Research as a resource in a high-school calculus curriculum

Tommy Dreyfus¹ · Anatoli Kouropatov² · Gila Ron³

Accepted: 31 January 2021 / Published online: 17 February 2021
© FIZ Karlsruhe 2021

Abstract

Documents specifying a national mathematics curriculum for grades K-12 have recently been written in Israel. We focus on the calculus component for the highest of three matriculation bound levels, and specifically on the influence of research on this component. In addition to issues of content, we identify three principles that have led the writing team, namely, the manner in which sample tasks in the curriculum document incorporate fundamental mathematical ideas and mathematical Reasoning, the Impact expected from connecting mathematics to its role in everyday life and science, and the Cultivation of fertile intellectual ground from which new concepts may emerge naturally. We demonstrate how these principles are implemented in the unit on integration. We show that mathematics education research, though not mentioned explicitly, has had a profound and pervasive influence on the content and principles of the curriculum document, from the design of entire units down to the formulation of sample tasks.

Keywords Calculus · High school · Curriculum · National curricular document · Influence of research · Israel

1 Introduction

Learning the basic concepts of calculus is an important part of high school and university mathematics curricula (Bresoud et al. 2016; Greefrath et al. 2016). Indeed, it is hard to imagine modern scientific culture without derivatives and integrals. These concepts constitute a tightly interconnected cluster of fundamental ideas and a basic tool for understanding the world (Newton 1678/1999). Issues in learning and teaching calculus have been in the worldwide focus of research in mathematics education for many years (e.g., Orton 1983a, 1983b; Tall 1986; Artigue 1998; Larsen et al. 2017).

Expressions of discontent with the kind of knowledge students acquire in the course of their calculus studies in high school or college are frequent in a large number of countries (e.g., Tallman et al. 2016; Törner et al. 2014). Israel, in addition to such discontent, did not have a mathematics

curriculum document for senior high school for many years (Fares 2018). This omission changed recently.

When mathematics educators talk about curriculum development, they refer to several stages, starting from establishing standards or a national curriculum document, and continuing with the design of instructional materials, teacher training, and multiple stages of implementation and revision. This paper relates to the first stage, the writing of a national curricular document in Israel. We focus on the calculus component of the mathematics curriculum document for the highest of three levels bound for matriculation; we ask whether research in mathematics education has influenced the content and nature of this document and how this influence came about. We consider these questions not only for the contents of the proposed curriculum but also for three principles, which we identified as having determined the work of the team writing the curriculum.

The paper is structured as follows: In Sect. 2, after a brief comment about research on curriculum development, we provide background about the Israeli school system and its matriculation examination; we also describe the place of mathematics, and specifically calculus, in the system. Moreover, we give some information about the process by which a team was tasked with writing the curriculum document. In Sect. 3, we discuss the rationale of the document and present and exemplify the three principles, in light of

✉ Tommy Dreyfus
tommyd@tauex.tau.ac.il

¹ Tel Aviv University, Tel Aviv, Israel

² Levinsky College, Tel Aviv, Israel

³ Tel-Hai College, Upper Galilee, Israel

relevant research literature. In Sect. 4, we similarly discuss the contents of the curriculum. In Sect. 5, we focus more narrowly on the unit addressing integration, and show how content and principles interweave. We come back to the matriculation examination and its curriculum in Sect. 6, and close the paper in Sect. 7 with conclusions about the influence of research on the writing of the national curriculum document and remarks about limitations of our study.

2 Background

2.1 Research and curriculum development

In his classical paper on research in mathematics curriculum development, Clements (2007) agreed that not all considerations towards establishing educational goals involve scientific knowledge, and that in order to answer questions such as why given curricular goals are important, values are taken into account. Clements identified several stages of curriculum development, as follows: the writing of standards for national curricular documents, the development of instructional materials, the implementation of these instructional materials at various scales, teacher training for the curriculum, evaluation of instructional materials, and teaching and learning. Some of these stages are more likely to be research-based, others less. The work we describe in this paper concerns the writing of a national curriculum document, and this is the least overtly research-linked of all stages. One reason for this paucity is that it is more than the others subject to normative decisions, preferences, and policies, because people have underlying commitments and beliefs about the nature of mathematics, mathematical activity, and how mathematics is learned; these commitments and beliefs may be influenced by lifelong research and/or teaching experiences. For example, while research shows that inquiry-based learning improves undergraduate student outcomes in mathematics courses (Laursen et al. 2014), many mathematicians teaching at the undergraduate level believe that ‘chalk and talk’ allows them to best achieve their goals (Woods and Weber 2020).

One of the few papers in the literature that focus on the first stage is the work by Rojano and Solares-Rojas (2018). Their aim is a comparative evaluation between the national curricular documents of four countries and one dimension that they consider is “research in mathematics education and the curriculum”. They found that in the documents of all four countries, the lack of explicit references to research literature is notable but that, on the other hand, all four programs showed clear signs of implicit influence of mathematics education research.

2.2 The Israeli school system

The Israeli school system is centralized with most decisions being taken by the Ministry of Education, and specifically in the case of mathematics, by the superintendent of mathematics. State supported compulsory schooling is split into kindergarten (3 years, the last of which is compulsory), elementary school (6 years, grades 1–6), middle school (3 years, grades 7–9) and high school (3 years, grades 10–12); students typically start compulsory kindergarten at age 5 and finish grade 12 at age 18; roughly 75% of the population study in grades 11 and 12 in matriculation-oriented schools; roughly 50% of the population pass the matriculation examinations, one of two main criteria used for admission to tertiary studies. Mathematics is a compulsory subject in these examinations.

2.3 Mathematics in the Israeli school system

The ministry has a superintendent and a ‘professional committee’ for mathematics. The superintendent is responsible for everything that happens in the field, such as approval of mathematics textbooks and a wide-ranging system of support for school mathematics departments. The task of the professional committee is to initiate and approve major changes. Currently, the committee has 17 members, 4 of whom are professors of mathematics; the others are involved with mathematics education in a variety of roles such as academic researcher, teacher educator, supervisor of mathematics instruction, teacher guide, or lead teacher.

In matriculation-oriented high schools, students learn mathematics at three different levels. The levels differ in the number of hours mathematics is studied, in difficulty of tasks assigned to the students, in required depth of understanding, and to some extent in content. They also differ in the amount of matriculation credit they award. According to ministry instructions, the number of units corresponds to the number of weekly mathematics lessons, but many schools increase the number of lessons, mainly at the 5 units level. Traditionally, about 60% of students study at the 3 units level, 30% at the 4 units level, and 10% at the 5 units level.

The previous mathematics curriculum was written in the 1970s. What actually has been taught in recent years, however, is determined by a document called *examination program*. This document lists in a rather detailed manner the topics that will appear in the matriculation examination, and is updated from time to time by the superintendent. The document includes numbers of hours to be allocated to each topic; however, it does not provide a rationale, aims, and standards to teachers and textbook

writers; it does not include examples, nor recommendations about teaching style, or views of mathematics that might shape instruction. As a consequence, the influence of the matriculation examination on mathematics classes is overwhelming. Textbooks are, to a large extent, collections of exercises intended to prepare students for examinations; they typically include explanations of formulas and worked examples, but little or no material intended to connect mathematics to everyday life or to science, to develop conceptual understanding by means of questions for discussion, or similar goals.

The professional committee initiated the writing of a mathematics curriculum for the entire K-12 school system. The writing of the high school mathematics curriculum constituted the last stage this long-term project, after the curricula for elementary and middle school had been completed.

2.4 Calculus in the Israeli curriculum

Approximately 25% of the senior high school (grades 10–12) mathematics hours are devoted to calculus. The topics taught in calculus are conventional: the derivative and its uses for polynomial, rational, trigonometric, power (rational exponent), exponential and logarithmic functions, and integral calculus for a similar set of functions. The curriculum that forms the topic of this paper introduces only minor changes in the content to be taught but more substantial changes to the style of teaching this content, the connections between different components of the content, the kind of uses of the derivative and integral, and in the case of the integral, the overall approach. In order to appreciate the nature of these changes, it is therefore necessary for the reader to become acquainted with the ways in which typical Israeli students know the above content. This is one of our reasons for presenting research done in the Israeli educational context in Sect. 3.

2.5 Israeli students' understanding of calculus

In spite of the large variety of innovative research-based teaching proposals, high school calculus is taught in many countries as a collection of skills in answering a standard collection of exercises; this fact has been documented for many years (e.g., Orton 1983a, b) and is still valid as noted, for example, by Biehler (2019). Similarly, Israeli researchers point out that Israeli students encounter many difficulties in the study of calculus (e.g., Tsamir 2007). We illustrate the situation by means of some findings of an empirical study of (N=292) high school students' knowledge about the integral (Kouropatov 2016):

- Most students identify the definite integral with area, understood in a global, geometrical sense;

- students can use antiderivatives for calculations; however, there is no evidence that they can properly express the connection between antiderivatives and area;
- students can use some techniques in selected applications including calculations of area and distance, but this usage is not articulated; in particular, there is little evidence that students see the computation of an integral as in any way related to a process of accumulation;
- given a function and its accumulation function, there is no evidence that students connect the derivative of the accumulation function to the given function.

These findings show that students are much more successful following an algorithm or manipulating symbols than they are when dealing with the concepts of derivative and integral. Some students are aware of this problem. The following two quotes from student interviews support this result:

We really don't know what the integral is. But it's interesting to know what this concept is.
... the teachers show us different stuff all the time like derivative, area, volume... And that's the reason why we don't know what the integral really is...

These findings are well aligned with the situation elsewhere; for example, Orton (1983a) found in the UK that "The most striking feature of their responses was that many students appeared to know what to do, but, when questioned about their method, didn't really know why they were doing it" (p. 8). And Ely (2017) summarized the recent literature about students in the USA as follows: "Sum-based interpretations of the definite integral are much more productive in general for supporting student reasoning than area and anti-derivative interpretations" while "area and anti-derivative interpretations are very commonly displayed by calculus students, and sum-based interpretations occur more rarely" (p. 153).

2.6 Context and guidelines for the curriculum

At about the time when the development of the curriculum was initiated, the government acknowledged that Israel's economy and security depend on a high level of mathematical literacy, and hence declared mathematics and science education a national priority with the hope of educating a new generation of creative developers. According to a study by the Taub center (2015), the number of mathematics [matriculation] units obtained by students has a substantial influence on their wage level.

In view of the government decision and the Taub center report, the Israel Ministry of Education's Professional Committee for Mathematics prepared 'guidelines' (2010) and nominated a curriculum committee to write a high

school mathematics curriculum. The guidelines specified that the curriculum committee was to prepare a national curriculum document that would include the following: a set of aims, a rather detailed syllabus including topics to be taught and the order in which they could be taught, and above all, and approaches to these topics, according to which instructional materials including textbooks could later be developed. Mathematics education research was not mentioned in the guidelines.

The guidelines specified that the 5 units curriculum is intended to prepare students for studying STEM subjects at the tertiary level. Increasing the percentage of students studying mathematics at the 5 units level from about 10% to about 20% was specified as an aim, to be combined with social mobility by providing opportunities for students from less advantaged social backgrounds to study mathematics according to the 5 units program.

The guidelines noted that the previous high school mathematics curriculum was written more than 30 years earlier, and that what is taught in the school system is not that curriculum but rather is determined by the examination program published by the superintendent. Hence, the structure of the examinations determines what is taught rather than the opposite. The guidelines specified that the curriculum committee should stress the need for understanding, the relationships between topics, and the relevance of mathematics. The task of the committee was to write a curriculum document specifying, in addition to content, a rationale, standards for skills and reasoning processes, and advice on ways of examining these.

A boundary condition imposed by the ministry was to plan the 5 units curriculum for 450 lessons (of typically 45 min) of mathematics, equally distributed over three years; this condition corresponds to 5 h of mathematics in the students' weekly schedule.

Finally, the guidelines required the curriculum committee to ensure continuity with the new middle school curriculum that was just being implemented. The middle school curriculum (Israel Ministry of Education 2012) includes an approach to the notion of function and rate of change that later become central for high school calculus. The main goal of this approach is to conceive of a function as describing a connection between two quantities, which can be represented in different settings (numerical, graphical, verbal, algebraic). The curriculum defines the rate of change of a function as "the ratio between the change in its y-values divided by the change in its x-values"; this is followed immediately by: "if the same ratio is obtained for any two values of x, then the rate of change is uniform; in each other case, it is not uniform" (translated by the authors); the difference between uniform and non-uniform rate of change is to be stressed in all settings. While this description is technical, the accompanying examples relate

to extra-mathematical as well as to purely mathematical situations, and demand interpretation.

2.7 Team

The high school mathematics curriculum committee was chaired by a university mathematics professor and included mathematicians and mathematics educators teaching at universities and teacher training colleges, professionals with extensive experience in developing curricular materials, in advising high school teachers and in teaching themselves, and relevant ministry officials.

The committee set up a team to write the 5 units curriculum. This team included a university professor in applied mathematics, a university professor in mathematics education, and 4 senior mathematics education professionals with extensive experience in high school teaching, pre- and in-service teacher education, development of instructional materials, and in-the-field guidance of teachers, as well as research experience in mathematics education. The first and third authors of this paper were members of the team.

According to the team's interpretation of the guidelines, the task of the team included issues such as the following: the extent to which, and how, connections should be established within or between topics and between different representations of the same mathematical objects, what role to give to extra-mathematical situations in different contents, what role to give to the history of mathematics, to communication, whether and how to distinguish between proof and justification, and when and where to stress inquiry-oriented activities. The team also decided to exemplify such decisions by sample tasks in order to decrease the ambiguity that any general description necessarily leaves. The team aimed for a document that was coherent in terms of the ideas underlying the kind of experiences students make along the three years of instruction, but on the other hand left freedom to developers of instructional materials in the manner of creating opportunities for these kinds of experiences, and to teachers in how to implement the instructional materials.

No explicit reliance on research was expected in the guidelines. It is this situation that we examine in the current paper: Given the result of the study by Rojano and Solares-Rojas (2018), that research does clearly have an indirect influence, a main aim of this paper is to tease out where and how this influence has come to bear in the case of the Israeli high school calculus curriculum.

3 Rationale and principles

The rationale in the 5-units curriculum document posits as a main goal students' familiarity with the role of mathematics in everyday life, science and technology, as well as the

recognition of mathematics as part of human culture. This is followed by the goal of providing the tools for studying STEM disciplines at the tertiary level. In order to achieve these goals, the rationale stipulates the following emphases: the development of mathematical ways of reasoning, which include logical, critical, algorithmic thinking, and inversion (of function, operations and theorems); the notions of definition, claim, and proof; the use of examples; the use of different representations; the development of the ability to explain, justify and prove; basic technical skills; and mathematical literacy, with the aim of seeing the relevance of mathematics in science and society.

This rationale is based on the guidelines (Sect. 2.6), which focus on the relevance of mathematics, on the need for understanding, and on preparing students for the study of STEM disciplines, while at the same time expanding the student population and ensuring continuity with the middle school curriculum. In the course of their work on the curriculum document, the team progressively transformed these declarations into three principles, namely, Reasoning, Impact, and Cultivation. Each principle relates to several guidelines.

In this section, we discuss separately each of these three principles. While the principles informed the design of the entire 5 units high school curriculum, we illustrate them with examples from calculus. In the next sections, we briefly describe the content of the calculus curriculum, and then use the unit on the integral to show how the three principles are often interwoven in the design of tasks for students.

3.1 Reasoning

Reasoning is an abbreviation for the ways in which the curriculum elicits, fosters and supports mathematical ways of reasoning as well as fundamental mathematical ideas such as justification. Mathematical ways of reasoning include the following: logical, critical, and algorithmic thinking, and inversion (of function, operations and theorems); the notions of definition, claim, and proof; the use of examples; the use of different representations; and the development of

the ability to explain, justify and prove. Reasoning is intimately connected to opportunities for students to deepen their understanding. Reasoning also includes limiting the requirement of technical skills.

Some aspects of Reasoning have entered the school system over the past years. Since it is difficult to describe this phenomenon quantitatively, it is present but not yet as systematic as called for by the 5-units curriculum document.

In each part of the curriculum document, the description of aims, content, and emphases is liberally supported with examples of activities and tasks for the students. These include a considerable number of questions whose answers require mainly understanding the concepts and relationships, and little or no technique. An example of such a problem is the task in Fig. 1, which is included in the chapter on points of inflection in grade 11.

This task constitutes an example of the deeply rooted connection between research and curriculum task design. This connection is not mentioned in the curriculum document and the team members may not even have been currently aware of it when inserting the task into the curriculum document. On the other hand, several members of the team had been exposed to the relevant research studies at some point in their careers. The connection is twofold: it concerns students' concept images of inflection points and their understanding of the roles of examples in proving.

The research of Ovodenko and Tsamir (2017; see also Tsamir and Ovodenko 2013) enriches the existing body of knowledge about students' conceptions of the notion of inflection point, and about possible sources of related common errors. The results of this study offer a broad collection of related correct and incorrect conceptions; for example, when students think graphically, they tend to conceive of points of inflection as being horizontal; and a function necessarily has a point of inflection if its second derivative equals zero at the point and the function is monotonically increasing (or decreasing) in a neighborhood of the point.

The research of Buchbinder and Zaslavsky (2019) examined strengths and weaknesses in students' understanding of the status of examples in proving. They noted several types

Fig. 1 An example for reasoning (grade 11)

In each statement below, choose the correct option and justify your claim.

- a. A second degree polynomial has a point of inflection—always / sometimes / never.
- b. A third degree polynomial has a point of inflection—always / sometimes / never.
- c. A fourth degree polynomial has a point of inflection—always / sometimes / never.

of tasks, each of which is intended to address some aspects of the role of examples in proving or rejecting the truth of a mathematical claim. One of these types, called ‘always, sometimes or never’ was implemented in the task in Fig. 1. Solving this task requires an understanding of what counts as evidence supporting claims, including one of the quantifiers ‘for all’, ‘for some’, and ‘for none’, as well as a sufficient example space of low degree polynomials. The technical knowledge needed is minimal; no algebraic manipulation is involved all. Hence the task focuses on the concept of point of inflection, its definition, and its basic properties, as well as on some important aspects of justifying, without any technical burden. It illustrates quite a few aspects of Reasoning that have been mentioned above, including critical thinking, the notions of definition and justification, and the use of examples, and teachers can use it to introduce aspects of proof and help students develop the ability of explaining. Even the notion of inversion (of a claim) can be brought in by means of a slight extension of the task, to discuss claims like the following: if a polynomial has a point of inflection, then it is of odd degree; if a polynomial has no point of inflection, then it is of even degree.

3.2 Impact

Impact refers to the ways in which the curriculum incorporates the function and position of mathematics in today’s world. Impact means to strengthen the connection of mathematics to its applications in everyday life, science and engineering. Impact is intended to show the relevance of mathematics, and to help in preparing students for the study of STEM disciplines. Mathematics as a cultural tool may be seen as the sum total of Impact and Reasoning.

The team decided to stress the role of mathematics in the everyday and scientific realm for several reasons. One is that the majority of students in the 5 units mathematics curriculum are expected to study STEM disciplines in the tertiary sector, and therefore the role of mathematics was also stressed in the guidelines.

A second reason is that incorporating the use of mathematics in everyday and scientific applications might motivate students to study mathematics at the 5 units level, thus increasing the number of students, as required by the guidelines. As Freudenthal (1968) stated in his paper entitled “Why to teach mathematics so as to be useful”, which was the very first paper to appear in the journal *Educational Studies in Mathematics*, “In its first principles mathematics means mathematizing reality, and for most of its users this is the final aspect of mathematics, too” (p. 7).

The third reason is conceptual: The team members believe that extra-mathematical situations are apt to support the development of an in-depth understanding of the central notions of derivative as rate of change, and integral as accumulation. In turn, it is expected that such an understanding then supports students when they are asked to apply the mathematical notions of derivative and integral in extra-mathematical situations. This belief is supported by the research literature. Kaput (1994) stressed the importance of “creating viable, functional connections between the world of authentic human experience and the formal systems of mathematics” (p. 379), and described an environment “to link the phenomenologically rich everyday experience of motion in a vehicle to more structured and formal representations and to provide exciting and intensely experienced contexts for reasoning about change, accumulation, and relations between them” (p. 391), i.e., the fundamental theorem of calculus. Stump (2001) concluded from her research on students’ understanding of slope as a measure of steepness and slope as a measure of rate of change that “real-world situations provide meaningful opportunities for students to develop their understanding of mathematics” (p. 88) under the condition that teacher questioning encourages them to think of slope as a ratio.

While the team was neither aware of, nor directly based their work on, these or other research results, they did use their personal experiences from learning, teaching and research in applied mathematics and the sciences to design examples that link extra-mathematical experiences to the mathematical concepts to be learned. We present an example

Fig. 2 An example for Impact

in Fig. 2. It offers an opportunity to discuss the meaning of convexity and point of inflection in a situation from economics.

This example deals with calculus properties in an economically meaningful situation. Hence it offers an opportunity for students to appreciate the impact of mathematics to describe a situation in the real world, to associate the very real notion of rate of price increase with the rate of change of the curve, and to discuss the meanings of the point of inflection and of the concavity (convexity from below) of the curve after the point of inflection. At the time of our writing of this paper, in 2020, it would be absurd not to mention the parallel case of the logistic function modeling the number of people infected with the Covid-19 virus as a function of time, what it means to slow the rate of infection and to flatten the curve, and how important the point of inflection and its interpretation are in this context.

The question of the incorporation of applications in a mathematics curriculum is, of course, very wide. We position our approach within the excellent review by Niss et al. (2007). According to them, we clearly use " 'applications and modelling for the learning of mathematics', i.e. on the actual or potential ways in which applications and modelling may be a vehicle for facilitation and support of students' learning of mathematics as a subject" (p. 5, italics in original).

3.3 Cultivation

Cultivation refers to ways of creating intellectual ground fertile for a new concept to emerge naturally. This is done,

whenever feasible, through activities using situations familiar to the students, in which the new concept occurs prominently, and through setting tasks in these situations that create a need for the new concept by asking questions for whose answer the new concept is helpful, or even necessary. Cultivation enables continuity with the middle school curriculum and supports students' realization of the value of understanding in mathematics. It also contributes to enlarging the proportion of the population who learn mathematics at the 5 units level.

We illustrate Cultivation by the task in Fig. 3.

The questions in Fig. 3 are designed to cultivate the notions of concavity and point of inflection in grade 11, after students have become familiar with the derivative, and its relationship to the slope of the tangent in grade 10. The questions begin from an everyday situation familiar to students; they build on notions well known to students from middle school and grade 10 (function, graph, increasing, decreasing, tangent, slope of tangent); and they focus attention on central properties of concavity and points of inflection. This affords students opportunities to communicate about these properties by speech, gestures and other means before defining them. The terms concave, convex and point of inflection can then be offered in order to aid students in expressing themselves, and at the same time to link the new terms right away to the relevant properties.

Cultivation is related to research at several levels. Cultivating a new concept means introducing it in a manner that is coherent and resonates with earlier learning; it also means creating a need for the new concept; such a need

Fig. 3 An example for Cultivation (grade 11)

The picture shows a path leading up a mountain.
A person walks up a mountain along the path.

- Where is the path steepest?
- Where is the path getting steeper?
- Where is the path getting less and less steep?

Now, consider the path as the graph of a function associating the height of the path with the horizontal distance from its beginning. Consider the slope of the tangent to this graph.

- Where is the slope of the tangent maximal?
- In which interval is the slope of the graph increasing?
- In which interval is the slope of the graph decreasing?
- In which interval is the tangent above the graph of the function?
- In which interval is the tangent below the graph of the function?
- What is the position of the tangent with respect to the graph of the function at the point where its slope is maximal?

plays a central role in the construction of knowledge (e.g., Dreyfus et al. 2015).

Movshovitz-Hadar and Hazzan (2001) established the principle of introducing a term only after giving an appropriate example, as one of six principles for good lecturing, even at the tertiary level. They elaborated: “don’t define a notion before you show a particular instance worthy of the definition; don’t state a general phenomenon before demonstrating it on a few (sufficiently large number of, yet not too many) particular cases” (p. 818). They also recommended gradually proceeding from informal forms that are familiar and concrete for the audience, towards a more abstract and more formal presentation.

The idea of the Cultivation principle is rooted in Davydov’s (1990) work concerning generalization. One of Davydov’s claims was that the mental function of conceptual generalization is basic for mastering a new (for the learner) concept. “In the process of learning and practical activity a person uses various *rules* of operation. A condition for the application of a rule in a specific situation or to an individual object is that they first be attributed to a certain *general* class. Therefore, one must know how to ‘see’ this general aspect in every specific and individual case” (p. 11). According to Davydov, students should be led to develop concepts by implementing this mental function of conceptual generalization thoroughly: “Having mastered it, the students bridge the gap between the concrete and the abstract which exists originally in their consciousness. A basic means of bridging this gap is to enrich the child’s *sensory experience*.” (p. 12).

4 Selecting content

The choice of mathematical topics included in the curriculum (Israel Ministry of Education 2019) is largely similar to what was studied in the past by Israeli high school students, but with new emphases corresponding to the three principles. The curriculum includes calculus, geometry, statistics and probability, complex numbers, and some elements of discrete mathematics. It does not contain any units dedicated to algebraic or other techniques; required techniques are taught (and repeatedly exercised) in a suitable context within the relevant chapters.

The calculus component of the curriculum is distributed over the three high school years for a total of 170 h out of the 450 mathematics hours, or approximately 38%. It can be seen as composed of the units on precalculus, derivative, and integral. While again no research is mentioned in the curriculum document, in this section we consider it through the lenses of a researcher, and show that it is in fact strongly research-linked.

With respect to precalculus, the curriculum suggests deepening the understanding of the concept of function and

developing a ‘sense of functions’—identifying and analyzing functions in a qualitative manner, including the notions of linear and quadratic functions (that were learned in various contexts in the middle school). Developing a sense of function is suggested as an important step towards meaning-making; it was introduced into the research literature by Eisenberg (1992) and further developed by Artigue (1998), Shternberg (2001), Thompson and Carlson (2017), and others. Next, new families of functions are introduced (e.g., absolute value functions, power functions) together with operations on functions (transformations, arithmetic operations, composition); this development is accomplished using mainly graphical considerations as this register has been found to be effective for learning (Sever 2007). Expanding the database of familiar functions and actions performed on them has been found to constitute a solid base for constructing calculus concepts (Akkos and Tall 2002).

The curriculum presents the derivative as an expression for the rate of change of the function. This is coherent with research literature suggesting rate of change as the link between the mathematical concept of derivative and mathematics as a tool for describing continuously changing phenomena in everyday life and scientific contexts (e.g., Schneider 1992; Thompson 1994; Byerley and Thompson 2017; Greefrath et al. 2016). The development of the concept of the derivative goes from constant rate of change to variable rate of change. Step-functions are used as a tool for analyzing the graph of a function as well as its rate of change.

The derivative is defined after basing the idea on contextual situations in everyday life and on the graphical presentation of values that change with time or place, thus establishing links to what has been learned in middle school. Later, the second derivative is presented as a change in the rate of change of a function, including the notions of concavity, convexity and inflection points.

The use of dynamic software for the demonstration and investigation of processes of change is recommended in the curriculum. And indeed, research studies have shown that students are more likely to understand the notion of derivative when learning incorporates technology (e.g., Kaput 1994; Nemirovsky et al. 1998; Niss et al. 2007).

The curriculum proposes accumulation as the core idea for the concept of the integral. In the next section we take the unit on integration as an example in order to describe and exemplify how the emphases expressed in the three principles are interwoven with the content to create a coherent whole.

5 The concept of the integral

5.1 Research on integration as accumulation

The idea of accumulation is foundational for integration (Thompson and Silverman 2008). It also has didactical benefits related to (i) a direct link to the uses of integration in everyday life and science; (ii) the elimination of the compartmentalization of the integral into definite and indefinite; (iii) the interconnection between the two processes of integration and differentiation. A complete development incorporating accumulation for college level calculus in the USA was led by Thompson; it resulted in a complete calculus course that is heavily technology-dependent (Thompson and Ashbrook 2015). The underlying idea is that calculus was invented to solve two foundational problems, namely, given a rate of change function, finding the function describing the quantity that accumulates according to the given rate of change, and vice versa, given an amount function finding the rate at which it changes (Thompson et al. 2013). Considering that the operations that solve these two problems are inverse to each other, leads directly to the fundamental theorem of calculus (FTC).

Building on this background, and adapting it to the high school level, Kouropatov (2016) suggested a curriculum dedicated to the integral calculus, where the notion of integral was based on the idea of accumulation in its plain meaning: an accumulating sum that has a very large number of very small terms. The proposed approach is easier to handle than the classical one, even at the high school level, and yields both notions of integral (definite and indefinite) and the FTC at the same time, all in one interlinked package. The details of this curriculum were described by Kouropatov and Dreyfus (2013a); students' construction of knowledge according to this curriculum were empirically tested on a small scale (5 pairs of students in a laboratory situation) and found to be feasible (Kouropatov and Dreyfus 2014).

5.2 Structure of the unit on integration

In light of the research on integration as accumulation, the team decided to introduce the integral into the curriculum via the idea of accumulation, making adaptations from the proposals of the research, in order to take into account the background, knowledge and experience of teachers.

The unit on integration is thus a special case in the curriculum; it is the only unit in which the basic mathematical-didactical approach is radically different from what has been taught in the vast majority of classes previously. The

two main reasons why the idea of accumulation was convincing to the team were the applicability of accumulation in extra-mathematical problems, and the link accumulation establishes between the definite and the indefinite integral. This latter link leads to the proof of the fundamental theorem, which links accumulation and rate of change.

The integral is introduced as the accumulation function of a given rate of change function, which is not necessarily positive, using mathematical and extra-mathematical examples, and, in the beginning of the unit in particular, motion on a straight line with speed as the given rate of change function. The unit culminates when the conceptual foundation laid earlier with the meaning of the derivative as rate of change links with the view of the integral as accumulation by means of the fundamental theorem of calculus.

The curriculum proposes the following 5-part structure for the unit:

1. Accumulation of a constant function and of a step function
2. Approximation to the accumulation of a general (continuous) function by means of accumulation of step functions
3. Definition and properties of the accumulation function
4. The fundamental theorem of calculus
5. Applications of the fundamental theorem, including the notion of primitive function and computations of simple integrals in applied situations

5.3 Integrating the principles in the implementation of the unit on integration

The curriculum proposes a sample introduction to the unit, which starts with an activity presenting various exercise regimes for a bicycle rider in the form of speed as a function of time, starting from a constant function, via step functions, to linear and more general functions; students are asked to investigate what distance the rider has covered. Two early examples for speed as a function of time and sample questions students might be asked are included in Fig. 4.

In terms of mathematical content, this task introduces the notions of accumulated value and accumulation function graphically, numerically and verbally. It exemplifies the Cultivation principle by using a situation that is likely to be familiar to students in order to make the accumulating quantity emerge naturally, specifically by partitioning the domain into intervals so that the accumulated quantity is a sum of products represented by rectangles under the graph of the rate of change function, and thus by the area under this graph. Making the time steps shorter (last bullet), may raise questions related to issues of approximation and limits, without yet requiring the students to deal with these issues.

Fig. 4 Sample activity for introducing the idea of accumulation function (T) refers to technology as explained below (Sect. 5.4)

This aspect prepares students for a later task in which the rate of change function is linear, as well as for a discussion about the sense in which the area under the graph represents the distance covered in this case. We also note that the term ‘accumulated distance’ is introduced as a common language term in this specific situation, which can later be given a more technical and more generally valid definition—another feature of Cultivation.

At the same time, the tasks in Fig. 4 implement the Impact principle as they are set in an extra-mathematical context, a situation from motion that is very likely to be familiar to the students. Within this situation, another aspect of Cultivation is nested, namely a first appearance of the fact that the contributions to the accumulated quantity are products (here of speed and time), as they are in a majority of applied situations where integration is used to find an accumulated quantity.

The notions initiated in the sample activity in Fig. 4 can then be expanded in several directions, for example in the following: Other applied contexts as well as mathematical examples without extra-mathematical context; linear and then more general continuous given rate of change functions (part 2 above); rate of change functions with negative values, for example for the volume of water in a tank or the amount of money in a bank account; properties of accumulation functions and their relationship to the properties of the

For the speed function given by both graphs,

- What’s the total distance the rider covered?
- Draw a graph that associates the accumulated distance the rider has covered with the time she has ridden (T)
- What’s the relationship between the distance covered and the area between the graph and the x-axis?

For the speed function given by the lower graph, in addition,

- How will this graph change when the time steps are made shorter and shorter (T)

given rate of change function (part 3); and later the introduction of the algebraic representation.

Properties of the accumulation function can be introduced using the same bicycling situation with a different (graphically) given speed function, for example,

$v(t) = 600 - 20 \cdot |t - 30|$, $0 \leq t \leq 60$, with t measured in minutes and v in meters per minute. This and similar examples afford opportunities to discover relationships such as the following:

- if the rate of change function is positive, the accumulation function is increasing;
- if the rate of change function is increasing, the accumulation function is concave from above;
- if the rate of change function has a maximum, the accumulation function has a point of inflection.

These relationships are obtained before students know that the accumulation function is an antiderivative of the given rate of change function; on the other hand, activities leading to these relationships may be used to conjecture this deep connection, for example in the form that the given rate of change function is the derivative of its accumulation function—another instance of Cultivation and Reasoning. This prepares students for the fundamental theorem of calculus (part 4). At this stage, the operations of finding the

accumulation function given the rate of change function, and finding the rate of change function given the accumulation function, may be presented as operations inverse to each other, illustrating the Reasoning principle.

Once the knowledge that the accumulation function is an antiderivative of the rate of change function has been institutionalized, computations of elementary integrals and standard applications, such as finding the volume of a solid of revolution or the average value of a function over an interval, become accessible (part 5). However, the curriculum document warns against a transition to well-known standard exercises that disconnects from the first four parts of the unit and would turn integration into a collection of techniques. This warning is not merely abstract, but illustrated by tasks that make the connection between the conceptual approach and more technical and algebraic standard exercises. The task in Fig. 5 exemplifies how this connection is made. As such it shows an important aspect of the Reasoning principle.

The graph in Fig. 5 is introduced as showing the rate at which snow (in cubic meters per minute) accumulated during a storm in Mas'ada near the Hermon mountain, time being measured in minutes. The task opens with four questions about properties of the accumulation function, ending with 'when was the amount of snow maximal?' Then the algebraic representation of the given graph is introduced, and two additional questions are asked:

$$f(x) = \begin{cases} -0.1(x-30)^2 + 90, & 0 \leq x \leq 30 \\ 3x, & 30 \leq x \leq 60 \\ -0.1(x-60)^2 + 180, & 60 \leq x \leq 90 \\ -0.05(x-90)^2 + 90, & x > 90 \end{cases}$$

Fig. 5 Accumulation of snow according to $y = f(x)$

- What's the quantity of snow that accumulated from minutes 0–120?
- Express algebraically the function $V(x)$ that describes the amount of snow that accumulated until time x for $0 \leq x \leq 120$; note appropriate units.

5.4 Summary

The principles of Impact, Reasoning and Cultivation play a central role in this unit as in all others, and specifically in the tasks in Fig. 4. The given situation is close to the everyday experience of the students; the question how far the cyclist rode, altogether or in the first t minutes of the ride, is completely natural, and prepares the intellectual ground for the idea of accumulation to emerge (Cultivation). The tasks set for the students puts this notion in the focus of attention by asking questions for whose answer the concept of accumulation is needed. All of this is happening in a situation allowing students to experience the power of mathematics and of the idea of accumulation, in order to answer relevant questions arising in the real world (Impact). The principle of Reasoning is implemented in the use of several connected representations. The task in which properties of the accumulation function are explored presents excellent opportunities for explanation and justification (why is the accumulation function concave where the rate of change function is increasing?), which are other central aspects of Reasoning. Reasoning is also important during the introduction of the definition in part 3, not treated in detail here.

The accumulation approach to integration is based on a dynamic process of approximation, which is most evident in part 2; as a consequence, it is essential that learning be supported by a broad use of technological tools, especially at the beginning of the unit. For example, the two questions marked by (T) in Fig. 5, as well as the situation where properties of the accumulation function are explored, are meant to become technology-based investigations; the curriculum not only mentions this intent, but provides links to a suitable application program. Similar pointers to the use of technological tools are inserted throughout the curriculum. Swidan and Yerushalmy (2014; see also Swidan 2015) explored the learning process of the concept of integral in an interactive technological environment. They considered technology as a dynamic artefact and concluded that its use may lead to the change of the role of the students—from accomplishing a mathematical mission to “objectifying the historical, cultural knowledge deposited in the artifact through a mediated and reflected activity” (p. 530).

Of course, the curriculum only provides a framework—it is not a textbook. Textbook authors will need to make many decisions concerning the order of instruction, the details of approaching the new concepts, the choice of tasks, and the formulation. Here we focused on the contents of the

curricular document and the manner in which the principles find their expression in it.

6 The matriculation examination

One of the persistent demands of the ministry representatives on the curriculum committee was their insistence that everything that was included in the curriculum should be examinable. This demand can be explained by the tradition of examination programs (see Sect. 2.3). This tradition preserved the structure of the examination, and as a consequence, prevented including issues and aspects that are difficult to examine given the current structure.

Consider, for example, the Cultivation principle. This principle stands in direct opposition to current practice in the sense that it asks where students come from, whereas current practice asks where students need to get to, in order to pass the examination. Similarly, several aspects of the Reasoning principle, including basic understanding of concepts, and justifications of simple properties are not well aligned with complex sequences of questions, which characterize the current examination structure.

In order to enable examination of aspects of Reasoning, the 5-units team proposed to the professional committee for mathematics the inclusion in the matriculation examination of some short questions, adapted to mathematical ways of reasoning corresponding to the Reasoning principle. Such questions would give opportunities to demonstrate mathematical ways of reasoning with little need for computation. Moreover, questions that require little or no computation provide time and opportunities to deepen conceptual understanding, and enable the linkage between different aspects of a concept (such as slope and concavity); this is particularly relevant for matriculation examination questions, since many

teachers spend a considerable amount of time discussing them in class.

For example, an item could ask students to find a property such as the location of extrema of a certain function $f(x)$ and then ask about the corresponding property of $g(x) = f(3x)$ or some other transform. The proposal to include short questions in the matriculation examination has been approved and various potential short questions have been proposed.

For the task in Fig. 6, no technical algebraic work is required; while students may produce $f(x) = (x - 2)^3 + 3x$ if they are algebraically proficient, this is not expected; in fact, they are explicitly told that a graphical justification is acceptable; on the other hand, conceptual knowledge about points of inflection is addressed, and so is flexibility. Neither the term ‘point of inflection’ is mentioned, nor the common definition (change from concave to convex or the opposite); answering correctly requires recognition that the tangent at a point of inflection need not be horizontal; and an acceptable justification must be produced, which could be an example, graphical or algebraic. Points of inflection with non-horizontal tangents do not only occur frequently in the curriculum (Figs. 1, 2 and 3), but the manner in which this issue has been cultivated (e.g., in Fig. 3), and supported by the Reasoning (Fig. 1) and Impact principles (Fig. 2) gives students many opportunities to acquire the required flexibility concerning the notion of inflection point and the required ability to justify (which is implemented systematically as a matter of Reasoning, throughout all topics, not only calculus. By the way, in the task in Fig. 5 the accumulation function giving the amount of snow also has a non-horizontal inflection point (at $x = 60$), and the version that actually appears in the curriculum document asks for the domains where this function is concave up and concave down.

The example in Fig. 7 similarly relates to the Reasoning principle, and to the notion of accumulation

Fig. 6 A short question on points of inflection

Is the following claim true or not? Justify! You may justify by sketching graphs.

Given a function $f(x)$ whose graph passes through the point $P(2, 6)$. If the second derivative changes sign at $x = 2$, then the equation of the tangent to the graph of the function at P cannot be $y = 3x$.

Fig. 7 A short question on accumulation

The sketch presents the graph of the function . For which point P , among A, B, C, D, E, F, G is the value of $\int_{x_A}^{x_P} f(x) dx$ largest? Justify your answer.

(Subsection 4.7). Again, no technical work is required, or even possible. An identification of integral with area, as is common (Sect. 3.1), is likely to produce the wrong answer G. A conception of integral as signed area might produce the correct answer E, but a view of the integral as accumulating elements of $f(x)\Delta x$ from left to right will make this answer immediately accessible.

Of course, it is expected that many new examination questions, and possibly new types of questions will be proposed, tried and then accepted or rejected, as the curriculum will be gradually implemented.

7 Concluding remarks

In this paper, we took the finished curriculum document as a point of departure and proceeded backwards to search for evidence of the principles that may have guided the team in writing the document. We found that these principles are rooted in the team members' knowledge, experiences, beliefs and views; and these, in turn, are based to a large extent upon the team members' familiarity with mathematics education research. The question of the influence of research on national curriculum documents is relevant since Rojano and Solares-Rojas (2018) recently noticed that in each of the four countries they considered, and which did not include Israel, research had clearly influenced the national curriculum documents. Yet in none of these documents was this influence actually explicit. We found that, with the single exception of adopting an accumulation approach to integration, the research influence was similarly implicit in the Israeli document, and we asked how this influence came about.

In particular, we found (Sect. 3) that research had a powerful influence on the three principles according to which the team designed the document, and hence the curriculum: Reasoning, by means of which sample tasks in the curriculum document incorporate fundamental mathematical ideas and ways of mathematical reasoning; the Impact expected from connecting mathematics to its role in everyday life and science; and the Cultivation of fertile intellectual ground from which new concepts may emerge naturally.

Since research is not mentioned explicitly in the document, we infer that the manner by which research influenced the work of the team is rooted in the previous experience of a majority of the team members with research in mathematics education. While much of their own research was not directly linked to the learning and teaching of high school calculus, they were mathematics teachers and mathematics teacher educators and the usefulness of research for the practice of learning and teaching the high school curriculum in general and calculus in particular was compelling to them. They were particularly knowledgeable about research that had been done in Israel with Israeli high school students,

a population with which they are intimately familiar. This background naturally influenced their work in the team. For the purpose of demonstrating this, we selected the case of research on the multiple aspects of convexity of functions and points of inflection (see Sect. 3, Fig. 1), and alternative conceptions of students that this research had investigated (Tsamir and Ovodenko 2013). This led the team closely to connect the idea of concavity and second derivative to variation of the rate of change (as in Figs. 2 and 3 above), and to define points of inflection as points where the convexity changes while the function is continuous, and hence extremal points of the rate of change. This connection allows the emphasis that the change of convexity has nothing to do with the value of the first derivative, and hence that inflection may occur at a point with any rate of change, positive, negative or zero.

The team members' research backgrounds similarly influenced many other topics such as the use of shift and stretch transformations of functions in precalculus (Sever 2007), an emphasis on the relationship between derivative and rate of change, shown to be crucial for the fundamental theorem of calculus (Kouropatov and Dreyfus 2013b), the formulation of questions encouraging appropriate use of examples for justification (Buchbinder and Zaslavsky 2019), specific recommendations for using technology (Swidan and Yerushalmy 2014), and more.

The report presented in this paper has several limitations. One is that the process of writing the curriculum was not accompanied by any formal research. For example, no systematic information is available with respect to the interactions between different groups of professionals involved in the development of the curriculum, and the ensuing differences of opinion and tensions as pointed out by Potari et al. (2019).

Another limitation is that we report on work in progress; the curriculum document has been completed, submitted to the authorities, and approved as of this writing, but implementation has only started experimentally on a limited scale of about 30 schools. Even in these schools, the Ministry of Education has decided not to implement the accumulation approach to integration for the first cohort of students but only for the second cohort, in order to limit the demands on teachers who might not be familiar with this approach; this decision can be considered as a case of tension between professionals with different tasks in introducing a new curriculum.

As a consequence of these limitations, this paper is more a report about how a calculus curriculum came into being and how research has influenced this process, hopefully an interesting and relevant report, but still a report rather than a research study. In spite of these limitations, we feel confident to have shown a variety of informative ways by which research may find its way into national curriculum

documents. From our presentation on roles research studies have played in designing the curriculum, we may conclude that research can have a powerful influence in making a curriculum document coherent, stressing mathematical processes and reasoning, as well as intra- and extra-mathematical connections. This research influence need not be explicit. It may be based in team members' varied background as pure and applied mathematicians, and as mathematics education researchers and practitioners, and it may be manifested through their interaction in developing and consistently implementing a comprehensive and coherent set of principles: Cultivation, Reasoning and Impact. And these principles may provide a basis for the analysis of not only curricula but also a broad variety of instructional materials including textbooks and websites.

Acknowledgements The research reported in this paper has been partially supported by the Israel Science Foundation under grant 1743/19.

References

- Akkos, H., & Tall, D. (2002). The simplicity, complexity and complication of the function concept. In A. D. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th Conference of the International Group for the psychology of mathematics education* (Vol. 2, pp. 25–32). Norwich: PME.
- Artigue, M. (1998). Teaching and learning elementary analysis. In C. Alsina, J. M. Alvarez, B. Hodgson, C. Laborde, & A. Perez (Eds.), *8th International Congress on Mathematical Education: selected lectures* (pp. 15–30). Sevilla: Sociedad Andaluza de Educación Matemática Thales.
- Biehler, R. (2019). The transition from calculus and to analysis—Conceptual analyses and supporting steps for students. In J. Monaghan, E. Nardi, & T. Dreyfus (Eds.), *Calculus in upper secondary and beginning university mathematics—Conference proceedings* (pp. 4–17). Kristiansand, Norway: MatRIC. <https://matric-calculus.sciencesconf.org>. Accessed 8 Mar 2020.
- Bressoud, D., Ghedamsi, I., Martinez-Luaces, V., & Törner, G. (2016). *Teaching and learning of calculus*. Cham: Springer.
- Buchbinder, O., & Zaslavsky, O. (2019). Strengths and inconsistencies in students' understanding of the roles of examples in proving. *Journal of Mathematical Behavior*, 53, 129–147.
- Byerley, C., & Thompson, P. W. (2017). Secondary mathematics teachers' meanings for measure, slope, and rate of change. *Journal of Mathematical Behavior*, 48, 168–193.
- Clements, D. H. (2007). Curriculum research: Toward a framework for "research-based curricula". *Journal for Research in Mathematics Education*, 38(1), 35–70.
- Davydov, V. (1990). On the potential for implementing the idea of theoretical generalization in solving problems in educational psychology. In J. Kilpatrick (Ed.), *Types of generalization in instruction: Logical and psychological problems in the structuring of school curricula Soviet studies in the psychology of learning and teaching mathematics* (Vol. 2). National Council of Teachers of Mathematics: Reston.
- Dreyfus, T., Hershkowitz, R., & Schwarz, B. (2015). The nested epistemic actions model for abstraction in context - Theory as methodological tool and methodological tool as theory. In A. Bikner-Ahsbahs, C. Knipping, & N. Presmeg (Eds.), *Approaches to qualitative research in mathematics education: Examples of methodology and methods* (pp. 185–217). Dordrecht: Springer, Advances in Mathematics Education series.
- Eisenberg, T. (1992). On the development of a sense for functions. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy. Notes Series* (Vol. 25, pp. 153–174). Washington, DC: Mathematical Association of America.
- Ely, R. (2017). Definite integral registers using infinitesimals. *Journal of Mathematical Behavior*, 48, 152–167.
- Fares, M. (2018). How did a crisis in mathematics education lead to a positive reform? In N. Movshovitz-Hadar (Ed.), *K-12 mathematics education in Israel: Issues and innovations* (pp. 21–28). Singapore: World Scientific Publishing.
- Freudenthal, H. (1968). Why to teach mathematics so as to be useful. *Educational Studies in Mathematics*, 1, 3–8.
- Greefrath, G., Oldenburg, R., Siller, H. S., Ulm, V., & Weigand, H. G. (2016). Aspects and "Grundvorstellungen" of the concepts of derivative and integral. *Journal für Mathematik-Didaktik*, 37(1), 99–129.
- Israel Ministry of Education, Professional Committee for Mathematics. (2010). *Guidelines towards a high school mathematics curriculum*. Unpublished document. Jerusalem, Israel: Ministry of Education. [in Hebrew]
- Israel Ministry of Education. (2012). *The new mathematics curriculum for grades 7, 8 and 9*. http://cms.education.gov.il/EducationCMS/Units/Mazkirut_Pedagogit/Matematika/ChativatBenayim. Accessed 1 Apr 2020. [in Hebrew]
- Israel Ministry of Education. (2019). *5-units mathematics curriculum for grades 10, 11 and 12*. https://cms.education.gov.il/EducationCMS/Units/Mazkirut_Pedagogit/Matematika/ChativaElyona/pituach.htm. Accessed 24 July 2020. [in Hebrew]
- Kaput, J. J. (1994). The representational roles of technology in connecting mathematics with authentic experience. In R. Biehler, R. W. Scholz, R. Strässer, & B. Winkelmann (Eds.), *Didactics of mathematics as a scientific discipline* (pp. 379–397). Dordrecht: Kluwer.
- Kouropatov, A. (2016). *The integral concept in high school: Constructing knowledge about accumulation*. Unpublished doctoral dissertation. Tel Aviv University, Israel.
- Kouropatov, A., & Dreyfus, T. (2013a). Constructing the integral concept on the basis of the idea of accumulation: Suggestion for a high school curriculum. *International Journal of Mathematics Education in Science and Technology*, 44(5), 641–651.
- Kouropatov, A., & Dreyfus, T. (2013b). Constructing the fundamental theorem of calculus. In A. M. Lindmeier & A. Heinze (Eds.), *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 201–208). Kiel: PME.
- Kouropatov, A., & Dreyfus, T. (2014). Learning the integral concept by constructing knowledge about accumulation. *ZDM*, 46(4), 533–548.
- Larsen, S., Marrongelle, K., Bressoud, D., & Graham, K. (2017). Understanding the concepts of calculus: Frameworks and roadmaps emerging from educational research. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 526–550). Reston: National Council of Teachers of Mathematics.
- Laursen, S. L., Hassi, M. L., Kogan, M., & Weston, T. J. (2014). Benefits for women and men of inquiry-based learning in college mathematics: A multi-institution study. *Journal for Research in Mathematics Education*, 45, 406–418.
- Movshovitz-Hadar, N., & Hazzan, O. (2004). How to present it? The rhetoric of an outstanding lecturer. *International Journal of Mathematical Education in Science and Technology*, 35(6), 813–827.
- Nemirovsky, R., Tierney, C., & Wright, T. (1998). Body motion and graphing. *Cognition and Instruction*, 16(2), 119–172.

- Newton, I. (1687/1999). *Philosophiae naturalis principia mathematica*. Oakland, CA, USA: University of California Press (Translated by I. B. Cohen & A. Whitman).
- Niss, M., Blum, W., & Galbraith, P. L. (2007). Introduction. In W. Blum, P. L. Galbraith, H. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education: the 14th ICMI study* (pp. 3–32). New York: Springer.
- Orton, A. (1983a). Students' understanding of integration. *Educational Studies in Mathematics*, 14(1), 1–18.
- Orton, A. (1983b). Students' understanding of differentiation. *Educational Studies in Mathematics*, 14(3), 235–250.
- Ovodenko, R., & Tsamir, P. (2017). To be or not to be an inflection point. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education* (pp. 2209–2216). Dublin: DCU Institute of Education and ERME.
- Potari, D., Psycharis, G., Sakonidis, C., & Zachariades, T. (2019). Collaborative design of a reform-oriented mathematics curriculum: Contradictions and boundaries across teaching, research, and policy. *Educational Studies in Mathematics*, 102(3), 417–434.
- Rojano, T., & Solares-Rojas, A. (2018). The mathematics curriculum design from an international perspective: Methodological elements for a comparative analysis. In Y. Shimizu & R. Vithal (Eds.), *School mathematics curriculum reforms: Challenges, changes and opportunities proceedings of the ICMI Study 24 Conference* (pp. 475–482). <https://www.mathunion.org/icmi/conferences/icmi-study-conferences>. Accessed 6 Aug 2020.
- Schneider, M. (1992). On learning the rate of instantaneous change. *Educational Studies in Mathematics*, 23(4), 317–350.
- Sever, G. (2007). To feel and to see linear transformations of functions in a multi-computerized environment. Unpublished MA thesis. University of Haifa, Israel. [In Hebrew]
- Shternberg, B. (2001). Effect of analysing of the change of a function on the understanding of the mathematical meaning of phenomena and of the concept of a function. Unpublished doctoral dissertation. University of Haifa, Israel.
- Stump, S. L. (2001). High school precalculus students' understanding of slope as measure. *School Science and Mathematics*, 101(2), 81–89.
- Swidan, O. (2015). Emergence of processes of accepted meanings of mathematical signs and their influence on usage: The case of the integral. Unpublished doctoral dissertation, University of Haifa, Israel.
- Swidan, O., & Yerushalmy, M. (2014). Learning the indefinite integral in a dynamic and interactive technological environment. *ZDM*, 46(4), 517–531.
- Tall, D. O. (1986). Building and testing a cognitive approach to the calculus using interactive computer graphics. Unpublished doctoral dissertation. University of Warwick, UK.
- Tallman, M. A., Carlson, M. P., Bressoud, D. M., & Pearson, M. (2016). A characterization of calculus I final exams in US colleges and universities. *International Journal of Research in Undergraduate Mathematics Education*, 2(1), 105–133.
- Taub center staff (2015). Do the math: The connection between mathematics matriculation units of study and salary level. <https://taubcenter.org.il/math-connection-math-bagrut-units-study-salary-level>. Accessed 28 Mar 2020.
- Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181–236). Albany: SUNY Press.
- Thompson, P. W., & Ashbrook, M. (2015). *Calculus: Newton, Leibniz, and Robinson meet technology*. <http://www.patthompson.net/ThompsonCalc/>. Accessed 15 Mar 2020.
- Thompson, P. W., Byerley, C., & Hatfield, N. (2013). A conceptual approach to calculus made possible by technology. *Computers in Schools*, 30(1–2), 124–147.
- Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421–456). Reston, VA: National Council of Teachers of Mathematics.
- Thompson, P. W., & Silverman, J. (2008). The concept of accumulation in calculus. In M. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics* (pp. 117–131). Washington, DC: Mathematical Association of America.
- Törner, G., Potari, D., & Zachariades, T. (2014). Calculus in European classrooms: Curriculum and teaching in different educational and cultural contexts. *ZDM-Mathematics Education*, 46(4), 549–560.
- Tsamir, P. (2007). Should more than one theoretical approach be used for analyzing students' errors? The case of areas, volumes and integration. *For the Learning of Mathematics*, 27(2), 28–33.
- Tsamir, P., & Ovodenko, R. (2013). University students' grasp of inflection points. *Educational Studies in Mathematics*, 83(3), 409–427.
- Woods, C., & Weber, K. (2020). The relationship between mathematicians' pedagogical goals, orientations, and common teaching practices in advanced mathematics. *The Journal of Mathematical Behavior*, 59. <https://doi.org/10.1016/j.jmathb.2020.100792>.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.