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ORIGINAL PAPER

Language-responsive support for multiplicative thinking as unitizing: results of an intervention study in the second grade

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Abstract
Multiplicative thinking involves the ability to coordinate bundled units on a more abstract level than additive thinking and implies the identification of the different meanings of the multiplier and the multiplicand. The transition from additive to multiplicative thinking, however, constitutes an obstacle for many children. Specific formulations that are typically used in classroom discourse for talking about multiplicative tasks and situations (e.g., '3 times 4' or '3 lots of 4') might inhibit meaning-making processes because they do not address the idea of unitizing. A language-responsive introduction to multiplication that addresses the core idea of unitizing and that uses phrases such as '3 times 4 means you have 3 fours' may help to overcome these problems. In the study presented in this paper, three second grade primary school teachers joined a teacher program to introduce multiplication in their classes ($n = 66$) by addressing meaning-making phrases. Another 58 second graders taught by teachers without this teacher program served as the control group. A specially developed multiplication test gave insight into the children's understanding of multiplication as unitizing immediately after the intervention (posttest) and nearly three months later (follow-up test). We found significant differences between the intervention and control groups in the multiplication posttest. These differences could be underlined in the follow-up test. Our results indicate that a language-responsive teaching intervention that focuses on meaning-making processes can lead to long-term insights and help to develop multiplicative thinking as unitizing.

Keywords Multiplication · Unitizing · Multiplicative thinking · Conceptual understanding · Language-responsive teaching · Meaning-making processes

1 Fostering multiplicative thinking: a big challenge

Multiplication is an important topic learned in mathematics classes. Because of the complexity involved in understanding the essence of what makes a situation multiplicative (Clark and Kamii 1996; Jacob and Willis 2003), the development of multiplicative thinking requires a long period of time (Clark and Kamii 1996). It is often described as a learning trajectory in four central phases: (1) direct counting, (2) rhythmic or skip counting, (3) additive thinking (possibly by saying the count-by sequence), and (4) multiplicative thinking (Anghileri 1989; Battista 1999; Downton and Sullivan 2017; Larsson 2016; Mulligan and Watson 1998; Ruwisch 1998; Siemon et al. 2005; Simon and Blume 1994; Sherin and Fuson 2005; Steffe 1992; Sullivan et al. 2001; Thompson and Saldanha 2003). In the initial stage, repeated addition is considered to be more sophisticated than counting all or counting by multiples; however, equating multiplication with repeated addition is limiting because beyond natural numbers this way of thinking is no longer viable (e.g., $5½ \times 4¾$ cannot be solved by additive thinking; Thompson and Saldanha 2003). In contrast, multiplicative thinking involves the ability to coordinate bundled units on a more abstract level than additive thinking and implies the identification of the different meanings of the multiplier and the multiplicand (Clark and Kamii 1996; Downton and Sullivan 2017; Larsson 2016; Singh 2000; Steffe 1992). This ability is often called 'unitizing' (Lamon 1994) or 'dealing with composite units' (Steffe 1992). The transition from additive to multiplicative thinking, however, constitutes an obstacle

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for many children (Ehlert et al. 2013; Gaidoschik et al. 2018; Götze 2019a, b; Moser Opitz 2013; Siemon et al. 2005). The complexity of multiplicative thinking is also apparent in the fact that even students in upper grades have difficulties solving (two-digit) multiplication tasks (Mulligan and Mitchelmore 1997; Siemon et al. 2006), providing a multiplication story even if the numbers are small (Moser Opitz 2013), and discriminating multiplicative from additive situations (van Dooren et al. 2010). Furthermore, Siemon (2019) showed in the analysis of nearly 7000 children's data from Grade 5 to 9 that there is a 7-year span in students' general numeracy competences in each grade. According to Siemon, it is multiplicative thinking that seems to be responsible for this 7-year span (Siemon and Breed 2006; Siemon 2019). In consequence, the introduction and learning of multiplication is considered a 'cutoff point' (Cawley et al. 2001) for students' future mathematical learning because multiplicative thinking as thinking in bundled groups is fundamental for understanding topics such as fractions, proportionality, and percentage calculation (Downton and Sullivan 2017; Pöhler and Prediger 2015; Prediger 2011, 2019; Siemon 2019).

The increasingly pressing question is how multiplicative thinking as unitizing might be supported in young children. Therefore, further research is needed that gives evidence on how multiplicative meaning-making processes can be designed and how children can learn to *think* multiplicatively. Such meaning-making processes can demonstrably be supported by relating both different mathematical representations (concrete, graphical, symbolic, and verbal) and language registers (everyday, academic, and technical) forwards and backwards (for an overview see Erath et al. 2021) with a focus on verbalizing multiplicative structures. This implies the following:

> Students should not only translate the multiplication 5×3 into an array model with five rows of 3 points each, but also explain how to see the unitizing structure in the rows ('five threes' or 'five sets of three') in order to verbalize the meaning of multiplication as unitizing. (Erath et al. 2021, p. 5)

These meaning-related expressions such as '5 threes' from the everyday register combined with different mathematical representations such as rectangular arrays might enhance multiplicative thinking as unitizing. The qualitative studies of Breed (2011) and Götze (2019a, b) indicated that such language-responsive instruction might help children to develop multiplicative thinking as unitizing. However, the topics of these studies are the multiplicative learning pathways of children who have already learned the topic of multiplication. Less is known about the effects of using language-responsive methods to fostering multiplicative thinking as unitizing right from the start. To pursue this research direction, three second grade teachers were supported to address meaning-related phrases when multiplication was introduced in their mathematics classes. Additionally, three other classes served as the control group, whose teachers taught multiplication in their usual manner. The multiplicative thinking as unitizing of all these children was tested and analyzed immediately after the intervention (posttest) and nearly three months after the posttest (follow-up test). In the sections that follow, multiplicative thinking as unitizing and the language-responsive approach for fostering this understanding are defined (Sect. 2). After reporting on the methodology of the intervention study (Sect. 3), selected research outcomes of the quantitative investigations are presented (Sect. 4) and discussed (Sect. 5).

2 Theoretical background: multiplicative thinking as unitizing and the role of language-responsive support

2.1 Multiplicative thinking as unitizing

Many studies have indicated that multiplicative thinking is more conceptually demanding than additive thinking. This demand is due to the required process of abstraction needed to understand a situation as multiplicative (Downton and Sullivan 2017; Steffe 1994). One aspect of particular importance is to understand the different meanings of the factors: The multiplier indicates the number of composite units and the multiplicand the size of each composite unit. In Germany, the common interpretation is that the first number represents the multiplier and the second number the multiplicand (Ruwisch 2002). Multiplication is often introduced as and thereby connected with repeated addition, e.g., $3 \times 4 = 4 + 4 + 4$, which can cause difficulties in later grades if it remains the only model used for multiplication (Thompson and Saldanha 2003). This difficulty is due to the fact that in the operation addition, the addends have the same meaning. If children tend to interpret multiplication only as repeated addition of equally sized groups, this understanding is limited since it is not applicable to multiplication beyond natural numbers (Larsson 2016; Verschaffel et al. 2007). Therefore, one aim of mathematical instruction in upper grades should be for students to be able to explain how multiplication and addition are related and that whole number multiplication can also be described as repeated addition (Moschkovich 2015). Nevertheless, the development of the ability to reflect this interrelation of addition and multiplication and the process of abstraction from additive to multiplicative thinking is a big challenge (Downton and Sullivan 2017; Jacob and Willis 2003; Siemon et al. 2005) and involves two central and directly related requirements (Singh 2000). The first is to understand the core idea of 'unitizing' (Lamon 1994) as "operating with singleton units to coordinating composite units" (Singh 2000, p. 273). The ability to imagine a collection of single elements as a composite unit has been regarded as a key stage in multiplicative thinking (Clarke et al. 2006; Götze 2019b; Lamon 1994; Steffe 1994). It is directly related to the second requirement, namely, the awareness of the different meanings of the multiplier and the multiplicand (Singh 2000). Therefore, when children are able to think and to argue not only about units of one but also about composite units of more than one, they start to think multiplicatively (Downton and Sullivan 2017). This is what Steffe (1992, 1994) has repeatedly shown in his in-depth analyses of different multiplicative thinking learning pathways of primary school children.

Consequently, multiplicative thinking as unitizing as a general ability to construct "a reference unit or a unified whole, and then to reinterpret a situation in terms of that unit, appears critical to the development of increasingly sophisticated mathematical ideas" (Lamon 1994, p. 92).

In primary school, such an increasingly sophisticated mathematical idea is, for example, the understanding of multiplicative decomposition strategies that are based on an understanding of multiplicative commutativity, associativity, and distributivity (Anghileri 2000; Barmby et al. 2009; Downton and Sullivan 2017; Larsson 2016). This understanding requires the reorganizing of multiplication facts in a more flexible way. For example, solving 7×8 using 5×8 and 2×8 involves the transfer of a part-whole concept to composite units (Downton and Sullivan 2017; Lamon 1994).

However, as mentioned above, many children have difficulties in multiplicative thinking as unitizing and do not understand how multiplicative tasks are interconnected. Many children up to Grade 2 solve multiplicative tasks using additive thinking (Moser Opitz 2013; Siemon 2019; Siemon et al. 2006), and it is common to find that they cannot use simple memorized times table tasks to solve more difficult tasks and thus they revert back to additive thinking or counting (Downton and Sullivan 2017; Götze 2019a; Moser Opitz 2013; Mulligan and Mitchelmore 1997; Siemon 2019). Another difficulty is that children tend to be confused as to why the 2 and 5 can be added but the 8 remains when the tasks 2×8 and 5×8 are combined to 7×8 (Baiker and Götze 2019). In the study by Moser Opitz (2013), many of the fifth and eighth graders tested had difficulties in solving tasks such as 20×30 even though it is related to 2×3, a task that they were able to solve by retrieval. This underpins that "retrieval is an efficient strategy only if the child knows the [underlying multiplicative] math fact" (van der Ven et al. 2012, p. 2), in other words, when they think multiplicatively.

In summary, multiplicative thinking as unitizing conceptually implies two core concepts:

- *Concept A*: Thinking in composite units (Lamon 1994; Steffe 1994). This is directly related to the awareness that the multiplier and the multiplicand have different meanings and the ability to coordinate them independently (Downton and Sullivan 2017; Götze 2019a, b).
- *Concept B*: Independent use of Concept A for multiplicative retrieval with recourse to decomposition strategies using associative, distributive, and commutative relationships (Baiker and Götze 2019; Downton and Sullivan 2017; Gaidoschik 2015; van der Ven et al. 2012).

Some existing empirical findings have indicated that specific formulations that are typically used in classroom discourse for talking about multiplicative tasks and situations from the technical and even the everyday register might inhibit the development of multiplicative thinking as unitizing (Breed 2011; Downton and Sullivan 2017; Götze 2019a, b; Larsson 2016; Thompson and Saldanha 2003). These are formulations such as '3 times 4' and '3 multiplied by 4' but also '3 lots of 4' and '3 groups of 4'. The first two expressions are technical terms and linguistically do not illustrate the idea of unitizing (Anghileri 1989, 1991). The expressions 'lots of' or 'groups of', however, facilitate demonstrably additive thinking because they focus on the idea of equal additive groups and thus of adding the groups step by step (Larsson 2016). Moreover, these expressions are often connected with typical groups-of models of separate units (Barmby et al. 2009; Breed 2011; Downton and Sullivan 2017; Greer 1992; Larsson 2016). As a consequence, the quantification of equal-sized groups does not become obvious (Thompson and Saldanha 2003). Larsson et al. (2017) observed that learners who tended to use a 'groups of' interpretation showed limited conceptual understanding of both using the commutative property and multiplying decimals, because they tried to handle them additively.

However, in order to develop conceptual understanding of a mathematical topic, it is meaning that matters (Moschkovich 2015; Erath et al. 2018). Such understanding can be supported by language-responsive teaching and rich discursive meaning-making processes (for an overview, see Erath et al. 2021). In this respect, empirical evidence is needed on how language-responsive teaching can support multiplicative thinking as unitizing. If typically used expressions such as 'times', 'multiplied by', 'lots of', and 'groups of' result in merely additive thinking in many children, then it should be considered which expressions can support multiplicative meaning-making processes.

2.2 Language-responsive support for multiplicative thinking as unitizing

Erath et al. (2021) extracted in their literature review six major design principles for designing materials (tasks, lessons, and units) and instruction on enhancing language in mathematics classrooms. The teaching program of the intervention described in this paper was based on two of these major design principles: (a) connecting registers and representations and (b) focusing on rich (oral) discourse practices for meaning-making processes.

Erath et al. (2021) stated that, in particular for lexical support and for connecting different registers, meaning-making learning processes should start "from students' everyday resources, but explaining meanings often requires collective explanations and therefore a common meaning-related language and vocabulary" (p. 7) as a language for the classroom discourse. Thus, for meaning-making processes it is important to establish a collective meaning-related language (Pöhler and Prediger 2015; Prediger and Wessel 2013). For the topic of multiplication, this means that meaning-related phrases are needed that directly address multiplicative thinking as unitizing. Such meaning-related phrases—as Thompson and Saldanha (2003) note—might start with the following:

> the basic meaning of 5×4 as 'five fours', then 5 $2/3 \times 4$ means '(five and two-thirds) fours'. The principal difference between 'add four five times' and 'five fours' is that the former tells us a calculation to perform while the latter suggests something to imagine. (p. 24)

Combining such expressions with the concrete and graphical representations of rectangular arrays allows understanding of commutative, associative, and distributive properties even beyond natural numbers (Larsson 2016; Thompson and Saldanha 2003).

In the third assessment round of the Australian Scaffolding Numeracy in the Middle Years Project (SNMY), the multiplicative learning pathways of 1,732 students from Grades 7 to 10 were analyzed (Breed 2011; Siemon 2019; Siemon and Breed 2006). For this purpose, a learning and assessment framework for multiplicative thinking as unitizing (Siemon et al. 2006) was used as the basis for developing and implementing a targeted teaching approach for improving students' multiplicative thinking (Siemon 2019). Although the importance of supporting meaning-making processes using meaning-related expressions was not mentioned directly, the analysis of the examples of the teaching materials indicates that meaning-related expressions were explicitly considered for enhancing rich discourse practices. Typical prompts for stimulating oral classroom discourses and for deepening the multiplicative thinking of the students were as follows: "Think of 6 fours as 5 fours and 1 more four…. 18 is 2 nines, 9 twos, 3 sixes, 6 threes" (Siemon 2019, p. 19). Simultaneously, these phrases were visualized by rectangular arrays with a shift away from the 'groups of' idea to exploring rectangular arrays (Siemon et al. 2006). The data analysis shows that this teaching approach seems to be effective, because after four months of intervention many students significantly improved their multiplicative thinking levels.

An accompanying intervention study by Breed (2011) with 14 low-achieving sixth graders, gave some more indications of how multiplicative thinking was fostered in detail. Design elements of the intervention were, for example, (a) fostering efficient and reliable strategies for counting in larger groups (not only twos, fives, or tens) and (b) understanding arrays by identifying and naming the group sizes (e.g., 2 fours, 2 fives, or 2 sixes). To emphasize the idea of multiplicative thinking as unitizing, the language used changed from the language associated with 'groups of' (or '3 lots of 4') to '3 fours' to emphasize the numbers as composite units (Breed 2011).

A multiplicative oral pre-test showed that the 14 low-achieving sixth graders solved multiplication tasks with primarily basic strategies such as modeling, count-all strategies, and skip counting only for groups of less than five (Breed 2011). Nine of these 14 students were assigned to the intervention group. After an 18-week intervention, they solved tasks similar to those in the pre-test in a more sophisticated way and showed a deeper understanding of multiplication as unitizing. They justified their responses by expressions such as "think times, 4 fours, 16, and 4 twos, 8" (Breed 2011, p. 181). The other five students formed the non-intervention group and joined the regular mathematics classes without focusing linguistically on the idea of multiplication as unitizing. Only one student of the non-intervention group showed some progress in the posttest and the other four students made no progress.

Similar findings can be seen in the single-case intervention study carried out by Götze (2019b). In a four-lesson intervention, two third graders were supported to develop multiplicative thinking as unitizing. One core idea of the intervention was to verbalize the connection between different mathematical representations (concrete, graphical, verbal, and symbolic) in a meaning-related way: "I see 2 fives. This is why the task 2 times 5 fits this array." Relating and connecting such different mathematical representations demonstrably results in rich meaning-making communications (Götze 2019a; Prediger and Wessel 2013). The in-depth analysis of the children's individual learning pathways showed that only one of these two children developed a multiplicative understanding of decomposition strategies. Success seemed to depend on whether the children made the meaning-related language of multiplication their language of thinking (Götze 2019a, b). This means that "understanding depends primarily on if and to what extent the students are able to internalize these meaning-related verbalizations and how they can contribute to the forming of mental models" (Götze 2019a, p. 170).

In summary, the few studies available that focus on using language-responsive approaches to meaning-making of multiplicative thinking as unitizing show that one way to overcome the obstacle between additive thinking and multiplicative thinking can be to address meaning-related phrases while talking about and connecting different mathematical representations. Such expressions help children to recognize the composite units in concrete and graphical multiplicative arrays (Breed 2011; Götze 2019a, 2019b; Larsson 2016; Larsson et al. 2017; Thompson and Saldanha 2003) and symbolic terms, and can still be used for understanding decomposition strategies (e.g., '7 sixes are 5 sixes and 2 more sixes together' or '7 sixes are 7 fives and 7 ones together'). However, most of the studies are single-case studies and—as mentioned above—less is known about how effective it might be to use language-responsive approaches to fostering multiplicative thinking as unitizing right from the start of the introduction of multiplication in the second grade and in whole classes. Therefore, the study presented in this paper addresses the following main research question:

> To what extent can a preventive and language-responsive intervention program have an impact on students' outcomes in relation to multiplicative thinking?

The next section gives insight into the design and methodology of our study. We then present some results related to more specific research hypotheses.

3 Methodological frameworks

In order to pursue the research question and investigate the role of meaning-related language for the development of multiplicative thinking as unitizing, this research study was conducted as a quantitative controlled experiment in primary school when multiplication is introduced. It was realized in a controlled trial with children's multiplicative thinking as unitizing as the dependent variable and their general mathematics proficiency as the control variable. For the independent variable, the teaching intervention varied between a meaning-related-phrases-focused intervention group and a control group without additional treatment (see Fig. 1).

The 3-month teaching intervention was carried out by the children's current mathematics teachers. The children of the intervention group ($n = 66$) belonged to three second grade classes in one school, and the children of the control group ($n = 58$) belonged to three classes in a different school in close proximity to the intervention group school.

In order to ensure comparability of the data, the general mathematical competences of all children were tested with the standardized test Basis-Math 2 + (Moser Opitz et al. 2019). This test examines basic arithmetical topics such as place value, addition, subtraction, and even simple multiplication and division. It is the only instrument in Germany that tests both the procedural calculation skills and the conceptual knowledge of basic mathematical skills of second graders, but since it is conceptualized and validated for the end of second grade, we were forced to implement this test after the intervention phase in June 2019.

Because the children had not been taught multiplication before our intervention and the objective was to test multiplicative thinking as unitizing, the multiplication posttest was implemented directly after the intervention. Moreover, we found it ethically unacceptable to ask the children to solve those conceptual tasks before the intervention because many of them would fail, which might have influenced the multiplicative learning pathways of the children negatively.

The multiplication posttest in a slightly modified version (see Sect. 3.3) was also used as a follow-up test in the third week of school after the summer holidays. At this time, basic principles of multiplication had already been repeated in approximately four hours of teaching in both schools.

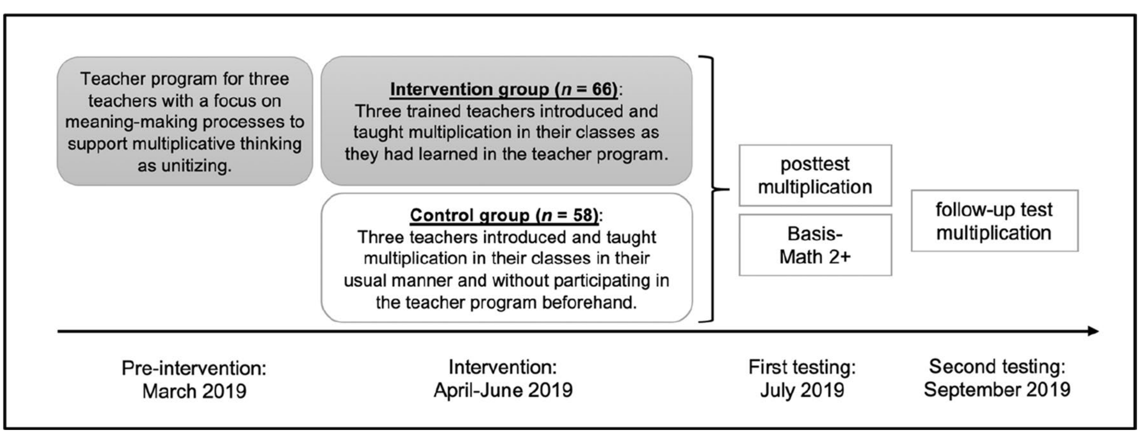


Fig. 1 Chronological design of the study

3.1 Design principles of the teaching intervention

Our second-grade intervention was based on two of the six major design principles of Erath et al. (2021). The main design principle of the study was as follows: relating different mathematical representations (concrete, graphical, verbal, and symbolic) and different registers by continuously interrelating and connecting these representations with a focus on oral and written meaning-related expressions. As the connection of representations and registers requires highly discursive processes, our study can also be assigned to the design principle of enhancing rich (oral) discourse practices.

The main design principle was taken from a well-evaluated German intervention for fostering conceptual understanding of fractions in Grades 7–9 (Prediger and Wessel 2013), which we adapted to an intervention for fostering conceptual understanding of multiplicative thinking as unitizing in primary school. Prediger and Wessel's (2013) study gives empirical evidence that in meaning-making processes in particular, different mathematical representations (concrete, graphical, verbal, and symbolic) and registers (everyday, academic, and technical language) need to be related forwards and backwards. "The emphasis is not on *changing*, but on *connecting* the registers and representations" (Erath et al. 2021, p. 5) with a focus on meaning-related expressions for supporting meaning-making processes. Such connections build the foundation for a language for thinking and talking about multiplication as unitizing (Götze 2019a, b) and, thus, for conceptual understanding.

Our intervention was based on the materials of the widely used German textbook *Das Zahlenbuch* (translated *The Number Book*, Nührenbörger et al. 2017) because this textbook already provides the connection of different representations and the development of decomposition strategies (Fig. 2). All teachers in the intervention and control group classes worked with this textbook. As usual in German textbooks, multiplication is initially introduced using everyday situations (groups-of and array models) with a focus on repeated addition. Simultaneously, the repeated addition is connected with the multiplicative expression 'times' and a multiplicative term (Fig. 2). The expressions 'lots of' and 'groups of' are not used because such expressions are uncommon in German. Progressively rectangular arrays are used. As Fig. 2 illustrates, students have to connect concrete, graphical, and symbolic representations (constructing arrays, interpreting graphical arrays, and finding symbolic terms). However, the connections of different registers with a focus on meaning-related expressions for meaning-making discourses are missing. The children in the textbook use expressions from the everyday register (e.g., "here are 6 and 6 and 6 and 6") or academic/technical register (e.g., "these are 4 times 6") and do not address directly multiplicative thinking as unitizing by means of meaning-related phrases. The design principle of connecting registers and representations (Erath et al. 2021) states that meaning-making processes need a common meaning-related language and vocabulary (Pöhler and Prediger 2015; Zwiers et al. 2017). In consequence, children should be supported to explain how to see the unitizing structure in rectangular arrays ('5 threes') in order linguistically to realize the meaning of multiplication as unitizing (Erath et al. 2021). Simultaneously, these phrases support the understanding of how multiplication tasks are interconnected. However, meaning-making phrases with a focus on multiplication as unitizing have been largely missing in German textbooks.

Therefore, for our intervention, the textbook materials were supplemented by additional hints for the teachers on how to connect representations and registers and how to enhance rich discourse practices. Figure 3 gives an example of our supplemental material for the intervention teachers. In a 1-day teacher program, the three teachers in the intervention group used our supplemental material to learn how to start from children's everyday language and move on to connecting different representations in oral classroom discourses. The teachers were further instructed to use meaning-related phrases when talking about multiplicative structures and to encourage the children to use meaning-related phrases such as '3 fours' individually and, thus, to support a "language for thinking and talking about structural relationships" (Prediger and Wessel 2013, p. 454).

The teachers of the control-group children did not join this teacher program and did not receive the supplementary material, but worked with the same textbook as the teachers of the intervention group.

3.2 Comparability of the intervention and the control group

As mentioned above, the overall mathematical competences of the children (as a control variable) were tested with the standardized test Basis-Math 2 + (Moser Opitz et al. 2019) as control variable. The scores of the children were used to conduct an independent sample *t*-test of the significance of the difference between the mean scores of the intervention and control groups. Regarding the basic mathematical competences at the end of second grade, the children of the control group ($M = 22.71$, $SD = 6.44$, $Med = 25$) performed slightly better, but with a greater standard deviation than the children of the intervention group ($M = 21.67$, $SD = 5.01$, $Med = 22$). The median of the control group showed that half of the children achieved 25 or more of the 30 points in the Basis-Math 2 +, or greater than 83%. Conversely, half of the children in the intervention group gained 22 or more points

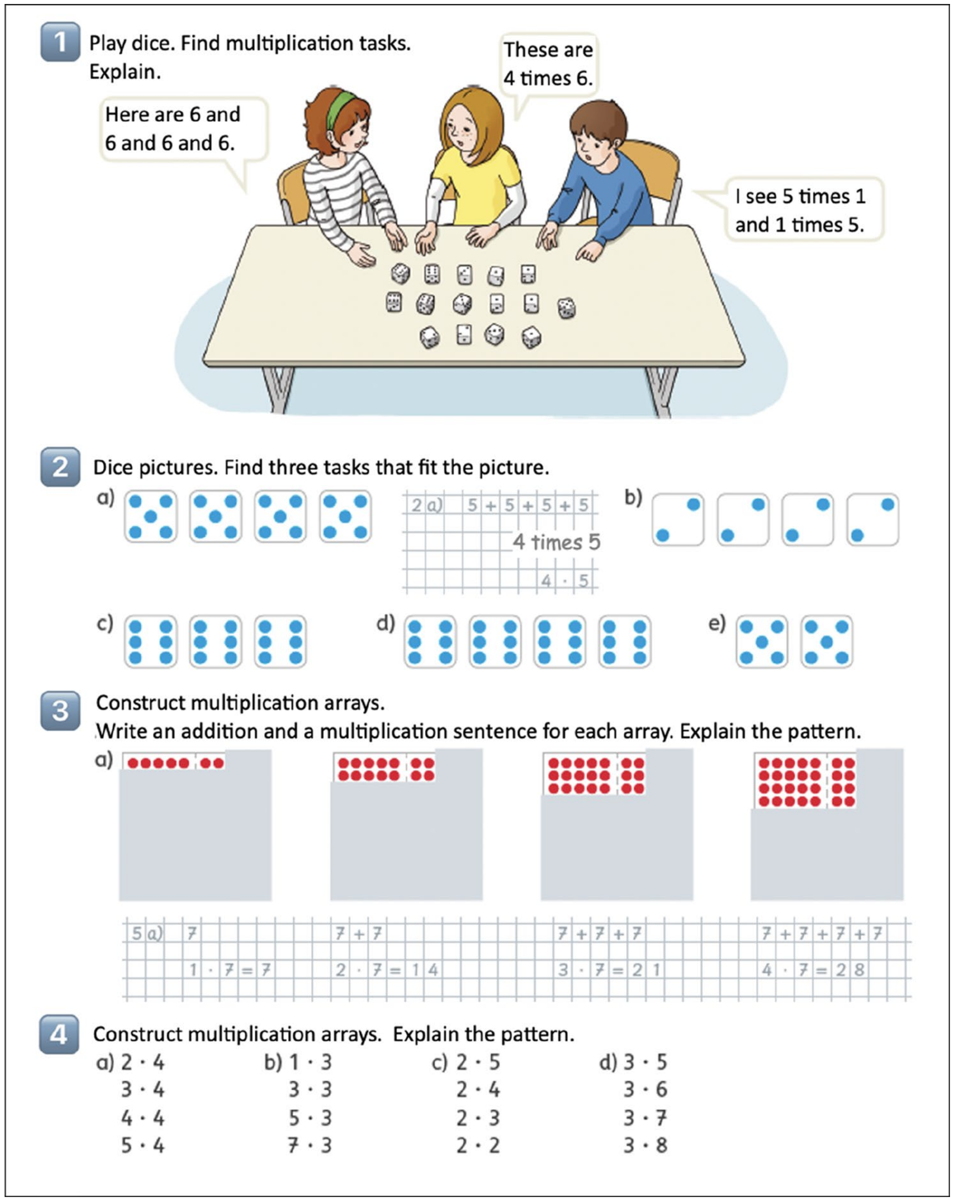


Fig. 2 Examples of multiplicative tasks from the textbook *Das Zahlenbuch* (Nührenbörger et al. 2017), translated by the authors. A dot is typically used as a multiplication sign in German textbooks

(> 69%). Only eight children of the control group (14%) and six children of the intervention group (9%) received scores of less than 50%. Nevertheless, the difference between the mean scores was not statistically significant ($t(122) = -1.22$, $p = 0.16$).

Since the children of the control group performed slightly better in the Basis-Math 2 + than the children of the intervention group, it is interesting to have a closer look at the differences between these children concerning multiplicative thinking as unitizing.

3.3 Design of the multiplication test

As all available standardized written tests primarily tested procedural knowledge or fundamental multiplication facts and not a conceptual understanding of multiplication as unitizing, a test for this purpose had to be generated. The items for this written test were based on the literature. Taking the current state of the research and the two core multiplicative concepts, A and B, mentioned in Sect. 2.1, three different item elements for controlling the dependent variable were implemented in the posttest:

- *Item 1*: In the first sub-item, 1.1, the children had to solve three simple tasks with 2, 5, or 10 as one factor, and

4 Construct multiplication arrays and calculate. Explain the pattern.

a)	b)	c)	d)
2 · 4	1 · 3	2 · 5	3 · 5
3 · 4	3 · 3	2 · 4	3 · 6
4 · 4	5 · 3	2 · 3	3 · 7
5 · 4	7 · 3	2 · 2	3 · 8

To support the use of meaning-related phrases in (a) let a child model the multiplication array for 2 · 4 on the board by using the strips of four. Circle with your finger the 2 fours (or let the child circle) and support the child to express the pattern multiplicatively: 2 times 4 means you have 2 fours. For example, by asking: "Why does it fit 2 times 4? Where can you see 2 fours in the array?"

To illustrate the relationship between 2 · 4 and 3 · 4, a child models other fours under the array of 2 · 4. Expected explanation: 3 · 4 is 1 fours more than 2 · 4. Oral discourse for (b) is similar to (a).

In (c), support meaning-related explanations that address commutative tasks (5 twos and then 4 twos) as well as the understanding of smaller groups (2 fives become 2 fours). Oral discourse for (d) is similar to (c).

Be aware that the children use this meaning-related language individually. To support this, ask the children whether they can imagine how the array changes from the second to the third task. The children must solve this task mentally. Afterwards you or a child construct(s) the array.

Fig. 3 A translated example involving explaining patterns in oral classroom discourse from the supplemental material for the intervention teachers

one square number task (in total 4 tasks). In the second sub-item, 1.2, they had to solve four more difficult tasks with 6, 7, 8, or 9 as factors (no square number tasks). No further explanation was requested.

- *Item 2*: In the second item, students had to draw a picture that fitted the task 3 × 6 (Sub-item 2.1) and explain why they were convinced that this picture fitted the task (Sub-item 2.2).
- *Item 3*: The third item was intended to indicate whether the children could use associative, distributive, and/or commutative insights for deriving tasks. A more difficult task and three simple tasks useful for deriving were provided (see Fig. 4). The children had to choose one of these simple tasks for decomposition and explain the decomposition strategy. They had to do this for the two tasks: 8 × 9 and 6 × 4 (Sub-items 3.1 and 3.2). Only the explanation part was evaluated.

Which task might help you solve 8 x 9?

8 x 10 ☐
4 x 9 ☐
8 x 8 ☐

How can this task help you solve 8 x 9? Explain.

Fig. 4 Item for testing decomposition strategies and explaining those strategies (Sub-item 3.1)

For the follow-up test, the items stayed the same but specific numbers were changed to hinder effects of memorization.

To evaluate our teaching intervention, the research question given in Sect. 2.2 had to be operationalized into the following hypotheses:

A language-responsive introduction of multiplication is more effective for:

- *H1*: Solving simple and more difficult times table tasks (results of Item 1).
- *H2*: Drawing multiplicative structured pictures for given tasks and interpreting those pictures multiplicatively (results of Item 2).
- *H3*: Expressing connections between multiplication tasks with recourse to multiplicative thinking as unitizing (results of Item 3).

Hence, we suggested that the teaching done by the teachers having followed the teaching program is more effective than a non-language-responsive introduction.

Children's answers to the multiplication test were evaluated quantitatively in a point rationing scheme with respect to their correctness. Thus, 124 complete datasets (66 in the intervention group and 58 in the control group) were evaluated. For all the sub-items of our multiplication test, scores (between 0 and 1), means of scores, and standard deviations were calculated. This was used to build the foundation for a one-sided paired *t*-test. Cohen's *d* values were calculated to interpret the relevance of the changes. Cohen's *d* effect sizes were interpreted as follows: $d > 0.2$ was a small effect, $d > 0.5$ was a medium effect, and $d > 0.8$ was a large effect (Cohen 1988).

3.4 Coding of the multiplication test

All six sub-items of the multiplication test were coded with scores between 0 and 1, as outlined in the following section. Moreover, all empty responses were coded as 0.

For the first item, each correctly calculated multiplication task received 0.25 points, meaning that a maximum of one point was possible on each sub-item of Item 1.

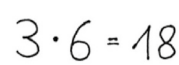
In the coding process of the second item, we distinguished between the drawing and the explaining part. Graphical representations of the children were coded as 1 if the multiplicative structure of the task was illustrated and, thus, if Concept A was clearly demonstrated: It did not matter whether the children drew the task 3×6 as three single groups of six things or drew it as an array. Furthermore, it did not matter whether they drew everyday objects or manipulatives. Moreover, even commutative drawings (6 threes instead of 3 sixes) were coded as a correct graphical realization, because Concept A could also be demonstrated in this way.

Conversely, many different inappropriate pictures were drawn that were coded as 0. Table 1 gives an overview of these drawings.

In the second part of this item (Sub-item 2.2), the children were asked to explain why their picture fitted the task. These explanations were used to determine whether the children expressed an idea of unitizing (Concept A). Many different explanations were given by the children. Expressions such as "I drew three dice with six points each" or "Because I drew 3 sixes" were coded as multiplicative and thus as 1. Some children wrote explanations such as "I drew 3 times 6 cars" or "I drew an array with 6 dots in the upper row and 3 rows downwards." These children did not write about sixes or 6 as unitized groups, but expressed a basic idea of unitizing (e.g., units of 6 cars or units in a row) and possibly lacked the words for expressing their idea in a more meaning-related manner. We therefore also coded such expressions as multiplicative and gave a score of 1. Expressions such as "because it fits 3×6," "first you see the 3 and then the 6," and "I see 6 plus 6 plus 6," however, were coded as not multiplicative and given a score of 0, because the children focused on addition or single numbers instead of interpreting the term 3×6 multiplicatively by expressing an idea of composite units or differentiating between multiplier and multiplicand.

In the third item, which examined H3, we wanted to test whether the children could derive formally difficult tasks such as 8×9 from simple tasks (see Fig. 4) and could thus explain distributive connections of multiplicative tasks (Concept B). Meaning-related explanations that indicate multiplicative thinking such as "8 times 10 are 8 tens and for 8 times 9 every 10 becomes a 9, so I have to subtract 8 ones," commutative argumentations such as "8 times 8 are 8 eights and for 8 times 9, I need another 8," and more formal explanations such as "I take 8 times 8 and I add another 8" were all coded as multiplicatively explained with a score of 1. We added the more formal explanations to this category as well because these children also explained the connection of multiplicative tasks distributively, which indicates multiplicative thinking as unitizing (Downton and Sullivan 2017; Jacob and Willis 2003; Steffe and Cobb 1994). Explanations

Table 1 Inappropriate graphical realizations for the task 3×6

Description	Example
Focusing on single elements (no differentiation between the factors)	
No graphical realization	
Only one factor was illustrated	
Both factors were illustrated as groups (sometimes the amount of the groups or the multiplicand was correct)	

such as “8 × 10 is an easy task,” “8 × 10 is 80, 8 × 9 is 72,” and “8 × 8 is the eighth rows” and wrong decompositions such as “8 times 8 is 64 and 8 times 9 is one more” were placed in the not multiplicatively or wrongly explained category and were given a score of 0.

4 Results from the analysis of the multiplication test

In order to test the hypotheses, the results of the written multiplication posttest and follow-up test of all the children were coded. Fifty-two multiplication tests were coded by two researchers (42.6%). The interrater reliability was quite high ($\kappa = 0.96$).

Table 2 presents the results of the *t*-test of the posttest, and Table 3 shows the results of the follow-up test.

The overall results of our multiplication test showed significant differences between the average scores of the intervention and the control groups in the posttest as well as in the follow-up test. It was evident that the children in the intervention group showed significantly more multiplicative thinking in the three items than the children in the control group. Moreover, the mean difference increased from posttest to follow-up test, and Cohen’s *d* effect size rose from medium to strong effect (from $d = 0.59$ to $d = 1.01$). The language responsive meaning-making processes in the intervention group seemed to have deepened multiplicative thinking from posttest to follow-up test and to have built a long-term basis for conceptual understanding of multiplicative thinking as unitizing.

The results of every specific item of the multiplication test showed heterogeneous results in the posttest, but unequivocal results in the follow-up test.

In the first main item, which examined H1, we found no significant differences between the second graders in solving simple tasks (Item 1.1). Nevertheless, more children in the control group had difficulties in calculating tasks with 6, 7, 8, or 9 as factors (Item 1.2) in the posttest and the follow-up test than children in the intervention group. The disparities between the intervention and the control groups from posttest to follow-up test increased to an effect size that was almost in the high range (from $d = 0.5$ to $d = 0.77$). Presumably, this was due to the fact

Table 2 Results from the multiplication posttest

Multiplication posttest	Intervention group ($n = 66$)		Control group ($n = 58$)		$t(122)$	p	Cohen’s d
	M	SD	M	SD			
Total test	4.80	1.31	3.94	1.58	3.32	< 0.001***	0.59
Item 1.1 (4 simple tasks)	0.99	0.05	0.97	0.10	1.65	0.06	0.30
Item 1.2 (4 difficult tasks)	0.90	0.20	0.76	0.33	2.79	0.004**	0.50
Item 2.1 (picture drawing)	0.85	0.36	0.74	0.44	1.48	0.07	0.27
Item 2.2 (explanation)	0.68	0.47	0.47	0.50	2.48	0.007**	0.45
Item 3.1 (derivation I)	0.73	0.45	0.55	0.50	2.07	0.02*	0.37
Item 3.2 (derivation II)	0.65	0.48	0.45	0.50	2.32	0.01*	0.41

* $p < 0.05$

**$p < 0.01$

*** $p < 0.001$

Table 3 Results from the multiplication follow-up test

Multiplication follow-up test	Intervention group ($n = 66$)		Control group ($n = 58$)		$t(122)$	p	Cohen’s d
	M	SD	M	SD			
Total test	5.06	1.24	3.65	1.54	5.66	< 0.001***	1.01
Item 1.1 (4 simple tasks)	0.98	0.08	0.96	0.10	0.96	0.17	0.17
Item 1.2 (4 difficult tasks)	0.92	0.18	0.71	0.34	4.28	< 0.001***	0.77
Item 2.1 (picture drawing)	0.95	0.21	0.69	0.46	4.16	< 0.001***	0.75
Item 2.2 (explanation)	0.77	0.42	0.50	0.50	3.30	< 0.001***	0.59
Item 3.1 (derivation I)	0.68	0.47	0.38	0.49	3.54	< 0.001***	0.63
Item 3.2 (derivation II)	0.76	0.43	0.41	0.49	4.16	< 0.001***	0.74

* $p < 0.05$

**$p < 0.01$

***$p < 0.001$.

that difficult tasks tend to be harder to memorize than simple tasks (Mabbott and Bisanz 2003; Moser Opitz 2013; van der Ven et al. 2012). Consequently, the children had to rely on decomposition strategies if they did not know the result of a difficult multiplication task. Doing this required internalization of Concepts A and B (see Sect. 2.1). The children in the control group may have had some difficult tasks memorized at the time of the posttest because they had trained in retrieving the results of times table tasks in their mathematics class immediately before, but at the time of the follow-up test they had difficulties in retrieving and decomposing.

In the second item, which examined H2, many children in both groups could draw an appropriate picture for a multiplicative task in the posttest (Item 2.1). However, in the follow-up test, 95% of the children in the intervention group and only 69% in the control group were able to draw a multiplicative picture. These differences were significant, with an effect size nearly in the high range ($t(122)=4.16$, $p<0.001$, $d=0.75$). Whereas the number of multiplicative drawn pictures (Item 2.1) increased in the intervention group from posttest to follow-up test, the number of multiplicative pictures decreased in the control group. Presumably, this was due to the fact that the children in the control group may have drawn multiplicative pictures in the posttest without really knowing what made the picture multiplicative, and may have forgotten during the summer break what a multiplicative picture looks like. The statistical results of Sub-item 2.2 underpinned this assumption. In both tests, 50% or less children in the control group were able to explain their picture multiplicatively (posttest $t(122)=2.48$, $p<0.01$, $d=0.45$; follow-up test $t(122)=3.30$, $p<0.001$, $d=0.59$), with the effect size in the medium range in the follow-up test.

Item 3, involving decomposition, produced results in terms of the control group's forgetting effects in the follow-up test that were similar to the results in Items 1 and 2. The *t*-test analyses of Sub-items 3.1 and 3.2 showed significant differences in the posttest; however, the effect sizes were small ($d=0.37$ and $d=0.41$, respectively). In the follow-up test, 38% and 41%, respectively, of the control-group children were able to interpret multiplicative interconnections of tasks, whereas 68% and 76%, respectively, of the intervention-group children interpreted the interconnection multiplicatively. The effect sizes for the intervention group increased from medium to nearly high ($d=0.63$ and $d=0.74$). In both tests, many children in the control group were not able to interpret the interconnection of multiplication tasks multiplicatively, thus showing less multiplicative thinking than the children in the intervention group.

5 Main results, conclusions and limitations

The presented study contributes to the research on the design of instructional approaches for supporting multiplicative thinking as unitizing and thus for supporting multiplicative meaning-making processes. For this purpose, the multiplication test contains items that uncover multiplicative thinking as unitizing. The quantitative results of the multiplication test cast light on the question of the extent to which the preventive intervention that focuses on connecting registers and representations and on discursive meaning-making processes has an impact on children's multiplicative thinking as unitizing. These data provide empirical evidence for a significant difference in achievement in the multiplication test between the intervention and the control groups. The results of the follow-up test strengthen the results of the posttest.

However, for each single item of our multiplication posttest, significant differences cannot be found for all items. Only the group differences in the items solving more difficult tasks (1.2), explaining multiplicative pictures (2.2), and explaining decomposition strategies (3.1 and 3.2) are significant, with small to medium effect sizes. In the follow-up test, all single items other than item 1.1 (solving simple multiplication tasks) became significant, with medium or nearly high effect sizes. The assumption is that the children of the control group perhaps learned how to draw multiplicative pictures or how to use multiplicative decomposition strategies more procedurally. In fact, some of the control group's written solutions strengthen this assumption. Frequently found expressions such as "and then one must calculate…." or "you have to…" indicate that the multiplicative explanations are not grounded on conceptual understanding but rather on procedures that have been taught. Children's multiplicative knowledge is not based on meaning-making processes and leads to the forgetting effects in the follow-up test. On the contrary, the children in the language-responsive intervention group developed multiplicative knowledge that is more conceptually based on multiplicative thinking as unitizing than the children in the control group.

Generally speaking, teaching multiplication using meaning-related phrases such as '3 fours' and connecting these expressions with different representations can help to develop conceptual understanding of multiplicative thinking as unitizing. For this reason, we recommend that primary school teachers be attentive to whether their students have in fact understood multiplication as unitizing by the end of second grade. Even if they seem to have reached the learning goals of multiplication, it must be determined whether this ability is grounded in multiplicative thinking or is procedural without this understanding.

As the study was conducted with only a medium-sized sample of 124 children, our results can give only a rough idea of how to support multiplicative thinking as unitizing. However, as in all written tests with free-text answers, the interpretation of the written documents is limited because many writing processes suffer from information loss. Furthermore, we have no indications when the children started to think multiplicatively. Moreover, we do not know exactly how the teachers of the intervention group put the materials and hints into practice and how they supported rich discourse practices, because we know only what they reported to us, and we saw neither the intervention nor the control group's teaching units.

While this study is limited in different aspects and the results call for further investigation, it is already encouraging to see that language-responsive and meaning-related teaching interventions can lead to conceptual understanding, meaning-making, and long-term insights for topics such as multiplication.

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