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ORIGINAL PAPER

Theoretical approaches to teachers' lesson designs involving the adaptation of mathematics textbooks: two cases from *kyouzai kenkyuu* in Japan

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Abstract
In this paper, we explore theoretical approaches to Japanese teachers' lesson designs involving the adaptation of mathematics textbooks for instructional change. In Japan, although most teachers use textbooks as the main resource for lesson planning, called *kyouzai kenkyuu,* the ways in which they use textbooks can differ. In a *kyouzai kenkyuu* practice, Japanese teachers sometimes attempt to design innovative tasks and/or learning trajectories beyond or parallel to the textbooks and curriculum guidelines. Such practice is often invisible to international researchers, since it is culturally situated in the local context. To make it more visible, in this study we address and advance theoretical approaches using two frameworks which are well-established for research in mathematics education, namely, Anthropological Theory of the Didactic (ATD) and Documentational Approach to Didactics (DAD). Two cases of lesson design regarding fractions using the same textbook chapter for 3rd graders in primary schools in Japan, are used as examples to illustrate crucial elements of each framework and ways of combining and coordinating both frameworks. As a result, we conceptualise teachers' *kyouzai kenkyuu* practice using two approaches: from a praxeological analysis based on ATD, the difference in teachers' mathematical and didactic knowledge was emphasised, while within DAD, the difference in the instrumentalisation process was emphasised. This analysis implies that the two theoretical approaches can be compatible, and used to gain deeper insight into the relationship between lesson design using the textbook and teacher knowledge.

Keywords Anthropological theory of the didactic · Documentational approach to didactics · Instructional change · *Kyouzai kenkyuu* · Lesson design · Textbooks

1 Introduction

In recent years, the study of mathematics textbooks has focused more on their design and usage rather than on the analysis of the textbook itself (Rezat et al., 2018). Many earlier studies investigated how teachers used mathematics textbooks (as well as other curriculum resources) to design and promote their classroom practices (e.g., Haggarty & Pepin, 2002; Pepin, 2018; Pepin et al., 2017; Remillard et al., 2009; Rezat, 2012). In textbook research, it is important to recognise textbooks as 'potentially implemented curriculums' (Valverde et al., 2002), which implies that textbooks are vital curriculum resources for classroom teaching, but the same textbook will not necessarily result in the same classroom practice. Thus, although textbooks can be used as agents of instructional change, it is important for researchers to understand the different conditions in which teachers' work in order to realise change in the classroom. In this paper, we consider 'teacher design' using textbooks as key to the implementation of change in an enacted curriculum. Although the implementation of change ranges from daily educational improvement to more drastic instructional innovation for realising curriculum reform, we address instructional change, which refers to change in teachers' ordinary practices that enriches their instruction by transforming the textbooks they use. This is related to teachers' design capacity in utilising curriculum materials or creating new materials (Brown, 2009). Since the notion of 'teacher design' can be characterised differently in international contexts (Pepin et al., 2017), it is important to take into account

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local contexts that may affect teachers' designs and textbook usage in a given country (Rezat et al., 2018).

In the present study we focused on Japanese teachers' design activities in relation to instructional improvement. We first describe an overview of the Japanese context related to teachers' use of textbooks and then address international literature reviews. We present two theoretical frameworks in mathematics education that are suitable for approaching such a complex phenomenon, illustrated by two local examples. Thus, in this paper we aim to explore theoretical approaches to Japanese teachers' lesson designs involving the adaptation of mathematics textbooks for instructional change.

2 Teachers' use of textbooks in Japan

In this section we briefly describe the local context regarding textbooks, professional development, and instructional change in Japan.

2.1 Textbook as a primary resource

In Japan, most teachers use textbooks as the primary recourse for teaching, for two key reasons. One is related to the educational system in Japan, in that all public school teachers have an obligation to use textbooks approved by the Japanese Ministry of Education (MEXT) according to the national curriculum (called the *Course of Study*). For primary schools, six different mathematics textbook series are currently approved for publication, and textbook companies are permitted to publish supplementary resources for textbook series, such as teachers' editions and teaching guides. Textbooks are selected by local education boards of cities or regions, not by each school or teacher.[1] According to an international survey (Japan Textbook Research Center, 2012), the Japanese mathematics textbook series used in both primary and secondary schools is relatively *thin* (a small number of pages), compared with textbooks in other countries. This is because it is assumed that almost all pages in Japanese textbooks are taught in classroom lessons, and the textbooks influence teachers' practice and expertise.

The second reason concerns the cultural context of Japanese teachers' professional development. The term '*kyouzai kenkyuu*',[2] meaning the "[s]tudy of topic, curriculum, learning, learning progression and related teaching materials" (Bahn, 2018, p. 167), is relevant. Fujii (2014) explained that "[Japanese] teachers consider *kyozaikenkyu* [*kyouzai kenkyuu*] inherent in a teacher's life so they are actively involved in this endeavour in the hope of improving their level of teaching" (p. 7). In this sense, *kyouzai kenkyuu* is considered a cultural phenomenon (Corey & Ninomiya, 2019). For most Japanese teachers, *kyouzai kenkyuu* and textbooks are closely related, because textbooks can be seen as a collection of *kyouzai* (Watanabe et al., 2008, p. 134). In designing mathematics lessons, Watanabe et al. (2008) explained that "studying textbooks is an important part of the *kyozaikenkyu* process". Furthermore, different usages of textbooks for *kyouzai kenkyuu* have been discussed. Takahashi (2011) mentioned the distinction between 'teaching the textbook (*kyoukasyo wo oshieru*)' and 'using the textbook to teach [mathematics] (*kyoukasyo de oshieru*),[3] which is well-known in the Japanese teacher community. Although there is only one particle difference ('*wo*' and '*de*'), the meanings are very different. According to Takahashi (2011), to teach textbooks, teachers need only to tell students what is described in the textbook, but when using textbooks to teach mathematics, teachers need to draw on the textbooks for designing lessons aimed at developing students' understanding though *kyouzai kenkyuu* (Ishii, 2020).

2.2 '*Kyouzai kenkyuu*' and instructional change

Kyouzai kenkyuu plays an essential role in Japanese lesson study as well as in daily teaching practice (e.g., Doig et al., 2011; Fujii, 2016, 2018; Takahashi & McDougal, 2016; Watanabe et al., 2008). Generally, lesson study is an opportunity for professional development and educational improvement (Fernandez & Yoshida, 2004; Stigler & Hiebert, 1999), and comprises the following five phases: (1) *goal setting*, (2) *lesson planning*, (3) *research lesson*, (4) *post-lesson discussion*, and (5) *reflection* (Fujii, 2018). *Kyouzai kenkyuu* can be conducted in the lesson planning phase, which involves task design to ensure an innovative lesson. Although Japanese mathematics textbooks are strongly regulated by the Course of Study, *kyouzai kenkyuu* may provide an opportunity for teachers to gain an understanding of the background and intent of textbook content, and to develop their didactic and mathematical knowledge in order to enrich their teaching in the enacted curriculum.

The word *kyouzai* (教材) literally translates into instructional materials, however, *kyouzai* has a meaning that is more concrete than subject matter content and can be represented as "tasks or problems (with their contexts) and instructional

[1] There are some exceptions: For 'private schools' or 'university schools,' textbooks can be selected by each school.

[2] The word *kyouzai* (教材) literally means instructional materials or curriculum materials, and *kenkyu* (研究) refers to research or study. To clarify the Japanese keywords in texts, we use an italic font *kyouzai kenkyuu* throughout the paper. Since different spellings have been used to transcribe this word in the existing literature, we did not change the original spelling used in citations: e.g. *kyozaikenkyu*.

[3] '*Kyoukasyo* (教科書)' refers to 'textbook', and '*oshieru* (教える)' means 'teaching' or 'to teach'.

Table 1 A search on *Web of Science* database in December 2020

Set	Topic	Field	Result (ZDM)
#1	lesson study AND mathematic*	All	142
#2		ME journals	42 (18)
#3	#1 AND textbook	All	0
#4	textbook AND mathematic* AND teacher	ALL	404
#5		ME journals	133 (13)
#6	#4 AND lesson	All	66
#7		ME journals	30 (4)
#8	#1 AND planning	All	49
#9		ME journals	15 (6)
#10	#4 AND planning	All	22
#11		ME journals	6 (0)

tools" (Watanabe et al., 2008, p. 133), which implies learning goals. In the Japanese context, *kyouzai* is often considered as a component of the so-called 'didactic triangle' (teacher-student-*kyouzai*) in a classroom setting.[4] Through *kyouzai kenkyuu*, a teacher designs a *kyouzai* that enables students to interact with and achieve a learning goal in the lesson. According to Watanabe et al. (2008), there are two types of *kouzai kenkyuu*: one is to understand if a particular pre-developed *kyouzai* is suitable for a teacher's classroom; the other is to develop a *kyouzai* with an emphasis on an in-depth investigation of the subject matter. Although both are important for professional development, we especially consider the latter kind of *kyouzai kenkyuu* as necessary to implement instructional change, by which teachers *adopt* or *adapt* mathematical and pedagogical aspects of textbooks to enrich their lesson designs and improve their ordinary practices. Fujii (2018) mentions that many aspects of *kyouzai kenkyuu* remain under-represented in international contexts:

> Although *kyozaikenkyu* is recognized as a critical practice of lesson study by Japanese educators, teachers outside Japan often neglect it. This may be because the effort involved is almost invisible, the same way that 90% of an iceberg is hidden underwater. (p. 16)

Although this comment seems metaphorical, it is challenging to reveal the implicit but important aspects of *kyouzai kenkyuu* in the international research context. Additionally, most research papers on lesson study, written by Japanese researchers and teachers comprise reports of descriptive or intervention research, but more theoretical descriptions are required (Winsløw et al., 2018). Accordingly, we conceptualise crucial aspects of teacher design by utilizing relevant theoretical frames.

3 Literature review

In this section, we explore a short bibliography based on the database, and discuss the kinds of theoretical frames used in the related studies.

3.1 A short bibliography

Table 1 shows results of a search on the database *Web of Science* [WoS] (https://www.webofknowledge.com) with advanced search options to ensure search terms in the title, abstract, and keywords. Each line ('set') indicates the searched 'topics' (or combination of topics), 'fields' (all journals/mathematics education journals/ZDM), and 'results' (number of papers).[5] We selected the topics related to our study using two search strands: 'lesson study' and 'textbook'. As no intersected results were found (see set #3), we searched additional topics (mathematic/teacher/lesson/planning). As a result, we identified 20 related papers in set #9 and #11 for review (one paper was removed from set #11 because of its irrelevance to our study). Although we could have reviewed more papers in the 'textbook' strand, set #7, we focused on papers related to 'planning' across the two strands.

3.2 Different theoretical frameworks

We reviewed the related papers to understand the diversity of the theoretical frameworks used, not only the number but also the different types and levels of theoretical frameworks. For example, we distinguished theoretical frameworks *within*

[4] In this sense, the notion of *kyouzai* has much in common with the notion of *milieu* in the theory of didactical situation (Brousseau, 1997).

[5] In Table 1, 'mathematic*' in the topic column denotes 'mathematics', 'mathematical', 'mathematician', and so on. 'All' journals include 'article' and 'early access' paper types. 'ME journals' refer to only mathematics education journals in the database (numbers in the brackets denote the number of ZDM papers).

Table 2 Examples of theoretical frameworks at different levels

Levels of frameworks	Examples	References
Local	FOCUS framework (Choy, 2016)	Choy (2016)
	Pedagogical design capacity (Brown, 2009)	Lim et al. (2018)
Intermediate	Mathematical knowledge for teaching (MKT) (Ball et al., 2008)	Appelgate et al. (2020), Leavy (2015)
	Technological pedagogical content knowledge (TPACK) (Koehler & Mishra, 2009)	Joubert et al. (2020)
	Interconnected model of teacher growth (Clarke & Hollingsworth, 2002)	Widjaja et al. (2017)
Grand	Anthropological theory of the didactic (ATD) (Chevallard, 2019)	Clivaz and Miyakawa (2020), Miyakawa and Winsløw (2019)
	Cultural-historical activity theory (CHAT) (Engeström, 2001)	Wake et al. (2016)

and *of* lesson study (Winsløw et al., 2018). This is also the case in other research contexts of teachers' practice (e.g., teachers' use of textbooks). Concerning the former, *learning trajectory* (Confrey, 2012; Simon, 1995) for instance, is described as a theoretical framework in some papers (Huang et al., 2016, 2019; Suh & Seshaiyer, 2015) indicating a particular process (mainly that of a student) involved in the context of lesson study. Concerning the latter, we identified three different levels of theoretical frameworks adapted from Kieran et al. (2015)[6]:

- *Local theoretical frameworks*: frames that specify a particular feature of teachers' knowledge or activities (e.g., teacher noticing, teacher design);
- *Intermediate theoretical frameworks*: frames that do not specify any particular feature, but rather a general aspect of teachers' knowledge or activities, (e.g., knowledge for teaching, professional growth);
- *Grand theoretical frameworks*: well-established frames for research inside and outside mathematics education that have been developed in a broader context of human activity (e.g., ATD, CHAT).

We identified nine papers in which certain theoretical frameworks were explicitly used. In the other eleven papers, although theoretical frameworks were mentioned in the literature review, these were not used to substantiate the study. For example, Corey et al.'s (2020) approach to teachers' instructional knowledge of lesson planning is similar (but not identical) to the approach used in our study. Although their analyses was based on rich empirical data about lesson plans and contains a particular model they used to analyse their data, their study pays scant attention to developing theoretical approaches within well-established frameworks beyond the literature review. It remains important for researchers to develop and elaborate theoretical aspects in this research area for the purpose of searching for common ground. Table 2 summarises the levels of theoretical frameworks used in these papers, with examples.

Based on our reviews of the identified papers in Table 2, we found that most studies used relevant frameworks to analyse teacher knowledge or capacities, but some also focused on the social and cultural aspects of teacher practice (e.g., Clivaz & Miyakawa, 2020; Miyakawa & Winsløw, 2019; Wake et al., 2016; Widjaja et al., 2017). We also identified studies which refered to textbooks (e.g., Lim et al., 2018; Miyakawa & Winsløw, 2019), but did not have teachers' adaptations of textbooks as a main focus.

On the other hand, as discussed by Prediger et al. (2008), the diversity of theoretical frameworks may cause challenges for research practice in mathematics education. A research methodology, such as networking theoretical approaches, can deal with such a diverse research situation by establishing relationships between parts of theoretical frameworks (Prediger et al., 2008). Our study attempts to advance theoretical approaches to research on lesson design and teachers' use of textbooks by considering different theoretical frameworks.

Following the abovementioned reviews in the Japanese and international context, we considered the following research question: In what ways can a teacher's lesson design adapting textbooks in *kyouzai kenkyuu* be conceptualized?

To address the question, we present two theoretical frameworks in mathematics education, namely, the Documentational Approaches to Didactics (DAD) (Gueudet & Trouche, 2009; Pepin et al., 2013; Trouche et al., 2020) and the Anthropological Theory of the Didactic (ATD) (Bosch et al., 2019; Chevallard, 2019; Chevallard & Bosch, 2020). We exemplify these theoretical approaches using two cases of *kyouzai kenkyuu*.

[6] For Kieran et al. (2015), three levels of theoretical frames (domain-specific, intermediate, and grand) are distinguished to argue for research on task design. We redefined each category for the present study.

4 Theoretical framing

4.1 Documentational Approach to Didactics

DAD[7] is considered an intermediate-level frame, which allows us to understand the teacher-resource interaction in a focused way and has been recognised as a suitable theoretical approach in the field of mathematics textbook research (Rezat et al., 2018).

DAD focuses on *teacher documentation work* in which teachers interact with curricula, textbooks, and other resources in their daily practice (Gueudet & Trouche, 2009). One of the main theoretical sources of DAD is the *instrumental approach* (e.g., Artigue, 2002; Vérillon & Rabardel, 1995). In DAD, a *document* is defined as the outcome of joint resources combined with knowledge guiding usage, and the process of developing the document is called *documentational genesis* (Gueudet & Trouche, 2012; Pepin et al., 2013). Trouche et al. (2020) describe two processes from which the genesis emerges:

> [T]he documentational approach is particularly pertinent to viewing the 'use' of resources as an interactive and potentially transformative process. This process works both ways: the affordances of the resource/s influence teacher practice (the *instrumentation* process), and the teachers' dispositions and knowledge guide the choices and transformation processes between different resources (i.e. the *instrumentalisation* process). (p. 239)

For our study, *instrumentation* is a process in which a teacher's lesson planning is affected by the designs, sequences, and tasks in textbooks, while *instrumentalisation* is a process in which a teacher's *kyouzai kenkyuu* brings different adaptations of the textbook to design a lesson. From a DAD perspective, a lesson plan produced by teachers to realise what they have learned through *kyouzai kenkyuu*, can be considered a 'document', and the process of developing the lesson plan undergoes documentational genesis (Corey et al., 2020). Therefore, we can understand the implementation of instructional change as an outcome of documentational genesis, such as effective mathematical tasks and design principles, different pedagogical approaches to learning, and alternative learning trajectories for students (cf. Pepin et al., 2019).

In this way, DAD provides a tool to characterise both process and product of teacher–textbook interaction in lesson design (see Fig. 1). Figure 1 is reconstructed from the work of Trouche et al. (2020) for particular cases, using

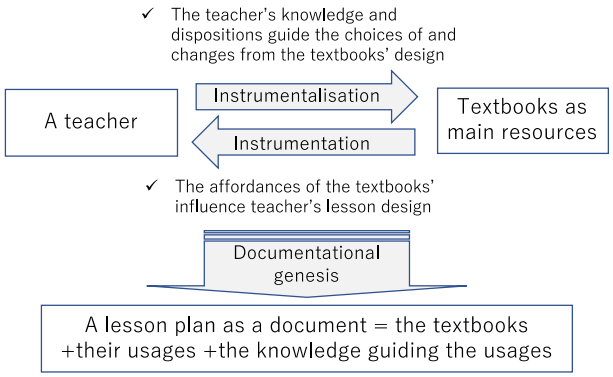


Fig. 1 Teacher documentation work on lesson design, adapted from Trouche et al. (2020, p. 240)

textbooks as their main resources, although DAD considers other analogue and digital resources. To illustrate aspects of Fig. 1, we analyse and interpret a teacher's lesson plan as a result of documentational genesis, focusing on the teacher's documentational work in terms of both instrumentation and instrumentalisation.

4.2 Anthropological Theory of the Didactic

ATD is regarded as a grand-level frame which theorises knowledge and practice in a given *institution*. According to ATD, knowledge exists in a certain institution in which conditions and constraints may shape the specificity of that knowledge. In this framework, curricula or textbooks are produced from a *didactic transposition* (Bosch & Gascón, 2006; Chevallard, 1985/1991), and textbooks are regarded as an empirical source of the 'knowledge to be taught' in a specific educational system (Kang & Kilpatrick, 1992). From this perspective, the instructional change using textbooks especially undergoes a transposition process from 'knowledge to be taught' to 'taught knowledge' in a given didactic system (e.g., a classroom), although other transposition processes can also be related to the implementation of change in various aspects (Fig. 2).

In this study, to conceive teachers' practice and knowledge in the transposition process, we consider the notion of *praxeology*. In ATD, every human activity is described as praxeology which comprises two blocks: praxis and logos (Chevallard, 2019; Chevallard & Bosch, 2020). Each block has two components: *task type* ($\mathbf{T}$) and *technique* ($\boldsymbol{\tau}$) for the praxis block ('know-how'), and *technology* ($\boldsymbol{\theta}$) and *theory* ($\boldsymbol{\Theta}$) for the logos block ('knowledge'). *Task type* indicates the problems of a given task, while *technique* is a way of performing the task. *Technology* is a form of explaining and justifying the technique, while *theory* is employed to explain or justify the technology. Although ATD is a grand theoretical framework, the concept of praxeology is applicable

[7] General information on DAD is available at the following link: https://hal.archives-ouvertes.fr/DAD-MULTILINGUAL/

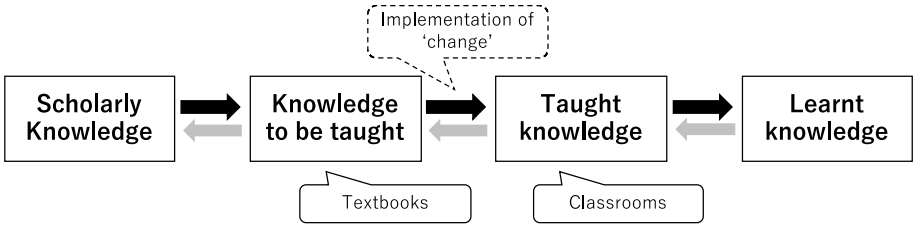


Fig. 2 Didactic transposition process and textbooks, adapted from Bosch and Gascón (2006, p. 56)

to a wide range of phenomena. Praxeological analysis does not necessarily offer all of the four elements above, and the distinction between two elements in the logos block is often unclear (cf. Winsløw et al., 2018). We therefore use $\mathbf{\Lambda}$ to denote '*technology* ($\boldsymbol{\theta}$) + *theory* ($\boldsymbol{\Theta}$)' for the characterisation of teachers' activities related to lesson preparation.

A praxeology of mathematical practice and knowledge called *mathematical organisation* (MO) is used to characterise mathematical activities and knowledge at stake, and the praxeology related to teachers' pedagogies of MO is called *didactic organisation* (DO). The specificities of MO and DO can be characterised in a school system, which is a specific institution. According to Miyakawa and Winsløw (2013, 2019), the system of teaching and learning activities (related to MO and DO) inside the classroom is called a *didactic system*, and the teacher's activity outside the classroom is called *paradidactic organisation.* Accordingly, teachers' praxeologies in the paradidactic organisation are called *paradidactic praxeology*. In this study, we focus mainly on teachers' paradidactic praxeologies as a theoretical frame, but also refer to their associated didactic praxeologies in the classroom in order to obtain a better understanding.

In earlier studies, these ATD notions were used to theorise lesson study and teacher knowledge (Miyakawa & Winsløw, 2013, 2019; Rasmussen, 2016; Winsløw, 2012; Winsløw et al., 2018). In our study we use such notions to conceptualise teachers' *kyouzai kenkyuu* guided by their didactic and mathematical organisation.

4.3 A way of networking the two frames

DAD and ATD describe teachers' activities in lesson design from different perspectives. In this study, we address *combining* and *coordinating* as strategies of networking theories (Prediger et al., 2008). Recently, Trouche et al. (2019) showed the theoretical combining of DAD and ATD (as well as CHAT) in their empirical study on teachers' collective works. Combining strategy does not necessitate the complementarity of different theories, but is important to obtain multi-faceted insight into the phenomenon in question (Prediger et al., 2008). In contrast, coordinating theories "usually should include a careful analysis of the mutual relationship between the different elements and can only be done by theories with compatible cores" (Prediger et al., 2008, p. 172). This strategy should not be understood as developing a new unified theory achieved by *synthesizing* or *integrating* strategies. Theoretical coordinating is a strategy that "can be a starting point for a process of theorising that goes beyond the better understanding of a special empirical phenomenon" (Prediger et al., 2008, p. 173). From different theoretical levels, we demonstrate how DAD and ATD frame teachers' lesson designs differently and can be combined and locally coordinated. This process offers new insight because most of the existing literature, as reviewed in 3.2, is substantiated by a single theoretical framework from a specific level. As we describe in the next section, DAD was first used to compare and describe the teacher (task and learning trajectory) and textbook design. ATD was then used to understand a teacher's didactic and mathematical knowledge guiding the design by means of praxeological analysis. Although we pay attention to teachers' preparation rather than classroom teaching, additional sources from the classroom are also considered to illustrate the associated didactic praxeologies.

5 Illustrative examples

In this section, two cases of Japanese primary school teachers' *kyouzai kenkyuu* for lesson design are analysed in retrospect to illustrate 'documentational work' in DAD and 'paradidactic praxeology' in ATD.

5.1 Contexts of two cases

Although the two lesson studies were implemented independently, the same textbook series, *Keirinkan*'s Fun with Math series (Shimizu et al., 2011/2015), and the same chapter for 3rd graders (ages 8–9) on fractions was selected for the research lesson. We considered two lesson studies as sufficient examples because their commonalities allowed us to compare the teachers' practices and effectively to illustrate different aspects. The topic 'fraction' is presented because it is relatively traditional and stable content in Japanese textbooks in primary schools and, thus, it might create a challenge for teachers to change or adapt concerning both the mathematical and didactic aspects of textbooks. We therefore considered this topic informative in demonstrating what

Table 3 The textbook chapter on fractions

Lesson	Section title	Content	Pages
L1	How to represent the size of the remaining part	How to represent "1/□" by partitioning	2
L2		Meaning and expression of a fraction	1
L3		How to represent a fractional quantity	1
L4	Size of fractions	Meaning of fraction as a number, and a collection of unit fractions	1
L5		Fractions in the number line	1
L6		Comparison of the size of fractions, and the expressions using equality and inequality signs	1
L7	Addition and subtraction of fractions	Addition of fractions with the same denominators	1
L8		Subtraction of fractions with the same denominators	1
L9		Exercise	1
L10	Practice and check	Basic points	2

Table 4 Unit plan in case 1

Lesson	Content and goal	Lesson number in the textbook
1	Thinking of ways to represent fractional lengths and understanding of unit fractions	L1
2	Understanding how to represent fractions for a certain number of unit fractions, through length of expression methods	L2
3	Understanding how to represent fractions for a certain number of unit fractions, through volume expression methods	L3
4	Learning the significance of fractions as numbers, and understanding their relationship with the number '1'	L4
5	Indicating fractions on a number line and expressing points on a number line in fractions	L5
6[a]	Thinking of size of fractions, inequality, and equality relations	L6
7	Thinking of ways to calculate the addition and subtraction of fractions with common denominators, and making calculations	L7–8
8	Ensuring proficiency of fractions	L8–9
9		

[a]The sixth lesson was selected as the research lesson

Table 5 Unit plan in case 2

Lesson	Content and goal	Lesson number in the textbook
1	Thinking of how to represent fractional lengths. Understanding how to read and write fractions	L1
2	Being able to view numbers in terms of how many unit fractions there are	L2
3	Thinking of how to represent 'quantity fractions' using figures and words	L3
4	Thinking of the size of fractions, using figures, formulas, and words	L4–5
5	Thinking of sizes of whole numbers and fractions, using figures, formulas, and words	L6
6[a]	Being able to think about the calculation methods of addition and subtraction of fractions, using figures, formulas, and words	L7–8
7		
8	Reflecting on one's learning of fractions	L8–9
9		

[a]The sixth lesson was selected as the research lesson

kind of adaptations can be effected through the teachers' *kyouzai kenkyuu*.

Table 3, 4, and 5 summarise the textbook chapter and what each teacher actually designed as 'unit plans (lesson sequences)' according to the chapter. It appears that the teachers designed unit plans that included and were consistent with all content in the textbook chapter, although there were some minor changes such as names of contents and lesson sequences (see also the column 'lesson number in the textbook' in Tables 4, 5). As mentioned in 2.2, *kyouzai* in

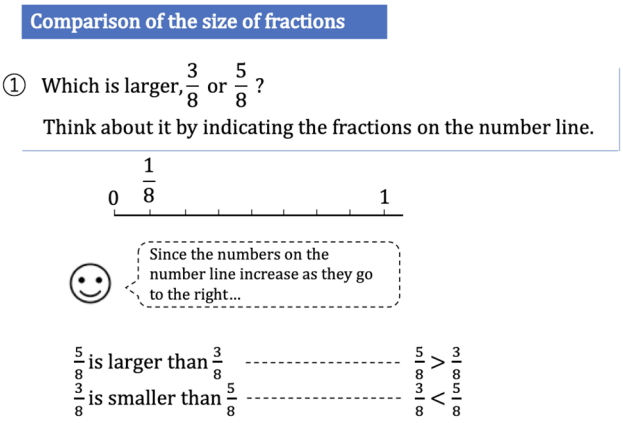


Fig. 3 Textbook page for Case 1 (reconstructed from Shimizu et al., 2011/2015, p. 52)

each case study is more concrete than the contents listed in the tables, but it can be represented as a task or problem in the textbooks, lesson plans, or actual lessons. Additionally, for Case 1, 'tapes' prepared by the teacher as a teaching tool (manipulatives), are also included as *kyouzai*.

It is important to note that second grade students already know simple fractions such as 1/2 and 1/4; this is merely an introductory lesson and is not connected to the meaning, size, or calculation of fractions. Students' conceptual learning of fractions, understanding of meaning, and representations of fractions, do not begin until 3rd grade with the following three goals in the Course of Study:

(a) To use fractions in expressing the size of fractional parts or quantities obtained as a result of equal partitioning. To get to know the representations of fractions.
(b) To comprehend that a fraction can be represented as a collection of unit fractions.
(c) To understand the meaning of addition and subtraction of fractions in simple cases,[8] and to explore ways of calculation. (MEXT, 2008)

5.2 Case 1

Case 1 is derived from the results of a lesson study (Maeda et al., 2016), in which the second author was involved as a collaborative researcher. At the time, the teacher had a master's degree in education and three years' experience at primary schools.

5.2.1 Teacher's documentational work

In Case 1, 'comparison of size of fractions' was selected for the research lesson. When the teacher engaged in the lesson design, the main resource used was the textbook page as shown in Fig. 3, including the task '*Which is larger, 3/8 or 5/8? Think about it by indicating the fractions on the number line.*' Fig. 3 involves students' learning processes of investigating the two fractions using the number line and expressing this relation using the inequality sign.

We analysed the lesson plan as a 'document' (the resources + their usages + the knowledge guiding the usages), allowing us to interpret how the teacher adapted the page in terms of both instrumentation and instrumentalisation. Figure 4 shows the lesson plan (simplified by the authors), in which the task was designed as follows: '*Consider how to compare the size of fractions without using tapes. Is there a fraction between 1/4 and 3/4?*' The teacher's task was more open-ended than the textbook task which was a seemingly more closed problem. Nevertheless, the teacher's task was affected by the textbook task, because basic elements of content (such as how to compare the size of fractions and how to represent the order of fractions) are linked with the intended curriculum. Although the teacher's task is described without using tapes, the teacher planned for students to use tapes as manipulatives for them to understand the task setting. We therefore interpreted this aspect as instrumentation. On the other hand, there are crucial adaptations from the textbook tasks which can be considered instrumentalisation. To understand this aspect, the three questions in Fig. 5 are informative.

The three questions express the teacher's disposition and knowledge which guided her task design adapted from the textbook. For example, 'the density or order of rational numbers' are important mathematical ideas not intended in the textbook or curriculum. While the textbook's task focused only on comparing or ordering two fractions, the teacher's designed task also focused on investigating different *intermediate* fractions which exist in-between two different fractions. This mathematical goal is important for the teacher, because a teaching approach that uses tapes as concrete objects is considered a way to teach fractions. From a didactic viewpoint, the teacher did not use the number line model as shown in the textbook. Instead, she considered tape-folding manipulation to indicate fractions and compare sizes. The tape-folding manipulation is a more concrete mediator than the number line model, although there are common features (such as the notion of distance). In these respects, we analyse the teacher's knowledge guiding such an adaptation from a praxeological perspective.

[8] 'Simple cases' refer to the addition and subtraction of 'proper fractions' as well as addition with sums up to 1 (MEXT, 2008).

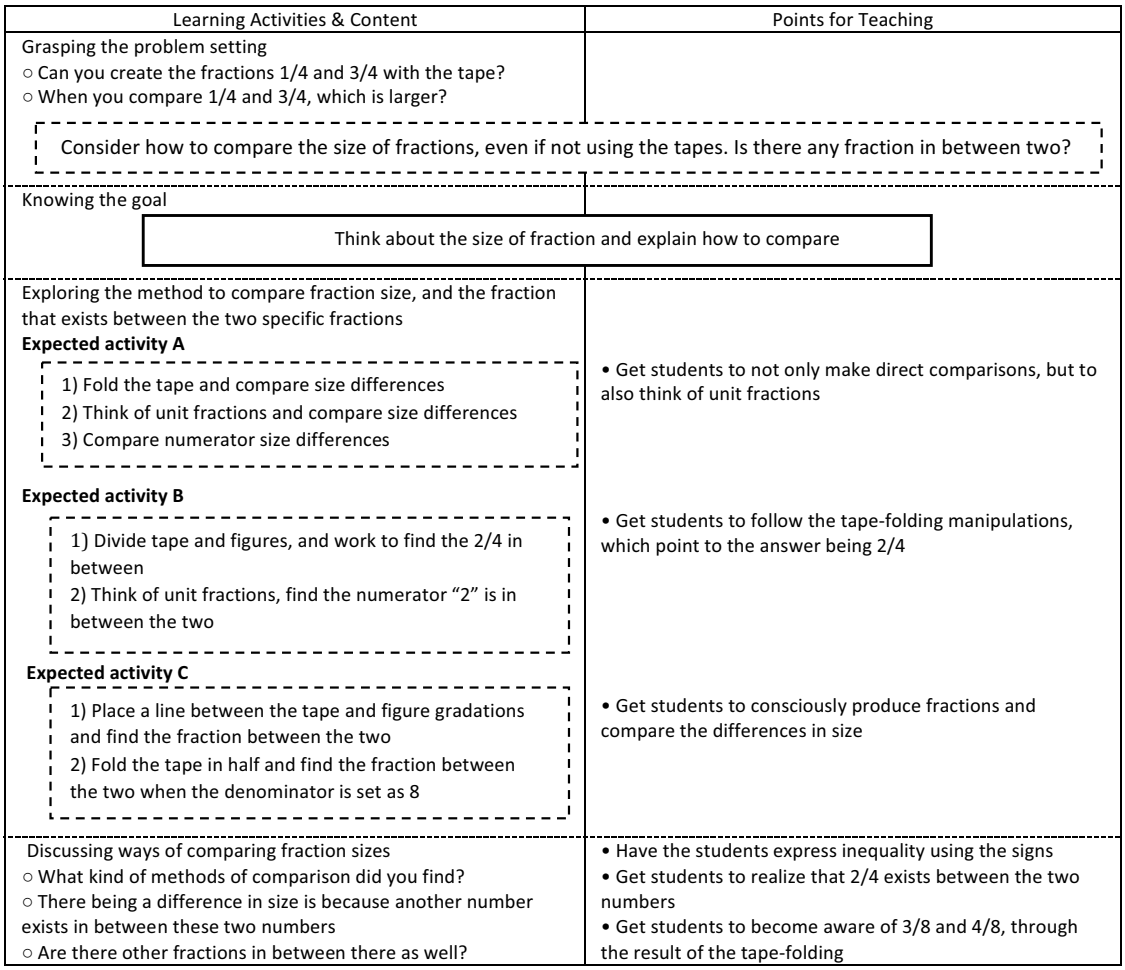


Fig. 4 A simplified version of case 1 lesson plan

1) In what ways can we teach fractions to 3rd graders based on the density or order of rational numbers?
2) How can students change their activity from manipulative actions to interpretative actions?
3) How can we support students' concept formation related to fractions?

Fig. 5 Questions for the task design in case 1 [extracted from Maeda et al. (2016)]

5.2.2 Teacher's paradidactic praxeology

Within a paradidactic organisation, the praxis part (**T1** and **τ1**) of the teacher's praxeology illustrates that the task and its setting were significantly changed from the textbook. The textbook's task asked students only to find one fraction in-between two fractions, while the teacher's task challenged students to find different intermediate fractions. The logos part (**Λ1-1** (DO), **Λ1-2** (MO)) explains the reasons for the changes. Concerning **Λ1-1** (DO), the teacher supplied tapes which allowed students to manipulate these objects and make sense of fraction sizes. **Λ1-2** (MO), which reflects the first question of Fig. 5, justifies the teacher's adaptation of the textbook task.

To understand how the teacher's paradidactic praxeology (Table 6) was related to her didactic praxeology in the classroom, we describe the outline of the lesson as follows: concerning DO, to support students' physical activities to find fractions between 1/4 and 3/4 (Fig. 6), the teacher prepared different coloured tapes so that students might become aware of the number of 'partitions'. Using the tapes, students first found 2/4 as an intermediate fraction, then gradually found additional fractions (such as 3/8, 4/8, 5/8). As shown in Fig. 7 (writing on the blackboard), the teacher helped students understand the relative relations

Table 6 Praxeological account of teacher design in Case 1

Case 1: Comparison of size of fractions	
T1-1	To design a research lesson to compare the size of fractions
T1-2	*Consider how to compare the size of fractions, even without the tapes. Is there a fraction between 1/4 and 3/4?*
τ1	Using the textbook task with significant change to the task setting for students to understand the substantial mathematical meaning of the order of rational numbers
Λ1-1 (DO)	This is an effective way to use a tape as a manipulative which allows students to understand that there are different intermediate fractions in-between two fractions
Λ1-2 (MO)	The density and order of rational numbers are essential mathematical ideas for deeper understanding of the comparison of fraction sizes

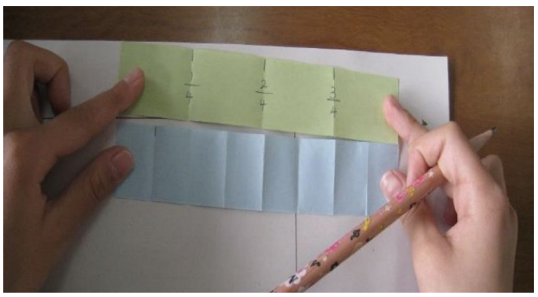


Fig. 6 Student folding tape (Maeda et al., 2016, p. 5)

between fractions using inequality signs ($1/4 < 2/4 < 3/4$; $2/8 < 3/8 < 4/8 < 5/8 < 6/8$). This teacher's practice is guided by what we characterize as MO. What she meant by 'interpretative actions' in Fig. 5 (see second question) may have been to represent mathematical expressions in relation to physical actions, which in this case was the tape-folding manipulation.

In summary, the teacher's *kyouzai kenkyuu* with textbooks can be characterised as follows:

- Documentational analysis: We illustrated the results of instrumentalisation and instrumentation to clarify which aspects of textbooks were adopted or adapted by the teacher. In Case 1, the teacher's task design and pedagogical approach were significantly changed from the textbook, guided by the teacher's disposition and knowledge regarding the rational number concept (instrumentalisation). On the other hand, the basic contents and teaching goal in the unit plan were aligned with the curriculum guideline (instrumentation).
- Praxeological analysis: We analysed the teacher's paradidactic praxeology to gain a better understanding of teacher knowledge behind the lesson design. In Case 1, although both DO and MO support the teacher's adaptation of the textbook, the MO associated with the density of rational numbers was emphasised in the lesson plan as well as classroom activities. The DO is comprised of ways of making such a mathematical idea accessible to students.

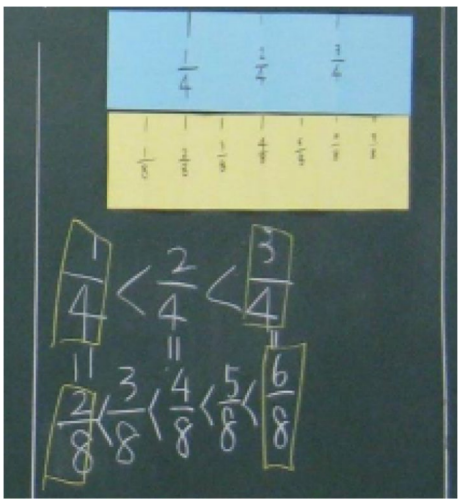


Fig. 7 Intermediate fractions (Maeda et al., 2016, p. 5)

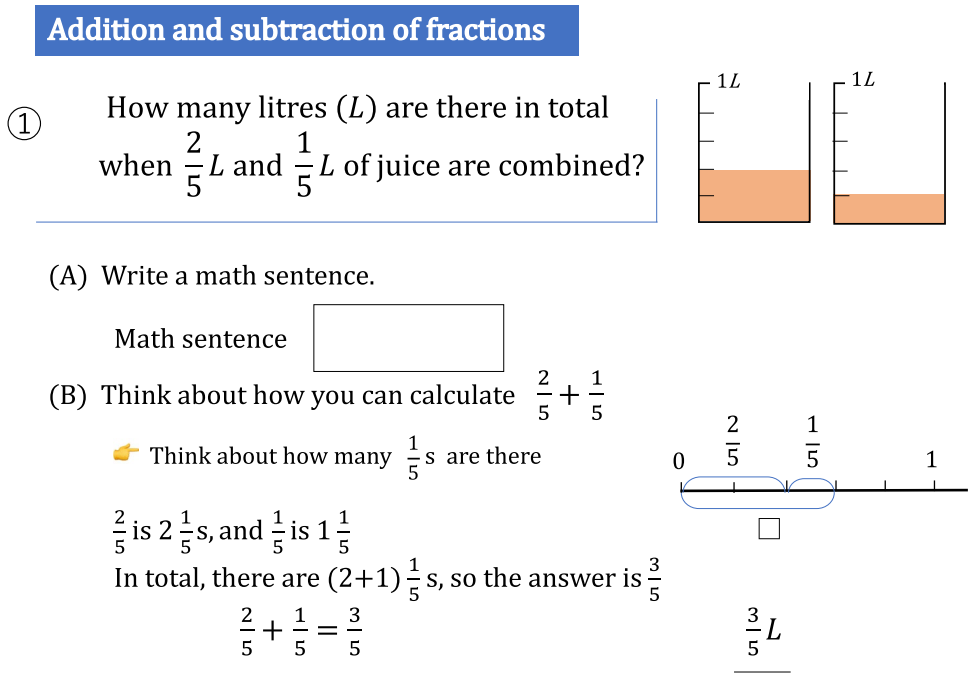
5.3 Case 2

The source of case 2 is from the results of another lesson study (Yamamoto et al., 2015), in which the first author visited the school to participate in the research lesson and post-lesson discussion but did not participate in the lesson planning. The primary school is attached to a national university, and students at such schools are relatively high-achievers. The teacher had ten years' experience in primary schools.

5.3.1 Teacher's documentational work

In case 2, 'addition of fractions with common denominator' was selected for the research lesson. Figure 8 shows the related textbook page. The task is: '*How many litres (L) are there in total when two-fifths of a litre of juice and one-fifth of a litre of juice are combined?*' The page implies students' learning by formulating an expression from the

Fig. 8 Textbook task [reconstructed from Shimizu et al. (2011/2015, p. 53)]



word problem and explaining the calculation using the number line model.

Figure 9 shows the lesson plan (as a 'document') for Case 2, simplified by the authors to focus on the teacher designed task and learning trajectory. The teacher's task was: '*There are four-tenths of a litre of juice in a cup, and three-tenths of a litre of juice in a bottle. How many litres of juice are there all together?*' This task is similar to that of the textbook, although the teacher slightly changed the numerical and linguistic setting. In terms of instrumentation, this numerical change seems to have made the problem easier for students to solve, because of using 'tenth' as the denominator. This implies that the textbook had a strong influence on the teacher's task design. The teacher also designed the students' learning process similarly to the task in the textbook; namely, students' activities start with formulating an expression and then explaining how to calculate it by means of a number line and other representations. In terms of instrumentalisation, a pedagogical approach using an incorrect expression, 4/10 + 3/10 = 7/20, was considered by the teacher, with the aim of conflicting it with the correct expression, 4/10 + 3/10 = 7/10 (in the lesson plan). While the textbook focused on adding fractions, the teacher intended students to understand why this was the correct method. Thus, the textbook usage in this case was guided by the teacher's didactic, rather than mathematical goals. This aspect may be influenced by the social context of university schools in Japan: students are relatively high-achievers, and the teacher's adaptation offers a deeper understanding of the textbook content, rather than merely calculation skills.

5.3.2 Teacher's paradidactic praxeology

For case 2, the praxis part (**T2** and **τ2**) of the teacher's paradidactic praxeology illustrates that the teacher mainly followed the textbook's task design but adapted it by considering an inappropriate method. We can analyse this adaptation as guided by DO in the logos part (**Λ2-1**). Although not included in Fig. 9, the teacher began the lesson plan as follows:

- Students tend to understand the procedure 'adding numerators only, keeping the denominator as it is', but I would like to encourage them to understand the reason by *developing a mathematical way of thinking*. [emphasis added] (Excerpt from lesson plan for case 2)

This point illustrates DO in Table 7, marking an essential difference between the textbook and lesson plan. The DO comprises the outcome of the *kyouzai kenkyu*, which made for a better pedagogical approach to promoting students' learning.

The following pictures from the classroom demonstrate the associated didactic praxeologies, which were consistent with the paradidactic praxeologies, as well as the relationship between DO and MO (in Table 7). Figure 10 shows the teacher's writing on the blackboard, observed at the start of the lesson, when the teacher facilitated students to consider whether denominators can be added or not. When the teacher wrote the incorrect expression, students immediately responded that denominators should not be added, but the

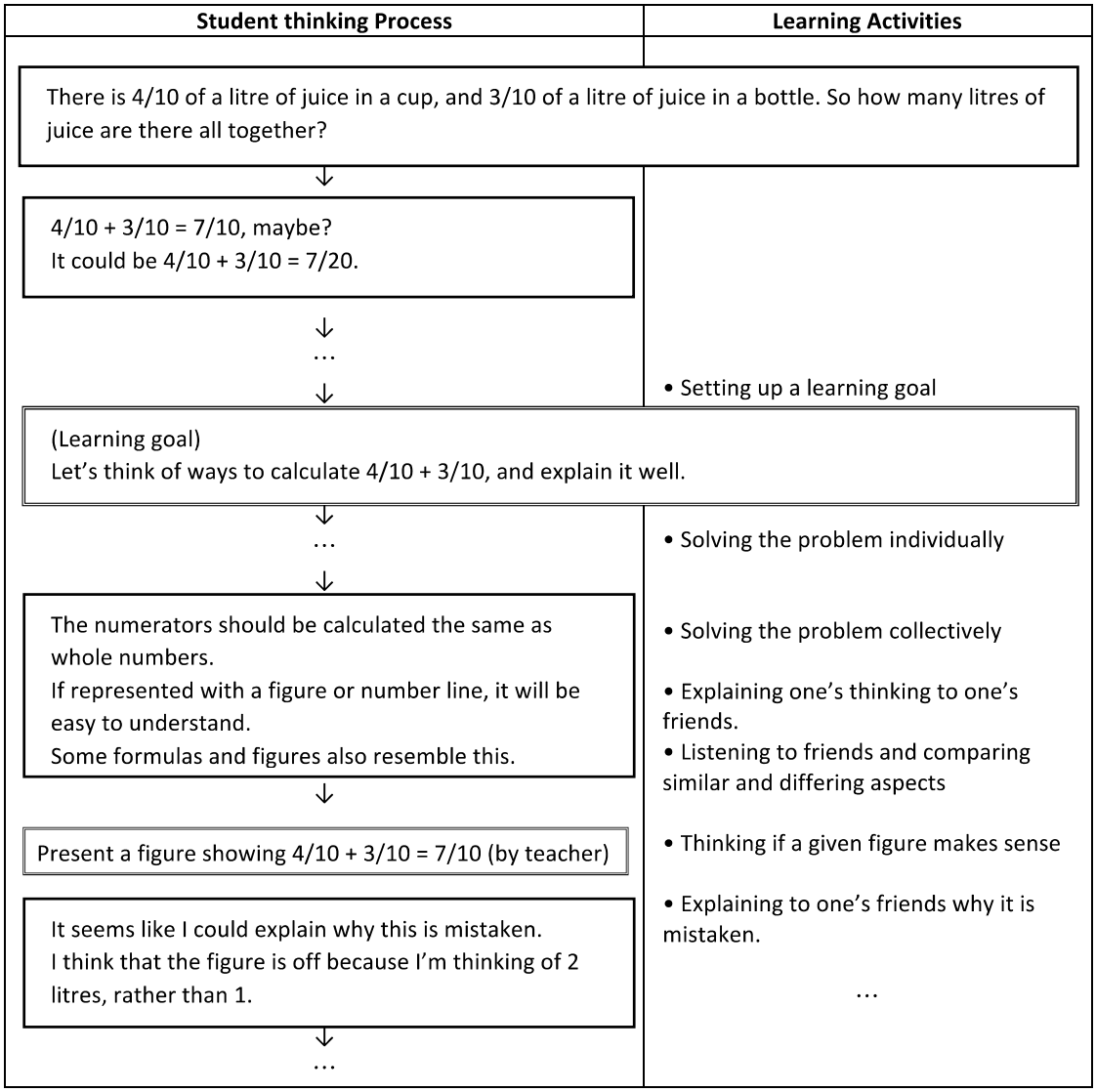


Fig. 9 Simplified version of the case 2 lesson plan

teacher intentionally responded with: *Why not add denominators?* (see Fig. 10).

Figures 11 and 12 show the drawing by a student, who answered '7/20 L.' When the student expressed this idea, she was already aware that it was wrong. However, the teacher responded by showing the representation (see Fig. 12), aiming to justify '4/10 + 3/10 = 7/20'. In this way, the teacher created an opportunity for students to see the incorrect expression '4/10 + 3/10 = 7/20' and to consider why it was wrong.

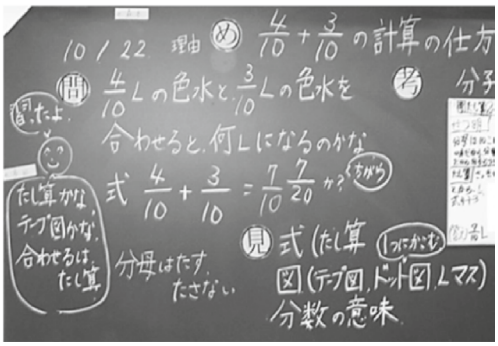
From these activities, we can understand that the teacher's DO is related to the MO, comprising different meanings of fractions. For the teacher, Fig. 12 represents fractions as 'partitioning' which might justify the incorrect expression '4/10 + 3/10 = 7/20'. As described in the lesson plan, this explanation can be refuted by referring to the fact that 'the representation (Fig. 12) supposes two litres as a unit volume, but the unit should be one litre'. Thus, it seems that the teacher's knowledge (MO) relates the fraction concept not only to 'partitioning' but also to 'measuring'.

In summary, we analysed the teacher's *kyouzai kenkyuu* using an alternative teaching approach, which can be seen as characteristic of instructional change in this case. The basic roles of the two theoretical frameworks are in common with those in the analysis of case 1, but the following points were found as specificities in case 2:

- Documentational analysis: the teacher's task and process were mostly designed according to the textbook (instru-

Table 7 Praxeological account of teacher design in Case 2

Case 2: Addition of fractions with common denominator	
T2-1	To design a lesson on the addition of fractions with common denominators
T2-2	*There is four-tenths of a litre of juice in a cup, and three-tenths of a litre of juice in a bottle. How many litres of juice are there all together?*
τ2	Using the textbook task with a minor change in the numerical values for students to understand why this way of adding denominators is inappropriate
Λ2-1 (DO)	This is an effective way to create an opportunity for students to notice conflicting ideas when developing their mathematical thinking
Λ2-2 (MO)	This way of adding denominators can be appropriate, if the meaning of fractions is restricted to 'partitioning' without referring to 'measuring'



Problem *There are four-tenths of a litre of coloured water and three-tenths of a litre of coloured water. How much water is there all together?*

Math sentence
4/10 + 3/10 = 7/10 or 7/20?

Wrong

Add denominators or not

Is it addition?
Bar-model is useful?
Addition means all together

Fig. 10 Teacher's writing on the blackboard (partly translated into English) (Yamamoto et al., 2015, p. 31)

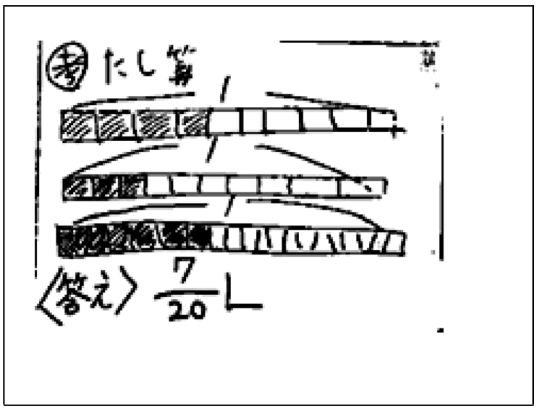


Fig. 11 Student's (S4) drawing on the blackboard (Yamamoto et al., 2015, p. 31)

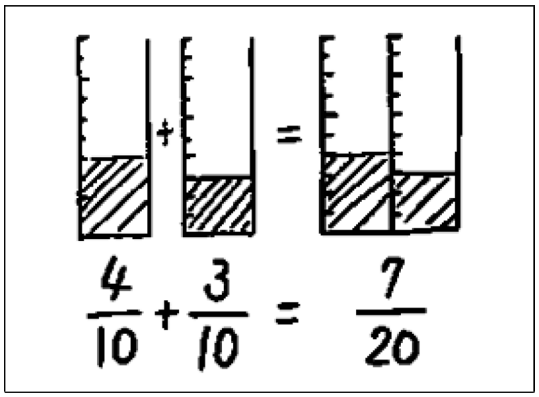


Fig. 12 Teacher's drawing on the blackboard (Yamamoto et al,. 2015, p. 31)

mentation), while the incorrect mathematical idea was considered an important approach to achieve the goal of the lesson (instrumentalisation).

- Praxeological analysis: the teacher's paradidactic praxeologies (DO) explain that the teacher values the opportunity for students to become aware of the reason why the denominators cannot be added, to develop students' mathematical thinking. This was also illustrated by the associated didactic praxeologies, allowing us to understand the relationship between DO and MO in the praxeologies.

6 Conclusion

6.1 Some impacts of the theoretical approaches

This paper explores how two theoretical frameworks can conceptualise essential aspects of teachers' lesson designs involving the adaptation of textbooks in the context of

lesson study in Japan. Within DAD, *kyouai kenkyuu* can be regarded as the teacher's documentational work to produce a lesson plan. DAD is an intermediate level frame which focuses on the teacher–textbook interaction in terms of instrumentation and instrumentalisation, allowing for analysis of differences in the two processes of each case. This approach may explain what kind of affordances are affected by the textbook as well as the kind of adaptations made by the teachers. As illustrated by the two cases, the unit plans designed by the teachers were based on the textbook chapters without any major changes, but the task and setting (learning trajectory) for the research lessons were changed in different ways.[9] On the other hand, ATD is a grand level frame offering an institutional approach, conceiving *kyouzai kenkyuu* as a vehicle of instructional change in the transposition process between 'knowledge to be taught' and 'taught knowledge,' as shown in Fig. 1. In addition, for Case 1, the transposition was also linked with 'scholarly knowledge,' because the mathematical idea (density of rational numbers) does not exist in 'knowledge to be taught.' Thus, MO in the textbook was reconstructed by incorporating a new mathematical idea in case 1. In contrast, for case 2, DO in the textbook was reconstructed by considering the incorrect expression, although MO was consistent with that in the textbook.

Exemplified by two cases, we offered the networking of two theoretical approaches, which can conceptualise teachers' *kyouzai kenkyuu* practice differently. Although authors of most previous research studies, as identified in 3.2, utilised a single theoretical framework to substantiate their studies, our theoretical framing offers a dialogue between different theoretical levels (intermediate and grand), which may lead to an approach to understanding the complexity of the phenomena (Prediger et al., 2008). In the praxeological account, the difference in the logos parts (DO and MO) was emphasised, while in the documentational account, the difference in the instrumentalisation process was emphasised. However, such different accounts can be considered as strategies of the networking of theories, namely, 'combining' and 'coordinating' (Prediger et al., 2008), because 'the knowledge guiding the usages'—a component of documents in the DAD framework—can be coordinated with the logos block of the paradidactic praxeology; some aspects of the teachers' documentation work can also be coordinated with the praxis block. This implies that the praxeological and documentational approaches are compatible, and these approaches can be used to gain deeper insight into the relationship between lesson design using textbook and teacher knowledge.

6.2 Implications and limitations

Our theoretical approach supports various descriptive studies of *kyouzai kenkyuu* conducted by Japanese researchers (e.g., Fujii, 2016, 2018; Takahashi, 2011; Watanabe et al., 2008) and contributes to theorising about some of their aspects. In fact, theorising about *kyouzai kenkyuu* within well-established frameworks for research in mathematics education is important, but it is challenging to make such cultural specificity visible in the international context. Most Japanese teachers use *kyouzai kenkyuu* to explain a method for implementing instructional change but may find it difficult to explain without using the words in researchers' discourse. The two theoretical frameworks used in this paper are helpful in making the implicit explicit. Such theoretical descriptions may also contribute to a deeper insight of the distinction between teaching the textbook (*kyoukasyo wo oshieru*) and using the textbook to teach mathematics (*kyoukasyo de oshieru*) (Takahashi, 2011), because this difference can be understood as the difference in the teacher's documentational work and paradidactic praxeologies. In this sense, our study may advance Miyakawa and Winsløw's (2013, 2019) theorisation of paradidactic infrastructure regarding teacher knowledge by specifically focusing on *kyouzai kenkyuu*.

There are some limitations to be addressed as future tasks. Although we based our analysis on constructs from ATD and DAD, we exemplified teachers' preparation work based on the results of the lesson study. However, further research is required to investigate both teachers' design processes (outside the classroom) and teaching processes (inside the classroom). Another important issue is how to theorise teacher-researcher interactions or teachers' collaborative works in lesson study (e.g. Jaworski et al., 2017; Robutti, 2018), because teacher design, teacher praxeology, and their evolutions can be affected by others in the community.

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[9] Generally, a Japanese teacher's change of the 'main problem' of the textbook (what we called 'tasks' in this paper) can be crucial, because the design of the 'problem' may shape Japanese mathematics lessons (cf. Fujii, 2018).

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