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ORIGINAL PAPER

Teachers' guidance of students' focus toward lesson objectives: how does a competent teacher make decisions in the key interactions?

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Abstract

In this paper we describe how a teacher makes decisions during interaction with students in order to guide their focus towards the lesson's objectives that the teacher has in mind. In particular, in the paper we examine how a mathematics teacher's knowledge connects with the teacher's noticing skill and interactive action. We present the case of a competent Japanese teacher, Mr. T, during a series of nine lessons on comparing fractions in a fifth-grade classroom. We first look at how Mr. T structured the lessons and interactions to help the students shift their focus from the procedural aspects to the quantitative aspects, when generating their explanations concerning how to compare fractions. Then, we analyze how Mr. T interpreted the students' responses and made decisions about his interactive actions in the two lessons. The results show that Mr. T structured the entire lessons by repeatedly eliciting students' ideas and focusing on the object of examination. Detailed analysis of three vignettes shows Mr. T's constant reference to the lesson objectives when attending to and interpreting the students' thinking both in verbal and written form. Furthermore, he strove to adjust his initial decisions according to the students' reactions. These processes of noticing the students' mathematical thinking were made possible by his extensive content knowledge and pedagogical content knowledge; in particular, his knowledge of figural representation had a significant influence on his decision-making.

Keywords Classroom social interaction · Teacher's decision-making · Lesson objectives

1 Introduction

Identifying what mathematics teachers do to deal with classroom discourse in a way that produces desirable student outcomes is the object of several recent research endeavors (Walshaw & Anthony, 2008). A growing topic of research is what teachers do in the interaction with students when they stimulate students' mathematical thinking and direct their attention to important mathematical content (e.g., Chazan & Ball, 1999; Lobato et al., 2005, 2013; Stockero & Van Zoest, 2013; Stockero et al., 2020). Research results have been gathered on significant teaching moves based on students' ideas and thinking. For example, Chazan and Ball (1999) and Lobato et al. (2005) claimed that teacher's "telling" needs to be reconceptualized by understanding what kind of telling it is and what it accomplishes in the interactive constitution of discourse. In this reconceptualization, considering what type of object the teacher is focusing on in the discussion is crucial. Lobato et al. (2013) proposed a highlighting, renaming, and quantitative dialog as the teacher's discursive practices contributing to creating a specific "center of focus" in the classroom.

For an interaction to be productive for the learning process, it needs to have "intellectual ferment" (Chazan & Ball, 1999), wherein disagreement can be an important catalyst (p. 7). Teacher's roles of supporting and sustaining the intellectual ferment by monitoring and managing classroom disagreement seem to be critical. In the process of orchestrating lessons, the teacher sometimes interrupts the flow of the lesson, which provides opportunities to expand or change the nature of the students' mathematical understanding. These moments are defined by Stockero and van Zoest (2013) as "Pivotal Teaching Moments (PTMs)." The teacher's actions in PTMs are strongly influenced by the teacher's knowledge and orientation, because the teacher is required to respond immediately to unexpected contingencies. In recent years,

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studies have been conducted to study teachers' orientations (Stockero et al., 2020) and to propose a framework that would capture the teachers' decision-making and consider the influence of the teachers' own conceptions (Kooloos et al., 2021).

In this research trend, in our study we attempt to provide more information on the teacher's conceptions and decision-making in dealing with the students' ideas about essential mathematics. Significant teaching moves are both closely and intricately connected with the ways the teacher attends to and interprets the students' responses. In this paper, we offer information from a teacher's case analysis, wherein we analyze the teacher's in-the-moment decision-making with its different aspects.

The previous studies have primarily focused on the teachers who were inexperienced in teaching or who had little or no experience in inciting students to discuss their diverse ideas in the whole classroom. Considering the fact that "teaching is a cultural activity" (Stigler & Hiebert, 1999), focusing on the educational practices of those who are identified as competent by the local mathematics education community is valuable, because they are expected to possess rich conceptions and culture-based orientations. Therefore, in this paper, we examine the decision-making of a competent Japanese teacher (Mr. T) as he was teaching the mathematical content of how to compare fractions with different denominators.

We approach the issue of the teacher's decision-making by building on the previous research on mathematics teachers' knowledge. Below, on the basis of a review of several related studies, we develop an analytical model of the teacher's decision-making by highlighting how the mathematics teacher's knowledge relates to the teacher's noticing skills and interactive actions. In the case study of Mr. T, we first capture his way of structuring the lessons and interactions. The analysis of three vignettes follows, in which we examine how Mr. T's knowledge connects with skills of paying attention, interpreting students' responses, and making decisions concerning the interactive action. Mr. T's case study advances existing research on teachers' decision-making during interaction by showing how teachers manage the gaps between their lesson objectives and the reality of student learning.

2 A review of selected research on teacher's knowledge in the act of teaching

2.1 Conceptualizations of professional knowledge

Research in mathematics education has explored mathematics teachers' knowledge from two angles (Stahnke et al., 2016). The first is a cognitive approach, which considers the teacher's professional knowledge as a collection of cognitive elements (e.g., Ball et al., 2008; Kaiser et al., 2017; Shulman, 1986). Shulman's (1986) notion of content knowledge and pedagogical content knowledge lays the foundation of this approach. Large-scale assessments (e.g., Kaiser et al., 2017) have substantially contributed to measuring mathematics teachers' professional knowledge from different facets.

Ball et al. (2008) elaborated Shulman's notion by focusing on the knowledge teachers use to teach mathematics effectively. Their efforts went into developing a practice-based theory of content knowledge for teaching by focusing on the work that teachers do in teaching mathematics. In their framework of "mathematical knowledge for teaching," they subdivided Shulman's content knowledge into common content knowledge and specialized content knowledge (SCK), and his pedagogical content knowledge into a knowledge of content and students (KCS) and knowledge of content and teaching (KCT). According to Ball et al. (2008), SCK is "the mathematical knowledge and skill unique to teaching" (p. 400). To make certain content features visible to and learnable by students, they say that teachers' work involves "an uncanny kind of unpacking of mathematics" (p. 400). Specialized content knowledge includes understanding the meanings and principles of mathematical content and justifying mathematical ideas. The KCS is "knowledge that combines knowing about students and knowing about mathematics" (p. 401). It includes typical students' conceptions and misconceptions about certain content of mathematics. The KCT is "an amalgam, involving a particular mathematical idea or procedure and familiarity with pedagogical principles for teaching that particular content" (p. 402). It includes different instructional models for teaching certain mathematical content and ways of sequencing particular content for instruction.

As the above domains indicate, mathematical knowledge needed for teaching is multidimensional and requires to be conceptualized in more detail than Shulman's original distinction. Nevertheless, Ball et al. (2008) noticed the differences among teachers in their ways of analyzing the specific situation and using knowledge. One of the future tasks they highlighted is to understand the mathematical reasoning that underlines teacher's decisions and moves made in teaching.

2.2 Teacher's situation-specific skills

The second approach to mathematics teachers' knowledge is a situational approach in which teachers' abilities to perceive and attend to essential classroom situations within the context are captured and examined (e.g., Schoenfeld, 2011; Sherin et al., 2011). The construct of teacher "noticing" addresses different aspects of such teacher expertise. Sherin et al. (2011) posit that teacher noticing involves two main processes that are interrelated and cyclical, namely, "attending to particular events" and "making sense of the events in an instructional setting" (p. 5). The former involves the aspect of teachers choosing and paying attention to some things and not to others in order to manage the complexity of the classroom; the latter involves the aspect of teachers' active motion of "interpret[ing] what they see, relating observed events to abstract categories and characterizing what they see in terms of familiar instructional episodes" (p. 5). Some researchers, for example Jacobs et al. (2010), took a broader view and included teachers' reasoning about how to respond, in addition to the processes of teachers' attention to and interpretation of classroom situations.

Researchers have investigated mathematics teachers' "noticing" in the classroom by setting specific research foci and methods (e.g., Dyer & Sherin, 2016; Jacobs & Empson, 2016; Stockero & Van Zoest, 2013). Teachers' perception, interpretation and decision-making are primarily accessed by asking teachers to respond to or plan actions based on classroom situations, using video clips or written materials (e.g., Bruckmaier et al., 2016; Kaiser et al., 2017). Some research directly approaches teacher's in-the-moment decision-making by using lesson observation together with teacher interview (e.g., Dyer & Sherin, 2016; Thomas & Yoon, 2014). For example, to base instruction on the substance of student thinking, or responsive teaching, Dyer and Sherin (2016) explored teachers' thinking in play that leads to responsive practices. Their target was instructional reasoning about interpretations of student thinking, which is "dimensions of teacher thinking related to student thinking that go beyond interpretations of student thinking" (p. 73). They identified three types of instructional reasoning used by two teachers who used responsive practices in the classroom. The teachers reasoned with specifics of particular moments, kept attentive to mathematics, or exercised creative thinking. Some of the types were also conjectured to require a considerable level of subject matter knowledge. However, it remains as a future task to explore empirically how instructional reasoning influences the enactment of responsive teaching practices.

To capture how and why teachers make the choices they do, Shoenfeld's (2011) study, which has been repeatedly cited, provides a framework of Resources-Orientations-Goals (ROGs). He suggests that most human behavior has a goal-oriented structure. According to Schoenfeld (2011), *resources* include facts, or isolated pieces knowledge such as procedural knowledge, conceptual knowledge, and problem-solving strategies. They also include material or social resources such as textbooks or class history. The term *orientations* is inclusive and encompasses related terms such as dispositions, beliefs, values, tastes, or preferences. The teacher's *goals* are something that the teacher wants to achieve during a lesson. Goals come in various sizes, shapes and degrees of importance and may be immediate or long term.

Teachers possess different goals or orientations and that they occurred simultaneously with the desire to maintain classroom force (e.g., Zimmerman, 2015). One major concern in the ROGs framework is how the tension of competing goals arising in specific situations is managed or resolved (Paterson et al., 2011; Schoenfeld, 2011; Thomas & Yoon, 2014). For example, Thomas and Yoon (2014) investigated the decisions made by a teacher during a lesson. Their detailed analysis revealed the mechanism of the teacher's decision-making that was intended to resolve the conflict between his competing goals. The teacher abandoned one of the goals in favor of an updated goal. The new goal was not a complete rejection of the previous goal but rather "both an amendment and blend of other goals in the system" (p. 240). They further stated that the main drive in determining the new goal was the teacher's orientations, in which he produced new orientations that worked within the constraints he encountered.

2.3 Analytical model

Although both approaches described in Sects. 2.1 and 2.2 have made progress in their respective fields, research on teacher expertise that integrates these two approaches has advanced in recent years (e.g., Blömeke et al., 2015; Kaiser et al., 2017; Kooloos et al., 2021). Blömeke et al. (2015) proposed a framework that viewed competencies as a continuum consisting of three layers. The first layer captures the facet of the teacher's knowledge as a set of relatively stable cognitive resources. The authors call it "disposition," which consists of cognitive and affect-motivational resources. The second layer captures facets of cognitive skills that relate more to specific classroom situations and, therefore, are inherently more variable. The authors call it "situation-specific skills," consisting of perception, interpretation, and decision-making (Kaiser et al., 2017, pp. 172–173). The third layer is the teacher's observable behavior or "performance." These constitute the instructional processes, including both general pedagogical quality and more subject-specific quality, implemented in classrooms. They consider that the first two layers underpin teacher performance. In particular, the second layer mediates between disposition and performance. Recently, by drawing on Blömeke et al.'s framework, Schoenfeld's ROGs framework and a sequence of studies on noticing, Kooloos et al. (2021) proposed a model for capturing the teachers' in-the-moment decisions based on two major components, namely, "what goes in their minds" and "what happens around them." The former comprises three core concepts, namely, conceptions, interpretation of thinking, and decision-making. They use the word “conceptions” as a general category containing constructs such as beliefs, knowledge, understanding, preferences, meanings, and views.

As the current study was motivated to obtain information on teachers’ reasoning and decision-making, which are both closely and intricately connected with their ways of interacting with students’ responses, we were inquisitive about the teacher’s expertise in paying attention to students’ responses, interpreting them, and making decisions based on the responses. Considering the accumulated results on teachers’ professional knowledge at work in the act of teaching, in particular in the act of constructing and maintaining interaction, we were also interested in the connection of such teacher knowledge with the teacher’s noticing skills. Therefore, we believed that the idea of viewing teacher competency as a three-layered continuum was useful in pursuing our query.

Therefore, in this paper, we view the teacher’s decision-making as a process in which the teacher pays attention, and wherein he interprets the students’ thinking, supported by the teacher’s content and pedagogical content knowledge, which results in certain decisions in the teacher’s interactive actions (Fig. 1). We analyze the teacher’s ways of paying attention, interpreting, and making decisions, which we call “noticing skills” in this paper; we also capture the resources possessed by teachers, especially their mathematical knowledge for teaching, which is working for such noticing skills. Using the idea of a three-layered continuum, we examine how the teacher’s knowledge connects with the teacher’s noticing skills and interactive actions.

3 Research questions

In this paper we describe how a competent Japanese teacher made decisions during interaction with students, in order to guide their focus towards his lesson objectives. Specifically, the following two research questions were addressed:

- How does the teacher structure the lessons and interactions that enable the students to generate new explanations by shifting their focus from the procedural aspects to quantitative aspects?
- In several interactions that the teacher considers “important,” how does the teacher exercise his noticing skills, along with his mathematical knowledge for teaching, to make a decision on his interactions to guide the shift of students’ focus?

4 Context of the study

4.1 Aim of the project: the Learner’s Perspective Study—primary

In the research described in this paper we utilized data collected from sources as part of the Learner’s Perspective Study—Primary project (LPS-P; Shimizu, 2011). This project originated from the Learner’s Perspective Study (LPS), an international comparative research project led by David Clarke. The Japanese team members designed the LPS-P project in the LPS. The difference between LPS-P and LPS is that LPS-P collected data from primary school teachers.

The LPS project was aimed to complement the survey-style approach, characteristic of the research of Stigler and his co-workers, with a more in-depth approach that accorded a more prominent voice to the perspective of the teacher and learners in complex social and cultural settings (Clarke et al., 2006). To this end, data were collected from mathematics lessons given by competent teachers in each country. The essential point is that the lesson structure reflects the purposeful decision-making of competent teachers who structured their lessons in recognition of the needs of their students as well as their priorities and strengths, and the situation and consequent purpose of the lesson in the instructional sequence (Clarke et al., 2007).

4.2 The teacher

In this study, we analyzed lessons that were taught by Mr. T. When the data were collected, he had more than 20 years of teaching experience and worked at a primary school affiliated with a national university (*fuzoku* school) in Tokyo. As a teacher of a *fuzoku* school, he actively conducted lesson

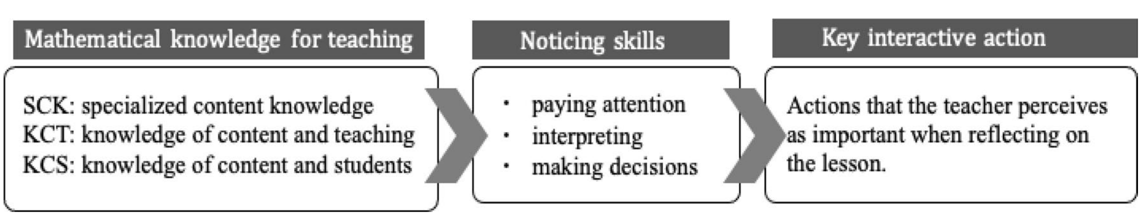


Fig. 1 Analytical model that captures the process of teacher’s decision-making

studies every year. He closely followed the trends in mathematics education research and published a number of papers and books, and was also involved in writing textbooks.

Based on his deep knowledge of the mathematical background of teaching materials, he emphasized the importance of structuring lessons by focusing on students' thinking processes. In order to do this, he stated that before class, he clarifies "what I want the students to say as evidence that they have achieved the goal of today's lesson" and then thinks about "what I can do to make that happen?" (Takahashi, 2021).

4.3 Classroom data and lesson objectives

Classroom data in this paper come from Mr. T's nine consecutive lessons that dealt with comparing fractions (Table 1) in his fifth-grade classroom. The lessons were conducted in November and December, 2011. Thirty-eight students participated in the lessons. The lessons were aligned with the textbook chapter on the addition and subtraction of fractions with different denominators. In the Japanese national Course of Study, fractions are first introduced in Grade 2. Addition and subtraction of fractions with the same denominators are taught in Grades 3 and 4, and unit fractions are also introduced by focusing on the units that make up a number. In Grade 5, students learn how to compare the size of fractions with different denominators and connect this to the addition and subtraction of fractions with different denominators. Mr. T spent many hours comparing fractions because he thought that the most critical part in the chapter was the unit fraction and its transformation into an equivalent fraction.

The overall objective of the lessons was twofold: (a) understanding that fractions can be compared if a common unit fraction is found and (b) understanding reasons for comparing fractions by finding a common denominator or numerator. In the interviews, Mr. T repeatedly emphasized that a key concept to explain equivalent fractions is a unit fraction. He stated that making an equivalent fraction is to "remeasure" (his word) the original fraction by a different unit fraction. For example, 9/12 is constructed by remeasuring the original 3/4 by changing the unit of measure from 1/4 to 1/12. Both numerator and denominator are tripled because each unit fraction is divided into three parts. He said that concepts of the unit fraction and measurement are crucial to understanding fractions as numbers, because once the new unit fraction is found, one can compare fractions and add or subtract them in the same way as whole numbers. These comments show Mr. T's robust SCK for comparing fractions with different denominators.

5 Data collection and analysis

By following the data collection procedure of LPS, data were collected from the lessons using three cameras (for the teacher, the focus students, and the whole class), and interviews with Mr. T and two focus students (for details of the data generation guidelines, see Clarke et al., 2006). We believe that these data are appropriate for our study for at least two reasons. First, in his lessons, Mr. T emphasized discussion among students with clear goals regarding instructional content. He stimulated and extended the students' reasoning and language on the use of unit fractions persistently in the series of lessons. These features provide evidence that Mr. T's case is appropriate for studying the teacher's decision-making in the interactions for the purpose of guiding students' focus toward essential mathematics. Second, Mr. T was interviewed after each lesson. In the interview, he watched videos of the lessons he taught and made detailed comments on his in-the-moment decision-making in the lessons (Clarke et al., 2006). These detailed data offer a valuable source of information, in particular to answer our second research question.

Table 1 Overview of tasks and main activities in nine lessons

Lesson	Task and activity	Lesson	Task and activity
1	Which is larger 2/4L, 3/4L, or 2/3L? Students explained $2/4 < 3/4$ and $2/4 < 2/3$	6	Students applied the same approach to 3/8 and justified the approach to make 6/16 from 3/8
2	For 3/4 and 2/3, a student provided a way of finding a common numerator	7	Students expressed and explained $2/5 = 6/15 = 12/30$ by using figural representations
3	For 3/5 and 2/3, students discussed how to justify a way of finding common numerators	8	Which is larger, 3/4 or 2/3? Students justified the method of finding common denominators
4	Which is larger, 2/5 or 3/8? Students tried to further justify the approach by clarifying the meaning of "$\times 3$" or "$\div 3$" to make 6/15 from 2/5	9	Which is larger, 1/2 or 1/3, and by how much? Students explained their methods and found that the common denominator better clarifies the difference between the two fractions
5	Discussion continued. The term *unit fraction* was introduced by Mr. T to clarify the object of discussion		

Among the data collected, we used videos (taken with the teacher's camera) and transcripts of the lessons and interviews with Mr. T for analysis. In the remainder of this section, we describe how we conducted the analysis in order to answer the two research questions. The analysis, which primarily comprises coding and categorizing the verbal data, was done collaboratively by two of us. In this process, first, we analyzed the data independently and then discussed our results. Whenever there were any discrepancies between our results, we re-examined the data and revised our interpretations until we reached agreement.

Concerning the first research question, we used the framework of "guided focusing pattern" (GFP) (e.g., Funahashi & Hino, 2014; Hino, 2018; Hino & Funahashi, 2021) to clarify the structure of Mr. T's nine lessons. This framework aims at capturing the classroom interaction, wherein students' thinking is placed at the center, along with the teacher's role in balancing the students' thinking with the objectives that the teacher wishes to achieve during the lesson. The four phases that comprise the pattern show the teacher's actions (Table 2; see Funahashi & Hino, 2014, for more details).

These phases were constructed by drawing on the results of studies on the characteristics of Japanese mathematics classrooms and the pedagogy of mathematical thinking and problem-solving in Japan (e.g., Becker & Shimada, 1997; Clarke et al., 2006; Koto et al., 1992; Stigler & Hiebert, 1999). In the analysis, we applied this framework and segmented the nine lessons by using the four phases of GFP.

To acquire the information concerning the students' progress and on where they focused when explaining the larger fractions, we identified and classified the students' explanations made in public in the situation of comparing two fractions (see also Hino, 2019). Thus, we used the data of public utterances together with the written work on the blackboard. This process was further necessary to decide on the lessons for further analysis, based on the changes in students' explanations.

By combining the information of segmentation of the lesson by GFP and of the classification of the students' explanations, we found the moments wherein a shift of focus from the procedural aspects to the quantitative aspects were observed in their explanations. Further, we searched Mr. T's interactive actions in the focusing phase of GFP and categorized them into three salient actions (Hino & Funahashi, 2021). Finally, we decided on the two lessons for the object of closer examination. The results are described in Sect. 6.

To answer the second research question, we used the stimulated-recall interview data with Mr. T, which was conducted after each of the two lessons. In the interview, the interviewer asked Mr. T, "Please fast forward the videotape until you find the sections of the lesson that you think were important. Please play these sections at a normal speed and describe for me what you were doing, thinking and feeling during each of these videotape sequences" (Clarke et al., 2006, p. 33). Mr. T paused the tape at 10 sections in each lesson. Every time he paused the videotape, he made comments regarding his reasons for pausing the tape and his actions, thinking, or feeling during the time.

In analyzing Mr. T's interview transcript with respect to the sections that he considered important, we examined the context of interaction wherein each comment was constructed using the lesson video and transcript. Based on those processes, we decided on 10 interactions in each lesson as the object of analysis and developed our interpretation of them in terms of Mr. T's knowledge and noticing skills in connection with the interaction with the students. In so doing, we generally made references to the lesson transcripts; we sometimes had to look back at the lesson video.

By comparing our interpretations of the content and process of Mr. T's thinking across the interactions in the two lessons, we developed conjectures on his decision-making mechanism behind his interactive actions. From the 20 interactions examined, we selected three (we call '"vignettes") and we present the results in Sect. 7.

6 The teacher's way of structuring the nine lessons

6.1 Students' four explanations on comparing two fractions

We begin by presenting the results of the students' explanations on comparing two fractions. From transcribed records

Table 2 Four phases of the guided focusing pattern

Phase	Brief description
Proposing the problem	Posing the task for the day; sharing approaches for exploring the task
Eliciting students' ideas	Asking a question that may elicit multiple ideas from students; accepting and/or elaborating on students' ideas
Focusing on the object of examination	Focusing on the important ideas that students proposed in the phase of Eliciting students' ideas; if students do not spontaneously produce the idea expected by the teacher, he or she facilitates or leads students' attention to the idea
Formulating the result	Formulating results and/or approaches as generally as possible, not limited to the given problem

of the lessons, the students' claims and explanations of larger fractions were classified into four types (see Table 3). Exp. A presents only the numerical manipulation and calculation, while Exp. C is the one that Mr. T expected; it describes the equivalent fraction in terms of remeasuring the fraction by changing the unit fraction. The other two explanations are considered transitional from Exp. A to C. Exp. B1 presents numerical manipulation, but includes visually backed-up information. Exp. B2 presents quantitative operations on fractions as a quantity, which contains an important shift from numerical manipulation to the quantitative aspect of the fractions. Nevertheless, compared with Exp. C, the idea of remeasurement is not explicit in Exp. B2. We can observe a shift of focus in the students' explanations from a narrative grounding in numerical discourse (Exp. A and B1) to a narrative grounding in more quantitative discourse (Exp. B2 and C).

6.2 Change of students' explanations and Mr. T's interactive actions

Figure 2 shows the placement and duration of the four phases of GFP over the nine lessons. Overall, we can see that the lesson proceeds from the proposing phase to the eliciting, focusing, and formulating phases. Notably, most of the time in the lessons was spent on the focusing phase. Figure 1 further shows the trajectory of four explanations over the lessons. In Lessons 1 and 2 (L1 and L2) at a macro-level, the students' explanations were about numerical manipulations (Exp. A and B1). It was at L3 that a student first proposed the explanation in terms of quantity (Exp. B2). In L4 and L5, they began to pay attention to unit fractions and the beginning of Exp. C appeared in the conversation. However, this Exp. C was not easily obtained by the students. They reverted to earlier explanations in L6 or L8 when a different representation or a different method was the object of discussion. Nevertheless, the students came to see the difference between the explanations and to coordinate them in L9.

We also inquired what interactive actions Mr. T made in the focusing phase in which the change of explanations took place. Due to space constraints, we only summarize three categories of interactions (for more information, see Hino & Funahashi, 2021). The first category is "proposing focus," which captures Mr. T's actions to draw the students' attention toward new ideas, words, or figural representations about unit fractions. Mr. T was sensitive to the students' utterances or figures they drew concerning unit fractions and carefully highlighted them by recording, repeating, underlining, or annotating them. The second is "modifying focus," which captures Mr. T's actions for (re)directing the students' attention to a focus closer to the lesson's objective. A notable feature of this aspect is his continuous leading of "quantitative dialog" (Lobato et al., 2013). For example, Mr. T communicated about the relationship between numerical symbols and their referents through the written figures on the blackboard. Moreover, he repeatedly assisted the students to think by figural representations and addressed precisely how they drew their figures to externalize their thinking about quantities. Another notable feature was Mr. T's frequent questioning of the students' argumentations. He pointed out gaps among the students' opinions or between his questions and the students' responses, which created certain perturbations among the students. The third is "narrowing focus," which captures his actions for making explicit the goal the students were asked to pursue. During the focusing phase, he made the task explicit or reformulated and occasionally reflected on the arguments from various viewpoints.

Based on the analysis, we decided on L3 and L4 for the further analysis of Mr. T's decision-making because it is in those lessons that the students generated Exp. B2 and C.

7 Mr. T's decision-making in context

7.1 Lesson objectives in L3 and L4

In the interview, Mr. T said that his first objective in L3 was to judge the larger fraction correctly, given that the two fractions have different denominators. His second objective was to let the students know that the whole must be the same size when judging the larger one as a fraction. He said, "[I want to let them know that] we cannot tell that the size of a

Table 3 Summary of the students' explanations to the question, "which is larger, a/b or c/d?"

Type	Description
Numerical manipulation (Exp. A)	Explanation presents numerical manipulation and how to calculate equivalent fractions
Numerical manipulation with visual backup information (Exp. B1)	Explanation presents numerical manipulation and claims visually that multiplying the same number to *both* numerator and denominator is indispensable
Operation on fraction as a quantity (Exp. B2)	Explanation shifts to the quantitative aspect of fraction and operation on quantity. It connects numerical manipulation with operation on quantity, even though the attention to re-measurement by changing unit fraction is not explicit
Re-measurement by unit fraction (Exp. C)	Explanation describes that making an equivalent fraction is to re-measure a fraction by changing the unit fraction as the unit of measure

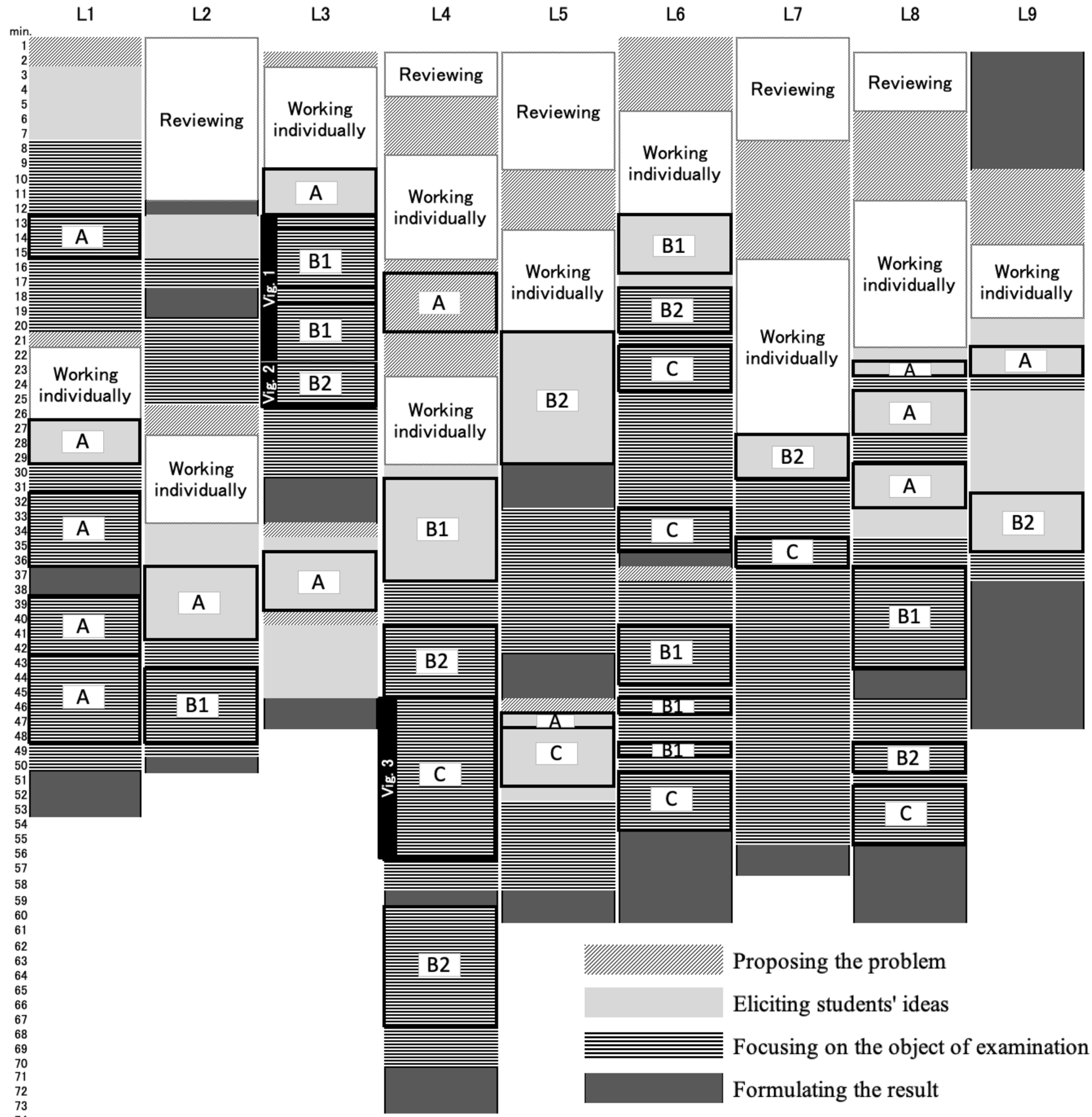


Fig. 2 GFP and children's explanations in the nine lessons

fraction is larger unless '1' remains the same. The fraction must be conceived [of] as a quantity, not as a ratio." For the objective of L4, Mr. T said, "[The goal is] to understand the meaning of constructing equivalent fractions by changing first the numerators. … [To let them know] tripling numerator 2 in 2/5 means to identify a unit fraction 1/15 and to remeasure 2/5 by the unit fraction." In L3 and L4, we can see that he intended the students to understand equivalent fractions quantitatively, that is, having an image of fraction as a quantity and conceiving the equivalent fraction as remeasurement of a fraction by a new unit fraction.

7.2 Analysis of three vignettes

We present three vignettes in which Mr. T carried out critical decision-making as follows: two in L3 and one in L4. Mr. T chose them as "important" and made comments on his thinking or feeling in the teacher interviews. The two vignettes in L3 connect with the generation of Exp. B2, including the first one that prepared the context for the generation. The third vignette in L4 connects with the generation of Exp. C. In each vignette, we first clarify the focused interactive actions by Mr. T from his utterances (the action is underlined in the transcript) and then interpret his noticing skills and knowledge from the data of the teacher interviews.

7.2.1 "But it does not necessarily represent this one, does it?": surfacing discrepancy between two explanations

7.2.1.1 Vignette 1 After proposing the task "which is larger, 3/5 or 2/3?" and asking the students to think about it in their own way for about five minutes, Mr. T nominated Ino to present her thinking (see Fig. 3, left).

> Ino: I found a common numerator. Six is the smallest number that divides both 2 and 3, so I used 6. Since 3 was doubled and became 6, 5 was also doubled. [She continued a similar explanation for 2/3.]

She explained how to calculate the equivalent fraction (Exp. A). Mr. T asked her, "Well, if the numerator is doubled, the denominator needs to be doubled, doesn't it? Why?" Ino responded, "because they are proportional." Mr. T repeated, "Is it because they are proportional?" Here, another student, Ida, took a turn and presented his thinking to clarify Ino's statement by drawing circles of 3/5 and 2/3 (see Fig. 3, right).

> Ida: First let this part 3/5 … and, what it means to make the numerators same is … since [it is] 6/10, so it is this part [for the part 6/10 in the upper right-hand circle in Fig. 3 (right), he drew slashes in red]. But if I only made the numerator … six, and if I didn't double the denominator, it became one and, something like 1/5… Well, if I doubled the upper part (meaning numerator), I must double the lower part (meaning denominator). Otherwise, the fraction becomes a fraction with a different size, doesn't it?

Ida's explanation is classified as Exp. B1 because it presents numerical manipulation that was visually backed-up by figural representation. Mr. T reflected on Ida's figure and the following interaction took place.

> Mr. T: What did you do here? [he meant the upper right-hand circle in Fig. 3 (right).] When you make 6/10 from 3/5, … what did you do first, first of all?
> Child: He doubled…
>
> Mr. T: He doubled, didn't he? He doubled and it became 6. Six, six [he repeated]. How about this one [pointing at Fig. 3 (left)]. Did it first divide evenly by 10? If Ino first divided evenly by 10, then this must be [the first one she did]. No, well what Ida drew [pointing at Fig. 3 (right)] makes sense. It makes sense as it is… <u>but, [pointing at Fig. 3 (left)], it does not necessarily represent this one, does it? (meaning Ino's way of constructing 6/10).</u>

In response to Mr. T's last utterance, Mochi presented her explanation, which again belonged to Exp. B1. As she did not connect it to the figures that had been written on the board, Mr. T requested her and other students to use figural representation to explain their reasoning.

7.2.1.2 Analysis from Mr. T's comment in the interview In the interview, Mr. T said, "When I was circulating among the students during the time of individual activity, I found that Ino was the only student who was changing numerators to the same number, instead of denominators." This comment tells us that Mr. T purposely chose Ino, in order to direct the students towards thinking about the reason for the calculation.

Mr. T paused the video where Ida's circular representation of 6/10 was drawn on the blackboard and said, "This is probably the most important point in today's lesson. He first divided it evenly into ten parts and shaded six of them." Mr. T also said that when Ino was explaining her thinking, he had already worried by saying, "She only explains

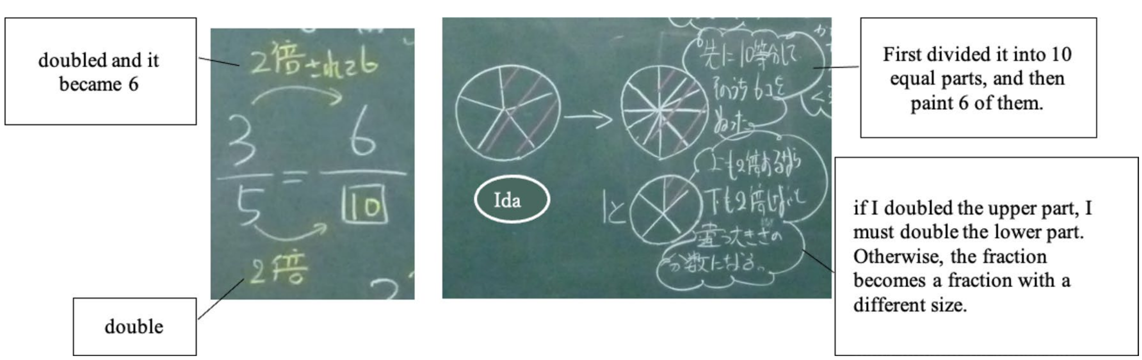


Fig. 3 Ino's idea (left) and Ida's figure (right)

the knowledge …, not a good situation." Moreover, he said, "So, I was thinking, by somehow eliciting figural representation from the students, hopefully, certain distortions or gaps would emerge."

Mr. T's comments revealed that he was dissatisfied with Ino's explanation and waited for certain "distortions or gaps" to emerge. Fortunately, he got Ida's explanation. Then he found an important moment when looking at Ida's way of drawing 6/10. Mr. T observed Ida's order of constructing an equivalent fraction to 3/5, i.e., first dividing a circle into 10 parts and then shading 6 of them, and developed an interpretation that he first calculated 6 and 10 in his head and then drew its representation. According to the second objective of L3, the original quantity (3/5) must remain the same size in equivalent fractions; therefore, the construction of the equivalent fraction should start by drawing 3/5 and carrying on quantitative operations such as dividing each piece further. This was not so in Ida's case. We conjecture that Mr. T connected his interpretation of Ida's thinking with this objective and developed a "goal-oriented interpretation," i.e., Ida's drawing had the potential to direct the students toward a discrepancy. He decided to contrast the orders of constructing 6/10 between Ida's and Ino's representations and pointed out the inconsistency between the two. Importantly, Mr. T referred to his lesson objective when he observed and interpreted Ida's behavior, and decided to create confusion among the students.

To highlight the distortion, Mr. T specifically used figural representation. He could indicate the discrepancy explicitly by way of making equivalent fractions because he attended to Ida's figure. From these observations, we conjecture that Mr. T's knowledge of figural representation was an important resource in his noticing skills. This knowledge of using figures developed by the student would give him a method of exploring students' thinking and prompted him to attend to features of Ida's thinking. We can think of this knowledge as KCT because it gives him a way to teach. Mr. T mentioned in another part of the interview his pedagogical principle of attending to figures developed by students: "By drawing a figure, I would say, I believe that [a] way of thinking will manifest itself in the figure." He also mentioned instructional advantages and disadvantages of different representations to teach the idea of equivalent fractions in particular. He said, "I did not want number line representation [at this moment] …. Since students often express the numerator and the denominator on different number lines, they do not look at both of them as a number. … So, I didn't want it." Here, we note that Mr. T's knowledge of figural representations has a relationship to his KCS. Figure 4 is a summary of his decision-making process.

7.2.2 "Is this OK as a fraction?": talking about new ways of constructing equivalent fractions

7.2.2.1 Vignette 2 Mr. T's request for students to explain their reasoning with figural representations in the last part of vignette 1 was an invitation for the students in the class to give further explanations. It was Naka who responded to the request. He created two circles (see Fig. 5, left):

> Naka: [To make 6/10 from 3/5] First is 6, so, now, here we have three equal parts, the red part is divided into three equal parts, so we make them six equal parts. [In place of Naka, Mr. T divided each of the three red

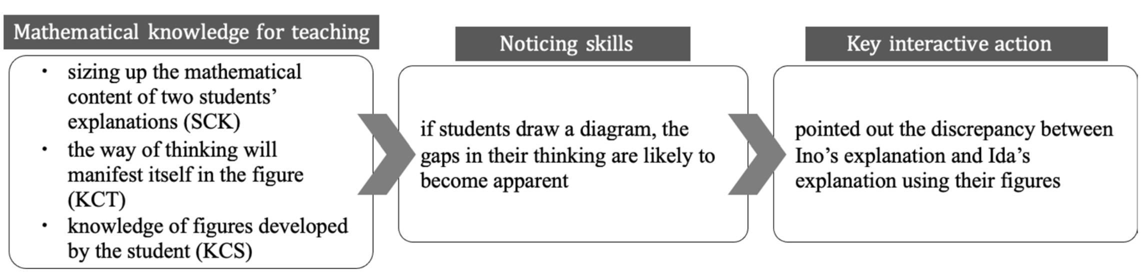
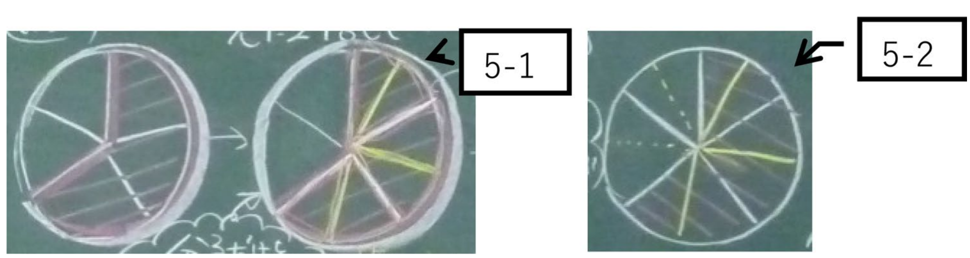


Fig. 4 Mr. T's decision-making process in Vignette 1

Fig. 5 Circles related to Naka's explanation



parts in half (see 5-1 in Fig. 5)] … Well, it became six equal parts. But I think this state (5-1 in Fig. 5) is 6/5.

Then, the following interaction took place:

Mr. T: <u>Is this OK as a fraction?</u>
Naka: No, it isn't.
Mr. T: Why isn't it OK?
A student: Because it is not divided evenly.
Naka: But this state expresses exactly that case (in a loud voice). It came to be 6/5, but we must do the same thing all over, doubling and tripling them, too (referring to 5 and 3 in 3/5 and 2/3.) We draw lines for these parts, too. [He moved his finger straight as if he was dividing each of the two unshaded parts in half.]
Mr. T: [By following Naka's instruction, he wrote 5-2 in Fig. 5.]

7.2.2.2 Analysis from Mr. T's comment in the interview Naka's explanation belongs to Exp. B2. It stated the process of remeasuring the fraction using a new unit fraction, i.e., first to find a new unit fraction and then to remeasure the original fraction by using the unit fraction. The Fig. 5-1in Fig. 5 shows the first step.

In the interview, Mr. T pointed out his question, "Is this OK as a fraction?" as important and commented on this action as follows: "By asking whether this is all right as a fraction, they will draw additional lines for the two 1/5, and as result, 10 [denominators] will come out … in brief, area, [as for] the part of the area we want to compare (meaning the part representing the numerator of 3/5), umm… the number, by changing the number of pieces from 3 to 6, we are remeasuring the '1' or a whole by using a new piece. I thought [this question] would connect to [such a line of reasoning]." This intricate comment shows Mr. T's close reasoning behind his action. Mr. T perceived and interpreted Naka's explanation by drawing on the second objective of L3. He developed a "goal-oriented" interpretation, namely, that the figure (5-1 in Fig. 5-1) can be a promising figure that will drive the students' attention to finding a new unit fraction to be used to remeasure the fraction 3/5. Then he decided to ask the question "Is this OK as a fraction?" Thus, Mr. T referred to his lesson objective in exercising his noticing skills. We can also note that Mr. T's knowledge of figural representation was used again to analyze how Naka constructed equivalent fractions, including the order of drawing and his effort to keep the whole unchanged.

Mr. T went on to say, "Here, I hesitated what to ask. Well, students are likely to recognize that a fraction is equal to partitive division, so I asked (the question) in this way. Then I got a response from the students, 'it is not divided evenly.' I felt like saying, 'thank you.' What would I do if I didn't get this response? Well, maybe I let them draw the circular representation one more time." This part of the utterance reveals that Mr. T was actually puzzled by what he ought to ask. He was carefully attentive to the students' responses to his question in order to make further decisions. Receiving a positive response from a student, he let Naka continue and complete his explanation. It is also noteworthy that Mr. T had the idea of an alternative teaching action in the event that he did not receive the expected response. These comments reveal that Mr. T's KCS (common student conception of a fraction as a partitive division) and KCT (different instructional viable models) were working behind his decision-making processes. Figure 6 shows a summary of his decision-making process.

7.2.3 "What do you mean when you say that the size of a whole doesn't change?": stimulating verbalization of unit fractions

7.2.3.1 Vignette 3 The discussion explaining the construction of equivalent fractions continued in L4. The task in L4 was, "Which is larger, 2/5L or 3/8L?" Mr. T emphasized using figures so that everyone in the class could make sense of the process of construction. They concentrated on 2/5 and explained their reasoning why the denominator 5 must be tripled once the numerator 2 is tripled.

Initially, the students presented explanations in the type of Exp. A or Exp. B1. When a student asked about the connection between the drawing and the calculation in Exp. B1, several students attempted to explain how to make 6/15 from 2/5 by using a circular representation similar to the one

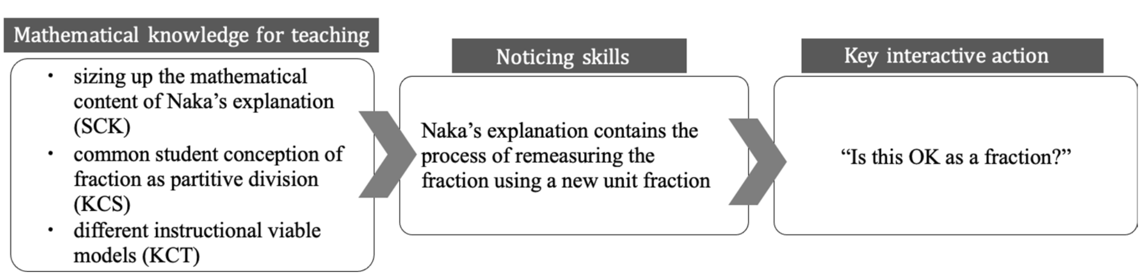


Fig. 6 Mr. T's decision-making process in Vignette 2

shown in Fig. 5. Among them, Ida raised his hand, went to the front of the classroom, and began to explain his reasoning. When Ida said, "Although the size of a whole doesn't change, (the) numerator becomes one-third…," the following interaction took place:

Mr. T: <u>What? What do you mean when you say that the size of a whole doesn't change?</u>
Mr. T: The two-fifths [2/5] don't change. What I mean by 'a whole' is this. A whole is…
Ida: The numerator, because 1/5 becomes, well, one-third of it… It means that 1/3 [sic] is divided evenly into three. Since the original 2/5 doesn't change, … let's stop the numerator, in the same way…
Mr. T: Wait a minute, wait a minute. … Numerator means [pointing at the part 2/5 (7-1 in Fig. 7)] …, what does this numerator mean in this figure?
At this point, Naka participated in the conversation. The transcript below details part of the interaction between Naka and the teacher.
Naka: The size of a whole (meaning 2/5) is not changing, but the size of one numerator is from 1/5 to …,
Mr. T: Well, you say this is the numerator, or it is a 'piece.' Yes, this is…
Naka: Well, it was, well, 1/5 was evenly divided by 3. [Mr. T shaded the part 1/5 in red (7-1 in Fig. 7).] … well, 1/5 became 1/15…

Mr. T: Yes, but you are talking about this, aren't you? [Mr. T drew arrows (7-2 in Fig. 7).]
Naka: 1/5, oh, one numerator … [Mr. T repeated 'one-fifth'] was divided evenly by 3.
Mr. T: Can we call this a numerator?
Naka: One numerator.
Mr. T: One numerator. [For the class,] Do you understand? Now you know? Do you understand what he is talking about?
A student: It is not a numerator.
Mr. T: It is not a numerator. What he said was this, one numerator…
Naka: [Is it] moto (meaning 'basis' in Japanese)?
Mr. T: 1/5 became 1/15. Did you say moto?
Naka: Yeah, moto, …, well, the left part, …, the 1/5… [Mr. T repeated 'moto.']

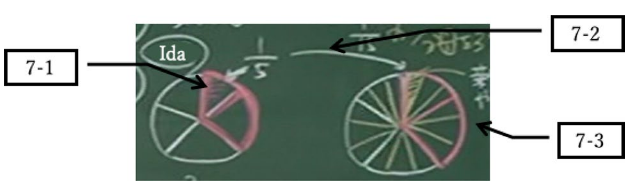


Fig. 7 Two circles by Ida

Naka:… I mean there are two 1/5s. Well, I think 1/5 is the moto [of 2/5] … There are two 1/5s … And, for the new one … [Mr. T pointed to the part 6/15 (7-3 in Fig. 7)], 1/15 is the moto, … there are six of them (meaning moto).

7.2.3.2 Analysis of Mr. T's comment in the interview In the interview, Mr. T chose the moment of asking Ida, "What do you mean when you say that the size of a whole doesn't change?" as important. He said, "Here, I pretended not to know. I wanted to let them pay attention to the fact that the size of a whole remains the same. … then I wanted to draw attention to the fact that the number of unit fractions has changed…" From this comment, we knew that Mr. T attended to Ida's utterance and interpreted that Ida thought a whole was the same in equivalent fractions. He developed a "goal-oriented" interpretation so that he could utilize what Ida would say to attract the students' attention to the constancy of the size of a whole, which becomes the object to be remeasured. To do so, he decided to ask the question in order to make explicit Ida's thinking about the constancy of the size of 2/5.

However, Ida did not give an expected answer. Rather, Ida tried to quit his explanation of constancy of size of 2/5. He hastened to the part of procedure without making explicit the meaning of "unit fraction" or "remeasurement." Mr. T said, "Ida said '1/5 becomes one-third of it.' It means 1/15. So, I thought that he was saying that the unit fraction had changed. I thought he would go [to the part of remeasurement] in one breath, but he didn't." Mr. T was puzzled at Ida's quick pace and interrupted to clarify the connection between language and figure. The quantitative dialog continued after Naka joined the conversation. Naka did not mention unit fractions explicitly either. While waiting for the students to make statements related to the unit fractions, Mr. T kept modifying his goal-directed interpretations and decisions by listening to the reactions from the students.

Consequently, Naka chose the word "moto" in Japanese. This word is in everyday use, which connotes origin, cause, or basis. Even though Naka used everyday language, he insisted on the unit fraction being a unit of measurement. Finally, Naka described 2/5 and 6/15 in terms of "moto," which shows his creation of an equivalent fraction by remeasurement using a new unit fraction (Exp. C). This pivotal verbalization was made possible in the quantitative dialog led by Mr. T that pressed Naka to convey the distinction between the fraction as a whole and the unit fraction.

Concerning Mr. T's knowledge, it is worthwhile that all these conversations were made possible by the figures developed by Ida and Naka. It was the teacher's knowledge of the figural representation that made it happen. Mr. T's way of leading and dealing with the word "moto" shows another instance of KCT with respect to language and metaphors

(Ball et al., 2008, p. 402). He said that "changing all these vague expressions such as 'moto' or 'new one' to sound expressions" is what he valued. Similar comments were referred to by Mr. T several times during the interviews. From these observations, we can see his pedagogical principle at work, namely, "it is imperative to share the process of refining the students' vague expressions." It seems that this pedagogical principle of teaching underpins and gives a grounding to his knowledge. Figure 8 is a summary of his decision-making process.

8 Discussion and conclusion

The purpose of this study was to investigate a teacher's decision-making during the interaction with students in order to guide their focus towards the lesson's objectives the teacher had set forth. We first viewed the nine lessons from the GFP perspective and unpacked a rich and intense deployment of the eliciting and focusing phases in Mr. T's teaching. We recognized his salient interactive actions of proposing, modifying, and narrowing the students' foci of attention to the quantitative relationships and unit fractions. Mr. T's continuous leading of "quantitative dialog" (Lobato et al., 2013) and frequent questioning of the students' argumentation and monitoring of classroom disagreement (Chazan & Ball, 1999) were some of the notable features.

Our analysis of the three vignettes contributes to disclosing the complexity of teacher's in-the-moment decision-making. In identifying the vignettes, we were challenged to find the critical moments of interaction, in the sense that such moments directly connected to the students' shift of focus from the procedural aspects to more quantitative aspects when comparing fractions.

Our analytical model (Fig. 1) gave us several insights into how a competent mathematics teacher's knowledge connects with his noticing skills and interactive actions. First, we found out that the teacher was attending to and interpreting the students' articulated mathematical thinking by consistently referring to the lesson objectives. In the teacher interview on his important moments, Mr. T talked about the gaps between his expectation and the reality of the students' thinking, and about his intentions to make the next discursive move towards the objectives. For him, the lesson objectives were regarded as pivotal in moving on the interaction.

Compared with other studies on deconstructing the teacher's decision-making, these results are compelling with respect to the roles of goals set by the teacher. In Schoenfeld's (2011) framework of ROGs, the element of goals was clearly present, and researchers using this framework explained the mechanism by identifying several goals and orientations and describing how the teacher made decisions to resolve the conflict between the competing goals (e.g., Thomas & Yoon, 2014). However, Mr. T's methodology was more suitably explained by the mechanism of constant reference to the specific lesson objectives and developing and adjusting his decisions by listening carefully to the students. Although we cannot easily generalize the results, it reminds us of the feature of an exemplary Japanese problem-solving lesson, in which the teacher's intention of positioning the particular lesson within the entire teaching plan is crucial (Funahashi & Hino, 2014; Shimizu, 2009). We can conclude that such intentions are actually working even in the midst of interaction with students.

The second notable result of Mr. T's noticing skills was his constant modification or adjustment of his initial decisions tailored to the students' reactions. Mr. T's interactive action often took the form of a question to the students. When the students did not respond as he had expected, Mr. T devised a new way of wording or changed the content of the question in light of the lesson objectives. In other words, Mr. T continued to interpret carefully what the students said and wrote, and then made decisions about what subsequent teaching actions to take.

These results provide a perspective on critical discursive moves that have been identified in previous studies. Lobato et al. (2013) identified highlighting, renaming, and quantitative dialog as significant discursive moves. They illustrated that highlighting, for instance, contributes to both the directing of students' attention towards and distracting students' attention from the intended objects. They also pointed out that teachers should be aware of the importance of subtle and fine-grained discursive moves. We build on their findings by proposing that teachers should also have clear objectives for

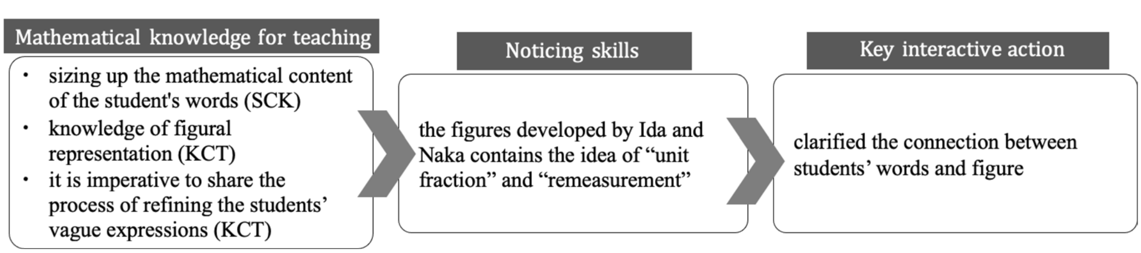


Fig. 8 Mr. T's decision-making process in Vignette 3

the lessons in order to facilitate subtle and fine-grained discursive moves to direct the students' attention to the intended mathematical content. One of the characteristic actions that constituted discursive moves in Mr. T's practice was pointing out or annotating specific information by writing on the blackboard. In Japan, these teaching acts of board writing are called "bansho" (Shimizu et al., 2021), which is said to function as a culturally valued artifact (Stigler & Hiebert, 1999). Mr. T used it purposefully not only to externalize the students' pathways of reasoning but also to involve different students in class in analyzing and refining their reasoning.

Furthermore, while the importance of goals in the teachers' decision-making has been recognized, it is a challenge for teachers to make goals conscious and take full advantage of the opportunities to build on the student's thinking (Kooloos et al., 2021). Kooloos et al. provided explanations why their teachers resorted to limited interpretation of student thinking. In their explanations from the teachers' conceptions, they claimed that having a clear mathematical goal requires the teachers to think how to enlarge the students' ideas toward the goal. Mr. T's practice provides a concrete picture that the key to this progress is how teachers find their expectations relevant to the lesson's objectives in the students' articulated mathematical thinking, and how they update such expectations by attempting different discursive moves with the students.

Our third result concerns the teacher's mathematical knowledge essential for teaching. Expert mathematics teachers have been found to possess a solid footing in content and pedagogical content knowledge and use both to teach flexibly and readily in response to students' thinking (e.g., Yang et al., 2020). As pointed out above, having explicit mathematical goals requires not only knowledge of mathematics, but also knowledge of how students understand the mathematics (Kooloos et al., 2021). It was obvious that Mr. T possessed substantial SCK. In the analysis of the vignettes, detailed comments by Mr. T on his decision-making processes indicated that he also held a rich knowledge of figural representations. We interpreted this knowledge as KCT because it is connected to pedagogical principles for mathematics teaching. Another KCT on using language and metaphors also functioned as he turned his attention to a student's use of everyday words. His consistent use and recognition of the value of figural representations indicates that Mr. T's KCT in utilizing students' expressions as a pedagogical principle seems to anchor his knowledge for teaching. It enabled him to make flexible and meaningful interactions, which may provide a reason for the concept of a "solid footing." Given that researchers began to pay greater attention to subject-specific facets of teacher knowledge and skills in the operationalization of instructional quality (Blömeke et al., 2020), Mr. T showcases that competent teachers possess particular knowledge that serves as an anchor for regulating their noticing skills. Interestingly, for Mr. T it is the knowledge of students' expressions that have been highly valued in the Japanese problem-solving approach (e.g., Koto et al., 1992).

We found an "inquiry stance" (Kooloos et al., 2021) toward the mathematics and students' thinking in Mr. T's interaction with the students. His valuing of the students' different, even unique, opinions as well as sharable explanations among the class members may reflect pedagogical values of the Japanese problem-solving lessons (e.g., Stigler & Hiebert, 1999) that he incorporated into his teaching. Needless to say, in reality, linking students' thinking with the objectives of the lesson is not straightforward. Exploring Mr. T's in-the-moment decision-making revealed that even a competent teacher struggled to interpret students' thinking and coordinate varied ideas. Nevertheless, it was also revealed that the teacher valued such a venture highly, backed by his knowledge and developing resources to control his decision-making. We think that a worthwhile future task will be to explore what teachers puzzle over and what they have enthusiasm for when exercising their interactive actions.

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