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ORIGINAL PAPER

Prospective Secondary Teachers' Emergent Knowledge and Beliefs: Inquiry-Oriented Differential Equations Contributing to Teacher Preparation

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Abstract

This article reports on the evolution of prospective secondary teachers' knowledge (meanings for $\frac{dy}{dt}$) and beliefs (about teaching and learning mathematics) in a semester-long inquiry-oriented differential equations class. Students entered the course with limited, primarily procedural, meanings for $\frac{dy}{dt}$. Throughout the semester, they engaged in collaborative mathematical inquiry using a research-based curriculum. As viewed through the emergent perspective, students' meanings for $\frac{dy}{dt}$ and their beliefs about teaching and learning mathematics co-evolved with community norms and practices through the classroom discourse. Students' end-of-term portfolios and portfolio presentations were analyzed for evidence of meanings for $\frac{dy}{dt}$ and beliefs about students' roles, instructors' roles, and the general nature of learning mathematics. In these, students expressed rich, multifaceted meanings for $\frac{dy}{dt}$ and beliefs about learning as an active process of meaning-making. While many prospective teachers do not see the relevance of advanced mathematics coursework to their career, these students reflected on their experiences in the course and volunteered ways in which their emerging knowledge and beliefs would influence their future practice. We emphasize that the classroom norms supported students in pursuing connections themselves, and conjecture that similar norms would support the development of reflective practitioners in other contexts.

Keywords Differential equations · Teacher education · Derivative · Inquiry-oriented instruction · Student meanings · Student beliefs

1 Introduction

Myriad factors contribute to a mathematics teacher's practice in the classroom. Some contextual factors, such as state or national content standards, class size, or school budget, are beyond the reach of any prospective teacher preparation program. Others, however, can be directly or indirectly impacted by such a program. We broadly group these malleable, and important, factors into *knowledge* and *beliefs* (AMTE, 2017; Tatto et al., 2008), which we use to illuminate how one differential equations course impacted prospective secondary mathematics teachers (PSTs). In this manuscript we use *knowledge* in reference to students' *mathematical meanings* and *beliefs* in reference to *beliefs and attitudes about teaching, learning, and mathematics.*

Differential equations is, generally speaking, a postsecondary mathematics course in the US which is taken after completing differential and integral calculus courses.[1] As such, the mathematical content of a differential equations course may be considered advanced, or at least *nonlocal* (as per Wasserman, 2018), in reference to the teaching of secondary mathematics. Knowledge of advanced mathematics is broadly considered important for teaching secondary mathematics, but individual knowledge alone is not sufficient support for high-quality teaching practice (AMTE, 2017; Darling-Hammond, 2000) and many PSTs

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[1] There is some variation in course sequencing across postsecondary institutions. Some institutions expect students to take a course in linear algebra or vector calculus before taking differential equations, others do not.

do not develop robust understandings in their coursework (Thompson, 2013). In addition, many PSTs see advanced mathematics courses as having little relevance for their intended career (Ticknor, 2012; Zazkis & Leikin, 2010). Differential equations requires the use of derivatives, a topic usually introduced in single-variable calculus courses, however it has been extensively documented that students' conceptual understandings of the derivative and underlying concepts are less developed than their ability to compute derivatives symbolically (Larsen et al., 2017). As such, and because it is a topic which bridges secondary and postsecondary mathematics, the derivative is a valuable concept for study—for PSTs and for scholars pursuing stronger PST preparation.

Many strategies for addressing these challenges have been proposed, such as explicitly incorporating secondary mathematics content and/or mathematics education topics into advanced mathematics curricula. In our case, we report on an advanced mathematics course which was, first and foremost, a *mathematics* course. While this was a section of differential equations offered just for PSTs, the instructor did not alter the core course content to specifically address teaching concepts from high school mathematics. He did, however, model inquiry-oriented pedagogy with an awareness that he might influence these future teachers. In inquiry-oriented classrooms, students actively work on coherent and challenging mathematical tasks and collaboratively process and discuss their mathematical thinking. In support of student inquiry, the instructor inquires into student thinking and seeks to use this thinking to further the mathematical agenda (Laursen & Rasmussen, 2019). The resulting microculture of this classroom afforded opportunities for these students to make many of *their own* connections to their future teaching endeavors. Our results have implications for any PST mathematics course, including those where a separate section designed explicitly for PSTs is not feasible.

In this manuscript, we focus on students' development of meaning(s) for the symbol $\frac{dy}{dt}$. The derivative is a topic which lies at the boundary of secondary and postsecondary mathematics, and is a persistent challenge for students around the world (Thompson & Harel, 2021). We also consider PSTs' beliefs about the role of students, instructors, and mathematical activity as related to this inquiry-oriented differential equations (IODE; Rasmussen et al., 2018) course and the PSTs' imagined future courses. These evolving conceptions and beliefs are viewed through the emergent perspective (Cobb & Yackel, 1996; Yackel & Cobb, 1996), meaning that we assume these changes coincided with the development of the classroom social and sociomathematical norms which supported mathematical argumentation and facilitated the development of a community of inquiry (Biza et al., 2014; Goodchild et al., 2021). Thus, we seek to answer the following research questions:

1. What (new) meanings for $\frac{dy}{dt}$ do prospective teachers report at the end of an inquiry-oriented differential equations course?
2. What (new) beliefs about learning and teaching mathematics do prospective teachers report at the end of that course?

We also report on the *relevance* of students' emergent knowledge and beliefs for their future teaching, both from our perspective and the PSTs' own observations.

2 Theoretical framing

We draw on the emergent perspective to conceptualize how students' mathematical conceptions and their beliefs about mathematics, teaching, and learning develop in an inquiry-oriented differential equations course. In this perspective, the individual (psychological) and social (collective) planes take on equal primacy in understanding the processes of learning. Learning is conceptualized as both individual construction *and* a process of negotiated enculturation into the practices of a community; more specifically, students' individual beliefs and knowledge are reflexively related to classroom mathematics practices and social and sociomathematical norms (Cobb & Yackel, 1996; Yackel & Cobb, 1996). The interpretive framework associated with the emergent perspective was developed through years of classroom-based research to explicitly name and relate social and psychological constructs which capture distinct, but dynamically interrelated, aspects of the complex learning processes which occur in interactive mathematics courses. The individual (psychological) constructs are (a) beliefs about one's own role, others' roles, and the general nature of mathematical activity; (b) mathematical beliefs and values (or mathematical disposition); and (c) mathematical meanings. The social (collective) constructs are: (d) classroom social norms; (e) sociomathematical norms; and (f) classroom mathematical practices. Our focus in this manuscript is on the individual constructs (a) and (c), thus we focus our theoretical discussion on these and how they relate to the social constructs. While we focus on individual meanings and beliefs, we never lose sight of the fact that these co-evolve with the negotiation of collective norms and practices.

Individuals' beliefs about their role, others' roles, and generally what it means to do mathematics are integrally related to *classroom social norms*, or the situation-specific patterns of discourse that are expected and accepted in a particular classroom community. The instructor is a high-status actor and has outsize influence over the development of social norms, but they cannot "set" these norms – they must be negotiated through repeated interaction. The dynamic co-evolution of beliefs and norms was originally developed in elementary school mathematics courses, and has been seen in undergraduate mathematics and science classes (Dixon et al., 2009; Johnson, 2000; Saglam et al., 2014; Yackel & Rasmussen, 2003; Yackel et al., 2000). In this manuscript, we focus on the PSTs' espoused beliefs about the students' roles, the instructor's role, and the nature of mathematical learning as they reflect on their experiences in this course and imagine their future teaching. This group of PSTs, encouraged by the instructor, evolved into a community of inquiry where reflecting on the nature of the enterprise of mathematics education, and the PSTs' current and future role in that enterprise, was valued (Goodchild et al., 2021).

Individual mathematical meanings refer to a person's ways of reasoning and their ideas about mathematical concepts and tasks. These meanings are enacted and developed as one interacts with others, through expression and interpretation of ideas related to a particular mathematical concept. That is, individual conceptions are reflexively related to *classroom mathematical practices*, which refer to ways of doing mathematics which are accepted and expected by members of the classroom community. Generally, these collective practices and their evolution are documented by identifying ways of reasoning which function-as-if-shared (Cole et al., 2012; Rasmussen & Stephan, 2008). Individual meanings have also been linked to *social* and *sociomathematical (socioscientific) norms* in undergraduate mathematics (Rasmussen et al., 2020), physics (Chang et al., 2020), and chemistry (Warfa et al., 2018).

3 Literature

Many factors influence the nature and quality of secondary mathematics teachers' instruction. In this manuscript we focus on aspects of mathematical knowledge and beliefs about teaching and learning.

3.1 Mathematical meanings related to $\frac{dy}{dt}$

A first order differential equation (DE) takes the form $\frac{dy}{dt} = f(t, y)$, where t is the independent variable and y is the dependent variable. Solving a differential equation is finding or approximating the functions y that satisfy the equation. The definition provided by the IODE curriculum elaborates this idea further by tying it to the actual meaning of the derivative, namely rate of change: "suppose $y = y(t)$ is some unknown function, then a differential equation, or rate of change equation, would express the rate of change, $\frac{dy}{dt}$, in terms of y and/or t […] given a rate of change equation for some unknown function, solutions to this rate of change are functions that satisfy the rate change equation" (Rasmussen et al., 2018, p. 1.2). In what follows, we present some literature relevant to student understanding of rate of change, covariational reasoning, and functions—ideas related to the derivative which were present in these students' meanings, but which many secondary teachers do not understand robustly (e.g., Byerley & Thompson, 2017; Thompson et al., 2017).

There are numerous studies in the literature that interpret student reasoning about rate of change, a concept mathematically denoted by $\frac{dy}{dt}$. For example, Zandieh and Knapp (2006) explain that the derivative concept when defined formally as $\frac{dy}{dt} = \lim\limits_{h \to 0} \frac{f(t+h)-f(t)}{h}$, consists of multiple process-object layers that include calculating a ratio and evaluating a limit. Herbert and Pierce (2012) identified a long-term, multi-year trajectory of student learning involving eight ways of understanding rate of change. Rasmussen and Keene (2019) identified five waypoints for reasoning about rate of change that might be reached in a single semester during a first course on differential equations: (1) a constant slope function where students view rate of change as a constant value; (2) a discretely changing ratio function where students go beyond the static slope interpretation and compute changing values of the rate of change; (3) a time invariant function where student reasoning focuses on invariant patterns in the rate of change, with an eye toward the family of solution functions; (4) an indexical function to determine the structure of the space of solutions; and finally (5) a parameterized function where students reason dynamically with the indexical function as a parameter in the rate of change equation varies. In all five waypoints, the rate of change function is foundational.

Sfard (2021) presented a framing for the concept of functions in which students reason about them operationally: one inputs a number and the function produces an output. Tall et al. (2000) assert that one may consider functions as a machine where an input produces an output. Related to functional thinking is covariational reasoning, which involves coordinating two varying quantities while attending to the way each variable changes with respect to the other (Carlson et al., 2002). Because an $\mathbb{R} \to \mathbb{R}$ function can be thought of as two quantities varying in tandem, it is productive for students to imagine or to coordinate how these quantities change in relation to each other. In the context of this research study, students sketch graphs of solution functions of a differential equation based on graphs of autonomous rate of change equations. Solving graphically an autonomous ODE $\frac{dy}{dt} = f(y)$ requires students to coordinate the graph of the rate of change function $\frac{dy}{dt}$ with the graphs of solution functions; the latter is in the y vs. t plane, while the former is in the $\frac{dy}{dt}$ vs. y plane. In other words, students are asked to coordinate the dynamics of $\frac{dy}{dt}$ vs. y with the dynamics of y vs. t, a complex endeavor which necessitates that students apply and coordinate covariational reasoning in two different coordinate planes (Rasmussen & Keene, 2019).

3.2 Teacher beliefs

Instructors' beliefs about mathematics, and the teaching and learning of mathematics, influence their instructional practices, albeit in nuanced and sometimes subtle ways (Marshman & Goos, 2018; Philipp, 2007; Stipek et al., 2001). While there is much variation in defining teachers' beliefs—and in how individuals' beliefs, affect, and knowledge are related to each other and to local contextual constraints—there is broad agreement that they *do* matter for teaching (Cooney & Wiegel, 2003; Eichler & Erens, 2014; Philipp, 2007; Tatto, 2008).

An early, but still relevant, framework comes from Ernest's (1989) work, in which he identified three (related) components of mathematics teachers' beliefs which might influence their instructional practice: beliefs about mathematics, beliefs about teaching, and beliefs about learning. Regarding mathematics, he notes three philosophies which had previously been theorized and observed: an instrumentalist view of mathematics as the accumulation of facts and procedures for obtaining solutions; a Platonist view of mathematics as a static body of unified knowledge to be discovered (not created); and a problem-solving view of mathematics which is dynamic and situated within its social and cultural context. Associated were three views of the teacher's role: instructor (correcting performance); explainer (of existing Platonic knowledge); or facilitator (posing problems). Further, he describes four categories of teachers' models of students' learning, which connect to the students' role in class: a behavioral model focused on procedural fluency; reception or transmission model of accepting knowledge; a constructivist model of active development of knowledge; and an exploration model of autonomous pursuit. More recent work has broadly been compatible with these categorizations (or use them explicitly, e.g., Marshman & Goos, 2018), though language has in many cases shifted to characterize teachers' belief systems and practices along a spectrum from 'instructor-centered' (teachers disseminate, students passively receive) to 'student-centered' (teachers facilitate collaborative problem-solving) (Conner et al., 2011).

Teachers' beliefs develop through their experiences, including experiences as learners in mathematics and teacher education programs (Beswick & Callingham, 2014; Conner et al., 2011; Stipek et al., 2001). PSTs enter teacher preparation programs with certain beliefs, which can evolve through critical inquiry into the nature of mathematics teaching and learning as well as engagement in mathematical discovery and exploration (Biza et al., 2014; Conner et al., 2011; Cooney & Wiegel, 2003; Goodchild et al., 2021; Jaworski, 2006). As such, this inquiry-oriented course in which students developed critical stances toward the teaching and learning of mathematics had the potential to support meaningful changes in PSTs' beliefs.

4 Research methodology

4.1 Context

This study is set in the context of a differential equations course at a research university in the United States. The inquiry-oriented approach in this class reflected core pillars of inquiry: students engage deeply with coherent and meaningful mathematical tasks; students collaboratively process mathematical ideas; and instructors inquire into student thinking and use their thinking to forward the mathematical agenda (Laursen & Rasmussen, 2019). The course differs from a conventional, technique-driven differential equations class by emphasizing a contemporary dynamical systems approach (Habre, 2013).

There were eight students in the class, all of whom were third-year or fourth-year mathematics majors. At this institution, prospective secondary mathematics teachers first complete a 4-year degree in mathematics and then apply for a 1-year credential program. This particular course, which was only open to PSTs, counted as one of their upper division mathematics electives. The instructional format of the course included cycles of students working collaboratively in small groups followed by whole-class discussions and student presentations of small group work. Very little class time was allotted to lecture, but the teacher regularly inserted information and connected the students' words to conventional or formal terminology.

In earlier work, we linked the inquiry-approach with the emergence of a *community of inquiry* among this same group of PSTs (Goodchild et al., 2021). This community of inquiry was characterized by the students' development of a critical stance toward the enterprise of mathematics education (Biza et al., 2014; Jaworski, 2006) and reflected particular social and sociomathematical norms (Yackel & Cobb, 1996). More specifically,

> The teacher was intentional in fostering the social norms that students explain and justify their thinking, however tentative, listen to and attempt to make sense of others' thinking, and indicate agreement or disagreement (with reasons) with others' reasoning.

> The primary sociomathematical norm of interest was that justifications be based on underlying concepts as opposed to appeals to procedures or external authorities such as the text or teacher (Goodchild et al., 2021, p. 8).

The mathematical goals for the course complemented the normative discursive practices and included students learning the content with meaning via a sequence of challenging problems. The IODE curriculum (Rasmussen et al., 2018) was inspired by the instructional design theory of Realistic Mathematics Education (Freudenthal, 1991; Gravemeijer, 1999) in which mathematics is principally conceptualized as a human activity of sense making and mathematizing, as opposed to an a priori collection of predetermined facts and procedures.

4.2 Data collection

We take a naturalistic approach to explore students' meanings for $\frac{dy}{dt}$, utilizing student artifacts produced in the regular activity of the classroom as research data. The three main data sources are students' beginning-of-semester background surveys; end-of-semester written portfolios; and videos of corresponding oral presentations of student-selected items from their portfolio. The background surveys asked students about their previous mathematical coursework and to describe all their different meanings for $\frac{dy}{dt}$. The students' end-of-semester portfolios included seven entries, based on work throughout the course, that would showcase their progress and accomplishments. Students were encouraged to select entries that they felt highlighted their creativity or inventiveness, mathematical growth, ability to connect ideas and concepts, deep understanding of concepts, and their ability to reason and model with mathematics. One entry was more specific, asking the students to discuss all their different meanings for $\frac{dy}{dt}$. For each entry, students were required to provide a rationale statement that explained the selected items as well as each item's personal significance. The 10-min oral presentation was an extension of the portfolio assignment in which the students were to present three of their entries, including their $\frac{dy}{dt}$ entry.

4.3 Data analysis

The oral presentations were transcribed and these transcripts and the written portfolios were analyzed using thematic analysis (Braun & Clarke, 2006) to characterize students' knowledge and beliefs. Our analysis was informed by prior work and our theoretical perspective. For students' meanings of $\frac{dy}{dt}$, the four authors (one of whom was the course instructor) began by open-coding responses to the question asking students to list their meanings for $\frac{dy}{dt}$ in the beginning of semester background surveys and end of semester portfolios. We read the participant responses, identified distinct ideas, and created codes that captured these meanings. Next, the team pulled quotes from both the written and oral portfolio entries, refining the operational definitions for each meaning and connecting each to the literature. Individual student entries were coded with multiple codes when multiple meanings were present in those entries. All coding was completed by at least two authors and vetted with the entire author team until consensus was reached. The first results section is organized according to the meanings which emerged: rate of change, calculations and calculus procedures, dynamic slope, and three groups of function meanings.

For students' beliefs, we leveraged the results of prior work on this data (Goodchild et al., 2021) and the constructs of the emergent perspective to focus our attention on the PSTs' beliefs about the general nature of learning mathematics and the roles of students and teachers in the mathematics classroom. Two authors (co-authors of the aforementioned) led this analysis, which considered the entirety of students' portfolios and presentations. Comments explicitly related to teaching and learning activities were excerpted, tagged, organized, and re-organized using the method of constant comparison to identify themes present across students' responses. Excerpts were coded with multiple codes when they related to multiple themes or the relationships between these themes; they were further tagged to identify excerpts in which PSTs explicitly noted *changes* in their beliefs or how these beliefs would influence their future teaching. Discrepancies were discussed until consensus was reached. The second results section is organized around these major themes, documenting beliefs about mathematics learning as an active process of meaning-making and connection-building, students' role as active collaborators who explain their thinking, and the instructors' role in developing a supportive classroom environment.

5 Mathematical meanings

Our first section of results describes students' individual mathematical meanings, organized based on the different interpretations for $\frac{dy}{dt}$ that we identified within students' written work and oral presentations.[2] When possible, we connect these meanings to specific items in the *Common Core State Standards for Mathematics* (CCSSM, 2010),[3] highlighting connections to the US secondary curriculum.

[2] Students' work is presented verbatim, unless noted, except for mathematical text which has been broadly reformatted for legibility (e.g., y^2 replacing y^2). All names are pseudonyms.

[3] At the time of writing, the full text of the CCSSM is available at https://learning.ccsso.org/common-core-state-standards-initiative; individual standards can be looked up by code.

5.1 Rate of change

The phrase "rate of change" is often associated with derivatives, but using this phrase is not in itself evidence of conceptual meaning. In the background questionnaire, four students mentioned rate of change, but only Jimena gave a concrete example suggesting what the phrase means to her: "Rate of change over time [for example] amount of water that a leaky faucet drips in two hours." In contrast, all eight students used the phrase in their end-of-term portfolios with meaning, for example:

Brad:	As a rate of change over time, how the population of the rabbits and foxes or the fish change over time
Jimena:	In this case $\frac{dP}{dt}$ is a growth in population, we've been calling it a decay. It's population changing over time
Denise:	We knew that when p was greater than 25, $\frac{dy}{dt}$ [sic] was negative, which meant the fish population was decreasing as time increased. When p was between 0 and 25 it was positive, so the fish population was increasing, and when p was less than zero, the rate of change was negative so the population was decreasing

Brad indicates that in two problems discussed in class, $\frac{dy}{dt}$ as a rate of change referred to the fact that the quantity of population (number of rabbits, foxes, or fish) changed in relation to the quantity of time elapsed. Jimena adds an element of directionality, noting that population changing over time might be growth or decay. Denise seamlessly switches between "$\frac{dy}{dt}$," "it," and "rate of change" as she speaks, indicating that she sees them as equivalent; she also indicates the covariational relationship between the population and time, noting that the two quantities change *together*.

Students' understanding of $\frac{dy}{dt}$ as a rate of change connects the content of the IODE course and content they are likely to teach as secondary mathematics instructors (CCSSM, 2010, 8.F.B.4, HSF-IF.B.6). While the association between the phrase and the symbolic expression was present in the background survey, in these portfolios students expressed new, more extensive understandings of these concepts.

5.2 Calculations and calculus procedures

In their previous calculus courses, students had experience finding the derivatives of functions (e.g., if $y = 3t^2$, then $\frac{dy}{dt} = 6t$) and using the first (and second) derivative of a function to identify critical points, regions of increase/decrease, and regions with positive/negative concavity to sketch a graph of the original function. Unsurprisingly, we saw language and procedures from calculus referenced by several students: three in the background survey and five in the final portfolios.

Many of the references in students' final portfolios were to procedures for checking whether a particular function is (or is not) a solution to a DE (or system of DEs). Denise describes this: "We just plugged in the function into the differential equation and matched it to the derivative of the function itself." In her final portfolio, Jimena again references the general power rule; Abel and Derrick both reference an analytic technique for solving linear DEs; Valencia describes using the first and second derivatives to determine critical points and concavity:

> The function $\frac{dy}{dt}$ is also the first derivative. Back when I was taking calculus one, when I asked about the increasing and decreasing points of a graph, I automatically thought about the first derivative, and when it came to concavity I related that to the second derivative of a function. Thus, by thinking about this as a first derivative, my focus went directly to finding my 'zeros' or my equilibrium solutions first, and then finding out in what intervals the graph was increasing and decreasing in order to obtain an overall result of the graph.

As evidenced by these excerpts, students connected their processes of solving DEs to their prior encounters with $\frac{dy}{dt}$ in calculus. Calculus has become more common in US high schools (Bressoud, 2021), so these connections should support students' teaching of mathematics local to their context. It is also evidence of students' deepening appreciation of the utility of their procedural skills for their own understanding.

5.3 Dynamic slope

In calculus courses, students are often asked to find the slope of a graph at a particular point using derivatives. These types of questions often focus on $\frac{dy}{dt}$ as a discrete or constant value, implicitly sending the message that the value is static and camouflaging the possibility of $\frac{dy}{dt}$ as a changing, or dynamic, slope. A dynamic slope meaning refers to the changing nature of slope, including indications that the slope could be calculated (and could be different) at different points.

In the background questionnaire only Enrique's response suggested dynamic slope as related to $\frac{dy}{dt}$. He wrote $\frac{dy}{dt}$ as "the rate of change [of] y at a given time" (emphasis added). The associated sketch he provided (Fig. 1) suggests that Enrique recognizes that slope changes, and that $\frac{dy}{dt}$ could be used to find the slope of a tangent line at a particular point (i.e., a particular value of t).

End-of-semester portfolio responses were more robust in explaining how $\frac{dy}{dt}$ represents the slope of tangent lines to a function y for varying values of t. Six of the eight students indicated that they associate $\frac{dy}{dt}$ with a dynamic slope. For example (emphasis added):

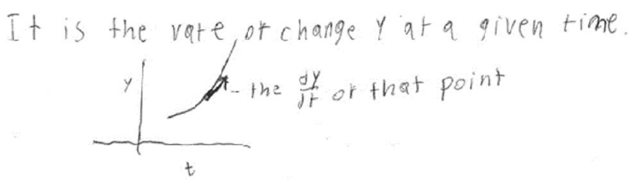

Fig. 1 Excerpt from Enrique's background survey

Brad:	When I learned about [slope] fields, I learned to not view $\frac{dy}{dt}$ as a point on a "y" vs "t" graph. Instead, I viewed $\frac{dy}{dt}$ as an *instantaneous slope, which was always changing*. The steepness of that slope relied on the outcome of $\frac{dy}{dt}$ given certain points
Carly:	Another way to think about $\frac{dy}{dt}$ is as the slope of a function. If we had the equation $\frac{dy}{dt} = 4y$ […] we think of dy as being the change in y and dt as being the change in t, we can say that $4y$, is the slope of the function $y(t)$. This would mean that *as y changes the slope changes*, which can be represented on a graph of $\frac{dy}{dt}$ vs. y

These students appear to associate slope fields and the idea of $\frac{dy}{dt}$ as a changing value. Brad explicitly describes $\frac{dy}{dt}$ as "always" changing, while Carly exhibits evidence of a covariational understanding in which the value of $\frac{dy}{dt}$, or slope, changes as y changes. This coordination is more complex than coordinating the dynamics of y with that of t. Other students displayed a dynamic slope meaning for $\frac{dy}{dt}$ as they explained concepts such as Euler's method, which relates to the second of Rasmussen and Keene's (2019) waypoints, in which students transition from thinking about rate of change as a discretely changing unit ratio to a discretely changing non-unit ratio. Moreover, these results show that students moved beyond a static view of $\frac{dy}{dt}$ (i.e., describing the slope of a tangent line to a function at a particular point) to view $\frac{dy}{dt}$ as something which varies continuously as y varies. We conjecture that students' more robust understanding of the concept in advanced courses will inform their future approach to teaching slope (or derivatives), which are topics relevant to secondary mathematics education in the US (CCSSM 2010, 8.EE.B). This dynamic view of slope may also support these students' conceptualization of $\frac{dy}{dt}$ as a function.

5.4 Function meanings

None of the students reported function meanings for $\frac{dy}{dt}$ on the background survey, but all eight included function meanings in their portfolios. Here we organize the results into three groups of functional meanings for $\frac{dy}{dt}$: input/output; operator; and indexical and parameterized functions.

5.4.1 Input/output

Several students indicated that they understood $\frac{dy}{dt}$ as representing a function in the sense that it describes a rule for relating inputs to outputs. This meaning of function resonates with descriptions in the literature and US mathematical education standards (CCSSM 2010, 8.F.A.1). Three students' end-of-term explanations made clear references to an input/output meaning of function, including:

Abel:	Interpreting $\frac{dy}{dt}$ as inputs and outputs is a meaning I had the first time I saw $\frac{dy}{dt}$, because with any variable, I plug numbers to get a result. Specifically on[e] such method we did this semester was Euler's method (tip to tail method) the steps required many different inputs with a step size which resulted in many outputs
Jimena:	In the case of $\frac{dy}{dt} = 0.3y$, $\frac{dy}{dt}$ is a function because y is treated as a variable, and $\frac{dy}{dt}$ is the output of the input y

While we have only three instances in which students explicitly draw on input/output language, this meaning of input/output relationship is often present in more sophisticated lines of reasoning about $\frac{dy}{dt}$ (Rasmussen & Keene, 2019).

5.4.2 Operator

Students also reported meanings for $\frac{dy}{dt}$ that we labeled a *function as operator* meaning, referring to $\frac{dy}{dt}$ as acting on (differentiating) some function[4] y, often in the context of checking whether a particular function was a solution to a DE. In this way, $\frac{dy}{dt}$ is a function in its own right, with another function as its input (and output).

Five of the eight students in the course expressed operator meanings for $\frac{dy}{dt}$ at the end of the term. For example, Derrick gave a general description of how one could think about $\frac{dy}{dt}$ as a function with another function y is its input:

> We were plugging in, checking whether something was a solution or not. We took the derivative on the left hand side of the equation and we plugged in the right hand side of the equation for $y(t)$. So I looked at a function as plugging two things in on each side of the equation and checking to see if that was a solution or not.

Some students reported initial struggles with the thinking that goes along with an operator meaning for $\frac{dy}{dt}$, but eventually taking command of the practice. For example, Denise said:

> I was kind of scared of this course when I figured out that we were plugging in functions into differential

[4] Note that the students tend to use y and $y(t)$ interchangeably, despite the fact that y is the name of a function and $y(t)$ refers to its range. Similarly, they describe dy/dt as an operator instead of the more conventional d/dt. A discussion of the affordances and constraints of using mathematically precise symbols in this context is beyond the scope of this manuscript and not germane to the topic of study.

equations just because I was so used to finding a variable [...] I thought this class is going to be really hard, so I want to talk about that. [...] one of the questions was we had to figure out if these functions were solutions to the differential equations.

The operator meaning of function highlights an analytic approach to determining or verifying the symbolic form of solution functions. In the following section we highlight meanings of $\frac{dy}{dt}$ that bring forth graphical approaches and address families of solution functions.

5.4.3 Indexical & parameterized functions

As students begin to reason about $\frac{dy}{dt}$ as a function, they can leverage graphs of $\frac{dy}{dt}$ to reason about the sign and magnitude of the rate of change, which serves to index the graphical space of solution functions (Rasmussen & Keene, 2019). For an autonomous differential equation, the graph of $\frac{dy}{dt}$ vs. y can be used to locate equilibrium points as well as regions where the solution functions increase or decrease. In this way, autonomous derivative graphs are used to make warranted inferences about graphs of solution functions. Students in this course regularly sketched these graphs to support their thinking, and they referred to these graphs as *Valencia graphs*, after one of their classmates. They also were explicit about the function they were graphing. For example, Jimena notes that a $\frac{dy}{dt}$ equation can be "looked at it as a function we could graph. We could graph $\frac{dy}{dt}$ on the Valencia graph."

The other students who discussed autonomous derivative graphs in their responses provided additional detail as to *why* they might create this graph and generally highlighted its importance for an overall graphical assessment of the solutions to an autonomous differential equation. Derrick noted that sometimes "the first thing I think of is $\frac{dy}{dt}$ vs y" because this graph "can help us for so many different things." Valencia wrote "my focus went directly to finding my 'zeros' or my equilibrium solutions first" while Enrique said "I think of a $\frac{dy}{dt}$ as a graph with multiple equilibrium solutions that could be seen on a $\frac{dy}{dt}$ vs. y graph." Valencia proceeds further to say that she can find out from the sketch in "what intervals the graph was increasing and decreasing."

An additional layer of complexity comes with the introduction of parameters. If students are reasoning about $\frac{dy}{dt}$ as a function, and a parameter is introduced into the rate of change equation, the Valencia graph now changes as the parameter changes, which in turn impacts the space of solutions, and "dynamic reasoning with a parameterized rate of change function leads to dynamic reasoning about the structure of the solution space" (Rasmussen & Keene, 2019, p. 11). In the portfolios, three students provided meanings for $\frac{dy}{dt}$ related to parameterized equations; none did so on the background survey. For example, Enrique expanded on his description of the autonomous derivative graph in the context of a class project about fish harvesting (referred to as the fish.net problem). In the context of viewing the $\frac{dy}{dt}$ vs. y graph as an index from which one can ascertain information about solution functions, Enrique described that graphing the parameterized function $\frac{dy}{dt}$ highlights how the graph characteristics change as the parameter changes: "So for that one, I was talking about fish.net, how graphically you can only see the bifurcation values and if you shift it down you can see the graph of each one." (see Fig. 2).

Carly also mentioned parameters and bifurcation diagrams in her portfolio:

> [$\frac{dy}{dt}$ is] considered as a function and it can change, like when there's a parameter changing, which we did recently in relation to, like, bifurcation diagrams and values and all that. […] From one of the homeworks $\frac{dy}{dt} = y^2 - ry + 1$ and while that r changes the whole function of $\frac{dy}{dt}$, changes.

Parametrized functions were described by Rasmussen and Keene (2019) as the fifth, and arguably most mathematically sophisticated, waypoint that students might encounter as they reason about rate of change in a differential equations course. Not only were students in this course able to reason about tasks and problem situations involving parametrized functions, but they also recognized this activity and the additional layers of meaning needed to reason about the solution space when parameters are involved.

While working with and analyzing rate of change equations as parameterized functions for the purpose of determining the changing structure to graphs of solution functions is nonlocal, there is a relatively close connection to local secondary school mathematics. Specifically, secondary school students are often tasked with analyzing how the roots to a quadratic function change as a parameter changes (CCSSM 2010, HSA-REI, HSF-IF.C). For example, secondary students might use graphing software to visualize quadratic functions such as $f(x) = x^2 - 25x - k$ and investigate how the roots change as the parameter k changes, much like how these undergraduate students investigated how the space of solution functions change as k changes.

6 Beliefs about teaching and learning

We now turn to our second research question, and report on students' *beliefs*. The emergent perspective draws our attention to students' beliefs about their role, others' roles, and the general nature of mathematical activity. In this context, we focus on PSTs' beliefs about their roles as learners, the

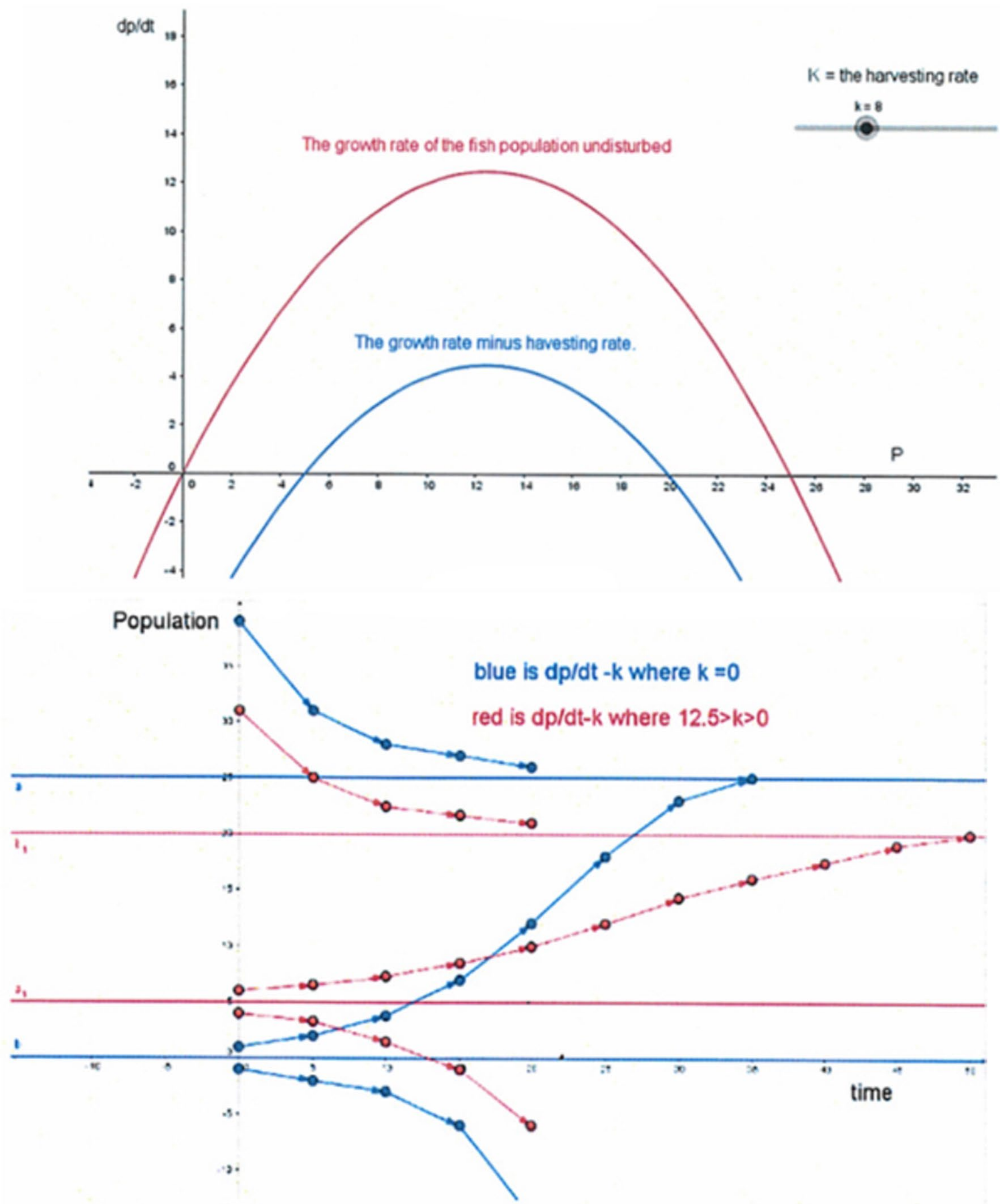

Fig. 2 Graphs for the fish.net problem using dP/dt = 2P(1 – P/25) – k as completed by the group of Enrique, Valencia, and Brad

teachers' role, and the general nature of learning mathematics. This analysis reveals students' shifting toward an understanding of learning as an active process of meaning-making. Not only do we see the potential impact of these shifting beliefs for teaching through the emergent perspective lens, but the students themselves made explicit connections to how these beliefs will influence their own future teaching.

6.1 Beliefs about students' role in mathematics classrooms

In their portfolios and final presentations, students evinced beliefs that students' role in class is an active one, including explaining their thinking, justifying their solutions, and engaging with other students. Some pointed to specific course activities wherein this role became clear to them, and often connected these roles to their own learning, several pointed to particular social norms of the class. For example, Abel described a task from class and then noted “the reason this problem was significant to me is [that] I had to give a reason for why I chose to go about a certain route for solving a problem, which strengthened my knowledge of what mathematical tools to use in certain situations.” In another entry, he reiterated this sentiment, noting that he “also realized that having to explain your answer in an essay format reinforces the techniques needed […] in future math classes I will write out explanations for my procedures when taking notes and doing homework,” suggesting that this realization impacted his view of his own role as a student in future courses.

These examples are connected to the social norm of justification with meaning. Students pointed their role in explaining and justifying solutions not only for their own benefit, but for others. Enrique noted that students didn’t just tell each other the answer when speaking, but “they had to convince everyone else of their own answer and […] if you were correct then you were teaching the whole class the correct idea, and by telling them your process and by telling them your process, they understood it the same.” This suggests that Enrique saw the students’ role in the class as not only explaining but teaching other students and helping them to understand the mathematical content; this sentiment was echoed by Brad, who noted that “twice in my reflection papers I wrote about how other classmates’ explanations really made it clear to me and brought a better understanding.” Enrique also connected this to classroom norms of explaining your reasoning, noting that “the class is designed so you have to verbally communicate your ideas to the rest of the class […] clearly, effectively, and convincingly.” Finally, he noted the role of students in engaging with others’ reasoning (connected to the social norm of indicating (dis)agreement with others), saying “the best way to know you are correct is to here [sic] others’ feedback whether or not you have convinced them of your thought process and your answer.”

6.2 Beliefs about the teachers’ supporting role

In addition to describing the roles of students, the PSTs noted the role of the instructor. This was not only in terms of describing the actions of their professor, but also in imagining their own future roles and responsibilities. Enrique was particularly explicit that he had learned about pedagogy as well as content, saying “this class wasn’t just differential equations to me. To me, this class was how to run a class properly.” Derrick, reflecting on how they had explored tasks and activities prior to mini-lectures on a topic, said “when I become a teacher I will make sure to replicate this type of learning in my classroom. I truly believe this is where students can start to reach [their] true potential.” The belief that the instructors’ role involves helping students reach their potential, was repeated by others. Brad noted that this course, in which he had reflected on his own learning process to develop “a better understanding of how I learn math,” would allow him to relate to students and support them. Specifically, “to be the best teacher you can be, you’ve got to be able to relate to your students.”

Students also referred to strategies that the instructor used to ‘get out of the way’ of their exploration and discovery of mathematical topics, which strengthened their understanding of the mathematical content and also gave them confidence in their own abilities. Enrique noted that “the class is designed so you have to verbally communicate your ideas to the rest of the class,” which is a design controlled by the instructor. He also said that the devolution of authority from the instructor (noted previously in regard to students questioning each other during explanations) was “the reason this course has so much understanding and knowledge.” Regarding confidence, Jimena pointed to a particular day in class and said, “I want to point out how cool it was that our class discovered Euler’s method on our own! That was a monumental day for me. Mathematicians like Euler just put their minds to something and eventually found what they were looking for, just like how we did in class.” She, as other students, pointed implicitly and explicitly to the instructors’ role in this course as being a ‘guide on the side’ while the students collaboratively discovered and re-invented mathematical concepts.

6.3 Beliefs about learning as an active process of making meaning

Many of the students’ comments about the roles of students and teachers in mathematics classrooms touched on emerging beliefs about learning mathematics as an active process of meaning-making and building connections. Brad was particularly effusive about building connections, noting in his portfolio that “once you learn a concept it does not just go away, rather you continue to build upon it and find that it ties together with other concepts” and saying in his presentation that “what, kind of, this class is about” was “building off of what we learn day in and day out.” Students also expressed new awareness of how to make meaning, and explore new concepts, using what they already understand, as in the following:

Derrick:	This made me realize that if you can simply breakdown [sic] all the little things and understand them, it can become very useful. This shows that if I know how to piece together tools and ideas, no problem is too difficult to start
Valencia:	I found this assignment to be extremely helpful because it required a lot of thought, teamwork, and dedication in order to figure it out and put it together […] this made the learning experience so much more interesting and I seemed to grasp the concept much better than usual

Valencia's comments also speak to the classroom norms of students working collaboratively and supporting each other (consistent with the PSTs' evolving views of students' roles in the math classroom), as well as to the positive impact of the experience for her own conceptual learning of mathematics. Views of learning mathematics as making meaning, and developing conceptual understanding, were widespread in the class, and half of the students explicitly mentioned reading or speaking 'with meaning,' which was a phrase the instructor used to push students to move beyond symbolic manipulation. Jimena in particular spoke about this in contrast to procedural understandings, noting that the instructor "began [class on Day 1] by encouraging us to 'read an equation with meaning'," and then reflecting at various points in her portfolio:

> I was thinking about how it's a habit for students to put a plus/minus sign in front of their solution to a square root. Do they even know what that means? I mean of course the answer to the square root of 4 is 2 or – 2, but do they know what it means to the problem? [...] The concept of understanding what you're writing rather than just putting it there because the teacher did it is really important to me.
> Some students are taught, 'memorize this formula and plug in the *a* here and the *b* here and so on…' [...] Little tricks to memorize formulas are helpful, but only when a student fully understands why he or she is using the formula.

We see not only emerging views of mathematics as making meaning, but also critical comparisons to prior experiences wherein learning mathematics is conceptualized as memorization, deployment of procedures, and focused on answers rather than understanding.

7 Discussion

Our theoretical perspective positions students' knowledge as reflexively co-created along with the collective, social aspects of the classroom. The first set of results demonstrate that this inquiry-oriented differential equations course supported PSTs' expansion of their individual meanings for derivatives. Students themselves provided corroborating evidence that their understandings had evolved, not only their ability to describe their thinking. For example, Valencia recalled the background survey and noted "I had never really taken the time to think about what [$\frac{dy}{dt}$] actually meant or represented," and Abel introduced his $\frac{dy}{dt}$ portfolio entry by saying, "when I came into this class I honestly thought $\frac{dy}{dt}$, what's so special about it? It's just a symbol we use and it's just a derivative, that's all it is." By the end of the semester, both these students (and their classmates) had developed rich, multifaceted understandings and the tools to communicate those meanings.

Additionally, our perspective on learning positions individuals' beliefs as reflexively co-negotiated along with the classroom social norms that characterize in-class mathematical discourse. Our second set of results demonstrate that the norms in this class supported the development of PSTs' beliefs about teaching and learning mathematics as an active process of making meaning. These beliefs were expressed generally as well as in contrast to prior beliefs and experiences, consistent with the development of a critical community of inquiry reported previously (Goodchild et al., 2021). The PSTs noted that these new beliefs would influence their future instructional practice. For example, Jimena noted that she was proud of her work on a particular activity and reflected: "I felt like I really understood the question and I understood my answer and I felt that I made really clear connections and I was really proud of it. So that's the experience I want to create for my future students one day."

Woven throughout both sets of results are an array of connections made to teaching secondary mathematics, despite the fact that this was a math-focused differential equations course – and these connections emerged from the PSTs themselves. This is noteworthy because PSTs often do not see the relevance of more advanced mathematics courses to their future careers without explicit discussion of connections. However, in this course PSTs were supported in making their *own* connections, recognizing the relevance of both their developing mathematical knowledge and beliefs about teaching and learning mathematics. These developments of knowledge and beliefs will support their futures as secondary mathematics teachers in how they handle both content and pedagogy.

The case reported here is an existence proof of an advanced mathematics course supporting PSTs' development into reflective practitioners (Schön, 1987). This group of PSTs enriched their own understandings of $\frac{dy}{dt}$, demonstrated shifts in their beliefs about teaching, and learning mathematics and connected to their future practice – all co-constructed through the emergence of a classroom community of inquiry. We call for future research to explore how other inquiry-oriented advanced mathematics courses might have similar impact on students' mathematical meanings for near local concepts as well as their beliefs about learning and teaching.

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Declarations

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