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SURVEY PAPER

History of mathematics in mathematics education: Recent developments in the field

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Abstract
Over the last 40 years, exploring the possible interrelations between the history of mathematics and mathematics education has gradually emerged as an interdisciplinary domain of educational research and practice, and entered a maturity stage during the last two decades. This development calls for an account of the general issues concerning the rationale and main themes underlying research and applications in its context, and the foundational issues addressed as a result of its interdisciplinary character. Therefore, after explaining the perspective characterizing work in this domain (what has been called the HPM perspective), we present the main themes along which current research is conducted and we analyze the main issues and concerns raised by current research in this domain, with due reference to recent publications. Papers in this issue further develop the key research strands shaped by these main issues and concerns from a variety of perspectives; in particular, papers address the theoretical points related to the interdisciplinary character of this domain (e.g., the role of history in promoting and developing STEM education more profoundly), and the design of innovative teaching approaches based on original sources (e.g., how non-didacticized resource material can motivate mathematically rich tasks without requiring too specialized knowledge in the history of mathematics). These papers also suggest fresh avenues for research, some bearing on potential methodological connections between mathematics, education, and history (e.g., the possibility of different readings of original sources that reveal the complex interrelations among historical knowledge, teaching objectives, and pedagogical practices), and some bearing on the implementation, evaluation, and dissemination of designs connected to the history of mathematics, in teaching at all levels of education (e.g., development of curricular material based on original texts, offering new learning opportunities in relation to core topics in university mathematics). We conclude with a brief description of each contributed paper.

Keywords History of mathematics · Mathematics education · History in mathematics education · HPM domain · HPM perspective

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1 Introduction

In the last few decades, exploring the possible relevance and significance of the history of mathematics in the context of mathematics education has evolved into an interdisciplinary domain of educational research and practice. Benefitting from or/and contributing to research both in mathematics education and the history of mathematics, work in this domain has provided innovative teaching approaches, has led to the design and production of helpful resource material and teaching aids, and has raised important theoretical issues concerning the nature of mathematics, and its teaching and learning in regard to its historical development[1].

[1] For developments up to 2000, see Fasanelli & Fauvel (2006). For an account of the various activities in this domain, see Clark et al. (2019,

This domain constitutes a meeting point of mathematics, history, and education, as three a priori distinct disciplines having their own epistemological and methodological characteristics and particularities. Therefore, it is expected that working and engaging in this domain requires facing many challenges that are not easy to resolve and which call for a constructive dialogue among the corresponding scientific communities based on mutual understanding and respect (Chorlay & de Hosson, 2016, Sect. 8.2, 8.4; Wang et al., 2018).

On this basis, one can distinguish between two closely interrelated lines of investigation, depending on whether history of mathematics—regarded as body of knowledge resulting from the professional endeavor of historians—is considered as a form of knowledge of mathematics which, as any form of knowledge of mathematics, supports research in mathematics education; or as a resource for didactic intervention in all aspects of mathematics education (curriculum design, classroom implementation, resource material, teaching aids, teacher education, etc.).

Along the first line, similarities between the professional endeavors of historians of mathematics and researchers in mathematics education have been highlighted:

Contrary to what research mathematicians do, the object of their investigation is not mathematics, and this object is not studied primarily mathematically. Rather, they study how agents *engage* with mathematics, in a context which can be described; mathematics is necessary to make sense of this engagement and this context, but cannot possibly be the only background tool (Chorlay & de Hosson, 2016, p. 159).

Artigue (1990) even argued for the "need" of epistemology and history of mathematics for mathematics education researchers, not only for them to become more knowledgeable about mathematics (as both a body of knowledge and an activity carried out along characteristic rules), but to enable them to distance themselves from tacit and contingent "epistemological representations" reflecting personal trajectories and professional communities. In this context, history of mathematics is regarded not only as a source of knowledge of mathematics, but as an opportunity to experience "*dépaysement*" (Barbin et al., 2020), "otherness" (Radford & Santi, 2022): "There are times in life when the question of knowing if one can think differently than one thinks, and perceive differently than one sees, is absolutely necessary if one is to go on looking and reflecting at all" (Foucault, 1990, as cited in Chorlay & de Hosson 2016, p. 187).

The second, more action-oriented line of investigation has long been called the *HPM domain* (*HPM* is an abbreviation for ***H****istory and* ***P****edagogy of* ***M****athematics*) because it has formed the core and main concern of the approaches adopted towards integrating elements from the history and epistemology of mathematics in mathematics education, namely, designing, implementing, and evaluating teaching approaches at all levels (including teacher education), producing educational aids and resource material, promoting and implementing interdisciplinary teaching, and connecting empirical educational research along these lines with research in other domains of mathematics education. This line of investigation emerged to a considerable extent in the context of the *HPM group*[2]. It has been an active field of research especially in the last two or three decades. A comprehensive account of the work done up to 2000 is presented in a highly collaborative and influential ICMI Study edited by Fauvel and van Maanen (2000), whereas a sufficiently comprehensive survey of the literature after 2000 is given by Clark et al. (2016, Sect.3), and a detailed discussion of selected examples is given in the collective volume edited by Clark et al. (2018b).

Research in this area has entered a maturity stage during the last two decades. This allows us to give an account of some general issues concerning the rationale and main themes underlying fundamental research and applications in its context, as well as the foundational issues that are or have to be addressed as a result of its interdisciplinary character. In particular, this special issue consists of contributions to this field that report on recent research along the above-mentioned two lines of investigation, at the levels of both theory and application.

Therefore, this paper is structured as follows: In Sect. 2 we address the question of why research in this area is important. In Sect. 3 we present an overview of research in this area. Specifically, we first present the rationale and the main themes along which research is conducted (and hence, the way this special issue is also structured), and then proceed to formulate, analyze, and comment on the main issues and concerns raised by recent research. Due to space limitations, we often refer the reader to the literature for further details. Finally, since the contributed papers in this special issue further develop the key lines of research shaped by these main issues and concerns from a variety of perspectives, Sect. 4 provides a brief description of the contributed papers, grouped according to the main themes of research presented in the preceding section on which they chiefly focus.

Sect. 3 and the appendix).

[2] The well-known abbreviated title for the *International Study Group on the relations between the* ***History*** *and* ***Pedagogy*** *of* ***Mathematics***, the ***HPM*** *Group*, which began as a Working Group at the second ICME (International Congress on Mathematics Education) in 1972, and at ICME-3 in 1976 became the first study group affiliated to ICMI (International Commission on Mathematical Instruction), together with the International Group for the Psychology of Mathematics Education (PME).

2 Connecting mathematics education and the history of mathematics

Perhaps it is not an exaggeration to say that above all sciences it is mathematics that places the greatest emphasis on and relies on the need for logical, rational, and intellectual rigor and consistency in our attempt to explore and understand better some aspects of the world we live in. And this is one of the main intrinsic reasons that mathematics has often been considered as a collection of definitions, axioms, theorems, and proofs, which—especially under the influence of the axiomatic approach and formalism as a philosophical thesis—became a common way of its presentation. Conceiving mathematics in this way, at least implicitly, suggests that mathematical knowledge coincides with the 'polished' products of humans engaged in mathematical activities. This emerging image is by no means specific to today's mathematics and, beyond differences in style, this impersonal 'coldness' rings true, for instance in Babylonian numerical tables and lists of worked-out problems, as well as in Euclidean deductions.

On the other hand, however, mathematics (like any other scientific discipline) is a living human intellectual enterprise with a long history and a vivid present, where "…knowledge of any value is never possession of information, but 'know-how'. To know mathematics means to be able to do mathematics" (Memorandum, 1962, p. 189). This perspective implies that mathematical knowledge is delimited not only by the circumstances in which it becomes a deductively structured corpus of knowledge, but also by the procedures that originally led or may lead to it. Therefore, what is acceptable as mathematical knowledge—like any other form of human knowledge—is not absolute, but 'time-dependent'. It is a continuously evolving, complex system of ways of thinking, reflecting, and acting (Radford & Santi, 2022, Sect. 2.3). Therefore, it is potentially subject to change. In other words, *historicity* is one of its epistemological characteristics (Barbin et al., 2020, p. 335; Radford et al., 2007, p. 107; see also Grabiner, 1974, Sects. 1, 5; Radford et al., 2014, pp. 105–106). Historicity can be experienced when learners engage in some activity in response to a sample of mathematics from the past. It is at this point that mathematics and history meet in a way beneficial for mathematics education: The integration of historical and epistemological issues in mathematics education is a way for exposing mathematics in the making, which may help learners to understand the following characteristics of mathematics:

- it has undergone changes over time, underscored by shifting views of what mathematics is and how it should be taught and learnt;
- it has been in fruitful dialogue with other scientific disciplines, technology, philosophy, and the arts;
- it has constantly stimulated and supported scientific, technical, artistic, and social developments; and
- it is the result of contributions from many different cultures (Clark et al., 2019, p. 3).

And it is at this point that the conception of mathematics as a human intellectual activity for the acquisition of knowledge either by the individual, or/and collectively together with the historical character of this knowledge becomes important for supporting the doing, learning, and teaching of specific pieces of mathematics and for appreciating the relation of mathematics with other intellectual and cultural pursuits all along its historical development (Clark et al., 2019, p. 4; Fried, 2014a, Sect. 21.3.3; Radford, 1997, Sects. 4, 5). In this perspective (that permeates both lines of investigation mentioned in Sect. 1 and for which the name *HPM perspective* has been coined; see Clark et al., 2018a, Sect. 1.1 for more details) mathematics is conceived both as a logically structured collection of intellectual products and as processes of knowledge production that lead to these products. As a consequence, learning mathematics becomes not only the process by which the learner gets acquainted with and competent in handling the symbols and the logical syntax of theories and in accumulating new results presented as finished products. It also encompasses acquaintance with the implicit motivations, the sense-making actions and the reflective processes of mathematicians that aim at the construction of meaning by linking old and new knowledge, and by extending and enhancing existing conceptual frameworks (Fauvel & van Maanen, 2000, p. 202).

Of course, though the above-mentioned connections between past and present mean that possible similarities between past and present knowledge exist, still there are also crucial differences and dissimilarities, due to the very different scientific, social, and cultural conditions in which this knowledge has emerged. However, ignoring either of the two will lead to a very limited and incomplete picture of mathematics (Nooney, 2002, p. 4; see also Thomaidis & Tzanakis, 2022, Sect. 2): Neglecting similarities will restrict mathematics simply to the (currently considered) acceptable results of mathematicians' activity deprived of the motivation behind them and without a deeper understanding of the processes that led to them. On the other hand, neglecting dissimilarities will lead to a distorted and biased view of the processes that led to current mathematical knowledge. The past will be forced "…through a sieve keeping out ideas foreign to a modern way of looking at things and letting through those that can be related to modern interests" (Fried, 2011, p. 16; see also Grattan-Guinness, 2004a, 2004b). Essentially this is what has been called a *Whig* or *anachronistic* approach to the past (Butterfield, 1965[3]). In this view, the past is studied in the light of our present knowledge (Kragh, 1989, ch. 9), so that the present becomes the measure of the past and therefore, "…what one considers significant in history is precisely what leads to something deemed significant today" (Fried, 2001, p. 395).

Therefore, although any similarities between past and present knowledge may serve as a motivation, guide, or resourceful aid for understanding specific pieces of mathematics and issues about mathematics, it is absolutely necessary to do so in a context that takes into account the quite different milieu in which teaching and learning takes place today. In other words, an effective development of a historical perspective in mathematics education necessarily has to take into account carefully that teaching and learning of mathematics today takes place under different social conditions, in the context of different cultural tradition(s), addressed to individuals and groups with varying characteristics and needs, and delimited by a variety of educational constraints (imposed by the curriculum, the instruction level, the learners' orientation, etc.). As a consequence, any educational approach aimed at the teaching and learning of a specific piece of knowledge along these lines can be efficient only by achieving a subtle balance between staying faithful to the historical developments, while paying due attention to the current conditions under which this teaching and learning takes place (Thomaidis & Tzanakis, 2022, Sect. 2). This is a difficult endeavor facing many challenges that do not admit easy nor general valid-for-all answers. Perhaps, this is the leitmotiv of research in and implementations of the HPM perspective, which, implicit as it was during the early development of this field, gradually became explicit, especially in the last 20 years or so.

3 Research in the *HPM domain*

3.1 Motivation, main themes, and rationale

Introducing elements from the history of mathematics in mathematics education has been advocated since the second half of the 19th century, by important mathematicians and historians such as De Morgan, Zeuthen, Poincaré, Klein, Tannery, and later Loria and Toeplitz, who showed an active interest in promoting the role the history of mathematics can play in mathematics education, though not all to the same degree and with the same motivation and rationale (Clark et al., 2019, Sect. 3; Furinghetti, 2020; Jahnke et al., 2022, and the bibliographies in these papers). Moreover, there were also some works referring to the history of mathematics in actual practice at school (Furinghetti, 2000, pp. 49–50; 2019, p. 110; 2000, pp. 969–970).

At the beginning of the 20th century this interest was revived as a consequence of the debates on the foundations of mathematics and became stronger after the *New Math* reform in the period 1960–1980, considering history a natural possible way to conceive mathematics as an evolving human activity, and in this way to help improve its teaching and learning (Lakatos, 1976; Memorandum, 1962, pp. 190–191; NCTM, 1969; see also Barbin et al., 2020, pp. 333–334 and references therein).

This led to the formation of the *HPM Group* in the 1970s. In the following decades the establishment of this group greatly stimulated and supported the interest and educational research in this area at an international level (Fasanelli & Fauvel, 2006), leading to the 4-year *ICMI Study* providing a survey of the work done in this domain and reporting on the main issues for further research, captured in the comprehensive collective volume cited in Sect. 1 above (Fauvel & van Maanen, 2000). This volume became a landmark in establishing and making widely visible the potential significance of the history of mathematics in mathematics education, stimulating and enhancing the international interest of the educational community, and inspiring and motivating further research and actual implementations in education[4]. In particular, several collective volumes and special issues of research and practitioner journals appeared after this ICMI study[5].

Therefore, aspects of the intensive research activity in this area in the last several years, as also recorded in important regularly organized international meetings[6], their proceedings, and collective volumes that resulted from them, deserve to be presented in a special issue of a journal like *ZDM – Mathematics Education*. This gives us the opportunity to communicate to a wide international readership recent research on the multifaceted role the history of mathematics

[3] "It is part and parcel of the Whig interpretation of history that it studies the past with reference to the present" (ibid., p. 11).

[4] For an indicative list of activities and publications after this ICMI study, see Clark et al., (2018a, Sect. 1.2) and Clark et al. (2019, Sect. 3 and appendix); for a list of (both old and recent) 'core publications' in the HPM domain that offers newcomers in the field the possibility to begin a survey of the literature see Katz et al. (2014, Sect. 3), whereas, for a sufficiently comprehensive and up to date bibliographical survey see Clark et al. (2016).

[5] These include Barbin (2018); Barbin & Bénard (2007); Barnett et al. (2014); Bekken & Mosvold (2003); Clark & Thoo (2014); Clark et al., (2018b); Furinghetti et al. (2007); Katz (2000); Katz & Michalowicz (2005); Katz & Tzanakis (2011); Katz et al. (2014); Knoebel et al. (2007); Matthews (2014); Siu & Tzanakis (2004); Sriraman (2012); Stedall (2010).

[6] Such as the *International Congress on Mathematical Education* (ICME) and the accompanying *HPM Satellite Meeting*, the *European Summer University on History and Epistemology in Mathematics Education* (ESU), and the *Congress of the European Research in Mathematics Education* (CERME).

can play in mathematics education at all levels of instruction, including teacher education. In this connection, a clear indication of the main areas of research interest and activity in the HPM domain are the recurrent and closely interrelated themes along which the main international meetings in the HPM domain mentioned above have been organized in the last two decades, which are as follows:

1. Theoretical and/or conceptual frameworks for integrating history in mathematics education; exploring how reflecting on the history of mathematics could enrich didactical research.
2. History and epistemology of mathematics in students' and teachers' mathematics education at all levels of instruction: Design and/or assessment of classroom experiments and teaching/learning materials (preferably based on empirical data), considered from various perspectives, e.g., cognitive, didactical, pedagogical, affective, etc.
3. Original historical sources and their educational effects: classroom implementations; enhancing and deepening reflections on the teaching and learning of mathematics.
4. Surveys on the existing uses of history or epistemology in curricula, textbooks, and/or classrooms in primary, secondary, or tertiary levels, and in teacher education.
5. History and epistemology of mathematics as a tool for an interdisciplinary approach in the teaching and learning of mathematics by unfolding its productive interrelations with science, technology, and the arts.

Although recent publications (including the papers in this issue) report on empirical investigations, surveys, and reflective and critical studies on the above themes, this does not mean that introducing historical elements into mathematics education should be regarded as a panacea to all problems in mathematics education. Instead, it should be seen as a possible way to improve the teaching and learning of (specific parts of) mathematics and to enable deeper reflection on the nature of mathematics and mathematical activity, which, in view of the discussion in Sect. 2, is worth exploring critically and carefully. Moreover, from this perspective and as research and implementations in this area are interdisciplinary in character, this fact (also reflected in the contributed papers either implicitly, or in some cases explicitly) stresses that a renewed discourse among the corresponding communities is desirable or even necessary; that is, among researchers in mathematics education, historians of mathematics, mathematics teachers, and mathematicians, and possibly historians, philosophers, and teachers of science and other disciplines (Siu, 2015, p. 43; cf. Thomsen et al., 2022, Sect. 7). Actually, being at the interface of mathematics, history, and education, the HPM domain points to, raises, and emphasizes several nontrivial questions and issues, as follows: Which history is suitable, pertinent, and relevant to mathematics education, why, and with which role(s)? In particular, can scholarly historical knowledge be useful for mathematics education, or is it necessary that it be didactically transposed (as is done for scholarly mathematical knowledge), and how this can be done? (Clark et al., 2018a; Furinghetti et al., 2006; Jahnke, 2014; Jankvist, 2014; Kjeldsen, 2011a, 2012a, Thomaidis & Tzanakis, 2022). Furthermore, do similarities between past mathematicians' creative work (including struggles or qualms among them) and students' ways of learning mathematics exist? What are the limitations imposed by the different scientific, sociocultural, and historical conditions between modern learners and past mathematicians? And to what extent could any such eventual similarities be beneficial for mathematics education and for understanding and exploring further the historical development (Bråting & Pejlare, 2015; Furinghetti & Radford, 2008; Jankvist, 2014; Schubring, 2011; Pejlare & Bråting, 2019; Thomaidis & Tzanakis, 2007)? At a more practical level, what can be the goal(s) and the objective(s) of a small-scale didactical intervention where historical elements have been integrated? And to what extent do or should such interventions commit to history (Barbin, 2022)? Can such interventions be designed so that they can be implemented at a larger scale? How can relevant large-scale didactical research be encouraged, enabled, and enlightened (Clark, 2019, p. 49)? Should historical elements be used in the same way and with similar goals in classroom teaching and in teacher education? If not, what are the differences, and the decisive factors shaping them that should be considered in designing teaching activities and providing teaching aids and resources? How can such interventions be evaluated and assessed and to what extent and in what sense do they contribute to the teaching and learning of mathematics (Clark et al., 2022)? In particular, whether and in which ways does a historical perspective contribute to or become necessary for the mathematical and pedagogical development of pre- and in-service mathematics teachers at all levels (Clark, 2019, pp. 49–50; Jankvist et al., 2020; Siu, 2015, p. 44)?

These questions and further theoretical discussions on the right scale, the right goals, and the appropriate assessment procedures of didactical innovations in this domain can be helped and promoted by tools, methods, and theoretical constructs developed in the context of research in mathematics education (see, e.g., Agterberg et al., 2022; Barnett, 2022; Bernardes & Roque, 2018; Chorlay, 2022; Gosztonyi, 2022; Jankvist, 2011; Jankvist & Kjeldsen, 2011 Kjeldsen, 2012b; Kjeldsen & Blomhøj, 2012; Moustapha-Corrêa et al., 2022; Spies & Witzke, 2018). More generally, the need for empirical studies on the actual impact of such didactical innovations calls for a fruitful dialogue with researchers in mathematics education working outside this domain (Chorlay & de Hosson, 2016). Similarly, new perspectives in historical research, as well as trends in mathematical research, call for an up-to-date scholarly discussion between historians of mathematics, researchers in mathematics education, and mathematicians on the relationship between the history of mathematics (and the history of sciences) and mathematics education (Barbin et al., 2020, p. 340; Fried, 2014b, p. 10ff; Radford et al., 2014); for instance, by putting emphasis on work collectives (their usually tacit norms, their shared practices and boundaries, the competition among collectives, the various sociological structures—the school, a theory, a discipline, etc.), exploring the interactions between 'high-mathematics' and 'low-mathematics' users (professional mathematicians, physicists, computer scientists, engineers, economists, calculators (i.e., persons who calculate), instrument makers etc.), reappraising non-Western mathematics (with new perspectives on proof, algorithms, the notion of 'problem'), etc.

Along these lines, further questions at a more fundamental level can be and have been raised: Can the communities of mathematics educators and historians of mathematics cooperate harmoniously by benefitting from the epistemological characteristics, aims, commitments, and methodologies specific to each discipline, or is this prevented by strong constraints due to crucial differences between these epistemological characteristics, aims, commitments, and methodologies, thus becoming a task that is almost unfeasible (Chorlay & de Hosson, 2016; Fried, 2001, 2007, 2011; Kjeldsen, 2011b; Radford et al., 2014, pp. 94–98)? In particular, can historical research and practice inspire, support, or supply explanatory frameworks and working tools for research in mathematics education? And conversely, can research and practice in mathematics education inspire, support, and broaden research in the history of mathematics? And if so, how this can be achieved (Furinghetti et al., 2006; Jankvist, 2014, Sect. 27.2, 27.7; Schubring 2011, p. 90; Thomaidis & Tzanakis, 2022; see also Barnett, 2022; Demattè & Furinghetti, 2022, Sect. 6; Moustapha-Corrêa et al., 2022)?

All these questions and issues that in one form or another have been addressed in the literature, reveal that in realizing the HPM perspective in practice, several intertwined factors related to research in history and mathematics education come into play, which cannot be ignored. Moreover, although there is no general consensus about their answers and much work is still to be done, we hope that the papers in this issue will contribute to their better understanding and will motivate and stimulate further work to this end.

3.2 On recent contributions and main current concerns

In the past few decades many mathematics education researchers and teachers all over the world have seriously considered and followed the general ideas outlined in the previous sections about the evolutionary nature of mathematical knowledge and the significance of adopting a historical perspective in unveiling these ideas, by exploring in one way or another the related questions and issues mentioned in the previous subsection. A number of recent surveys and overviews of the work done so far have appeared, addressing key issues from various perspectives and guiding the reader to the relevant literature. Below, we refer briefly to an indicative sample.

Jankvist (2009) focused on *why* and *how* history of mathematics may or should be used or integrated[7] in mathematics education according to the emphasis on history being a *goal*, or a *tool*, classifying the corresponding arguments accordingly, and categorizing the corresponding teaching and learning approaches into three categories from a methodological point of view, namely, the *illumination*, *module*, and *history-based* approaches. Furinghetti (2004, 2020) made the distinction between what she called the two main streams for exploring the role of the history of mathematics in mathematics education; these are "history for constructing mathematical knowledge" through an approach to the roots around which mathematical knowledge developed, and "history for promoting mathematics" as an activity of a community, thus appreciating mathematics as an integral part of human intellectual history and cultural development. Furinghetti and Radford (2008) and Schubring (2011) reviewed the work in this domain from the perspective of the connections and contrasts between the historical development and students' learning in a modern classroom. Fried (2014) argued that recent attempts for bringing the history of mathematics into mathematics education fall under three central themes, which he called "motivational", "curricular", and "cultural" themes. He also further argued that such attempts—especially under the cultural theme—could help enrich, deepen, and widen the main aims of mathematics education itself (this idea of transforming the main aims of mathematics education by connecting it with the history of mathematics is further elaborated by Fried (2018a). Clark et al. (2018a, pp. 1–2) argued that the interdisciplinary nature of work in the HPM domain is a result of the multifarious

[7] There has been a long discussion on different expressions conveying different connotations and meanings concerning the connections of history to mathematics education (hence, different understandings of these connections), including history *used* in, *integrated* in, *introduced* in, and *permeating* the teaching and learning of mathematics (e.g., Barbin, 2022, Sect. 1.2; Siu & Tzanakis, 2004, p. viii).

interrelations among history, mathematics, and education. They surveyed the work in this domain in relation to what they consider as the main issues of research in its context (ibid., p. 2) as expressed via the following questions: Which history is suitable, pertinent, and relevant to mathematics education? Which role can the history of mathematics play in mathematics education? To what extent has the history of mathematics been integrated in mathematics education (curricula, textbooks, educational aids and resource material, teacher education)? How can this role be evaluated and assessed and to what extent does it contribute to the teaching and learning of mathematics? From another perspective, Barbin et al. (2020) classified the contributions associated with experimental and theoretical work in the HPM domain into three interrelated types, namely, epistemological, cultural, and didactical. They further pointed to the need for a productive dialogue among the corresponding communities, with emphasis on the interdisciplinary character of history in an educational context, together with the need for developing appropriate and effective theoretical and conceptual frameworks, and conducting more in-depth empirical studies.

A more detailed study of all these surveys (and other publications, of course) indicates that despite the variety of perspectives from which research in the HPM domain is conducted, in the implementation of the results of international research both at the theoretical level and in the applications in actual didactical practice, the following strongly interconnected concerns play a central role (Clark et al., 2019, Sect. 5).

a. It is of significant importance to realize carefully designed and systematically applied *empirical investigations* in order to examine critically and evaluate the eventual effectiveness of a historical perspective in mathematics education on improving the teaching and learning of mathematics, as well as students' and teachers' awareness of mathematics as a discipline and their disposition towards it. Several such investigations have appeared, but much more work is still necessary that will shed more light on the issues briefly presented in the previous sections (see for instance, the surveys by Jankvist, 2007, 2012; also, Barbin, 2018; Clark et al., 2018b, parts III–V; Jankvist, 2011; Katz & Tzanakis, 2011, chs. 9–11, 13–16, 19; more references in Clark et al., 2019, p. 15).
b. For any integration of the history and epistemology in mathematics education to be possible at all, an adequate *pre- and in-service teacher education* in this area is necessarily presupposed, hence research on how to achieve this task is central. This point was recognized quite early and emphasized repeatedly (Fauvel & van Maanen, 2000, ch. 4 and Sect. 3.2; see also Barbin et al., 2011; Clark, 2019, pp. 48–50; Gazit, 2013, Sect. 4; Huntley & Flores, 2010, Sect. 1; Siu, 2015, p. 44). In this connection, many efforts have been made to educate teachers (Arcavi & Isoda, 2007; Bruckheimer & Arcavi, 2000; Clark, 2011; Clark et al., 2018b, chs. 4, 11, 14, 18; Povey, 2014; Smestad, 2011; Waldegg, 2004) and explore their attitudes, beliefs, and/or teaching approaches and possible ways they could be modified (Alpaslan et al., 2014; Buchholtz & Schorcht, 2019; Charalambous et al., 2009; Chevalarias, 2019; Furinghetti, 1997, 2007; Moyon, 2022, Sect. 2; Philippou & Christou, 1998; Smestad, 2011; Spies & Witzke, 2018; Vicentini et al., 2019). So far, however, there are no generally accepted results in relation to these issues. Further research is still needed to explore ways to educate teachers in this respect that, among other things, will be convincing enough concerning the effectiveness of introducing historical elements into teaching.
c. Closely related to (b) and of equal significance is the design, production, availability, and dissemination of diverse *didactical source material* in a variety of forms, e.g., anthologies of original sources, annotated bibliographies, description of teaching sequences or modules that could serve as a source of inspiration and/or as generic examples for classroom implementation, educational aids of various types, appropriate websites, etc. This has also been stressed in the literature (e.g., Fauvel & van Maanen, 2000, pp. 212–213; Panasuk & Horton, 2012, p. 16; Pengelley, 2011, pp. 3–4), and although the need for such material has been satisfied to a considerable extent in the last 20 years or so (for an indicative useful sample see Clark et al., 2018a, p. 12; Clark et al., 2019, pp. 14–15), further empirical investigation of the usefulness and appropriateness of this material is needed (Barnett, 2022; Clark et al., 2022; Danielsen et al., 2018; Moyon, 2022; Schorcht, 2018). For instance, an in-depth systematic exploration is needed of how teachers design, select (or reject), and/or use, modify, or misuse various resources in their teaching (Chorlay, 2022), why they do it and what this may reflect (their beliefs, a global state of the educational system in which they work, their knowledge of history of mathematics, their professional competencies, etc.).
d. Finally, in close relation to (a), (b), and (c), it is very important to identify and unfold theoretical ideas and concepts that underlie or enrich empirical investigations, classroom implementations, and the production of didactical material, and to develop them carefully into coherent theoretical frameworks and methodological schemes that will serve as a foundation and/or orientation for further research and applications in

this area. Several works, including the following, have been published in this connection, some of which have already been cited in the previous sections: Bråting & Pejlare, 2015; Clark et al., 2018b, chs. 2–4, 8; Chorlay & de Hosson, 2016; Fauvel & van Maanen, 2000, ch. 2; Fried, 2001, 2007, 2011, 2018a, 2018b, 2022; Furinghetti & Radford, 2008; Grattan-Guinness, 2004a, 2004b; Hanna et al., 2010; Jahnke, 1994, 1996, 2014; Jankvist, 2009, 2011Jankvist & Kjeldsen, 2011; Kjeldsen, 2011c, 2012b 2018, 2019; Kjeldsen & Blomhøj, 2012; Kjeldsen & Petersen, 2014; Radford & Santi, 2022; Radford et al., 2014; Schubring, 2011; Thomaidis & Tzanakis, 2007, 2022; Tzanakis & Thomaidis, 2012.

Elaborating on the many issues raised by the above concerns will improve the field's understanding of the benefits that possibly underlie the integration of the history of mathematics in mathematics education.

4 Structure and content of this issue

The papers in this issue are directly related to one or more of the four central concerns outlined in the previous subsection. Taken as a whole they refer in one way or another to issues in the HPM domain relevant to all levels of education, including pre- and in-service teacher education. Seen individually, each paper's main focus is situated within one of the five themes listed in Subsection 3.1. Therefore, for clarity we ground each set of papers according to the five themes. However, as these themes are strongly interrelated, almost all papers contribute to more than one of them.

In the first part on theme 1, six papers address important theoretical issues pertaining to historical research, and exploring how reflection on the history of mathematics could enrich didactical research.

Jahnke et al. (2022) present overviews of the writings of H. Poincaré, F. Klein, and H. Freudenthal on the role of the history of mathematics in mathematics education, since these important mathematicians' views have inspired or influenced subsequent research in mathematics education in general and the HPM domain in particular. The comparative study of these writings reveals these scholars' understanding of the role of the history of mathematics in mathematics education as a clear recommendation for history becoming part of mathematics teacher education, as well as history's placement within human culture.

Fried's (2022) paper links historical scholarship with the role of primary sources in mathematics education by considering an important historical text via which one sees a past mathematician looking at his own past and advocating the importance of studying such texts as an aid to refine one's own sense of possible postures towards historical material. This activity encourages self-reflection, and enables Fried to describe in detail different types of approach to the past (Fried, 2018b) that give a better understanding of a modern teacher's position in trying to integrate historical material into teaching. This self-reflection constitutes an important justification for the history of mathematics in mathematics education as a catalyst that could widen, deepen and enrich the main goals of mathematics education (Fried, 2007, 2014a, 2018a).

Thomaidis and Tzanakis (2022) argue that for understanding the interrelations between the history of mathematics and mathematics education, many relevant concepts are complementary rather than antagonistic. By analyzing a known students' misconception in elementary arithmetic and algebra in relation to an alleged mistake in Euler's *Elements of Algebra*, they illustrate how didactical and historical research can be mutually supportive, call for a constructive discourse between the two communities, and point to the need for a didactical transposition of historical knowledge that complements rather than distorts it.

Theoretical reflections on inquiry-based mathematics form the core of Gosztonyi's (2022) paper. She proposes an analysis of the structure of Clairaut's "*Éléments de Géométrie*" on the basis of an interdisciplinary project in which problem-centered texts from various cultures are analyzed, and ongoing didactical research on the "Hungarian guided discovery approach" where "series of problems" are central in designing and implementing teaching sequences. With the detailed presentation of two examples from this book, Gosztonyi shows how this approach aids reflection on constructing and analyzing inquiry-based teaching sequences in the context of mathematics education.

Radford and Santi (2022) aim to contribute to diminishing the gap between theoretical and empirical research in the HPM domain, by reconsidering and reconceptualizing mathematics learning and mathematical knowledge that students encounter at school. The former is theorized as a critical encounter with mathematical knowledge occurring via intertwined objectification and subjectification processes (Radford, 2021). Thus, the crucial educational point at stake is to offer students occasions for this encounter. In this perspective history becomes a necessary part and parcel of mathematics education. These ideas are exemplified by the analysis of two arithmetical problems from two original texts, and its actual implementation as a teaching-learning activity.

Approaching original texts is a real challenge for both teachers and learners because of a strong tendency to 'translate' these texts into modern mathematics, thus making difficult to understand and appreciate the historical development and one's own conceptualizations (Barbin, 2022; Fried, 2022). In this vein, Guillemette & Radford (2022) aim to go beyond the classical opposition between historians' *History* and *Whig History* (Grattan-Guinness, 2004b, Sect. 3.8). Drawing on inputs from linguistic theories, they highlight the theoretical and ethical potential of a "dialogical" perspective in which understanding an original text is seen as one of the ways to enter responsively in a dialogue with another human Being. They illustrate their position with an example from an empirical study and argue that proceeding along these lines reveals another possibility for approaching an original text, that of an educator.

The four papers in the second part (on theme 2), report on recent classroom experiments of introducing a historical perspective, and detailed discussion on the selection and design of the corresponding teaching material.

Clark et al. (2022) report on parts of a large-scale research project in progress, with focus on the use of curricular materials based on primary historical sources (Primary Source Projects; PSPs). Leaning upon a variety of theoretical tools, the authors provide a qualitative analysis of the data relevant for students' understanding of the mathematical meta-discursive rules, students' worldviews about mathematics and how these may change under the influence of PSPs, and for examining students' reports on the challenges and benefits of learning from primary sources resulting from their engagement with PSPs.

Demattè and Furinghetti (2022) report on an empirical study with upper secondary school students, who were given problems to solve from three Renaissance Italian abbacus treatises, aiming to explore their response to these texts and the concepts and solution strategies they used. The choice and implementation of the original excerpts focused both on their mathematical content and the characteristics of the societies these treatises served. It stimulated several students' interest in reflecting on cultural and scientific issues beyond the task of solving a problem and allowed the authors to explore their knowledge and beliefs concerning important pieces of mathematics.

Moustapha-Corrêa et al. (2022) present their "*historical zoom-in and zoom-out*" (ZIZO) approach for mathematics teacher education. That is, based on questions and problems capable of engendering and/or inducing concepts and results, the goal of bringing teachers in contact with the complex network of historical, social, and subjective processes leading to mathematical knowledge. ZIZO is structured according to teachers' alternating immersion in specific details of original sources and in historical overviews of the ideas in focus. The paper reports on implementing this approach with a group of in-service teachers with a focus on its influence on the teachers' awareness of what mathematics is and what teaching and learning mathematics means.

De Varent and Décamp (2022) report on an empirical study with grade 10 students, concerning the design and implementation of a teaching sequence based on a paleo-Babylonian cuneiform tablet showing the calculation of a square's area. In the paper they analyze from a history-as-a-goal perspective (Jankvist, 2009) the specifics of an unusual teaching sequence designed by a historian of mathematics. Both qualitative and quantitative data allow for a study of the tensions and discontinuities between the original design choices, the actual implementation by the historian, and the impact on students.

Original historical sources are an important means for introducing a historical perspective in actual didactical practice, and for enhancing and deepening meta-level reflections on the teaching and learning of mathematics. On the other hand, several nontrivial methodological, epistemological, and didactical questions and problems are related with their use, which are far from being settled (Barnett et al., 2014; Chorlay,2016; Fauvel & van Maanen, 2000, ch. 9; Furinghetti et al., 2006; Jankvist, 2014; Pengelley, 2011). The two papers in the third part (on theme 3) are specifically focused on particular aspects of these effects, and their dependence on design choices.

Having as a leitmotiv the unavoidable question, "why is it worthy to use original sources in teaching and learning mathematics when there are so many good contemporary textbooks prepared for dealing with the same topics using today's mathematical language, notation, and terminology?", Barnett (2022) focuses on the PSPs mentioned above. By using Sfard's commognitive approach, she analyzes three PSPs and discusses the new learning opportunities they offered, which are not shared by standard textbooks, at the same time pointing to the constraints imposed in this way.

Chorlay (2022) reports on a case study in which high school mathematics teachers autonomously designed and implemented classroom sessions based on an excerpt from Euler's *Elements of Algebra*. The paper focuses on teaching practices and not on the effect on students, while the historicity of the source is not regarded as a key factor. An interesting suggestion is that the dissemination of non-didacticized resource materials can motivate the design of mathematically rich tasks in ordinary teaching contexts without requiring the teachers to be specifically knowledgeable in the history of mathematics. This paper also suggests a complementary line of research, on the didactic potential of using historical sources in the classroom for purposes other than introducing a historical perspective in education (Chorlay, 2016).

Two of the three papers in the fourth part on theme 4 provide information on the existing uses of history in curricula, textbooks, and didactical sources.

In the new high school mathematics curriculum in France (2019) the history of mathematics is one of its three main guiding threads. However, an important long-lasting issue is the conflict between the requirement to integrate the history of mathematics into teaching, and many teachers' concerns or reluctance to do so because of their lack of knowledge of this history (Clark et al., 2018a, pp. 6–7; Fauvel & van Maanen, 2000, p. 203; Siu, 2006, pp. 268–269). Motivated by this conflict, Barbin (2022) explores the complex interrelations among historical knowledge, teaching objectives, and pedagogical practices. She illustrates these complex interrelations by analyzing different possible readings of two particular texts (of Diophantus and Euler), and two case studies in the context of teacher education based on original texts.

Also motivated by this reform, Moyon (2022) presents his survey conducted with French mathematics teachers, and a comparative study of textbooks used before and after the reform as to what they include concerning a typical important case, namely, Fibonacci. Though teachers appreciate the significance of the history of mathematics for its teaching and learning, they are hindered by lack of teaching time and lack of confidence on their historical expertise and textbooks provide insufficient, or inappropriate support for this. Moyon presents two cases of how the material in current textbook examples could be transformed so that this integration becomes historically accurate and didactically appropriate, and suggests how teachers could be helped to (re)design the tasks for the available textbooks.

The third paper by Thomsen et al. (2022) explores the main features of the "classical" subject of illuminating the mathematics learner through historical knowledge, with the "modern" subject of using digital technologies to support mathematics teaching and learning. To this end they survey the relevant literature in the last 30 years and examine to what extent it draws on theoretical constructs from mathematics education research and for what purposes. Based on a multifaceted theoretical framework, the authors analyze the surveyed publications, presenting in detail three examples of how this has been realized.

In the last part, two papers touch upon aspects of theme 5. Siu (2022) focuses on the promising and actively explored STEM approach in education, in which, however, mathematics seems to play an understated role (Maass et al., 2019, p. 869). Siu expresses this concern and argues that throughout history and across cultural traditions "one witnesses STEM at work". In his perspective "STEM embodies a spirit of exploration and a way of thought, combined with mathematical thinking and scientific spirit, to organize theories through experiments and observations in the search for knowledge and further innovation" (ibid, Sect. 2). Siu argues that historical knowledge can support this view and provides several important examples to this end, also highlighting their appearance under similar forms in different cultures.

Understanding the interrelations between mathematics and physics, especially by paying attention to their historical development, is a non-trivial subject, with far reaching educational implications and with ongoing discourse from several interconnected perspectives (Karam, 2015; Tzanakis, 2016). Liu (2022) takes the stance that these interrelations can be understood by adopting a *structuralist view*. To support this stance, he comments on some important historical examples, and reports on a historically motivated empirical study with college students about one of Galileo's inclined plane experiments, classifying students' beliefs and attitudes towards these interrelations as they are revealed via this didactical activity.

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