ZDM – Mathematics Education (2022) 54:1463–1478
https://doi.org/10.1007/s11858-022-01441-8

ORIGINAL PAPER

Series of problems in Clairaut's *Elements of Geometry*: interaction between historical analysis and mathematics education research

Katalin Gosztonyi[1,2]

Accepted: 29 September 2022 / Published online: 29 October 2022

Abstract
In this paper, I propose an analysis of the structure of Clairaut's *Éléments de Géométrie* (1741), a book presenting geometrical knowledge organized around problems, with rich metadiscourse connecting problems and other elements of mathematical content. This analysis is inspired both by the "Series of problems at the crossroad of cultures" interdisciplinary history of sciences project, in which texts from various cultures structured around problems were analysed, and by ongoing mathematics educational research on the Hungarian *Guided Discovery* approach, where "series of problems" play a central role in the planning and implementation of teaching trajectories. I show how a dialog between the analysis of Clairaut's historical text and mathematics educational research contributes to the reflection on the construction and analysis of inquiry based teaching trajectories in mathematics education.

Keywords History of mathematics · Series of problems · Inquiry based mathematics education · Clairaut

1 Introduction

In this paper I present an analysis of Clairaut's *Elements of Geometry* (1741) illustrating mutual interaction between research in history of mathematics and in mathematics education. I focus especially on questions related to Inquiry Based Mathematics Education (IBME) and more specifically on the example of the Hungarian Guided Discovery (GD) approach, which can be seen as a variety of IBME (Gosztonyi, 2020). I show how the analysis of Clairaut's historical text is inspired by earlier research in history of mathematics and in mathematics education, and how it contributes to reflections on IBME and to the analyses of GD.

Problem solving and IBME play an increasing role in mathematics educational policies in recent decades. Artigue and Blomhøj (2013) described IBME

> as an educational perspective which aims to offer students the opportunity to experience how mathematical knowledge can meaningfully develop. Thus, IBME becomes a powerful means of action, through personal and collective attempts at answering significant questions, making these experiences not just anecdotic but inspiring and structuring for the entire educational enterprise. (p. 12).

Nevertheless, this systematic integration of an inquiry based (IB) approach with the teaching of mathematical content determined by curricula is a challenging task. Artigue and Blomhøj considered the tension "between the development of inquiry habits of mind and the progression of mathematical knowledge paying necessary attention to curricular progression" as one of the inherent tensions in IBME (p. 13).

There exists a variety of approaches to IBME that address this challenge in different ways. They develop IB teaching trajectories[1] emphasising different principles related to each approach (Artigue et al., 2020). Teaching trajectories developed, for example, in the context of Realistic Mathematics Education, Study and Research Paths in the Anthropological Theory of Didactics, or Series of Problems in the Hungarian Guided Discovery approach, present different solutions to conceiving long and coherent IB trajectories. A panel discussion organized on this issue (Artigue et al., 2020) underlines that not only the conceptualisation, the construction,

[1] Although the term *learning trajectory* (Simon, 1995) seems to be more prevalent in literature, I use the term *teaching trajectory* to emphasise the focus on the teacher's perspective and on the analysis of the teacher's work.

✉ Katalin Gosztonyi
katalin.gosztonyi@gmail.com

1 Eötvös Loránd University of Budapest, Budapest, Hungary

2 Alfréd Rényi Institute of Mathematics, Budapest, Hungary

and the analysis of IB teaching trajectories are difficult, but the communication with teachers on the underlying principles and the support of teachers' work also represents a challenge.

In the specific case of the Hungarian GD approach, which is a research area for the author, a teaching practice session was developed on the basis of principles that were never described in a coherent didactical theory. GD is practiced today by a limited circle of teachers while the underlying principles remain implicit. This aspect makes the description, didactical analysis, and dissemination of the approach difficult.

In the frame of GD, long and complex teaching trajectories are developed by teachers based on 'series of problems' (SoP). By SoP I mean, in this specific context, a list of tasks or 'problems' with a consciously chosen ordering relevant for one or several educational purposes. The ordering is based on a complex network of links between problems which can be of various natures (including mathematical dependence, heuristic connections like different variations and analogies, etc.). Beyond this network of problems, the choice of a specific ordering is influenced by teaching purposes, specificities of the class, and other factors, and the ordering can even change on the spot. While a SoP serves as the skeleton of a teaching trajectory, the trajectory in its integrity is of course more complex, including various activities and the emergence or reappearing of mathematical knowledge.

Understanding the rationale and the underlying structure of SoP-based teaching trajectories is quite a challenging task (Gosztonyi, 2019). Convenient forms of description, vocabulary, and representations are lacking for understanding the expert teachers' strategies and choices in developing and implementing SoP on one hand, and for the education of non-expert teachers on the other. The aim in an ongoing research project is to understand and describe the underlying principles of the teachers' work with SoP, following a methodology partly inspired by Gueudet et al. (2012), named *reverse engineering* (Gosztonyi, 2019). I gain access to implicit aspects of teachers' work through collaboration with expert teachers who use the GD approach; this aspect of the investigation consists of teachers describing commented examples of SoP that have issued from their practice, and a collective development of adapted representation tools—most notably problem graphs. These representations play a crucial mediator role in the discussions with teachers, revealing implicit elements of their practice. Beyond research purposes, the collection in progress of commented examples of SoP is also meant to serve as a resource for other teachers, not for direct implementation but to elucidate the underlying planning strategies and to enable teachers to develop their own SoP.[2]

This ongoing study is deeply inspired by a preliminary historical analysis of the GD. Important elements for the study of GD, concerning its historical roots and some key principles, were developed (Gosztonyi, 2015) in the framework of the interdisciplinary history of science research project called "Series of problems at the crossroad of cultures" (Bernard, 2015a) (henceforth HistSoP). The term 'series of problems' itself was originally introduced in this framework. The project treated historical texts in the form of questions and answers or problems and solutions from various cultures and periods and with various content and goals. An important methodological principle was not to consider the notion of problem and the reification of problems in a series as self-understood, but as an object of historical inquiry.

In the present paper, I focus on a study developed as a continuation of this history of science project, namely, the analysis of the *Elements of Geometry* (EoG) written by the French mathematician Alexis Clairaut in the eighteenth century. Clairaut's explicit aim, explained in the preface, is to write an introduction to the 'elements' of geometry organised in a problem-based way. Clairaut's writing program thus shows interesting analogies with IBME approaches and, more specifically, with the Hungarian GD (cf. Sect. 2).

In this paper the following questions are addressed: how is this writing program realized in the construction of the book, and how are Clairaut's principles announced in the preface of EoG made effective in the structuration of his text? The expectation is that the analysis of EoG's structure can bring interesting insight to the reflection on the structuration of IB teaching trajectories and, more specifically, to the analysis of Hungarian SoP.

For this analysis, the methodology of graph representations, developed for analysing resources and teachers' work in the Hungarian context, was crossed with discursive analysis methods, developed on various other historical corpuses in the HistSoP project. The resulting methodology contributes to revealing the complex structure of Clairaut's text and how he effectively combines the presentation of mathematical content with a problem-based organizing strategy.

This work is based on collaboration with Alain Bernard,[3] historian of mathematics, although we were motivated by different research objectives. Beyond the motivations explained above, this analysis also has a purely historical interest for the comparison with other eighteenth century treatises of geometry (Bernard, in progress).

In the next part of the paper, I present Clairaut's writing project, situating it in its historical context and underlying analogies with modern IBME approaches. In the third part,

[2] This research project is described in detail by Gosztonyi & Varga (in progress).

[3] The use of 'we' in the paper refers to this collaboration.

I explain the methodology for analysing the structure of EoG and show it through the analysis of two excerpts from the text. In the fourth part, I discuss lessons from this analysis for modern mathematics education, concerning IB teaching trajectories and, more specifically, elements of inspiration for the development of research on Hungarian SoP.

2 Clairaut's *elements of geometry*: an interesting historical case study for reflection on the structure of inquiry based teaching trajectories

2.1 Clairaut's project: writing a problem-based elements of geometry

The famous opening words of Clairaut's EoG, first published in 1741, announce an ambitious project: "although Geometry be in itself abstract, it must be admitted that the difficulties felt by those who commence its study most frequently arise from the manner in which it is taught in ordinary elementary works." Facing 'dry' and unmotivated expositions of geometrical notions, continues Clairaut, "it commonly happens that beginners become wearied and discouraged before they get any distinct idea of what it is desired to teach them" (p. vii).[4] As a solution, Clairaut proposes a writing strategy to employ the readers "constantly in solving problems" (p. ix), allowing them to "acquire more easily the spirit of discovery" (p. x).

Motivating the exposition of mathematical notions, adopting problem solving as a structural strategy for the learning of mathematics, and acquiring "the spirit of discovery" beyond mathematical knowledge are all referred to as key principles of contemporary IBME approaches. Nevertheless, for a better understanding of Clairaut's writing project, it must be interpreted by considering its own historical context. For this purpose, I summarize the main conclusions of a preliminary historical study on the originality of Clairaut's work (Bernard, 2022).

Clairaut's book has an unknown genesis, but the editorial and didactic context in which it appeared allows us to formulate plausible hypotheses.

As an *editorial* product, the title and explicit contents of the treatise attach it to the genre of 'elements'. Since Antiquity, such a title announced an exposition of theoretical mathematics beginning with its most elementary components, namely, definition, axiom, proposition, theorem, etc. Clairaut's exposition follows a divergent and paradoxical model in which such components are apparently absent and which probably originated in the genre of practical geometries (Raynaud, 2015). In practical geometries, the essential emphasis is put on problem posing and solving through operational procedures applicable to practical tasks explained by standard (Euclidean) geometrical theorems. The work should thus be regarded as an original blending between two editorial traditions (elements of geometry and practical geometries) which were, so far, kept distinct, even though their contents were highly interdependent.

As for the *didactic* context, Clairaut was no professional teacher but a renowned astronomer and member of the *Académie des Sciences*. Nevertheless, his father was the private preceptor of officers and deeply interested in educational questions, as were the members of the *Society of Arts,* a group formed with the participation of the Clairaut father and sons. Alexis Clairaut himself plausibly tutored some high-ranking personalities like Émilie du Châtelet (Bernard, 2022; Lubet, 2020). We therefore presuppose here that he at least observed, and perhaps practiced, mathematical teaching through innovative ways. This implies that the text reflects actual practice comparable, albeit not identical, to contemporary problem-based strategies.

As a reflection of the editorial genesis outlined above, Clairaut's elegant preface to the *EoG* makes clear that he attempted to conciliate two seemingly contradictory ambitions, that is, to remain faithful to the genre of 'elements' and to dismantle the usual structuration of 'ordinary' elements in favour of a problem-based approach. This apparent paradox is solved through the definition of an original *purpose* and a particular *method of exposition* sustained by an underlying historical *narrative*.

Clairaut's philosophical and educative *purpose* is nicely captured by the remark, "Beginners will perceive at each step that they are made to take the reasons which guide the discoverer, and thus they can acquire more easily the spirit of discovery" (p. x). The notion of *problem* appears as the key idea on which the approach is based: "It has appeared to me more judicious to employ my readers constantly in solving problems, that is to say, in seeking the means of performing some operation or of discovering some unknown truth by determining the relation which exists between given magnitudes and magnitudes unknown which it is desired to discover" (ibid).

Accordingly, Clairaut's *method of exposition* gives a central role to the notion of *problem*. The term is taken in a general and philosophical sense, encompassing both 'problems' in a practical and operational sense and also 'theorems' of classical geometry. Clairaut insisted, furthermore, that the problems in question were not for practical purposes (as in land surveying treatises) but for the sake of discovering geometrical truths (p. xi).

Clairaut also suggested following a narrative based on a fictive history of mathematics, i.e., to guide the readers (the "beginners") through the way of "discoverers". That

[4] Page references are to Kaines' translation of Clairaut (1881).

translates to a 'natural history; of geometrical knowledge by which, beginning with practical *needs* and corresponding problems, the inventors of geometry came to consider more theoretical questions out of sheer *curiosity*. We see, in what follows, that these general themes are echoed within the deep logic of the treatise itself, and they also help in understanding the various themes and key questions that structure the treatise.

2.2 Clairaut from the perspective of IBME

This writing project, exposed in the preface of EoG, stimulated interest in connection with various reflections on mathematics education (Lubet, 2020), many of which are closely related to questions of IBME.

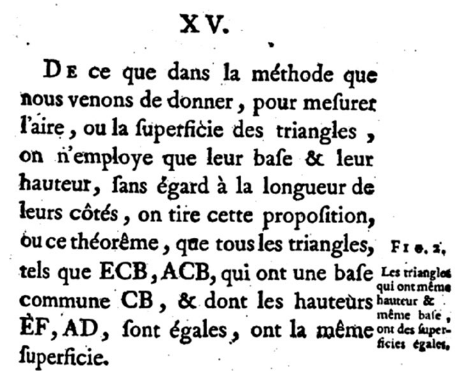
Felix Klein presented Clairaut's book as an interesting model for teaching geometry, applauding its style—"like a novel"—and the organisation of its content:

> Clairaut starts from practical problems of field surveying and leads gradually to general ideas, where the strict logical moment stands somewhat back. He argues in his very interesting preface, why he chose this approach: the practical problems of surveying incited mankind to develop a geometrical science; therefore, if one begins the book with them, one will succeed in interesting anybody much more in geometry—than by an abstract building of axioms and theorems, whose internal meaning nobody can easily understand. (Klein, 2016, p. 250)

EoG's rich reception history in modern mathematics education includes research concerning problem solving in mathematics education (Barbin, 1991; Sander, 1982) and the teaching of specific mathematical topics (Arnal-Bailera & Oller-Marcén, 2020; Chorlay, 2015). The analogies of Clairaut's writing project with principles of modern problem-based education are observed in several studies, but some studies (Barbin, 1991) also underlined the historical particularity of the text and the eventual tensions implied with the presuppositions of some modern problem-based approaches.

Clairaut's idea of following a fictive history of mathematics, which should not be confounded with the true development of geometry, reflects Toepliz's distinction between "genetic" and "historical" history. Several researchers discussed and debated Clairaut's possible role in the emergence of the genetic approach (Schubring, 2011). Furinghetti and Radford (2002) underlined proximities of Clairaut's ideas with principles of Toeplitz and Freudenthal.

In summary, we can see that proximities of Clairaut's preface with principles close to modern IBME approaches were observed by many readers, although some authors also reminded us of the limits of these analogies due to the


XV.

De ce que dans la méthode que nous venons de donner, pour meſurer l'aire, ou la ſuperficie des triangles, on n'employe que leur baſe & leur hauteur, ſans égard à la longueur de leurs côtés, on tire cette propoſition, ou ce théorême, que tous les triangles, tels que ECB, ACB, qui ont une baſe commune CB, & dont les hauteurs EF, AD, ſont égales, ont la même ſuperficie.

Fig. 2.

Les triangles qui ont même hauteur & même baſe, ont des ſuperficies égales.

Fig. 1 Example of an "article" (§I.15)

difference of historical context. Nevertheless, only a few of these readers entered into the analysis of the deep structure of EoG, exploring the effective realisation of the announced principles.

My interest in EoG was particularly stimulated by further analogies of its preface with principles of the Hungarian GD approach exposed in various historical texts related to GD (Gosztonyi, 2016). Such analogies, all recurrent elements of Hungarian texts, include the following: interesting the reader/learner instead of providing a dry presentation of mathematical knowledge; presenting mathematics in its development, like telling a story; guiding the reader through a 'rational reconstruction' of the history of discovering mathematics; accustoming the mind to discover and employing the reader in constantly solving problems; appealing to the curiosity of discoverers; and limiting rigorous formal exposition. The global form of the presentation of EoG also shows similarities with the Hungarian texts of GD as I demonstrate below.

These analogies suggested, despite the historical distance, that the inherent logic of structuration of EoG might share some characteristics with Hungarian SoP. Thus, the analysis of EoG's structure can have a heuristic value for further analysis of SoP, and more generally, for the reflection on possible structurations of IB teaching trajectories.

2.3 The structure of Clairaut's text

Clairaut proposed a project of initiating his readers into geometrical knowledge in a problem-based way. But, how was this project effectively realized in the book? At first glance, understanding EoG's structure and the role of problems in it is not an easy task.

Clairaut's treatise is divided into four main parts. Each one has its own introduction and title and presents itself as a continuous text, simply subdivided into numbered paragraphs (or "articles") with no title. As shown in the illustration (Fig. 1), the mathematical content introduced within these articles (here a proposition) is signalled through marginal notes that are recapitulated in turn within Clairaut's table of contents.[5]

The global structure can be recognized by the introductions and most visible transitions. Part I principally discusses the measure of distances and fields and is mainly structured[6] along six key problems of practical nature (or presented as such), each opening or 'generating' a series or sub-questions. For example, the third of these key problems is about the measurement of irregular fields such that the previously presented method (decomposition of the figure into triangles) cannot be applied due to practical obstacles. This sequence is discussed in more detail in Sect. 3.2.

Part II begins with a discussion of the notion of curiosity, already mentioned in the preface, and introduces the notion of geometrical exactness, based on ruler and compass constructions. The key problems are presented in the introduction and lead to the 'natural discovery' of the so-called Pythagoras theorem (Euclid I.47), as well as to a critical revision of the treatment of *proportions* introduced in Part I (§§37–45).

Part III synthesises the two first Parts by introducing a new key problem about the measurement of "circular figures" (as in Part I) through exact methods (as in Part II). As we can see through the second example studied below (Sect. 3.3), the problem itself is quickly solved. The bulk of this Part explores the consequences of the solution or, as the title announces, the properties of circular figures. While Part I is rather structured around practical problems, in Parts II and III, the "logic of curiosity" leading to theoretical results is dominant.

Part IV exhibits and justifies the measurement of solid figures. Here, the logic of problems and questions mainly follows the list of objects studied in order of complexity. The demonstrations are sustained by the explicit generalization of theorems about the proportions introduced from the beginning and deepened along the treatise.

The previous overview only broadly and incompletely summarizes the deep structure of the treatise. The table of contents, recapitulating the marginal notes, does not express the *structure* of the text either but only the *contents* introduced.[7] This crucial difference is reflected by the fact that not all paragraphs have such notes. The most strategic paragraphs, for example, discussing key problems, do not incorporate marginal notes. Paragraphs, in fact, constitute a flexible reference system, not only for the establishment of the table of contents but especially within the text from one article to another.

These features lead us back to the core of Clairaut's project and method of exposition. In close relation to this referential system, the contents of articles extend far beyond statements of problems or theorems and their solutions (procedures and demonstrations). They incorporate, most often at the beginning or end of the paragraphs, metadiscursive arguments serving as transitions. These explicit but always localized references constitute an indefinite network of 'heuristic' consequences, as if in a continuous discussion, rather than a hierarchical system of subthemes. Therefore, the deep ordering of articles is not explained by a 'table of chapters' but is immanent in this dense metadiscourse that connects the main elements of mathematical content. Making this heuristic logic clear and representing it is therefore an interpretative challenge and the purpose of Part III.

2.4 Analogies with the structure of texts related to the Hungarian guided discovery approach

Beyond the analogies in the principles of the writing project, discussed in Sect. 2.2, the form of EoG reflects some of the emblematic texts related to the GD approach that I analysed in my earlier research (Gosztonyi, 2015, 2016). Many texts of GD have a quasi-continuous writing style where problems motivate attempts of solutions which, in turn, raise new problems, and this process is made clear and coherent by a rich fabric of metadiscourse. This is especially true for Péter's *Playing with infinity* (1961), which was one of the first examples of 'Series of Problems' I analysed. I analysed the structure of two chapters of this book (Gosztonyi, 2015). Based on the interpretation of metadiscursive elements, I identified problems raised by the text and the transitions between the elements, and represented this underlying structure on a graph (Fig. 2).

[5] In what follows, I use the character "§" followed by numerals to denote a paragraph number, which is the same in Clairaut's original treatise and in Kaines' translation. I use the paragraph numbers instead of page numbers for references to the body text. As paragraphs are relatively short, this will help the reader to find the references easily in any edition of EoG, while I avoid ambiguities related to an important difference of editions, namely, that marginal notes are missing from Kaines' translation. Kaines translates though recapitulations of marginal notes in the table of contents; therefore, I use Kaines' table of contents for the English translation of the marginal notes.

[6] Except for the final §§73–82, which are explicitly introduced as a late addendum going back to the theme introduced in §64.

[7] Some nineteenth century editors of Clairaut have mistakenly confounded these contents with possible titles, as if the contents summarized the main theme or purpose of articles. In his survey of EoG, Sander (1982) makes the same mistake (probably because he used a late edition) and takes these marginal notes as „Leitsätze" (leading ideas) summarizing the contents, which is basically misleading.

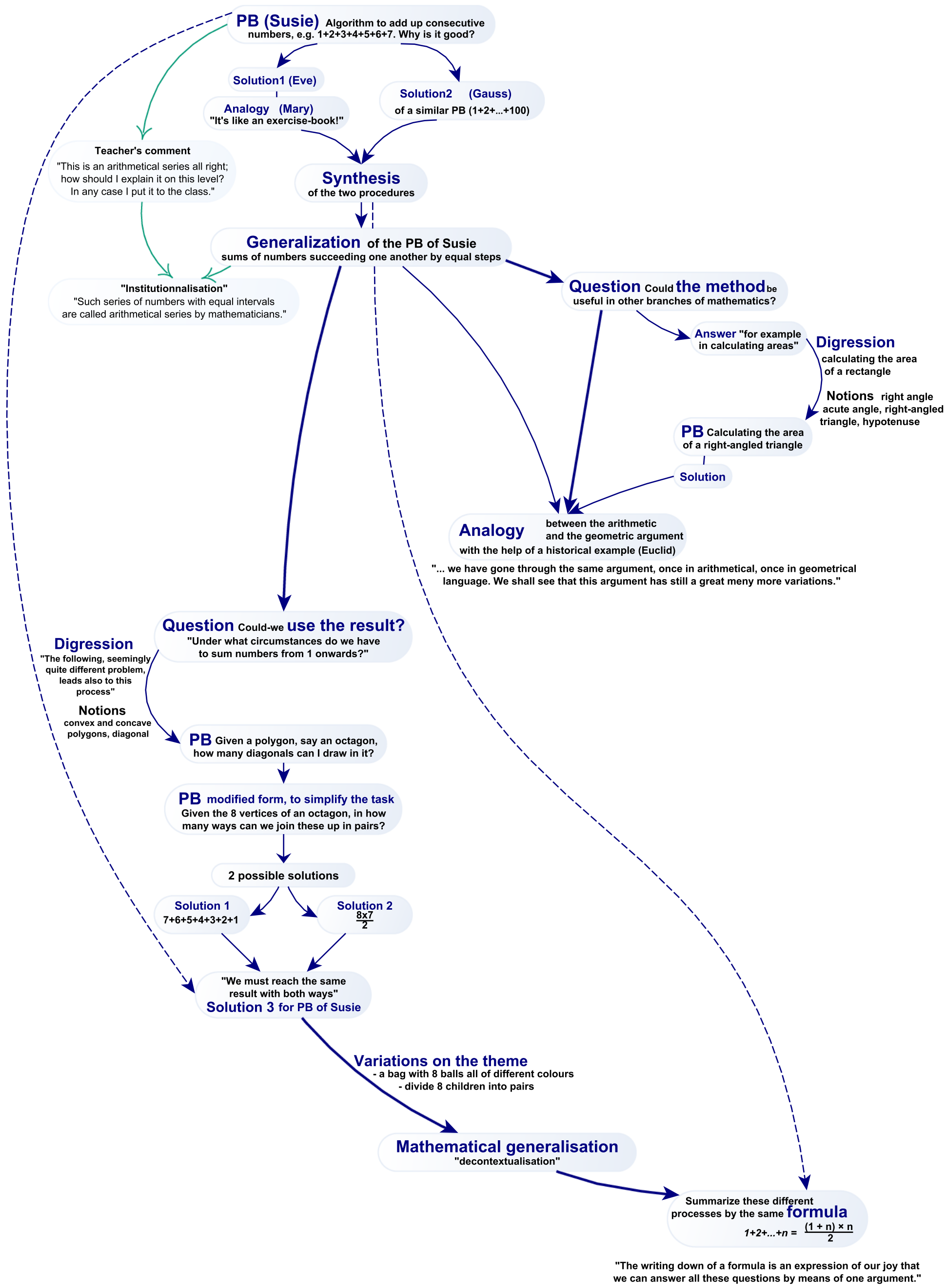


Fig. 2 Graph representing the structure of Péter's *Playing with infinity*

This representation reveals a complex structuration behind a relatively short text (Gosztonyi, 2015). It exposes the ordering of problems and the transitions between them which assure the coherence and the continuous interest of the text. Nevertheless, it was created in a quite intuitive way without a strict methodology on the identification of problems and transitions.

For the analysis of Clairaut's EoG, a similar graph representation of the underlying structure appeared to be promising due to the metadiscursive reference system described above. Seeking a more solid establishment of the analysis, the idea of a graph representation was crossed with discursive analysis methodology which has been developed for the analysis of historical texts by different participants in the frame of the HistSoP project.

3 Analysis of EoG: two examples

3.1 Methodological considerations

I represent the deep structure of EoG in the form of graphs, making connections among the contents of the paragraphs visible. This enterprise requires defining both the basic connected elements and the nature of their relationships.

The identification of the main constitutive elements is based on a discursive analysis of the text. Following the methodology of the HistSoP project (Bernard, 2015b, p. 11), the characterisation of basic elements was obtained from a preliminary analysis of key terms in Clairaut's vocabulary. The underlying presupposition is that their meaning is unfamiliar to the modern reader and needs clarification. The details of this analysis were presented by Bernard (in progress)—here I only summarize some key points necessary to the understanding of the following argumentation.[8]

We identified the following categories of constitutive elements of the text: *problem, proposition* (with the synonyms *property* and *theorem*), *method* (with the synonym *manner*), *definition*, and *demonstration* (with the synonyms *proof* and *justification*).[9]

While the term *problem* (PB) is given an extended meaning in the preface (see Sect. 1), in the body of the treatise, it restrictively refers to construction or measurement tasks. These problems are expressed by a specific and recurrent verbal form, including an action verb (to construct, to do, to divide, to measure, etc.) followed by one or several geometrical objects and the constraints of the task. This verbal core is most often preceded by introductory phrases like 'the problem is the following', 'in order to', 'when it comes to', etc. For example (§III.9): "When we have to [introductory formula] measure [verb] a figure Y, composed of arcs of different circles and straight lines [objects and constraint]".

Proposition (Prop) refers to attributive statements of truths such as "the three interior angles of a triangle are equal to two right angles". Clairaut emphasises in the preface that he wants to show how these truths can be discovered.[10] Thus, many of Clairaut's *propositions* are introduced in a problematized form, similarly to the format of problems, namely, [introductory formula] + [verb] + [proposition / property]. The verb is typically 'to demonstrate', 'to prove', 'to search', etc. In this form, propositions look like questions that one is meant to answer. This is coherent with the highly general notion of problem, explained in EoG's preface, corresponding not only to operations but also to 'truths'. We therefore call these kinds of problematized propositions '*proposition-problems*' (PrB).

Similarly, specific forms of expressions can also be identified for the other categories (which I will not detail here further).

Method (Met) designates a concrete procedure, appearing as a solution to a specific problem.

Demonstration (Dem) most often refers to the 'solution' of a proposition (or proposition-problem), just as *methods* refer to *problems,* by which we mean an explanation based on (deduced from) pre-established definitions, principles, or propositions.[11]

Definition corresponds to a clear and common meaning, i.e., an explanation of a technical term. Such *definitions*, when found in the text, are usually restated in the margin. Generally speaking, what is found in Clairaut's margin, and therefore in the table of contents, is a *definition*, *proposition*, or *method*.

In what follows, I thus use the word *content* to designate *definition*, *proposition*, or *method*, and the word *elements* in a larger sense to include all the key terms listed above.[12]

We introduced a coding system to designate the different elements found in the text and to represent them on a graph. Each element is identified by the number of the paragraph in which it is found and a complementary letter if needed. Thus, PB29b is the second *problem* found in §29, while Met29b is the corresponding method. If a problem statement appears several times (which happens typically because of the formulation of transitions), it is referred by the number of the paragraph in which its solution is discussed.

These different elements are connected in EoG through a rich fabric of 'metadiscourse'. Many of the paragraphs are introduced or ended by a transitional passage motivating the emergence of new problems, raising critical remarks about earlier methods or demonstrations, and establishing various links between different elements of the discourse. For example:

> As it often happens that in measuring a figure such as Y, the centre of the arc HIK is unknown, and yet without this centre the figure cannot be measured, since the preceding method requires the radius to be known, it

[8] This terminological analysis is based on the French version of the text. In this article, I present it in English using Kaines' corresponding terms to the French originals.

[9] In French, respectively: problème, proposition, propriété, théorème, méthode, manière, définition, demonstration, preuve.

[10] "I will carefully abstain from giving any propositions under the form of theorems (that is to say, of those propositions by which truths are demonstrated) unless I show at the same time how this discovery of the truth has been arrived at." (p. ix).

[11] In some cases, *demonstrate* or *prove* refer to *methods* justifying their validity. We keep the term *justification* for the larger concept.

[12] Which use is not identical with the sense of the word in the title "Elements of Geometry".

Table 1 Representations used in the graphs

Problem (PB)……………………	PB
Proposition-problem (PrB)…….....	PrB
Problem (PB) with its method of solution (Met)…………............	PB
Proposition-problem (PrB) with its demonstration (Dem)................	PrB
Proposition (Prop)………………	Prop
Definition (Def)………………….....	Def
Elliptic elements and links are represented with broken lines……	

is necessary to find means of ascertaining the centre of any circular arc. (§III.11)

Similar (although, sometimes very short) metadiscursive remarks also appear in the middle of paragraphs, contributing to, for example, the explanation of the details of a method. Some specific paragraphs are almost entirely of a metadiscursive nature. This rich metadiscursive texture assures the coherence and the continuity of the text and plays a key role in motivating and problematizing the emerging new mathematical knowledge.

This is one of the main focuses of the study. After identification of the 'elements' of the text described above, we analysed the transitions explicated by Clairaut's text by identifying which elements are connected and characterising the nature of these connections. In the following analysis, I highlight these transitions and indicate them by the sign (×).

Based on this analysis, I constructed graphs representing the structure of (a sequence of) the text. The different elements appearing in the sequence are represented in the following way (Table 1).

The transitions between these elements appear as arrows. The labels of the arrows characterize the transitions based on the analysis of the metadiscursive passages. Transitions to definitions appeared to be always 'need'-based; thus, these are not labelled individually. Some notable transitions from external parts of the text to the analysed sequences are represented with dotted lines.

After a global reading of the book, two relatively compact sequences, already mentioned above, were chosen for analysis, namely, I.25–32 and III.9–22. The first sequence, from Part I, is mainly structured around construction and measurement problems referring to practical questions regarding "measurement of lands". The second sequence, from Part

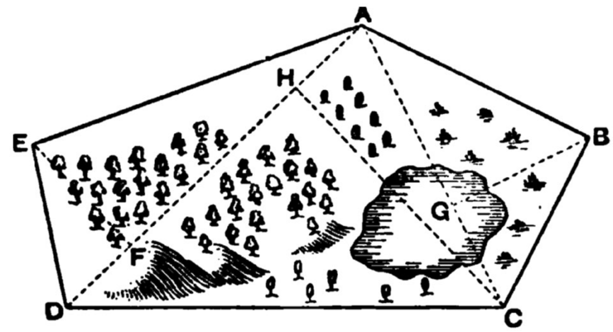


Fig. 3 PB25b

III, although introduced by a similar problem to the above, is structured mostly by questions that Clairaut characterised as "theoretical" and "'curiosity-driven" in the preface; besides, propositions and demonstrations play a more structuring role in this second example. Thus, these two examples illustrate the global progressiveness of Clairaut's writing strategy, described in Sect. 2.3.

3.2 First example: EoG I.25–32

The first example is from sections §25–32 from Part I of EoG. After having discussed questions concerning the "measurement of lands" in the previous paragraphs, this sequence focuses on problems related to the measurement of lands in which some obstacles hinder the implementation of methods explained earlier. The introduction in §25 makes explicit that what follows is a variation of problems and methods seen earlier (§13–14), concerning the measurement of lands which are fully accessible:

> Returning to the measurement of lands, there are many cases where the forms are such that their measurement cannot be effected by the methods already described.

This key problem (PB25a) is developed in an instantiated form (×) (PB25b), referring to a specific figure (Fig. 3):

> Let ABCDE be the figure of a field or an enclosure, etc., which it is desired to measure. Following what has preceded, let it be divided into triangles such as ABC, ACD, ADE, and let these triangles be measured after having drawn the perpendiculars EF, CH, BG. But if in the space ABCDE there should be some obstacle, an elevation, for example, a wood, a pond, etc., which prevents the necessary lines from being drawn, what then must be done?

Clairaut thus proposes a revision and critique of earlier methods (×) and points out the step that forbids their application, in this case, thus creating the need for new methods. He then outlines a strategy of solution, namely that triangles equal and similar to the original figures should be drawn on a well accessible ground (PB25c) where the measurements can easily be carried out. Thus, the introductory key problem is reduced (×) to PB25c, the drawing of "equal" (meaning isometric) triangles.

The introduction in §26 implicitly suggests that the problem is to be unravelled in the form of several sub-problems (×):

> Supposing in the first place that the obstacle occurs within the triangle ABC, of which the sides are known.

The first is the case where the three sides of a triangle are all accessible and, therefore, can be measured (PB26). Then, the procedure of drawing a congruent triangle is described and justified.

§27 introduces the first variation on PB26 (×), where only two sides of a triangle can be measured (PB27). Clairaut again follows the strategy of revising the method of solution applied earlier (×), in order to find out what should be modified. He concludes that the relative position of the two known sides should be determined. This motivates (×) the definition of the notion of an angle and reduces PB27 (×) to the problem of reproducing an angle (PB28), treated in the paragraph that follows. Once the method for PB28 is described, the solution of PB27 can be completed. At the end of the paragraph, Clairaut introduces a proposition (stated also in the margin) as the grounding principle underlying and justifying the method (×):

> This simple procedure[13] takes for granted this obvious principle, that a triangle is determined by the length of two of its sides, and by their angular opening; or what amounts to the same, that one triangle is equal to another when two of their sides are respectively equal and when the angle contained by those sides is the same.

Before continuing the discussion of various positions of obstacles, Clairaut proposes, in §29, an alternative method (×) for reproducing an angle (PB29a). This method raises the need (×) for placing a point on an arc at a given distance to its end (PB29b) which allows for the opportunity (×) to introduce the definition of a chord.

§30 presents a new variation on PB26 and 27 (×), namely the problem of reproducing a triangle of which only one side and two angles on that side are accessible (PB30). Similarly to §28, Met30 is described and then reduced (×) to a proposition in the margin stating, "two angles and one side being given, the triangle is determined".

In §31, a variation on the previous problem (×) is posed (PB31), as a specific case of PB30, concerning isosceles triangles (the notion being introduced together with the problem ×). We know one side (the basis), but we also know that the two other sides are equal. In this case, Clairaut says, we need only to measure and reproduce one angle (a simplified version of the method for PB30 described in a short, elliptic way) because of the property of isosceles triangles (×) which is introduced and demonstrated at the end of the paragraph: "The angles that its sides make with the base are equal".

§32 makes clear that we have arrived at the end of a sequence of the text:

> To return to the measurement of lands, it is seen that whatever may be the obstacles presented in their interior, it will be easy, by the preceding method, to transfer to clear land all the triangles dividing the space to be measured.

The introducing expression is the same as the one that appears in §25, closing syntactically the sequence (×). The expression "the preceding method", refers to the entirety of methods presented in §25–31 (×). Then, a new instantiated version of PB25a (×) is discussed (PB32) with reference to a complex figure representing a land of various obstacles. Met32 is described as a synthesis of the methods presented in the preceding sequence.

Fig. 4 shows the graph representing the structure of the sequence.

3.3 Second example: EoG III.9–22

This sequence introduces the notion of circular segments and discusses some initial properties.

§9 starts with a measurement problem (PB9), quickly reduced (×) to a second one (PB10):

When we have to measure a figure Y, composed of arcs of different circles and straight lines, or a figure Z, wholly composed of circular arcs, the whole difficulty reduces itself to the measurement of segments of circles.

This statement calls for the definition of a segment of a circle which is then given (×). Then, the reduction from PB9 to PB10 is justified and stated in the margin as the proposition (×), "the measurement of all circular figures may be reduced to that of the segment".[14]

§10 gives the solution (Met10) to PB10 (the measurement of segments of circles) which, in turn, calls for the introduction of a new definition (×), that of a sector, as well as for the reinvestment of the measurement of the circle (§III.1) (×).

As Clairaut explains at the beginning of §11, Met10 requires finding the centre of a circular arc when it is not

[13] Clairaut: "simple pratique", mistranslated as "this simple problem" in Kaines.

[14] Kaines' erroneous translation was corrected by us.

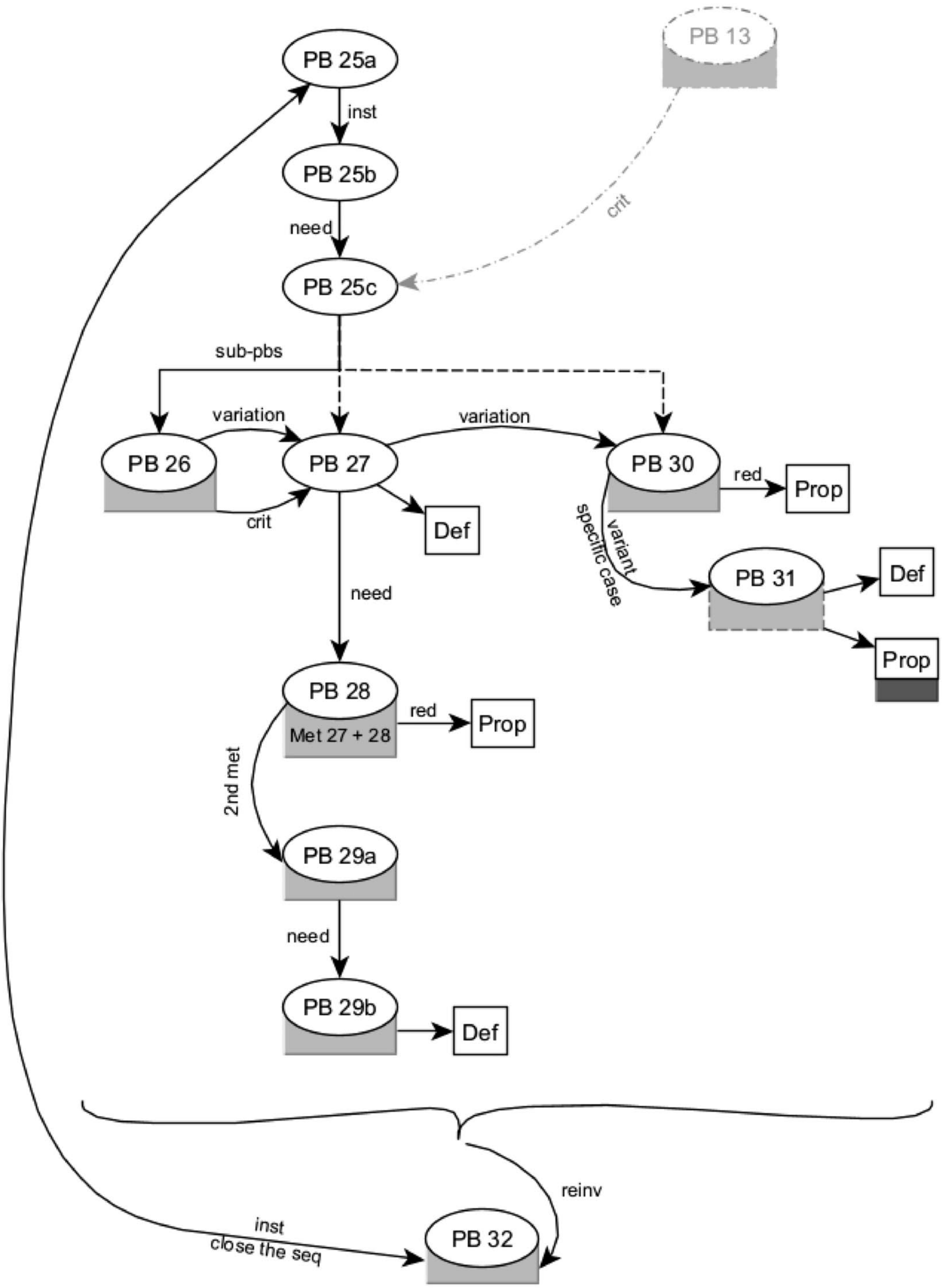


Fig. 4 Graph for §§I.25–32

given in advance (PB11) (×). The solution (Met11) is then described and justified.

The short, purely metadiscursive §12 offers the following reflection on Met11:

Thus, however three points be arranged, provided they are not in one straight line, they can always be connected by an arc of a circle, or, what comes to the same, whatever be

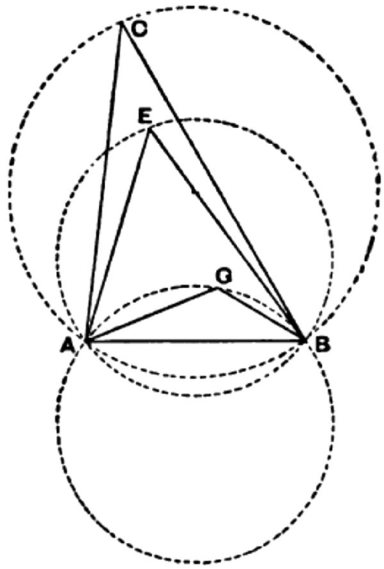


Fig. 5 Illustrating §13

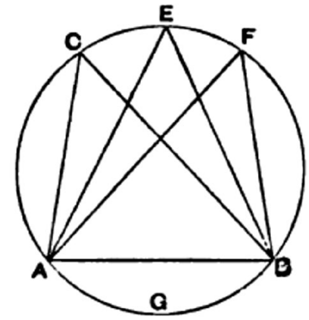


Fig. 6 PrB14b

the proportion of the sides AC and BC of a triangle ACB to its base, a circle can always be circumscribed about that triangle.

This paragraph introduces a change of perspective on Met11. While PB11 considered the arc of a circle and three points A, B, C taken arbitrarily on the arc as tools of construction in the solution, §12 suggests applying the same operations of Met11 on another object (×), namely a triangle built on three points A, B, C (PB/Met12a),[15] and then on a second object (×), allowing the construction of a circumscribed circle about the triangle (PB/Met12b).

§13 begins with a reminder of PB12b and proposes a thought experiment (×) in which Met12b is applied to various triangles having the common base AB (Fig. 5). Clairaut suggests observing the position of the centre of the circles in question as the location of the third point of the triangle varies. This leads to the following proposition-problem (PrB13a):

> Now, seeing this centre pass below AB after having seen it above, should, it seems, suggest an inquiry as to what kind of triangle AFB it is that has the centre of the circumscribed circle exactly upon AB.

The corresponding proposition (the triangle in question is a right triangle, Prop13b) is indicated in the margin, and the demonstration follows.

§13 curiously ends with the reciprocal of Prop13b which focuses on right angled triangles, not on semi-circles, and which is not proved but presented as the consequence of the former demonstration.

§14 is introduced by the following remark, accompanied by (Fig. 6):

> This property of the circle, that the angle in the half-circumference subtended by the diameter is always a right angle, leads us to inquire if the other parts of the circle have some analogous property; if, for instance, the angles ACB, AEB, and AFB, taken in a segment ACEFB, be not all equal as those in the semicircle are.

Thus, the reader is invited to think about the generalisation of the preceding theorem by analogy (×) through the proposition of a general hypothesis, namely that other (non-right) inscribed angles subtending the same arc of a circle might be equal (PrB14a). This is reduced to an instantiated version (×) of the proposition-problem (PrB14b), referring to the figure above (Fig. 6).

Clairaut suggests treating the demonstration in two steps (×):

> To make sure of this, we may begin by ascertaining the value of one of these angles, and we can afterwards see if the others are of the same value.

The first step is a measurement problem (PB14c) solved in the remaining part of §14 with explicit reference to problems and propositions from Part I, while the second step is a proposition-problem (PrB15a), solved in §15 by reduction (×) to another proposition-problem (PrB15b), completing the demonstration of the equality in question. The obtained proposition (Prop15 corresponding to PrB14a) is formulated in the margin as "all the angles at the circumference of the same arc are equal to each other and to half the arc in which they are."

§16 starts with the following consideration:

[15] It is unclear, whether the elliptic formula contained in §12 corresponds to the statement of a problem, to the underlying method or (less probably) to the statement of a condition of possibility.

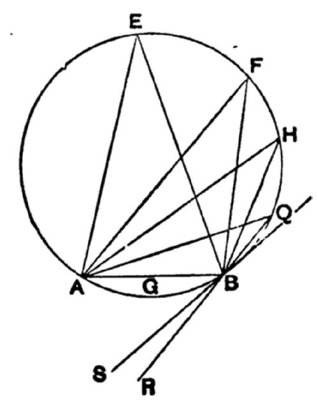


Fig. 7 PrB18a

> Among the different angles in the arc ACEFB, there are some which might at first sight appear not to be included in the preceding demonstration: these are angles like AFB such as the line FDG, drawn trough the centre, passes outside of the angle ADB.

Indeed, Clairaut relies on a figure (Fig. 6) in the instantiated form of the original hypothesis (PrB14b), and he uses the line introduced in Met14c and Dem15b to cut the examined angles into halves. However, PrB14a (reformulated as Prop15, cited above) has a more general value since the arrangement of some angles does not appear to be covered by the demonstration. Clairaut, thus, proposes a critique and revision of the demonstration (×) and shows how it can be extended to the new case (PrB16). A similar introductory formula is used in §17 (×) in relation to the case of segments of circles smaller than a semicircle (PrB17).

§18 begins with a reminder of the proposition which has just been proven; the figure (Fig. 7) recapitulates the corresponding configurations:

> Having seen that in the same segment the angles AEB, AFB, and AHB, at the circumference, are all equal, we are induced to inquire what the angle AQB becomes when its vertex coincides with the point B.

Following both the lead of the initial problem (PrB14a) and the logic of exploration of various arrangements (×), Clairaut questions the extreme case represented by the point B (PrB18a). However, the argument cannot follow the logic of criticism of the previous methods for reasons that Clairaut explains at length by introducing a discussion of tangents and limits, in the sense of infinitesimal calculus or, in Clairaut's terms, the *Géométrie de l'infini*. The paradox of the existence of a constant angle while one of its sides vanishes is highlighted in the form of questions, solved through the introduction of two definitions (tangent and angle of a segment) (×), and concluded by the proposition (Prop18b) that the "measure [of an angle of a segment] is half the arc of the segment" (×).[16]

In a concluding remark, Clairaut suggests that the demonstration described, despite its anticipated difficulty for beginners, might be relevant for readers who wish to continue their studies in the "Geometry of the Infinite to be accustomed early to considerations of this kind". Therefore, he invites readers (×) to discover another, simpler demonstration of the same proposition, which requires (×) the introduction of "the chief property of tangents".

§19 describes this property (a tangent is perpendicular to the corresponding diameter of a circle) and its demonstration. §20 then describes the promised alternative demonstration to Prop18b (×) based on Prop19 (×).

§21 introduces a new construction problem (PB21) which might be solved by the latter demonstration:

> The second demonstration just given of this property of the circle, that the angle ABS is measured by half the arc AGB, furnishes the solution of the following problem.
> To describe upon AB a segment of circle to contain a given angle L, that is to say, a segment AFB in which all the angles AFB at the circumference shall be equal to the angle L.

The statement of the problem introduces the definition (×) of a circular segment containing a given angle. Met21 is then described and followed by a proof calling for the previous articles.

§22 closes the sequence regarding properties of circular segments by giving an example of a practical application (×) and calling to the two main themes of curiosity and utility.

> The discovery of the properties of circular segments that we have just explained was probably due to the mere curiosity of geometers; but with this discovery it has been, as with many others: what was not at first believed to be useful became so afterwards.

Just as with the previous sequence analysed, these meta-discursive remarks close the corresponding sequence of paragraphs by referring back to the opening paragraph (here §9). The problem itself (PB22) is to determine the position of a point D in relation to three locations (points) A, B, C, knowing the respective distances between the three points, and the angles by which AB, BC, or AC are seen from D. Met22 reinvests (×) the methods and properties explored

[16] This paragraph, for the kinetic and experimental introduction of the notion of tangent, was treated in earlier studies in mathematics education (Arnal-Bailera & Oller-Marcén, 2020; Chorlay, 2015).

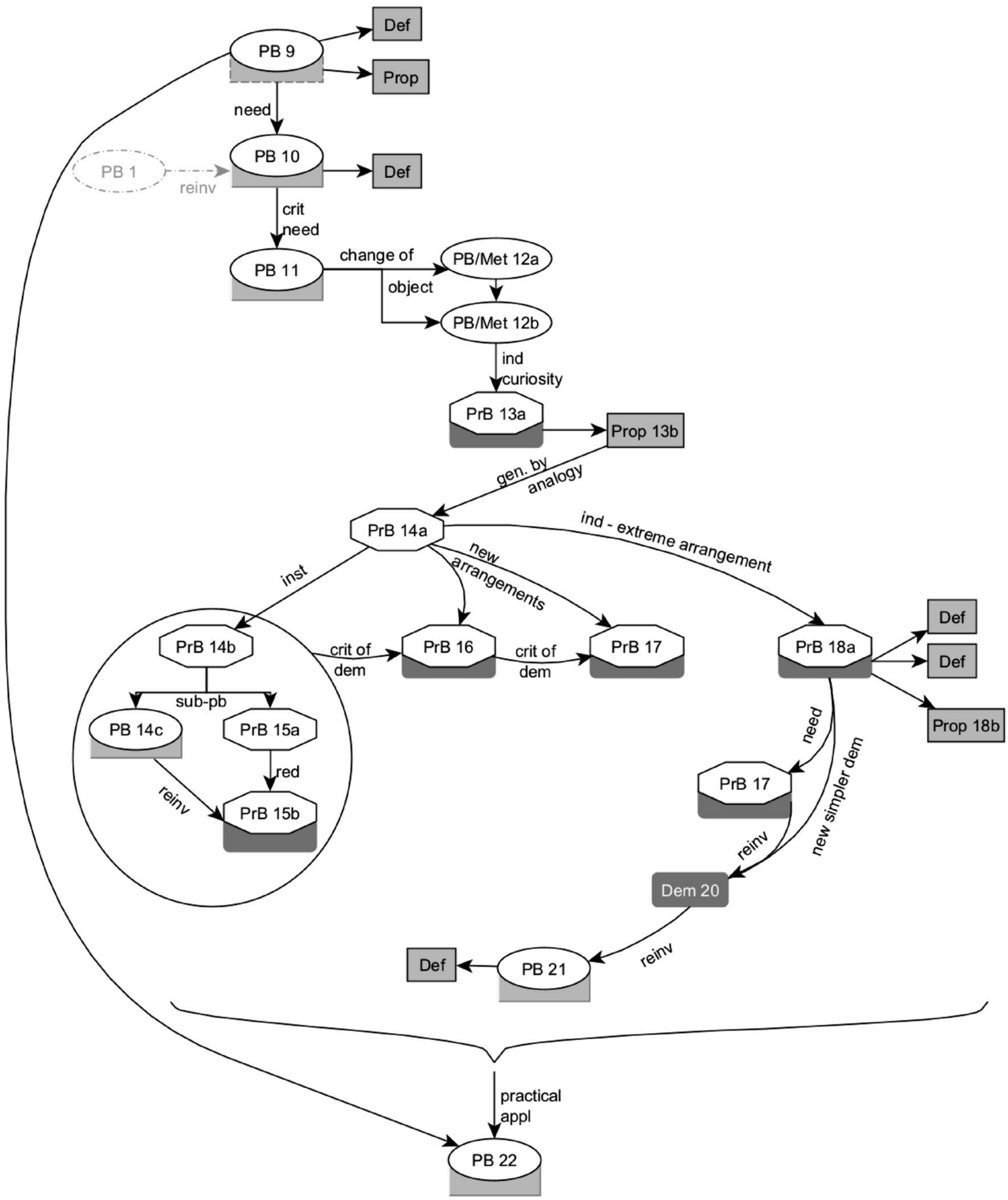


Fig. 8 Graph for §§III.9–22

in the preceding paragraphs, offering a synthesis of the sequence. The demonstration is presented as obvious based on the previous paragraphs.

Figure 8 below shows the graph representing the structure of the sequence.

4 Lessons from the analysis of EoG's structure

4.1 Clairaut's writing strategy as an inspiration for inquiry based teaching trajectories

The analysis presented above with the representation in form of graphs highlights many elements of Clairaut's writing strategy and contributes to the understanding of the structure of the book. It reveals a rich and complex network of references between problems and different elements of content, thus illustrating how the coherence and continuity of the text is assured and how the emergence of new mathematical content is problematized and motivated. This analysis also helps us to understand how Clairaut guides the reader through the discovery process of geometry. In this section, I discuss some of these elements, which can provide insights for the possible choices of structuration of IB teaching trajectories.

In the preface, I cited Artigue and Blomhøj's (2013) work in reference to inherent tensions in IBME. Beyond the tension "between the development of inquiry habits of mind" and the curriculum-based progression of mathematical knowledge, they underline "tension between internal and external sources of mathematical activities, and tension between scientific and real-life interests" (p. 13). These questions are explicitly discussed in EoG's preface, and our analysis offers elements to reveal how Clairaut deals with them both globally and locally in the construction of his text.

The two sequences chosen as examples are clearly delimited by metadiscourse. The last article summarises the sequence in each case and refers to the beginning of the sequence, creating a loop. In the first example, the text is driven mostly by practical problems. Many connections are need-based: definitions and propositions appear motivated by need during the solving of practical problems. The second example starts with a similar practical problem, but an important role is given to proposition-problems and their demonstrations; the connections in this sequence are more of a theoretical nature, and many of them are motivated by "curiosity". This difference between the two examples illustrates the global progression of Clairaut's text explained in Sect. 2, moving progressively from practical problems and a need-based motivation towards more theoretical problems following a logic of curiosity.

The strategies of structuration are, however, coherent in the two examples. While a well-chosen ordering of the elements creates the potential for conceptual transitions between them, the rich fabric of metadiscourse allows Clairaut to make those transitions explicit, and thus, motivates the introduction of new elements at each step. Beyond the duality of 'need' and 'curiosity', diverse types of connections appear, and the graph representation helps us to observe recurrences among them, including sub-problems, variation of problem statement, critique of a method, analogy, instantiation, reinvestment of a method, etc. While these observed recurrences suggest a possible categorisation of connections, surprising and atypical transitions also appear, for example the change of view on the treated objects. This implies that building an open inventory of types of transitions would be more meaningful than a closed categorisation.

Our analysis also helps to highlight several local writing choices which allow an IB-type organisation of specific subjects. A remarkable example is the repeated application of inductive methods in §§III.14–18. As we have seen, the demonstration is not presented with an a priori cut to subcases but as an inductive process progressively revisiting and enlarging a demonstration originally based on one generic figure. Another example is the emergence of propositions about the equality of triangles as parts of solutions to practical problems in §§I.28–31.

Of course, our methodology also has its limits. We chose to focus on links among 'elements' of the text described above, namely, problems, propositions, methods, demonstrations, and definitions. This analysis (and the related graph representation) does not illuminate the internal structure of methods and demonstrations; although during our analyses, it appeared that this would also contribute to the understanding of the dense network of references in the text. We also observed a micro-dialectic of methods and their justifications in the description of the methods, which our current analysis cannot take into account. Furthermore, we did not consider the role of figures in our methodology, although this appeared to be crucial in understanding the logic of certain passages, such as §§III.14–18. Finally, we analysed quite short sequences. In the case of longer sequences, the complexity of networks would increase, and the transparency of the graph representation would be compromised. Our methodology thus revealed important information about EoG's structure, relevant for reflection on modern IBME, but also showed that complementary analyses and/or a development of our methodology would be necessary to elucidate further characteristics of Clairaut's writing strategy.

4.2 Methodological inspiration for the analysis of Hungarian series of problems

The analysis presented above also offers important insights for the development of didactical research on SoP in the context of the Hungarian GD. As we have seen above, there is a significant analogy between the analytical challenges in the two cases. While the underlying organising logic of the text/teaching trajectories is based on complex networks of elements, this profound structure is not immediately transparent in Clairaut's book nor in Hungarian SoP-though for different reasons. In EoG, this is due to the quasi-continuous writing structured by articles. In the work with Hungarian teachers, descriptions of SoP often present only problems (with or without solutions), considered as the main pillars of a teaching trajectory, while other components of the planned trajectory remain implicit.

A crucial step for the analysis of EoG was the identification of the different elements constituting the text. In our first attempts at analysis, our focus was on 'problems', but it became clear that (1) the meaning of the term itself is not self-evident, and (2) the structure of the text cannot be described solely with links between problems.

The first point is coherent with basic ideas of the HistSoP project, namely that 'problem' can have various meanings in different contexts, and it also reflects present-day mathematics education where the term 'problem' carries different meanings in different educational contexts.[17] A specificity of EoG is that 'problem' bears two different meanings in the same text, one referring to precise mathematical tasks (construction and measurement problems) and another, more general one, referring to intellectual challenges. This ambiguity, which appeared to be crucial to understand for the analysis of EoG, sheds light on a similar ambiguity concerning the use of this term in the Hungarian context. Although I talk about 'series of problems' with teachers following GD, 'problem' (*probléma*) is not, in fact, a technical term in Hungarian mathematics education. What is given to students in textbooks or by teachers is generally called a 'task' (*feladat*). Nevertheless, teachers following GD claim, in discussions about their practice, that 'problems' play a central role in their teaching, and they characterise the tasks they propose to students as problems (in opposition to other teachers' practices who do not follow GD). The meaning implied by the term 'problem' in GD seems to be quite close to Clairaut's general sense (Gosztonyi, 2015).

The second point, concerning links between problems and mathematical content, refers to the core question proposed in the introduction, the combination of IB approaches with the teaching of curricular content, and has important implications for the Hungarian SoP project. Our descriptions of SoP and the graph representations we developed, until now, mainly present problems and their relationships; highlighting the various connections between problems and the treated mathematical content remained difficult for the Hungarian examples (Gosztonyi & Varga, in progress). At the same time, the graphs developed for EoG, based on a discursive analysis of the text, allowed us to highlight the links between problems and mathematical content and to reveal the role of problems in the emergence of mathematical knowledge. This can provide insight for the development of the analysis of Hungarian SoP and may help to represent the emergence of mathematical content—although the meaningful way of representation might be different, as I demonstrate below.

A third important point concerns the characterisation of links between the different elements, represented by arrows on the graphs. For the Hungarian SoP's graphs, we did not specify the meaning of arrows. In the case of EoG, the analysis highlighted the role of metadiscourse, the study of which served as a basis for the characterisation of links and opened the way towards the building of an inventory. This can offer heuristic inspiration for the building of a similar inventory of links in the Hungarian context and to better characterise the structuring links between elements of SoP.

4.3 Limits of analogies—understanding ourselves through the lenses of history

Of course, these analogies between EoG and modern mathematics education have their limits due to the difference in contexts, as was underlined by several authors (Barbin, 1991; Schubring, 2011). In the case of GD, for example, we can observe interesting epistemological discrepancies with EoG.[18] First, while Clairaut suggests entering into the study of geometry by practical problems, this is not a preference of GD.[19] Second, according to GD, new mathematical knowledge emerges slowly, through progressive generalisation and exploration by problem solving. While in the analysed sequences of EoG, definitions always appear related to a specific need, GD seems to establish new notions as a result of a progressive generalization process (Gosztonyi, 2020). This suggests that it might be less meaningful to represent mathematical content in the form of discrete elements in the context of GD.

This difference seems to be related to the different epistemological status of mathematical notions in the two historical contexts, although this hypothesis should be confirmed by further research. This implies, however, that Clairaut's solutions to IBME challenges are probably not entirely transposable to a modern context, and a direct transposition of analysis or representational tools might not be meaningful. However, as reminded by Artigue et al. (2020), differences also exist among modern approaches to IBME, and investigation of similarities and differences can help in understanding how these impact didactic strategies and realizations.

This idea is coherent with Fried's suggestion (Fried, 2007) that the study of history of mathematics through a mixture of familiarity and strangeness has a heuristic value and helps us to understand ourselves and our own culture better. The confrontation of Clairaut's text with modern IB teaching trajectories has a similar value in understanding our modern teaching strategies, while the dialogue between disciplines—history of mathematics and mathematics education research—contributed, in this study, to the development of research methodologies.

[17] This kind of inquiry into educational vocabulary was recently taken in charge by the Lexicon Project (Mesiti et al., 2021).

[18] Not exactly the same as what was underlined by Barbin (1991) in the French context of the 1990s.

[19] Contrary to some other modern IBME-type approaches, as RME or modelling (Artigue & Blomhøj, 2013).

Acknowledgements This study was supported by the MTA-ELKH-ELTE Research Group in Mathematics Education founded by the Scientific Foundations of Education Research Program of the Hungarian Academy of Sciences. I am grateful for the common work with Alain Bernard, without which this article would not have been possible.

Funding Open access funding provided by Eötvös Loránd University.

Declarations

Conflict of interest The author has no competing interests to declare that are relevant to the content of this article.

References

Arnal-Bailera, A., & Oller-Marcén, A. M. (2020). Prospective secondary mathematics teachers read Clairaut: Professional knowledge and original sources. *Educational Studies in Mathematics, 105*(2), 237–259. https://doi.org/10.1007/s10649-020-09988-7

Artigue, M., & Blomhøj, M. (2013). Conceptualizing inquiry-based education in mathematics. *ZDM - the International Journal on Mathematics Education, 45*(6), 797–810. https://doi.org/10.1007/s11858-013-0506-6

Artigue, M., Bosch, M., Doorman, M., Juhász, P., Kvasz, L., & Maass, K. (2020). Inquiry based mathematics education and the development of learning trajectories. *Teaching Mathematics and Computer Science, 18*(3), 63–89. https://doi.org/10.5485/TMCS.2020.0505

Barbin, E. (1991). Les Eléments de Géométrie de Clairaut: Une géométrie problématisée. *Repères IREM*, *4*, 119–133. https://www.univ-irem.fr/reperes/articles/4_article_24.pdf

Bernard, A. (Ed.). (2015a). *Les séries de problèmes, un genre au carrefour des cultures*. EDP Sciences. https://www.shs-conferences.org/articles/shsconf/abs/2015/09/contents/contents.html

Bernard, A. (2015). Les séries de problèmes, un genre au carrefour des cultures: Une première synthèse. *SHS Web of Conferences, 22*, 00001. https://doi.org/10.1051/shsconf/20152200001

Bernard, A. (2022). Les Éléments de géométrie de Clairaut: Rupture ou héritage ? *Philosophia Scientiæ*, *26–2*(2), 19–42. https://www.cairn.info/revue-philosophia-scientiae-2022-2-page-19.htm

Bernard, A. (in progress). *Demonstration and explanation in Clairaut's Elements of Geometry*.

Chorlay, R. (2015). Making (more) sense of the derivative by combining historical sources and ICT. In E. Barbin, U. T. Jankvist, & T. H. Kjeldsen (Eds.), *Proceedings of 7th European Summer University on the History and Epistemology in Mathematics Education* (pp. 485–498). Aarhus University.

Clairaut, A. C. (1741). *Éléments de Géométrie*. Lambert et Durand.

Clairaut, A. C. (1881). *Elements of Geometry* (J. Kaines, Trans.). Kegan.

Fried, M. N. (2007). Didactics and history of mathematics: Knowledge and self-knowledge. *Educational Studies in Mathematics, 66*(2), 203–223. https://doi.org/10.1007/s10649-006-9025-5

Furinghetti, F., & Radford, L. (2002). Historical conceptual developments and the teaching of mathematics: From philogenesis and ontogenesis theory to classroom practice. In L. English (Ed.), *Handbook of international research in mathematics education.* Routledge.

Gosztonyi, K. (2015). Séries de problèmes dans une tradition d'enseignement des mathématiques en Hongrie au 20ᵉ siècle. *SHS Web of Conferences, 22*, 00013. https://doi.org/10.1051/shsconf/20152200013

Gosztonyi, K. (2016). Mathematical culture and mathematics education in Hungary in the XXst century. In B. Larvor (Ed.), *Mathematical cultures. The London Meetings 2012–2014* (pp. 71–89). Springer Birkhauser.

Gosztonyi, K. (2019). Conceiving teaching trajectories in the form of series of problems: A step for the theoretical reconstruction of the Hungarian Guided Discovery approach. In U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Eleventh congress of the European Society for Research in Mathematics Education* (Vol. TWG17, Issue 17). Freudenthal Group. https://hal.archives-ouvertes.fr/hal-02418160

Gosztonyi, K. (2020). Tamás Varga's reform movement and the Hungarian Guided Discovery approach. *Teaching Mathematics and Computer Science, 18*(3), 11–28. https://doi.org/10.5485/TMCS.2020.0475

Gosztonyi, K., & Varga, E. (in progress). Teachers' practices and resources in the Hungarian "guided discovery" approach to teaching mathematics. Presenting and representing "series of problems". *ZDM – Mathematics Education*.

Gueudet, G., Pepin, B., & Trouche, L. (Eds.). (2012). *From text to 'lived' resources: Mathematics curriculum materials and teacher development*. Springer Netherlands. www.springer.com/us/book/9789400719651

Klein, F. (2016). Elementary mathematics from a higher standpoint. *Springer*. https://doi.org/10.1007/978-3-662-49445-5

Lubet, J.-P. (2020). Les Éléments de Géométrie de Clairaut: La longue histoire d'un petit livre. In G. Jouve, A.-M. Marmier, M. Moyon, F. Recher, R. Tazzioli, & D. Tournes (Eds.), *Mathématiques en perspectives: Hommage à Rudolf Bkouche* (pp. 247–257). PULIM.

Mesiti, C., Artigue, M., Hollingsworth, H., Cao, Y., & Clarke, D. (Eds.). (2021). *Teachers talking about their classrooms: Learning from the professional lexicons of mathematics teachers around the world*. Routledge.

Péter, R. (1961). *Playing with infinity* (Z. P. Dienes, Trans.). Dover Publications.

Raynaud, D. (2015). *Géométrie pratique: Géomètres, ingénieurs et architectes*. Presses Universitaires de Franche Comté.

Sander, H.-J. (1982). *Die Lehrbücher « Eléments de Géométrie » und « Eléments d' Algèbre » von Alexis-Claude Clairaut: Eine Untersuchung über die Anwendung der heuristischen Methode in zwei mathematischen Lehrbüchern des 18. Jahrhunderts* [Doctoral thesis]. University of Dortmund.

Schubring, G. (2011). Conceptions for relating the evolution of mathematical concepts to mathematics learning—epistemology, history, and semiotics interacting. *Educational Studies in Mathematics, 77*(1), 79–104. https://doi.org/10.1007/s10649-011-9301-x

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114–145. https://doi.org/10.2307/749205