered as *ds* and *dt* on the graph of a distance function. He elaborates:

> That *dt* is a small step to the right, since time is on the horizontal axis, and that *ds* is the resulting change in the height of the graph, since the vertical axis represents the distance traveled. So $\frac{ds}{dt}$ is something you can think of as the rise-over-run slope between two very close points on this graph.

Then the box starts moving on the graph of function while still on the bigger box in the right top corner of the screen, viewers could see how the slope changes as a result (*G. F. Tangent*). Simultaneously, he addresses *V. F.* and *S. F. Leibniz*:

> … There's nothing special about the value $t = 3$, we could apply this to any other point in time, so we can consider this expression $\frac{ds}{dt}$ to be a function of *t*, something where I can give you a time *t*, and you can give me back the value of this ratio at that time; the velocity as a function of time.

He draws the corresponding velocity function (*G. F. f and f′*). Afterwards, Grant addresses *P. R.* and *P. F.* (Fig. 15), following by one of its numerical realizations with pointing out how he programmed the computer to draw the velocity function (*N. F.*):

> when I had the computer draw … the velocity function … first, I chose a small value for *dt*, I think in this case it was 0.01 [*P. R.*]. Then, I had the computer look at whole bunch of times *t* between 0 and 10 [*P. F.*], and compute the distance function *s* at $t+dt$, and then subtract of the value of that function at *t*. In other words, that's the difference in the distance traveled between the given time *t*, and the time 0.01 s after that. Then you can just divide that difference by the change in time, *dt*, and that gives you the velocity in meters per second around each point in time [*N. F.*] [on the screen viewers can see $v(t)$ as $\frac{ds}{dt}(t) = \frac{s(t+dt)-s(t)}{dt}$ (the right hand side is added later and addresses *S. R. DQ*-Δx)].

Grant continues by presenting several distance functions and their corresponding velocity functions on the screen to help viewers better realize their associations (*G. F.* f and f').

5.5 The formal definition of the derivative

Grant while revisiting *G. R. Secant*, *S. L. Leibniz*, and *G. F. f and f′* begins to discuss the formal definition of the derivative by highlighting that "in pure math, the derivative is not this ratio $\frac{ds}{dt}(t) = \frac{s(t+dt)-s(t)}{dt}$ [*S. F. Leibniz* and *S. R. DQ*-Δx] for specific choice of *dt* [*P. R.*]. Instead, it is whatever that ratio approaches as your choice for *dt* approaches 0 [on the screen, *dt* takes closer values to zero starting from 0.1 (i.e., $\frac{ds}{dt}(t) = \frac{s(t+0.1)-s(t)}{0.1}$) to 0.0000001 (i.e., $\frac{ds}{dt}(t) = \frac{s(t+0.0000001)-s(t)}{0.0000001}$)]" [*N. L. Right*]. His formal definition of the derivative is "$\frac{ds}{dt}(t) = \underbrace{\frac{s(t+dt)-s(t)}{dt}}_{dt \to 0}$" without using the visual mediator *lim*. In terms of our realization tree, it could be count as both addressing *S. F. Leibniz* and *S. F. DQ*-Δx. Grant in this video probably intentionally does not refer to limit, and also even does not use the *lim* visual mediator when presenting the symbolic

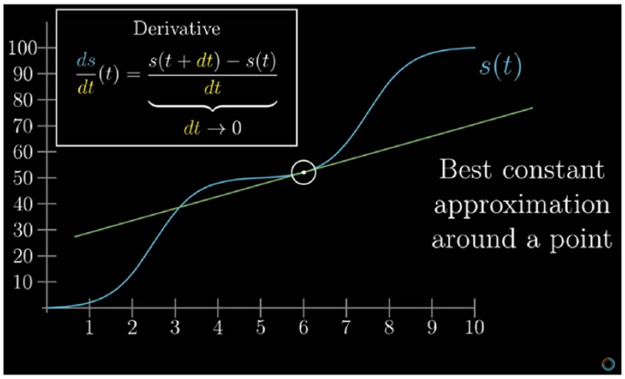


Fig. 16 A screenshot when Grant introduces *V. L. Best approximation* (9:42)

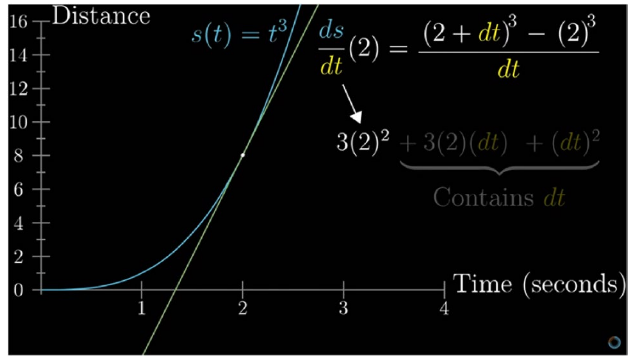


Fig. 17 A screenshot when Grant discusses how the derivative could be calculated at a specific time t (12:35)

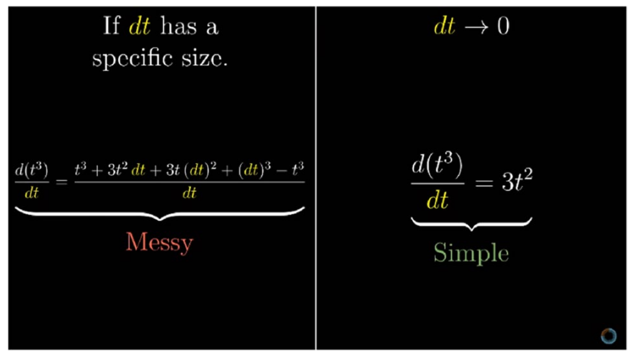


Fig. 18 A screenshot when Grant discusses why he has done the calculation of the derivative earlier (14:15)

realization of derivative. This could be because he wants to make the video more inclusive, especially for viewers with varying levels of participation in the discourse on limits.

Grant then moves back to the visual realizations of the derivative starting from *G. R. Secant* and then discusses what this ratio means graphically when dt approaches zero (*G. L. Tangent*).

He emphasizes further the limit realization of the derivative:

> … I'm not saying that the derivative is whatever happens when dt is infinitely small, whatever that would be, nor am I saying that you plug in 0 for dt. This dt is always a finitely small, nonzero value, it's just that it approaches 0, is all [on the screen moving between *G. L. Tangent* and *G. R. Secant*].

5.6 Proposing a new verbal realization for the derivative

Grant introduces a new verbal realization for the derivative, proposing almost halfway in the video (*V. L. Instantaneous* and *V. L. Best approximation*):

> because change in an instant still makes no sense, I think it's healthiest for you to think of this slope not as some "instantaneous rate of change", but instead, as the "best constant approximation for rate of change" around a point [Figure 16].

5.7 Calculating the derivative using symbolic realizations

Grant utilizes an example to show viewers how "$\frac{ds}{dt}(t) = \underbrace{\frac{s(t+dt)-s(t)}{dt}}_{dt \to 0}$" could be used to calculate the derivative at a specific time t. He revisits the earlier context, presenting a distance function for a car on a Cartesian plane, and also providing its symbolic realization, $s(t) = t^3$. Grant sets the task of calculating the velocity at $t = 2$ (on the screen first addresses *G. R. Secant*, then focuses on *S. L. Leibniz* and *S. L. DQ-*Δx). He then calculates $\frac{ds}{dt}(2)$ algebraically as follows and points out that "as dt approaches 0, representing the idea of looking at smaller and smaller change in time, we can just completely ignore those other terms $[3(2)(dt)$ and $(dt)^2]$ [on the screen revisiting *G. R. Secant* and then *G. L. Tangent*]" (Fig. 17).

"$\frac{ds}{dt}(2) = \frac{s(2+dt)-s(2)}{dt} = \frac{ds}{dt}(t) = \frac{(2+dt)^3-(2)^3}{dt} = \cdots = 3(2)^2 + 3(2)(dt) + (dt)^2$".

Grant then focuses on the realization of the derivative at the function level by saying "there is nothing special about the time $t = 2$; we could more generally say that the derivative of t^3, as a function of t [*V. F.*], is $3t^2$ [on the screen, $\frac{ds}{dt}(t) = \frac{(t+dt)^3-(t)^3}{dt}$ and $\frac{ds}{dt}(t) = 3t^2$ (*S. F. Leibniz* and *S. F. DQ-*Δx) and then moving the tangent line of $s(t) = t^3$ addressing *G. F. Tangent*]".

Later in the video, Grant discusses why he shows the details calculation of the derivative algebraically/symbolically:

> When you consider the tiny change in distance caused by a tiny change in time for some specific value of dt,

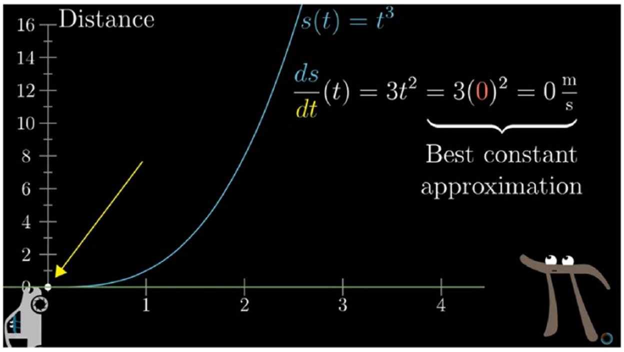


Fig. 19 A screenshot when Grant revisits *V. L. Best approximation* for a concrete example (15:35)

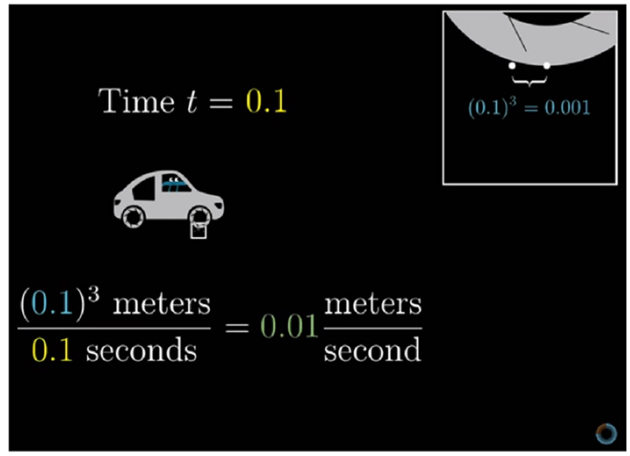


Fig. 20 A screenshot when Grant revisits *V. L. Best approximation* (16:06)

> you would have a kind of mess, but when you consider what that ratio approaches as dt approaches 0, it lets you ignore much of that mess, and it really does simplify the problem. That right there is kind of the heart of why calculus becomes useful [Figure 18].

5.8 Revisiting the paradox and promoting the use of new verbal realization

Towards the end of the video, on the screen Grant revisits the paradox with *V. L. Instantaneous* and returns to the example about the car movement with the velocity of $v(t) = 3t^2$ again to provide another reason why he discusses the concrete example and to discuss the paradox from another perspective. He points out:

> Consider its motion at moment $t = 0$… ask yourself whether or not the car is moving at that time. On the one hand, we can compute its speed at that point using the derivative …. But on the other hand, if it doesn't start moving at time 0, when does it start moving? … Do you see the paradox? [*V. L. Instantaneous*].

Grant then again discusses the new verbal realization for the derivative that he proposed (*V. L. Best approximation*): "What it means for the derivative of the distance function to be 0 is that the best constant approximation for the car's velocity around that point is $0\,\frac{m}{s}$" (Fig. 19).

Grant continues with further elaboration:

> Between $t = 0$ and $t = 0.1$ seconds, the car does move. It moves 0.001 m … [Fig. 20] What it means for the derivative of this motion to be 0 is that for smaller and smaller nudges in time, this ratio of $\frac{m}{s}$ approaches 0, but that is not to say the car is static. Approximating its movement with a constant velocity of 0 is after all just an approximation [returning to the previous example on the screen for a moment revisiting *G. L. Tangent* and *S. F. Leibniz* and *V. L. Best approximation*]. So, whenever you here people refer to the derivative as 'an instantaneous rate of change', … think of that as a conceptual shorthand for 'the best constant approximation for the rate of change' [*V. L. Instantaneous* and *V. L. Best approximation*].

6 Discussion

The findings suggest that online YouTube resources could provide many versatile opportunities for students to learn about different realizations of the derivatives by integrating digital technology. For example, the video that has been analysed from the *3Blue1Brown* channel regarding the derivative have addressed all the five main realizations of the derivative as pointed out by Roundy et al. (2015). As shown in the results section, Grants utilizes digital technology very effectively to transition between different realizations. More specifically, as shown in Fig. 8b, there are numerous utilizations of graphical realizations in this learning resource, as suggested in the literature (e.g., Ryberg, 2018). Furthermore, it is worth pointing out the care taken in this video to address the *paradox* with *V. L. Instantaneous*. Past literature noted challenges with realizing *V. L. Instantaneous* (e.g., Kertil et al., 2023), and it is interesting to observe how Grant developed this learning resource, beginning with *V. L. Instantaneous* and concluding by addressing this paradox. This is one of the reasons that makes the video *a must-watch* for those involved in learning or teaching calculus. Additionally, it seems Grant intentionally did not discuss the keyword *limit* in his discourse and did not use the *lim* visual mediator when presenting the symbolic realization of the derivative at the limit and function levels. That could be because of several reasons, such as making the video more accessible to viewers with different levels of participation in the discourse about limits. Grant might avoid using heavy mathematical language in the video since past research (Esparza Puga & Aguilar, 2023) reported that students use YouTube resources when they need *help*.

Although the role of lecturers and other learning resources such as textbooks in facilitating student learning should not be overlooked, online learning resources such as YouTube could also play a key role here by providing further insight into different realizations of mathematical objects and bringing different perspectives that might not have been captured in the textbooks and/or teaching. For instance, recently Haghjoo et al. (2023) reported that the physical realizations of derivative were not discussed well across several high school and university calculus textbooks; however, in our study, we have seen that such a realization of derivative is discussed in this learning resource at the ratio and function levels. As seen in the results section, certain physical and numerical realizations (i.e., *P. R.* and *N. R. Right*) are discussed early in this learning resource. This approach aligns, to some extent, with Radmehr et al.'s (2023) suggestion of focusing on physical and numerical realizations of derivative early in teaching this topic.

Finally, while our analysis mainly focuses on how Grant moves between different realizations, it is noteworthy that Grant's discourse in this video, consistent with other YouTube learning resources on his YouTube channel to the best of our knowledge, could be more aptly characterized as explorative discourse rather than ritualized discourse (Sfard, 2017). He places minimal emphasis on *how* to calculate derivatives through the manipulation of mathematical symbolic signifiers and instead prioritizes what the derivative as a mathematical object is.

7 Conclusions

In this study, we have developed further a recent RT for derivative (Haghjoo et al., 2023) to account for derivative at the function level. This study is also seeming to be one of the first attempt to analyse YouTube learning resources using an RT. We have identified 35 roots for the derivative reflecting on the relevant literature and our teaching and research experiences. The RT was used as a lens to analyse a popular YouTube resource regarding derivative. We believe that calculus lecturers and mathematics education researchers could benefit from the developed RT as an analytical tool to critically reflect on teaching and learning of derivative in calculus courses as well as using them for designing activities and tasks. As Sfard (2017) noted, students need to be supported in establishing connections between various realizations of mathematical objects. We believe that reflecting on thorough RTs grounded in the relevant literature could be a good starting point for thinking about improving the teaching and learning of mathematics, as supported by others (e.g., Haghjoo et al., 2023).

We would like to conclude that past research reported that YouTube videos are among the learning resources accepted and used by many undergraduate students to learn or reinforce their mathematics learning (e.g., Esparza Puga & Aguilar, 2023). Some YouTube learning resources go beyond discussing procedural routines and delve into different realizations of mathematical objects and their connections. They emphasize moving between multiple realizations which could facilitate student participation as past research have underlined (e.g., Hong & Lee, 2022). Digital technology has impacted the quality of many YouTube resources that have been developed in the past decade. At least it is evident from the learning resource analyzed in this study that these technologies have created opportunities for YouTube developers to discuss various realizations of mathematical objects simultaneously. Hence, we want to draw the attention of key stakeholders in undergraduate mathematics, such as mathematics lecturers and teacher assistants, to these resources. These resources can serve different purposes and could be possibly integrated into teaching and learning of undergraduate courses depending on students' level of participation in mathematics discourse and the intended learning outcome(s).

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Declarations

Conflict of interest There is no conflict of interest in the manuscript.

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