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COMPLETE INFINITE-TIME MASS AGGREGATION IN A QUASILINEAR KELLER–SEGEL SYSTEM

BY

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ABSTRACT

Radially symmetric global unbounded solutions of the chemotaxis system

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (uS(u)\nabla v), & \\ 0 = \Delta v - \mu + u, & \mu = \frac{1}{|\Omega|}\int_\Omega u, \end{cases}$$

are considered in a ball $\Omega = B_R(0) \subset \mathbb{R}^n$, where $n \geq 3$ and $R > 0$.

Under the assumption that D and S suitably generalize the prototypes given by $D(\xi) = (\xi+\iota)^{m-1}$ and $S(\xi) = (\xi+1)^{-\lambda-1}$ for all $\xi > 0$ and some $m \in \mathbb{R}, \lambda > 0$ and $\iota \geq 0$ fulfilling $m+\lambda < 1-\frac{2}{n}$, a considerably large set of initial data u_0 is found to enforce a complete mass aggregation in infinite time in the sense that for any such u_0, an associated Neumann type initial-boundary value problem admits a global classical solution (u, v) satisfying

$$\frac{1}{C} \cdot (t+1)^{\frac{1}{\lambda}} \leq \|u(\cdot,t)\|_{L^\infty(\Omega)} \leq C \cdot (t+1)^{\frac{1}{\lambda}} \quad \text{for all } t > 0$$

as well as

$$\|u(\cdot,t)\|_{L^1(\Omega\setminus B_{r_0}(0))} \to 0 \quad \text{as } t \to \infty \quad \text{for all } r_0 \in (0, R)$$

with some $C > 0$.

1. Introduction

Detailed descriptions of taxis-driven singularity formation can rarely be found in the literature concerned with Keller–Segel systems. In fact, already the mere detection of blow-up phenomena has been achieved only in a comparatively small fraction of cases for which they can be conjectured (cf. [19], [26], [33], [34], [3], [45], [9] for some examples, and [23], [30] as well as [1] for some broader

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surveys addressing this). Qualitative or even quantitative characterizations of irregular solution behavior, however, seem limited to few and fairly particular settings.

Even for the simplest representatives of such cross-diffusion systems, such as the classical parabolic-elliptic Keller–Segel model

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), \\ 0 = \Delta v - \mu + u, \end{cases} \tag{1.1}$$

and some of its close relatives, the understanding of blow-up mechanisms appears yet far from complete, although deep analysis has provided some partially quite far-reaching information on, inter alia, mass quantization phenomena and finiteness of blow-up sets in two-dimensional domains ([41], [35], [4]), on the occurrence or exclusion of boundary blow-up in planar cases ([41], [42]; cf. also [22]), and on the existence of spatial blow-up profiles and two-sided pointwise estimates therefore in three- and higher-dimensional radial frameworks ([40], [48]). The knowledge about temporal asymptotics near explosions so far mainly reduces to findings on particular examples of solutions which either exhibit so-called type I blow-up, or undergo a somewhat faster singularity formation in the sense of type II blow-up (cf. [15], [16], [17], [18], [36], [38] and [39]), and to a result asserting a certain non-degeneracy feature of singular points with respect to respective blow-up rates, as actually derived for a fully parabolic variant of (1.1) in [32]. For solutions corresponding to initial data from larger sets, a quantitative description of temporal blow-up behavior seems to have been comprehensively accomplished only for a particular Neumann–Dirichlet type boundary value problem for (1.1) in which an assumption on precise mass criticality enforces the occurrence of blow-up in infinite time ([27]).

Yet considerably less is known in this direction for more complex representatives of the model class proposed by Keller and Segel in [28]. The few available results going beyond basic issues of blow-up detection ([8], [9], [6], [7], [31], [44]) apparently concentrate on variants of (1.1) involving nonlinear diffusion in the first component, and address one-sided estimates for blow-up profiles ([13], [14], [12]), rough upper estimates for blow-up rates in the presence of nonlinearities exhibiting certain types of exponential decay ([46]), and the occurrence of type II blow-up ([25]).

Infinite-time blow-up in a quasilinear Keller–Segel system. Tying in with the latter observation, the present work attempts to give a detailed characterization of singularity formation in the context of a quasilinear generalization of (1.1), upon complementation by initial and no-flux boundary conditions given by

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (uS(u)\nabla v), & x \in \Omega,\ t > 0, \\ 0 = \Delta v - \mu + u, \quad \mu = \frac{1}{|\Omega|}\int_\Omega u, \quad \int_\Omega v = 0, & x \in \Omega,\ t > 0, \\ (D(u)\nabla u - uS(u)\nabla v)\cdot \nu = 0, \nabla v \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

with suitably regular functions D and S. The application relevance of such nonlinearly modified diffusion and cross-diffusion operators has been underlined in prominent places in the modeling literature concerned with refinements of (1.1) based, inter alia, on the inclusion of volume-filling effects (cf., e.g., [37], [20], [21]), and as instructive prototypical choices therefore one may think of the examples determined by

$$D(\xi) = (\xi + \iota)^{m-1} \quad \text{and} \quad S(\xi) = (\xi + 1)^{-\lambda-1}, \quad \xi > 0, \tag{1.3}$$

with $m \in \mathbb{R}$, $\lambda \in \mathbb{R}$ and $\iota \geq 0$. In this concretized setting, namely, it appears maximally transparent how the potential to enforce blow-up depends on the respective system ingredients: When posed in a smoothly bounded domain $\Omega \subset \mathbb{R}^n$ with $n \geq 2$, under the assumption that $\iota > 0$ and

$$m + \lambda > 1 - \frac{2}{n} \tag{1.4}$$

this problem is known to possess a global bounded classical solution for any nonnegative $u_0 \in C^1(\overline{\Omega})$; see [11] for a proof detailed under the additional restriction $m \leq 1$ but actually carrying over without any substantial change also to larger m (cf. also [5] and [10] for precedents addressing special cases, as well as [24] and [43] for an extension to a fully parabolic relative). On the other hand, if $\iota > 0$, if $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ is a ball, and if

$$m + \lambda < 1 - \frac{2}{n} \quad \text{as well as} \quad \lambda < 0, \tag{1.5}$$

then it is known that some solutions to (1.2)–(1.3) blow up in finite time ([11]). Here a strong indication for necessity of the latter assumption on negativity of λ can be found in [29] and [47], where in the case when $m + \lambda < 1 - \frac{2}{n}$ and $\lambda \geq 0$, global solvability for widely arbitrary initial data but, apart from that, the existence of some solutions exploding in infinite time has been asserted for the two variants of (1.2)-(1.3) containing $\tau v_t = \Delta v - v + u$ with either $\tau = 0$ or $\tau = 1$ as their second equations.

Main results. Beyond rigorously confirming the occurrence of the latter type of grow-up phenomenon also for (1.2) and in a framework slightly more general than the above one, inter alia allowing for singular diffusion operators and hence for the choice $\iota = 0$ in (1.3), the main objective of this study now consists in describing such infinite-time blow-up mechanisms in detail, throughout a considerably large set of trajectories.

Our main result in this direction reveals that in three- or higher-dimensional radial situations, for any choice of $\lambda > 0$ and $m \in \mathbb{R}$ fulfilling $m + \lambda < 1 - \frac{2}{n}$, and under a mild condition on sufficiently strong concentration of the initial data, the corresponding solution indeed undergoes an infinite-time explosion, that this singularity formation occurs at an explicitly computable algebraic rate, and that actually the entire population distribution asymptotically aggregates in its center of mass; in particular, in this strictly supercritical parameter setting none of these solutions exhibits any type of mass quantization in the flavor of those detected in the parameter-critical two-dimensional version of (1.1) ([41]).

More precisely, the most substantial part of this outcome is contained in the following main result of this manuscript.

Theorem 1.1: *Let $n \geq 3$, $R > 0$ and $\Omega = B_R(0) \subset \mathbb{R}^n$, and suppose that*

$$D \in C^2((0,\infty)) \quad \text{is positive,} \tag{1.6}$$

that

$$S \in C^2([0,\infty)) \quad \text{is such that } \limsup_{\xi \to \infty} \{\xi S(\xi)\} < \infty, \tag{1.7}$$

and that

$$D(\xi) \leq K_D \cdot \xi^{m-1} \quad \text{for all } \xi > 0 \tag{1.8}$$

as well as

$$S(\xi) \geq K_S \cdot (\xi + 1)^{-\lambda - 1} \quad \text{for all } \xi > 0 \tag{1.9}$$

with some $K_D > 0$, $K_S > 0$, $m \in \mathbb{R}$ and $\lambda > 0$ satisfying

$$m + \lambda < 1 - \frac{2}{n}. \tag{1.10}$$

Then for any choice of $\mu>0$, *one can find a nondecreasing* $M=M^{(\mu)}\in C^0([0,R])$ *such that* $\sup_{r\in(0,R)}\frac{M(r)}{r^n}<\infty$ *and* $M(R)=\mu|\Omega|$, *and such that whenever*

$$u_0\in W^{1,\infty}(\Omega) \text{ is positive in } \overline{\Omega} \text{ and radially symmetric with } \frac{1}{|\Omega|}\int_\Omega u_0dx=\mu \tag{1.11}$$

and

$$\int_{B_r(0)} u_0dx \ge M(r) \quad \text{for all } r\in(0,R), \tag{1.12}$$

the problem (1.2) *possesses a global classical solution* (u,v), *uniquely determined by the requirements that*

$$\begin{cases} u\in C^0(\overline{\Omega}\times[0,\infty))\cap C^{2,1}(\overline{\Omega}\times(0,\infty)), \\ v\in C^{2,0}(\overline{\Omega}\times(0,\infty)) \end{cases} \tag{1.13}$$

which is such that $u>0$ *in* $\overline{\Omega}\times[0,\infty)$, *and that* (u,v) *undergoes an infinite-time complete mass aggregation at the spatial origin, in the sense that with some* $C>0$ *we have*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \ge C\cdot(t+1)^{\frac{1}{\lambda}} \quad \text{for all } t>0, \tag{1.14}$$

but that for all $r\in(0,R)$,

$$\|u(\cdot,t)\|_{L^1(\Omega\setminus B_r(0))}\to 0 \quad \text{as } t\to\infty. \tag{1.15}$$

Remark: It can readily be verified that whenever $M\in C^0([0,R])$ is nondecreasing and such that $\frac{M(r)}{r^n}\le c_1$ and $M(R)=\mu|\Omega|$ with some $c_1>0$ and $\mu>0$, then any radial $u_0=u_0(r)\in C^1(\overline{\Omega})$ which is such that beyond the identity $\frac{1}{|\Omega|}\int_\Omega u_0dx=\mu$ we have

$$u_{0r}(r)\le 0 \quad \text{for all } r\in(0,R)$$

as well as $\operatorname{supp} u_0\subset B_{r_0}(0)$ with any $r_0\in(0,R)$ such that $r_0^n\le\frac{\mu|B_1(0)|R^n}{c_1}$, necessarily satisfies

$$\int_{B_r(0)} u_0dx\ge M(r) \quad \text{for all } r\in(0,R).$$

Accordingly, in the situation of Theorem 1.1 the requirement (1.12) on initial mass concentration indeed is satisfied within a considerably large set of initial data.

The second part of our results now makes sure that (1.14) indeed cannot significantly be improved, actually for arbitrary solutions to (1.2), under a correspondingly complementary hypothesis on decay of S; in particular, together with Theorem 1.1 the following provides a fairly comprehensive description of algebraic grow-up rates for (1.2) with system ingredients as in (1.3).

PROPOSITION 1.2: *Let $n \ge 3$, $R > 0$ and $\Omega = B_R(0) \subset \mathbb{R}^n$, and assume that* (1.6) *and* (1.7) *are satisfied, and that moreover there exist $\widehat{K}_S > 0$ and $\lambda > 0$ such that*

$$S(\xi) \le \widehat{K}_S \cdot \xi^{-\lambda-1} \quad \text{for all } \xi > 0. \tag{1.16}$$

Then whenever (1.11) *holds, one can find $C > 0$ such that the problem* (1.2) *admits a unique global classical solution (u, v) from the class in* (1.13)*, besides satisfying $u > 0$ in $\overline{\Omega} \times [0, \infty)$ having the property that with some $C > 0$,*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \le C \cdot (t+1)^{\frac{1}{\lambda}} \quad \text{for all } t > 0. \tag{1.17}$$

CONCLUSION. In summary, the outcomes of Theorem 1.1 and Proposition 1.2 may be interpreted as indicating that the assumptions (1.6)–(1.10) and (1.16) together with the requirement $\lambda > 0$ describe a regime of dominant but mildly destabilizing chemotaxis: The cross-diffusive action then remains too weak to enforce any finite-time explosion, but in all respects outweighs diffusion in the sense that not only all mass is asymptotically transported to the origin, but that beyond this also the corresponding rate of infinite-time blow-up is exclusively determined by the taxis-related contribution to (1.2).

MAIN IDEAS. While Proposition 1.2 can be verified by an essentially straightforward maximum principle based argument applied to (1.2) directly, our derivation of Theorem 1.1 will rather operate at the level of the scalar parabolic equation

$$w_t = n^2 s^{2-\frac{2}{n}} D(nw_s) w_{ss} + \Big(w - \frac{\mu}{n} s\Big) \cdot S(nw_s) \cdot nw_s \tag{1.18}$$

satisfied by the mass accumulation function w given by

$$w(s,t) := \frac{1}{n|B_1(0)|} \int_{B_{s^{\frac{1}{n}}}(0)} u(x,t)dx, \quad s \in [0, R^n], t \ge 0$$

(Lemma (2.2)); indeed, the essentially well-known fact that the resulting Dirichlet problem allows for a conveniently handy comparison principle (Lemma 2.3) has influenced crucial parts of blow-up detections in the literature on closely related problems ([26], [10], [11]). In contrast to the latter type of reasonings which according to their purely contradiction-focused motivation widely disregard quantitative aspects of the actually present finite-time explosion mechanisms in the respective situations, in the current context a major challenge consists in an appropriate design of comparison functions with behavior sufficiently close to that of the unknown solution w.

In fact, it will turn out that the optimal grow-up rate appearing in (1.14) coincides with that exhibited by all members of a three-parameter family of subsolutions $\underline{w} = \underline{w}^{(\mu,a,\tau)}$ to the problem in question, besides on the arbitrarily prescribed mass level μ depending on two free coefficients a and τ. As to be substantiated in Lemma 3.3, the composite structure of these subsolutions will firstly reflect expected dominance of the Burgers-type hyperbolic part in (1.18) near the point $s = 0$ of diffusion degeneracy in the presence of suitably strong mass accumulations (see (3.15)); a similarly simple, essentially linear functional form of $\underline{w}$ will be relied on to adequately capture solution behavior within an outer space-time domain suitably far from both the point $s = 0$ and the region of large spatial gradient $\underline{w}_s$ (cf. (3.17)). The core of our approach, however, is found in the construction of $\underline{w}$ in a corresonding intermediate range of variables, as determined by the choice

$$\underline{w}(s,t) := \frac{\gamma}{\gamma-1}A(t) - \frac{\gamma}{\gamma-1}B(t)\cdot(at+\tau)^{-\frac{\gamma-1}{\lambda}}s^{1-\gamma}, \tag{1.19}$$

with some suitably fixed parameter $\gamma > 1$ (see (3.16)). In fact, it will be seen in Section 3 that if here the function B, playing the role of a minor correction by being bounded from above and below by positive constants, as well as the numbers a and τ are carefully selected, then through (1.19) indeed a transition between the aggregation hotspot and regions of small densities can be achieved in a manner compatible with the asymptotic features claimed in (1.14) and (1.15).

2. Preliminaries

2.1. Global classical solvability. Let us first make sure by means of a comparison argument that the assumptions on D and S from Theorem 1.1, and especially the growth hypothesis contained in (1.7), warrant global existence of classical solutions in (1.2), actually without any restriction on the spatial dimension $n \geq 1$.

Lemma 2.1: *Let $n \geq 1, R > 0$ and $\Omega = B_R(0) \subset \mathbb{R}^n$, let $\mu > 0$, and assume that D, S and u_0 satisfy* (1.6), (1.7) *and* (1.11) *with* $\frac{1}{|\Omega|}\int_\Omega u_0 dx = \mu$. *Then there exist uniquely determined functions u and v fulfilling* (1.13) *and such that $u > 0$ in $\overline{\Omega} \times [0,\infty)$, that $(u(\cdot,t), v(\cdot,t))$ is radially symmetric with respect to $x = 0$ for all $t > 0$, and that (u,v) solves* (1.2) *in the classical sense. Moreover,*

$$\int_\Omega u(x,t)dx = \int_\Omega u_0 dx \quad \textit{for all } t > 0. \tag{2.1}$$

Proof. By adapting a well-established reasoning ([11]) to the present situation of diffusivities possibly exhibiting a degeneracy near points where u is small, one can readily verify that there exist $T_{\max} \in (0,\infty]$ and a classical solution (u,v) of (1.2) in $\Omega \times (0, T_{\max})$, unique within the class of functions from

$$(C^0(\overline{\Omega} \times [0,T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\max}))) \times C^{2,0}(\overline{\Omega} \times (0,T_{\max})),$$

such that $u > 0$ in $\overline{\Omega} \times [0, T_{\max})$, that (2.1) holds and $(u(\cdot,t), v(\cdot,t))$ has the intended symmetry feature for all $t \in (0, T_{\max})$, and that

$$\text{if } T_{\max} < \infty, \quad \text{then} \quad \limsup_{t \nearrow T_{\max}} \left\{ \|u(\cdot,t)\|_{L^\infty(\Omega)} + \left\| \frac{1}{u(\cdot,t)} \right\|_{L^\infty(\Omega)} \right\} = \infty. \tag{2.2}$$

To see that actually $T_{\max} = \infty$, assuming the opposite we would infer from (1.2) that if for $\varphi \in C^{2,1}(\Omega \times (0,T_{\max}))$ and $(x,t) \in \Omega \times (0, T_{\max})$ we let

$$(\mathcal{Q}\varphi)(x,t) := \varphi_t(x,t) - \nabla \cdot (b_1(x,t)\nabla\varphi(x,t)) + b_2(x,t) \cdot \nabla\varphi(x,t) - h(\varphi(x,t)),$$

with

$$b_1(x,t) := D(u(x,t)) \quad \text{and} \quad b_2(x,t) := (S(u(x,t)) + u(x,t)S'(u(x,t)))\nabla v(x,t),$$

and with

$$h(\xi) := -\mu\xi S(\xi) + \xi^2 S(\xi), \quad \xi \geq 0,$$

then

$$(\mathcal{Q}u)(x,t) = 0 \quad \text{for all } (x,t) \in \Omega \times (0, T_{\max}). \tag{2.3}$$

To derive a contradiction on the basis of this and (2.2), we note that (1.7) ensures that h is locally Lipschitz continuous with $h(0) = 0$ and $\limsup_{\xi\to\infty} \frac{h(\xi)}{\xi} < \infty$, from which it especially follows that both

$$\begin{cases} \underline{y}'(t) = h(\underline{y}(t)), & t > 0, \\ \underline{y}(0) = \inf_\Omega u_0, \end{cases}$$

and

$$\begin{cases} \overline{y}'(t) = h(\overline{y}(t)), & t > 0, \\ \overline{y}(0) = \sup_\Omega u_0, \end{cases}$$

admit globally defined solutions $\underline{y} \in C^1([0,\infty))$ and $\overline{y} \in C^1([0,\infty))$ fulfilling $0 < \underline{y}(t) \le \overline{y}(t)$ for all $t > 0$, because $u_0 > 0$ in $\overline{\Omega}$ by (1.11). Letting

$$\underline{u}(x,t) := \underline{y}(t) \quad \text{and} \quad \overline{u}(x,t) := \overline{y}(t)$$

for $(x,t) \in \overline{\Omega} \times [0, T_{\max})$, we thus obtain functions $\underline{u}$ and $\overline{u}$ which belong to $C^{2,1}(\overline{\Omega} \times [0, T_{\max}))$ and satisfy

$$(\mathcal{Q}\underline{u})(x,t) = (\mathcal{Q}\overline{u})(x,t) = 0 \quad \text{for all } (x,t) \in \Omega \times (0, T_{\max}).$$

As therefore $\underline{u} \le u \le \overline{u}$ in $\overline{\Omega} \times (0, T_{\max})$ by a comparison argument, we conclude that under the current hypothesis we would have

$$0 < \inf_{t \in (0,T_{\max})} \underline{y}(t) \le u \le \sup_{t \in (0,T_{\max})} \overline{y}(t) < \infty \quad \text{in } \Omega \times (0, T_{\max}),$$

which would evidently contradict (2.2), however. ∎

2.2. The evolution of accumulated densities. A comparison principle. The following transformation of the radial version of (1.2) to a one-dimensional scalar parabolic problem adapts a meanwhile quite classical observation ([26]) to the present framework.

Lemma 2.2: *Suppose that $n \ge 1, R > 0$ and $\Omega = B_R(0) \subset \mathbb{R}^n$, that $\mu > 0$, and that* (1.6), (1.7) *and* (1.11) *hold with $\frac{1}{|\Omega|} \int_\Omega u_0 dx = \mu$. Then letting*

$$w(s,t) := \int_0^{s^{\frac{1}{n}}} r^{n-1} u(r,t) dr, \quad s \in [0, R^n], t \ge 0, \tag{2.4}$$

defines a nonnegative function $w \in C^0([0,\infty); C^1([0,R^n])) \cap C^{2,1}([0,R^n] \times (0,\infty))$ with

$$w_s(s,t) = \frac{1}{n} u(s^{\frac{1}{n}}, t) \quad \text{for all } s \in [0, R^n] \text{ and } t > 0 \tag{2.5}$$

as well as

$$w(0,t) = 0 \quad \text{and} \quad w(R^n, t) = \frac{\mu R^n}{n} \quad \text{for all } t > 0, \tag{2.6}$$

and we have

$$(\mathcal{P}w)(s,t) = 0 \quad \text{for all } s \in (0, R^n) \text{ and } t > 0, \tag{2.7}$$

where for open sets $G \subset (0,\infty)^2$ and positive functions $\varphi \in C^{2,1}(G)$ fulfilling $\varphi_s(s,t) > 0$ for all $(s,t) \in G$, we have set

$$(2.8)\qquad \begin{aligned}(\mathcal{P}\varphi)(s,t) := {}& \varphi_t(s,t) - n^2 s^{2-\frac{2}{n}} D(n\varphi_s(s,t))\varphi_{ss}(s,t) \\ & - \Big\{\varphi(s,t) - \frac{\mu}{n}s\Big\} \cdot S(n\varphi_s(s,t)) \cdot n\varphi_s(s,t)\end{aligned}$$

for $(s,t) \in G$.

Proof. While (2.5) and (2.6) directly result from (2.4) and (2.1), the identity in (2.7) can be verified in a straightforward manner by differentiating in (2.4) and using (1.2) (cf. also [11]). ∎

The following comparison principle will form an indispensable fundamental fact for our derivation of Theorem 1.1. Its argument will need to be arranged in such a way that, on the one hand, a possible degeneracy-induced limitation of second order regularity of the solution w of (2.7) from (2.4) near the boundary point $s = 0$ can adequately be coped with, and that, on the other hand, also comparison functions exhibiting interior jump discontinuities in their second order spatial derivatives can be included. For the former reason abstaining from an integration-based approach, we hence employ a classical pointwise maximum principle argument which, in view of the latter of these ambitions, however, will require slight modifications near such discontinuity points; for completeness, we therefore include a brief proof.

Lemma 2.3: *Let $n \ge 1, R > 0$ and $\Omega = B_R(0) \subset \mathbb{R}^n$, and suppose that $D \in C^0((0,\infty))$ and $S \in C^0([0,\infty))$, that $\mu > 0$ and $T \in (0,\infty]$, and that $\underline{w}$ and $\overline{w}$ belong to $C^0([0,T); C^1([0,R^n]) \cap C^1((0,R^n) \times (0,T))$ and are such that*

$$(2.9)\qquad \underline{w}_s(s,t) > 0 \quad \text{and} \quad \overline{w}_s(s,t) > 0 \quad \text{for all } s \in [0,R^n] \text{ and } t \in [0,T)$$

as well as

$$(2.10)\quad \underline{w}(\cdot,t) \in W^{2,\infty}_{\mathrm{loc}}((0,R^n)) \quad \text{and} \quad \overline{w}(\cdot,t) \in W^{2,\infty}_{\mathrm{loc}}((0,R^n)) \quad \text{for all } t \in (0,T).$$

Then under the assumption that for all $t \in (0,T)$ and each common differentiability point $s \in (0,R^n)$ of $\underline{w}_s(\cdot,t)$ and $\overline{w}_s(\cdot,t)$ we have

$$(2.11)\qquad \underline{w}_t \le n^2 s^{2-\frac{2}{n}} D(n\underline{w}_s)\underline{w}_{ss} + \Big(\underline{w} - \frac{\mu}{n}s\Big) \cdot S(n\underline{w}_s) \cdot n\underline{w}_s$$

and

$$\overline{w}_t \ge n^2 s^{2-\frac{2}{n}} D(n\overline{w}_s)\overline{w}_{ss} + \Big(\overline{w} - \frac{\mu}{n}s\Big) \cdot S(n\overline{w}_s)\cdot n\overline{w}_s, \tag{2.12}$$

and that furthermore

$$\underline{w}(0,t) \le \overline{w}(0,t) \quad \text{and} \quad \underline{w}(R^n,t) \le \overline{w}(R^n,t) \quad \text{for all } t \in (0,T) \tag{2.13}$$

as well as

$$\underline{w}(s,0) \le \overline{w}(s,0) \quad \text{for all } s \in (0,R^n), \tag{2.14}$$

it follows that

$$\underline{w}(s,t) \le \overline{w}(s,t) \quad \text{for all } s \in [0,R^n] \text{ and } t \in [0,T). \tag{2.15}$$

Proof. Given $T_0 \in (0,T)$, using that

$$c_1 := \max_{(s,t)\in[0,R^n]\times[0,T_0]} \overline{w}_s(s,t) \tag{2.16}$$

is finite by continuity of $\overline{w}_s$, and that $h(\xi) := S(n\xi)\cdot n\xi$, $\xi \ge 0$, is continuous, we can pick $\beta > 0$ large enough such that

$$\beta > \max_{\xi\in[0,c_1]} |h(\xi)| =: m_1. \tag{2.17}$$

Then for fixed $\varepsilon > 0$, the function $z \in C^0([0,R^n]\times[0,T_0]) \cap C^1((0,R^n)\times(0,T_0))$ defined by letting

$$z(s,t) := \underline{w}(s,t) - \overline{w}(s,t) - \varepsilon e^{\beta t}, \quad (s,t) \in [0,R^n]\times[0,T_0], \tag{2.18}$$

satisfies $z(0,t) < 0$ and $z(R^n,t) < 0$ for all $t \in [0,T_0]$ as well as $z(s,0) < 0$ for all $s \in [0,R^n]$, so that if $\max_{(s,t)\in[0,R^n]\times[0,T_0]} z(s,t)$ were not negative, we could find $s_0 \in (0,R^n)$ and $t_0 \in (0,T_0]$ such that

$$z(s_0,t_0) = \max_{(s,t)\in[0,R^n]\times[0,t_0]} z(s,t) = 0 \tag{2.19}$$

and that hence, necessarily,

$$z_t(s_0,t_0) \ge 0 \quad \text{and} \quad z_s(s_0,t_0) = 0. \tag{2.20}$$

Now since $\underline{w}(\cdot,t_0)$ and $\overline{w}(\cdot,t_0)$ both belong to $W^{2,\infty}_{\rm loc}((0,R^n))$ by assumption, we can choose a null set $N \subset (0,R^n)$ such that $\underline{w}_s(\cdot,t_0)$ and $\overline{w}_s(\cdot,t_0)$ are differentiable in $(0,R^n)\setminus N$, and since $s_0 \in (0,R^n)$ and $(0,R^n)\setminus N$ is dense in $(0,R^n)$, using (2.19) together with (2.20) we can fix $(s_j)_{j\in\mathbb{N}} \subset (0,R^n)\setminus N$ and $c_2 > 0$ such that $s_j \to s_0$ as $j \to \infty$, that

$$|\overline{w}_{ss}(s_j,t_0)| \le c_2 \quad \text{for all } j \in \mathbb{N}, \tag{2.21}$$

and that

$$z_{ss}(s_j,t_0) \le 0 \quad \text{for all } j \in \mathbb{N}, \tag{2.22}$$

for if the latter was false, then according to (2.20), $z_s(s,t_0) = \int_{s_0}^s z_{ss}(\sigma)d\sigma$ would be positive for all $s \in (s_0, s_0+\eta)$ with some $\eta > 0$, hence contradicting (2.19).

But evaluating (2.11) and (2.12) at (s_j,t_0) we obtain that

$$\begin{aligned} z_t(s_j,t_0) =& \underline{w}_t(s_j,t_0) - \overline{w}_t(s_j,t_0) - \beta\varepsilon e^{\beta t_0} \\ \le& n^2 s_j^{2-\frac{2}{n}} D(n\underline{w}_s(s_j,t_0))\underline{w}_{ss}(s_j,t_0) \\ &+ \Big(\underline{w}(s_j,t_0) - \frac{\mu}{n}s_j\Big)\cdot h(\underline{w}_s(s_j,t_0)) \\ &- n^2 s_j^{2-\frac{2}{n}} D(n\overline{w}_s(s_j,t_0))\overline{w}_{ss}(s_j,t_0) \\ &- \Big(\overline{w}(s_j,t_0) - \frac{\mu}{n}s_j\Big)\cdot h(\overline{w}_s(s_j,t_0)) - \beta\varepsilon e^{\beta t_0} \\ =& n^2 s_j^{2-\frac{2}{n}} D(n\underline{w}_s(s_j,t_0)) z_{ss}(s_j,t_0) \\ &+ n^2 s_j^{2-\frac{2}{n}}\cdot\{D(n\underline{w}_s(s_j,t_0)) - D(n\overline{w}_s(s_j,t_0))\}\cdot\overline{w}_{ss}(s_j,t_0) \\ &+ (\underline{w}(s_j,t_0) - \frac{\mu}{n}s_j)\cdot\{h(\underline{w}_s(s_j,t_0)) - h(\overline{w}_s(s_j,t_0))\} \\ &+ (z(s_j,t_0)+\varepsilon e^{\beta t_0})\cdot h(\overline{w}_s(s_j,t_0)) - \beta\varepsilon e^{\beta t_0} \quad \text{for all } j\in\mathbb{N}, \end{aligned} \tag{2.23}$$

where thanks to the positivity of $\underline{w}_s(\cdot,t_0)$ and $\overline{w}_s(\cdot,t_0)$ in $[0,R^n]$, as asserted by (2.9), we may rely on the continuity of D on $(0,\infty)$ to infer that

$$D(n\underline{w}_s(s_j,t_0)) - D(n\overline{w}_s(s_j,t_0)) \to D(n\underline{w}_s(s_0,t_0)) - D(n\overline{w}_s(s_0,t_0)) = 0 \quad \text{as } j\to\infty$$

because of (2.20) and (2.18). Likewise,

$$h(\underline{w}_s(s_j,t_0)) - h(\overline{w}_s(s_j,t_0)) \to h(\underline{w}_s(s_0,t_0)) - h(\overline{w}_s(s_0,t_0)) = 0 \quad \text{as } j\to\infty,$$

whence in line with (2.22), (2.21) and (2.19) we infer from (2.23) on taking $j\to\infty$ that

$$z_t(s_0,t_0) \le \varepsilon e^{\beta t_0}\cdot\max_{s\in[0,R^n]}|h(\overline{w}_s(s,t_0))| - \beta\varepsilon e^{\beta t_0}$$

and that thus, by (2.20) and (2.16),

$$0 \le \varepsilon e^{\beta t_0}\cdot m_1 - \beta\varepsilon e^{\beta t_0},$$

which is absurd in view of (2.17). Consequently, z actually must remain negative throughout $[0,R^n]\times[0,T_0]$, from which (2.15) results upon letting $\varepsilon \searrow 0$ and then $T_0 \nearrow T$. ∎

3. Construction of subsolutions

The following lemma selects two parameters which will enter our construction of comparison functions for (2.7) in decisive places, and which will be kept fixed hereafter. Their choice crucially relies on our assumptions from Theorem 1.1 not only with regard to the parameters m and λ, but also on the spatial dimension.

LEMMA 3.1: *Let $n \ge 3$, and suppose that $m \in \mathbb{R}$ and $\lambda > 0$ are such that $m+\lambda < 1-\frac{2}{n}$. Then there exist $\gamma > 1$ and $\kappa \in (0, \frac{\gamma-1}{\gamma})$ such that*

$$1 - \frac{2}{n} - (m+\lambda)\gamma > 0 \tag{3.1}$$

and

$$(m\alpha+1)(\gamma-1) + \alpha\kappa \cdot \Big[1 - \frac{2}{n} - (m+\lambda)\gamma\Big] > 0 \tag{3.2}$$

as well as

$$(m-1)(\gamma-1) + \kappa \cdot \Big[1 - \frac{2}{n} - (m-1)\gamma\Big] > 0, \tag{3.3}$$

where

$$\alpha := \frac{1}{\lambda}. \tag{3.4}$$

Proof. According to our hypothesis,

$$\lim_{\gamma\to 1} \Big\{1 - \frac{2}{n} - (m+\lambda)\gamma\Big\} = 1 - \frac{2}{n} - (m+\lambda)$$

is positive, so that (3.1) can be achieved upon choosing $\gamma > 1$ suitably close to 1. Keeping this selection fixed, we can then assert (3.2) and (3.3) by observing that since $\alpha\lambda = 1$ by (3.4), we have

$$\begin{aligned}
&(m\alpha+1)(\gamma-1) + \alpha\kappa \cdot \Big[1 - \frac{2}{n} - (m+\lambda)\gamma\Big] \\
&\quad \to (m\alpha+1)(\gamma-1) + \frac{\alpha(\gamma-1)}{\gamma} \cdot \Big[1 - \frac{2}{n} - (m+\lambda)\gamma\Big] = \frac{\alpha(\gamma-1)}{\gamma} \cdot \Big(1 - \frac{2}{n}\Big)
\end{aligned}$$

and

$$\begin{aligned}
&(m-1)(\gamma-1) + \kappa \cdot \Big[1 - \frac{2}{n} - (m-1)\gamma\Big] \\
&\quad \to (m-1)(\gamma-1) + \frac{\gamma-1}{\gamma} \cdot \Big[1 - \frac{2}{n} - (m-1)\gamma\Big] = \frac{\gamma-1}{\gamma} \cdot \Big(1 - \frac{2}{n}\Big)
\end{aligned}$$

as $\kappa \to \frac{\gamma-1}{\gamma}$, and that both these latter limits are positive due to our assumption that $n \ge 3$. ∎

As a second preparation for our definition of subsolution candidates, let us introduce a family of functions A, as announced near (1.19), which will play the role of correcting factors of asymptotically minor influence that are adjusted in such a manner that said candidates will enjoy convenient continuity properties.

Lemma 3.2: *Let $n \geq 3$, $R > 0$, $m \in \mathbb{R}$ and $\lambda > 0$ be such that* (1.10) *holds, and let γ, κ and α be as determined by Lemma 3.1. Then for all $\mu > 0$ there exists $\tau_0 = \tau_0^{(\mu)} \geq 1$ such that whenever $a > 0$ and $\tau > \tau_0$,*

$$A(t) := \frac{\frac{\mu R^n}{n}}{\frac{\gamma}{\gamma-1} + R^n \cdot (at+\tau)^{-\alpha[(1-\kappa)\gamma-1]} - \frac{\gamma}{\gamma-1} \cdot (at+\tau)^{-\alpha(1-\kappa)(\gamma-1)}}, \quad t \geq 0, \tag{3.5}$$

defines a positive function $A = A^{(\mu,a,\tau)} \in C^1([0,\infty))$ which is such that for all $t > 0$,

$$\begin{aligned} A'(t) = & \frac{\frac{\mu R^n}{n} \cdot a}{[\frac{\gamma}{\gamma-1} + R^n \cdot (at+\tau)^{-\alpha[(1-\kappa)\gamma-1]} - \frac{\gamma}{\gamma-1} \cdot (at+\tau)^{-\alpha(1-\kappa)(\gamma-1)}]^2} \\ & \times \{\alpha[(1-\kappa)\gamma-1] R^n \cdot (at+\tau)^{-\alpha[(1-\kappa)\gamma-1]-1} \\ & \quad - \alpha(1-\kappa)\gamma \cdot (at+\tau)^{-\alpha(1-\kappa)(\gamma-1)-1}\}, \end{aligned} \tag{3.6}$$

that

$$A'(t) \geq 0 \quad \textit{for all } t > 0, \tag{3.7}$$

and that

$$\underline{A} \leq A(t) \leq \overline{A} \quad \textit{for all } t > 0, \tag{3.8}$$

where

$$\underline{A} \equiv \underline{A}^{(\mu)} := \frac{\frac{\mu R^n}{n}}{\frac{\gamma}{\gamma-1} + R^n} \quad \text{and} \quad \overline{A} \equiv \overline{A}^{(\mu)} := \frac{\gamma-1}{\gamma} \cdot \frac{\mu R^n}{n}. \tag{3.9}$$

Proof. Using that $(1-\kappa)\gamma > 1$ due to the inequality $\kappa < \frac{\gamma-1}{\gamma}$, we see that the number

$$\tau_0 \equiv \tau_0^{(\mu)} := \max\Big\{1, \Big\{\frac{(1-\kappa)\gamma}{[(1-\kappa)\gamma-1] \cdot R^n}\Big\}^{\frac{1}{\alpha\kappa}}\Big\} \tag{3.10}$$

is well-defined, and that if $a > 0$ and $\tau > \tau_0$, then

$$\begin{aligned} \frac{\gamma}{\gamma-1} + R^n \cdot (at+\tau)^{-\alpha[(1-\kappa)\gamma-1]} & - \frac{\gamma}{\gamma-1} \cdot (at+\tau)^{-\alpha(1-\kappa)(\gamma-1)} \\ & \geq R^n \cdot (at+\tau)^{-\alpha[(1-\kappa)\gamma-1]} > 0 \end{aligned} \tag{3.11}$$

and

$$\frac{\gamma}{\gamma-1} + R^n \cdot (at+\tau)^{-\alpha[(1-\kappa)\gamma-1]} - \frac{\gamma}{\gamma-1}\cdot(at+\tau)^{-\alpha(1-\kappa)(\gamma-1)} \le \frac{\gamma}{\gamma-1} + R^n \tag{3.12}$$

as well as

$$\begin{aligned}
&\alpha[(1-\kappa)\gamma-1]R^n\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]-1} - \alpha(1-\kappa)\gamma\cdot(at+\tau)^{-\alpha(1-\kappa)(\gamma-1)-1} \\
&\quad = \alpha[(1-\kappa)\gamma-1]R^n\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]-1}\cdot \\
&\qquad \times \Big\{1 - \frac{(1-\kappa)\gamma}{[(1-\kappa)\gamma-1]R^n}\cdot(at+\tau)^{-\alpha\kappa}\Big\} \\
&\quad \ge \alpha[(1-\kappa)\gamma-1]R^n\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]-1}\cdot \\
&\qquad \times \Big\{1 - \frac{(1-\kappa)\gamma}{[(1-\kappa)\gamma-1]R^n}\cdot\tau^{-\alpha\kappa}\Big\} \\
&\quad \ge 0
\end{aligned} \tag{3.13}$$

for all $t \ge 0$. Here, (3.11) and (3.12) ensure that (3.5) indeed determines a function $A \in C^1([0,\infty))$ fulfilling the left inequality in (3.8), while after verifying (3.6) by straightforward differentiation, from (3.13) we immediately obtain (3.7). The latter thereupon implies the right inequality in (3.8), because in view of the positivity of both $(1-\kappa)\gamma-1$ and $(1-\kappa)(\gamma-1)$, according to (3.5) we have $A(t) \to \overline{A}$ as $t \to \infty$. ∎

We are now prepared to precisely specify our prospective subsolutions for (2.7).

Lemma 3.3: *Suppose that $n \ge 3$, $R > 0$, let $m \in \mathbb{R}$ and $\lambda > 0$ satisfy* (1.10), *let γ, κ and α be as in Lemma 3.1, and for $\mu > 0$, let $\tau_1 \equiv \tau_1^{(\mu)} := \max\{\tau_0, , R^{-\frac{n}{\alpha\kappa}}\}$, with $\tau_0 = \tau_0^{(\mu)}$ as introduced in Lemma 3.2. Then for any choice of $a > 0$ and $\tau > \tau_1$, letting*

$$\underline{w}(s,t) \equiv \underline{w}^{(\mu,a,\tau)}(s,t) := \begin{cases} \underline{w}_{\text{in}}(s,t), & t \ge 0,\ s \in [0,(at+\tau)^{-\alpha}), \\ \underline{w}_{\text{mid}}(s,t), & t \ge 0,\ s \in [(at+\tau)^{-\alpha},(at+\tau)^{-\alpha\kappa}), \\ \underline{w}_{\text{out}}(s,t), & t \ge 0,\ s \in [(at+\tau)^{-\alpha\kappa}, R^n], \end{cases} \tag{3.14}$$

with

$$\underline{w}_{\rm in}(s,t) \equiv \underline{w}_{\rm in}^{(\mu,a,\tau)}(s,t) := A(t)\cdot(at+\tau)^\alpha\, s, \qquad s\ge 0,\ t\ge 0, \tag{3.15}$$

and

$$\underline{w}_{\rm mid}(s,t)\equiv\underline{w}_{\rm mid}^{(\mu,a,\tau)}(s,t):=\frac{\gamma}{\gamma-1}A(t)-\frac{1}{\gamma-1}A(t)\cdot(at+\tau)^{-\alpha(\gamma-1)}s^{1-\gamma}, \qquad s>0,\ t\ge0, \tag{3.16}$$

as well as

$$\underline{w}_{\rm out}(s,t)\equiv\underline{w}_{\rm out}^{(\mu,a,\tau)}(s,t):=\frac{\mu R^n}{n}-A(t)\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]}\cdot(R^n-s), \qquad s\ge0,\ t\ge0, \tag{3.17}$$

defines a nonnegative function $\underline{w}\in C^1([0,R^n]\times[0,\infty))$ *such that*

$$\underline{w}(\cdot,t)\in W^{2,\infty}((0,R^n)) \quad \textit{for all } t>0,$$

that $\underline{w}(0,t)=0$ *and* $\underline{w}(R^n,t)=\frac{\mu R^n}{n}$ *for all* $t\ge0$, *and that for all* $s>0$ *and* $t>0$ *we have*

$$\begin{cases} \partial_t\underline{w}_{\rm in}(s,t)=A'(t)\cdot(at+\tau)^\alpha s+\alpha a A(t)\cdot(at+\tau)^{\alpha-1}s, \\ \partial_s\underline{w}_{\rm in}(s,t)=A(t)\cdot(at+\tau)^\alpha, \\ \partial_s^2\underline{w}_{\rm in}(s,t)=0 \end{cases} \tag{3.18}$$

as well as

$$\begin{cases} \partial_t\underline{w}_{\rm mid}(s,t)=\frac{\gamma}{\gamma-1}A'(t)-\frac{1}{\gamma-1}A'(t)\cdot(at+\tau)^{-\alpha(\gamma-1)}s^{1-\gamma} \\ \qquad\qquad +\alpha a A(t)\cdot(at+\tau)^{-\alpha(\gamma-1)-1}s^{1-\gamma}, \\ \partial_s\underline{w}_{\rm mid}(s,t)=A(t)\cdot(at+\tau)^{-\alpha(\gamma-1)}s^{-\gamma}, \\ \partial_s^2\underline{w}_{\rm mid}(s,t)=-\gamma A(t)\cdot(at+\tau)^{-\alpha(\gamma-1)}s^{-\gamma-1}, \end{cases} \tag{3.19}$$

and also

$$\begin{cases} \partial_t\underline{w}_{\rm out}(s,t)=-A'(t)\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]}\cdot(R^n-s) \\ \qquad\qquad +\alpha\cdot[(1-\kappa)\gamma-1]\cdot aA(t)\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]-1}\cdot(R^n-s), \\ \partial_s\underline{w}_{\rm out}(s,t)=A(t)\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]}, \\ \partial_s^2\underline{w}_{\rm out}(s,t)=0. \end{cases} \tag{3.20}$$

Proof. Since the requirements $\tau_1 \ge R^{-\frac{n}{\alpha\kappa}}$ and $\tau_1 \ge \tau_0 \ge 1$ ensure that for any choice of $a > 0$ and $\tau > \tau_1$ we have $(at+\tau)^{-\alpha} < (at+\tau)^{-\alpha\kappa} < R^n$ for all $t \ge 0$, in view of the definitions in (3.15), (3.16) and (3.17) we only need to observe that

$$\begin{aligned}
&\underline{w}_{\mathrm{in}}((at+\tau)^{-\alpha},t) - \underline{w}_{\mathrm{mid}}((at+\tau)^{-\alpha},t)\\
&\qquad = A(t)\cdot(at+\tau)^{\alpha}\cdot(at+\tau)^{-\alpha}\\
&\qquad\quad - \Big\{\frac{\gamma}{\gamma-1}A(t) - \frac{1}{\gamma-1}A(t)\cdot(at+\tau)^{-\alpha(\gamma-1)}\cdot(at+\tau)^{-\alpha(1-\gamma)}\Big\}\\
&\qquad = A(t) - \frac{\gamma}{\gamma-1}A(t) + \frac{1}{\gamma-1}A(t)\\
&\qquad = 0 \quad \text{for all } t>0,
\end{aligned}$$

that thanks to (3.5) we have

$$\begin{aligned}
&\underline{w}_{\mathrm{mid}}((at+\tau)^{-\alpha\kappa},t) - \underline{w}_{\mathrm{out}}((at+\tau)^{-\alpha\kappa},t)\\
&\qquad = \Big\{\frac{\gamma}{\gamma-1}A(t) - \frac{1}{\gamma-1}A(t)\cdot(at+\tau)^{-\alpha(\gamma-1)}\cdot(at+\tau)^{-\alpha\kappa(1-\gamma)}\Big\}\\
&\qquad\quad - \Big\{\frac{\mu R^n}{n} - A(t)\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]}\cdot\big(R^n-(at+\tau)^{-\alpha\kappa}\big)\Big\}\\
&\qquad = \Big\{\frac{\gamma}{\gamma-1} + R^n\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]} - \frac{\gamma}{\gamma-1}\cdot(at+\tau)^{-\alpha(1-\kappa)(\gamma-1)}\Big\}\\
&\qquad\quad \times A(t) - \frac{\mu R^n}{n}\\
&\qquad = 0 \quad \text{for all } t>0,
\end{aligned}$$

and that according to (3.18), (3.19) and (3.20),

$$\begin{aligned}
&\partial_s\underline{w}_{\mathrm{in}}((at+\tau)^{-\alpha},t) - \partial_s\underline{w}_{\mathrm{mid}}((at+\tau)^{-\alpha},t)\\
&\qquad = A(t)\cdot(at+\tau)^{\alpha} - A(t)\cdot(at+\tau)^{-\alpha(\gamma-1)}\cdot(at+\tau)^{\alpha\gamma}\\
&\qquad = 0 \qquad\qquad \text{for all } t>0
\end{aligned}$$

and

$$\begin{aligned}
&\partial_s\underline{w}_{\mathrm{mid}}((at+\tau)^{-\alpha\kappa},t) - \partial_s\underline{w}_{\mathrm{out}}((at+\tau)^{-\alpha\kappa},t)\\
&\qquad = A(t)\cdot(at+\tau)^{-\alpha(\gamma-1)}\cdot(at+\tau)^{\alpha\kappa\gamma} - A(t)\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]}\\
&\qquad = 0 \qquad\qquad \text{for all } t>0
\end{aligned}$$

as well as

$$\begin{aligned}
&\partial_t \underline{w}_{\rm in}((at+\tau)^{-\alpha},t) - \partial_t \underline{w}_{\rm mid}((at+\tau)^{-\alpha},t) \\
&\quad = \{A'(t)\cdot(at+\tau)^{\alpha}\cdot(at+\tau)^{-\alpha} + \alpha a A(t)\cdot(at+\tau)^{\alpha-1}\cdot(at+\tau)^{-\alpha}\} \\
&\qquad - \Big\{\frac{\gamma}{\gamma-1}A'(t) - \frac{1}{\gamma-1}A'(t)\cdot(at+\tau)^{-\alpha(\gamma-1)}\cdot(at+\tau)^{-\alpha(1-\gamma)} \\
&\qquad + \alpha a A(t)\cdot(at+\tau)^{-\alpha(\gamma-1)-1}\cdot(at+\tau)^{-\alpha(1-\gamma)}\Big\} \\
&\quad = \Big\{A'(t) + \alpha a A(t)\cdot\frac{1}{at+\tau}\Big\} - \Big\{A'(t)+\alpha a A(t)\cdot\frac{1}{at+\tau}\Big\} \\
&\quad = 0 \qquad\qquad \text{for all } t>0
\end{aligned}$$

and

$$\begin{aligned}
&\partial_t \underline{w}_{\rm mid}((at+\tau)^{-\alpha\kappa},t) - \partial_t \underline{w}_{\rm out}((at+\tau)^{-\alpha\kappa},t) \\
&\quad = \Big\{\frac{\gamma}{\gamma-1}A'(t) - \frac{1}{\gamma-1}A'(t)\cdot(at+\tau)^{-\alpha(\gamma-1)}\cdot(at+\tau)^{-\alpha\kappa(1-\gamma)} \\
&\qquad + \alpha a A(t)\cdot(at+\tau)^{-\alpha(\gamma-1)-1}\cdot(at+\tau)^{-\alpha\kappa(1-\gamma)}\Big\} \\
&\qquad - \{-A'(t)\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]}\cdot(R^n-(at+\tau)^{-\alpha\kappa}) \\
&\qquad + \alpha\cdot[(1-\kappa)\gamma-1]\cdot aA(t)\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]-1}\cdot(R^n-(at+\tau)^{-\alpha\kappa})\} \\
&\quad = A'(t)\cdot\Big\{\frac{\gamma}{\gamma-1} + R^n\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]} - \frac{\gamma}{\gamma-1}\cdot(at+\tau)^{-\alpha(1-\kappa)(\gamma-1)}\Big\} \\
&\qquad - aA(t)\cdot\{\alpha\cdot[(1-\kappa)\gamma-1]\cdot R^n\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]-1} \\
&\qquad\qquad - \alpha(1-\kappa)\gamma\cdot(at+\tau)^{-\alpha(1-\kappa)(\gamma-1)-1}\} \\
&\quad = 0 \qquad\qquad \text{for all } t>0,
\end{aligned}$$

with the latter identity readily resulting from (3.6) and (3.5). ∎

3.1. Subsolution properties: Inner region. The following identification of a subsolution property of $\underline{w}$ in the subregion from (3.14) adjacent to the spatial origin essentially relies on the choice of α made in (3.4), and hence on the growth of $\|\underline{w}_s(\cdot,t)\|_{L^\infty((0,R^n))}$ thereby quantified.

Lemma 3.4: *Let $n\ge 3$ and $R>0$, assume (1.6), (1.7), (1.8) and (1.9) with some $m\in\mathbb{R}$ and $\lambda>0$ fulfilling (1.10), and let γ,κ,α and $(\tau_1^{(\mu)})_{\mu>0}$ be as introduced in Lemma 3.1 and Lemma 3.3. Then for all $\mu>0$ there exist $a_{\rm in}=a_{\rm in}^{(\mu)}>0$ and $\tau_{\rm in}=\tau_{\rm in}^{(\mu)}\ge\tau_0^{(\mu)}$ such that for any choice of $a\in(0,a_{\rm in})$*

and $\tau > \tau_{\rm in}$*, the functions* $A = A^{(\mu,a,\tau)}$ *and* $\underline{w}_{\rm in} = \underline{w}_{\rm in}^{(\mu,a,\tau)}$ *defined in* (3.5) *and* (3.15) *satisfy*

$$(\mathcal{P}\underline{w}_{\rm in})(s,t) \le 0 \quad \text{for all } s>0 \text{ and } t>0, \tag{3.21}$$

where $\mathcal{P}$ *is as in* (2.8).

Proof. With $\underline{A}=\underline{A}^{(\mu)}$ and $\overline{A}=\overline{A}^{(\mu)}$ taken from (3.9), we choose $a_{\rm in}=a_{\rm in}^{(\mu)}\in(0,1]$ and $\tau_{\rm in}=\tau_{\rm in}^{(\mu)} \ge \tau_1^{(\mu)}$ in such a way that

$$\frac{2^{\lambda+3}n^\lambda\alpha\overline{A}^\lambda}{K_S}\cdot a_{\rm in} \le 1 \tag{3.22}$$

and

$$\frac{1}{2}\underline{A}\cdot\tau_{\rm in}^\alpha \ge \frac{\mu}{n} \tag{3.23}$$

and

$$n\underline{A}\cdot\tau_{\rm in}^\alpha \ge 1 \tag{3.24}$$

as well as

$$\frac{2^{\lambda+3}n^{\lambda+1}\alpha[(1-\kappa)\gamma-1]\overline{A}^{\lambda+1}}{K_S\mu}\cdot\tau_{\rm in}^{-\alpha[(1-\kappa)\gamma-1]} \le 1. \tag{3.25}$$

Then assuming $a\in(0,a_{\rm in})$ and $\tau>\tau_{\rm in}$, for $A=A^{(\mu,a,\tau)}$ and $\underline{w}_{\rm in}=\underline{w}_{\rm in}^{(\mu,a,\tau)}$ as accordingly defined in (3.5) and (3.15) we may use (3.18) to obtain that for all $s>0$ and $t>0$,

$$\begin{aligned}(\mathcal{P}\underline{w}_{\rm in})(s,t) =& A'(t)\cdot(at+\tau)^\alpha s + \alpha a A(t)\cdot(at+\tau)^{\alpha-1}s \\ & -\Big\{A(t)\cdot(at+\tau)^\alpha s - \frac{\mu}{n}s\Big\}\cdot S(n\partial_s\underline{w}_{\rm in}(s,t))\cdot n\partial_s\underline{w}_{\rm in}(s,t).\end{aligned} \tag{3.26}$$

Here since $\tau>\tau_1^{(\mu)}$, we may utilize (3.8) to see that thanks to (3.23),

$$\begin{aligned}A(t)\cdot(at+\tau)^\alpha s - \frac{\mu}{n}s &= \frac{1}{2}A(t)\cdot(at+\tau)^\alpha s + \Big\{\frac{1}{2}A(t)\cdot(at+\tau)^\alpha - \frac{\mu}{n}\Big\}\cdot s \\ &\ge \frac{1}{2}A(t)\cdot(at+\tau)^\alpha s + \Big\{\frac{1}{2}\underline{A}\tau^\alpha - \frac{\mu}{n}\Big\}\cdot s \\ &\ge \frac{1}{2}A(t)\cdot(at+\tau)^\alpha s \quad \text{for all } s>0 \text{ and } t>0,\end{aligned} \tag{3.27}$$

while due to the same token, (3.24) along with (3.18) ensures that

$$n\partial_s\underline{w}_{\rm in}(s,t) = nA(t)\cdot(at+\tau)^\alpha \ge n\underline{A}\tau^\alpha \ge 1 \quad \text{for all } s>0 \text{ and } t>0.$$

As $(\xi+1)^{-\lambda-1} \ge 2^{-\lambda-1}\xi^{-\lambda-1}$ for all $\xi \ge 1$, in light of (1.9) this shows that

$$S(n\partial_s \underline{w}_{\rm in}(s,t)) \cdot n\partial_s \underline{w}_{\rm in}(s,t) \ge 2^{-\lambda-1} K_S \cdot (n\partial_s \underline{w}_{\rm in}(s,t))^{-\lambda}$$
$$= 2^{-\lambda-1} n^{-\lambda} K_S A^{-\lambda}(t) \cdot (at+\tau)^{-\alpha\lambda} \qquad \text{for all } s>0 \text{ and } t>0,$$

whence from (3.26) and (3.27) we infer that

$$\begin{aligned} &\frac{1}{A(t)\cdot(at+\tau)^{\alpha-1}s} \cdot (\mathcal{P}\underline{w}_{\rm in})(s,t) \\ &\le \frac{A'(t)}{A(t)} \cdot (at+\tau) + \alpha a \\ &\quad - \frac{1}{2}\cdot(at+\tau)\cdot 2^{-\lambda-1}n^{-\lambda}K_S A^{-\lambda}(t)\cdot(at+\tau)^{-\alpha\lambda} \\ &= \frac{A'(t)}{A(t)}\cdot(at+\tau) + \alpha a - 2^{-\lambda-2}n^{-\lambda}K_S A^{-\lambda}(t) \quad \text{for all } s>0 \text{ and } t>0, \end{aligned} \tag{3.28}$$

because $\alpha\lambda = 1$ by (3.4). We now recall (3.5) and (3.6) and use the restriction $a \le 1$ together with (3.8) and (3.25) to estimate

$$\begin{aligned} &\frac{\frac{A'(t)}{A(t)}\cdot(at+\tau)}{\frac{1}{2}\cdot 2^{-\lambda-2}n^{-\lambda}K_S A^{-\lambda}(t)} \\ &= \frac{2^{\lambda+3}n^{\lambda}A^{\lambda+1}(t)}{K_S}\cdot\frac{A'(t)}{A^2(t)}\cdot(at+\tau) \\ &= \frac{2^{\lambda+3}n^{\lambda}A^{\lambda+1}(t)}{K_S}\cdot\frac{a}{\frac{\mu R^n}{n}} \\ &\quad\times\{\alpha[(1-\kappa)\gamma-1]R^n\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]} - \alpha(1-\kappa)\gamma\cdot(at+\tau)^{-\alpha(1-\kappa)(\gamma-1)}\} \\ &\le \frac{2^{\lambda+3}n^{\lambda}\overline{A}^{\lambda+1}}{K_S}\cdot\frac{1}{\frac{\mu R^n}{n}}\cdot\alpha[(1-\kappa)\gamma-1]R^n\cdot\tau^{-\alpha[(1-\kappa)\gamma-1]} \\ &= \frac{2^{\lambda+3}n^{\lambda+1}\alpha[(1-\kappa)\gamma-1]\overline{A}^{\lambda+1}}{K_S\,\mu}\cdot\tau^{-\alpha[(1-\kappa)\gamma-1]} \\ &\le 1 \qquad\qquad \text{for all } t>0, \end{aligned}$$

while combining (3.8) with our additional restriction (3.22) on a guarantees that

$$\frac{\alpha a}{\frac{1}{2}\cdot 2^{-\lambda-2}n^{-\lambda}K_S A^{-\lambda}(t)} \le \frac{2^{\lambda+3}n^{\lambda}\,\alpha\overline{A}^{\lambda}}{K_S}\cdot a \le 1 \quad \text{for all } t>0.$$

Therefore, (3.28) implies (3.21). ∎

3.2. Subsolution properties: Outer region. Due to the spatially linear structure of $\underline{w}$ throughout the subregion in (3.14) which contains the outer boundary point $s = R^n$, similar to that in the previous lemma also the following argument actually ignores the diffusive contribution to the operator $\mathcal{P}$.

Lemma 3.5: *Let $n \ge 3$ and $R > 0$, and suppose that* (1.6), (1.7), (1.8) *and* (1.9) *hold with some $m \in \mathbb{R}$ and $\lambda > 0$ fulfilling* (1.10). *Then letting γ, κ, α and $(\tau_1^{(\mu)})_{\mu>0}$ be as in Lemma 3.1 and Lemma 3.3, for all $\mu > 0$ one can find $a_{\rm out} = a_{\rm out}^{(\mu)} > 0$ and $\tau_{\rm out} = \tau_{\rm out}^{(\mu)} \ge \tau_0^{(\mu)}$ with the property that if $a \in (0, a_{\rm out})$ and $\tau > \tau_{\rm out}$ and if $A = A^{(\mu,a,\tau)}$, $\underline{w}_{\rm out} = \underline{w}_{\rm out}^{(\mu,a,\tau)}$ and $\mathcal{P}$ are as in* (3.5), (3.17) *and* (2.8), *then*

$$(\mathcal{P}\underline{w}_{\rm out})(s,t) \le 0 \quad \text{for all } s > 0 \text{ and } t > 0. \tag{3.29}$$

Proof. Again relying on the inequality $(1-\kappa)\gamma - 1 > 0$, we pick $\tau_{\rm out} = \tau_{\rm out}^{(\mu)} \ge \tau_1^{(\mu)}$ large enough such that with $\overline{A} = \overline{A}^{(\mu)}$ as in (3.9) we have

$$n\overline{A}\tau_{\rm out}^{-\alpha[(1-\kappa)\gamma-1]} \le 1 \tag{3.30}$$

and

$$\overline{A}\tau_{\rm out}^{-\alpha[(1-\kappa)\gamma-1]} \le \frac{\mu}{2n}, \tag{3.31}$$

and we then choose $a_{\rm out} = a_{\rm out}^{(\mu)} > 0$ suitably small fulfilling

$$\alpha \cdot [(1-\kappa)\gamma - 1] \cdot a_{\rm out} \le 2^{-\lambda-2} K_S\, \mu. \tag{3.32}$$

Then letting $a \in (0, a_{\rm out})$ and $\tau > \tau_{\rm out}$, and taking $A = A^{(\mu,a,\tau)}$ as well as $\underline{w}_{\rm out} = \underline{w}_{\rm out}^{(\mu,a,\tau)}$ from (3.5) and (3.17), we first observe that according to (3.20) and (3.8),

$$\begin{aligned} n\partial_s \underline{w}_{\rm out}(s,t) &= nA(t)\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]} \\ &\le n\overline{A}\tau^{-\alpha[(1-\kappa)\gamma-1]} \\ &\le 1 \qquad\qquad \text{for all } s > 0 \text{ and } t > 0, \end{aligned}$$

whence using (1.9) and estimating

$$(\xi+1)^{-\lambda-1} \ge 2^{-\lambda-1} \quad \text{for } \xi \in [0,1]$$

we see that

$$\begin{aligned} S(n\partial_s\underline{w}_{\rm out}(s,t))\cdot n\partial_s\underline{w}_{\rm out}(s,t) &\ge 2^{-\lambda-1}nK_S\,\partial_s\underline{w}_{\rm out}(s,t) \\ &= 2^{-\lambda-1}nK_S\,A(t)\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]} \\ &\qquad\qquad \text{for all } s>0 \text{ and } t>0. \end{aligned}$$

Since furthermore

$$\begin{aligned} \underline{w}_{\rm out}(s,t)-\frac{\mu}{n}s &= \frac{\mu R^n}{n} - A(t)\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]}\cdot(R^n-s)-\frac{\mu}{n}s \\ &= \Big\{\frac{\mu}{n} - A(t)\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]}\Big\}\cdot(R^n-s) \\ &\ge \Big\{\frac{\mu}{n} - \overline{A}\tau^{-\alpha[(1-\kappa)\gamma-1]}\Big\}\cdot(R^n-s) \\ &\ge \frac{\mu}{2n}\cdot(R^n-s) \qquad\qquad \text{for all } s>0 \text{ and } t>0 \end{aligned}$$

because of (3.17), (3.8) and (3.31), thanks to the inequality $A'\ge 0$ asserted by (3.7) we obtain that according to (2.8) and (3.20) and due to $\tau\ge 1$,

$$\begin{aligned} (\mathcal{P}\underline{w}_{\rm out})(s,t) =& -A'(t)\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]}\cdot(R^n-s) \\ &+\alpha\cdot[(1-\kappa)\gamma-1]\cdot aA(t)\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]-1}\cdot(R^n-s) \\ &-\Big\{\underline{w}_{\rm out}(s,t)-\frac{\mu}{n}s\Big\}\cdot S(n\partial_s\underline{w}_{\rm out}(s,t))\cdot n\partial_s\underline{w}_{\rm out}(s,t) \\ \le& \alpha\cdot[(1-\kappa)\gamma-1]\cdot aA(t)\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]}\cdot(R^n-s) \\ &-\frac{\mu}{2n}\cdot(R^n-s)\cdot 2^{-\lambda-1}nK_S\,A(t)\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]} \\ =&\{\alpha\cdot[(1-\kappa)\gamma-1]\cdot a - 2^{-\lambda-2}K_S\,\mu\} \\ &\times A(t)\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]}\cdot(R^n-s) \end{aligned}$$

for all $s>0$ and $t>0$. The claim thereby becomes a consequence of (3.32). ∎

3.3. Subsolution properties: Intermediate region. The core of our reasoning can now be found in the following argument which takes full advantage of the choice not only of γ but also of the number κ from Lemma 3.1. Indeed, all five inequalities (3.1), (3.2), (3.3), $\gamma>1$ and $\kappa<\frac{\gamma-1}{\gamma}$ asserted therein will be of crucial importance in our verification of the fact that for suitably small a and appropriately large τ, the transition from steep to essentially flat behavior in the functional form specified in (3.14) and (3.16) is achieved in such a way that still some subsolution feature with respect to the operator $\mathcal{P}$ is retained, now to a crucial extent due to the diffusive contribution therein:

LEMMA 3.6: *Let $n \geq 3$ and $R > 0$, let* (1.6), (1.7), (1.8) *and* (1.9) *be satisfied with some $m \in \mathbb{R}$ and $\lambda > 0$ fulfilling* (1.10), *and let γ, κ, α and $(\tau_1^{(\mu)})_{\mu>0}$ be as provided by Lemma 3.1 and Lemma 3.3. Then for all $\mu > 0$ there exist $a_{\mathrm{mid}} = a_{\mathrm{mid}}^{(\mu)} > 0$ and $\tau_{\mathrm{mid}} = \tau_{\mathrm{mid}}^{(\mu)} \geq \tau_1^{(\mu)}$ such that whenever $a \in (0, a_{\mathrm{mid}})$ and $\tau > \tau_{\mathrm{mid}}$, with $A = A^{(\mu,a,\tau)}$, $\underline{w}_{\mathrm{mid}} = \underline{w}_{\mathrm{mid}}^{(\mu,a,\tau)}$ and $\mathcal{P}$ taken from* (3.5), (3.16) *and* (2.8) *we have*

$$(\mathcal{P}\underline{w}_{\mathrm{mid}})(s,t) \leq 0 \quad \textit{for all } t > 0 \textit{ and any } s \in ((at+\tau)^{-\alpha}, (at+\tau)^{-\alpha\kappa}). \tag{3.33}$$

Proof. We let $\overline{A} = \overline{A}^{(\mu)}$ and $\underline{A} = \underline{A}^{(\mu)}$ be as in Lemma 3.2, and fix $a_{\mathrm{mid}} = a_{\mathrm{mid}}^{(\mu)} \in (0,1]$ as well as $\tau_{\mathrm{mid}} = \tau_{\mathrm{mid}}^{(\mu)} \geq \tau_1^{(\mu)}$ such that

$$\frac{3 \cdot 2^{\lambda+2} n^\lambda \alpha \overline{A}^\lambda}{K_S} \cdot a_{\mathrm{mid}} \leq 1 \tag{3.34}$$

and

$$\frac{\mu}{n} \cdot \tau_{\mathrm{mid}}^{-\alpha\kappa} \leq \frac{1}{2}\underline{A}, \tag{3.35}$$

and that

$$\frac{3 \cdot 2^{\lambda+2} n^{\lambda+1} \alpha\gamma \cdot [(1-\kappa)\gamma - 1] \cdot \overline{A}^{\lambda+1}}{(\gamma-1) K_S\, \mu} \cdot \tau_{\mathrm{mid}}^{-\alpha[(1-\kappa)\gamma-1]} \leq 1 \tag{3.36}$$

and

$$\frac{3 \cdot 2^{\lambda+2} n^{m+\lambda+1} \gamma K_D \underline{A}^{m+\lambda-1}}{K_S} \cdot \tau_{\mathrm{mid}}^{-\delta_2} \leq 1 \tag{3.37}$$

as well as

$$\frac{3 \cdot 2^{\lambda+2} \gamma\alpha \cdot [(1-\kappa)\gamma - 1]}{(\gamma-1) K_S\, \mu} \cdot \frac{1}{\tau_{\mathrm{mid}}} \leq 1 \tag{3.38}$$

and

$$\frac{3 \cdot 2^{\lambda+2} \alpha}{n K_S \underline{A}} \cdot \tau_{\mathrm{mid}}^{-\alpha\kappa-1} \leq 1 \tag{3.39}$$

and

$$\frac{3 \cdot 2^{\lambda+2} n^m K_D\, \gamma \underline{A}^{m-2}}{K_S} \cdot \tau_{\mathrm{mid}}^{-\delta_1} \leq 1, \tag{3.40}$$

again noting that $(1-\kappa)\gamma-1$ is positive since $\kappa<\frac{\gamma-1}{\gamma}$, and observing that the numbers

$$\tag{3.41}\begin{aligned}\delta_2 &:= (m\alpha+1)(\gamma-1)+\alpha\kappa\cdot\Big[1-\frac{2}{n}-(m+\lambda)\gamma\Big],\\ \delta_1 &:= (m-1)(\gamma-1)+\kappa\cdot\Big[1-\frac{2}{n}-(m-1)\gamma\Big]\end{aligned}$$

are both positive according to Lemma 3.1.

Now if $a\in(0,a_{\rm mid})$ and $\tau>\tau_{\rm mid}$, and if $A=A^{(\mu,a,\tau)}$ and $\underline{w}_{\rm mid}=\underline{w}_{\rm mid}^{(\mu,a,\tau)}$ are as correspondingly defined in (3.5) and (3.16), then we can first use (3.6) and the fact that $a\le 1$ to estimate

$$\begin{aligned}\frac{\gamma}{\gamma-1}A'(t) &= \frac{\gamma}{\gamma-1}\cdot\frac{A'(t)}{A^2(t)}\cdot A^2(t)\\ &= \frac{\gamma}{\gamma-1}\cdot\frac{a}{\frac{\mu R^n}{n}}\cdot\Big\{\alpha\cdot[(1-\kappa)\gamma-1]\cdot R^n\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]-1}\\ &\qquad -\alpha(1-\kappa)\gamma\cdot(at+\tau)^{-\alpha(1-\kappa)(\gamma-1)-1}\Big\}\cdot A^2(t)\\ &\le \frac{n\gamma\alpha[(1-\kappa)\gamma-1]A^2(t)}{(\gamma-1)\mu}\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]-1} \qquad \text{for all } t>0,\end{aligned}$$

and rely on (3.16), (3.35) and (3.8) to see that in the considered region we have

$$\begin{aligned}\underline{w}_{\rm mid}(s,t)-\frac{\mu}{n}s &= \frac{\gamma}{\gamma-1}A(t)-\frac{1}{\gamma-1}A(t)\cdot(at+\tau)^{-\alpha(\gamma-1)}s^{1-\gamma}-\frac{\mu}{n}s\\ &\ge \frac{\gamma}{\gamma-1}A(t)-\frac{1}{\gamma-1}A(t)\cdot(at+\tau)^{-\alpha(\gamma-1)}\cdot\{(at+\tau)^{-\alpha}\}^{1-\gamma}\\ &\quad -\frac{\mu}{n}\cdot(at+\tau)^{-\alpha\kappa}\\ &= \frac12 A(t)+\frac12 A(t)-\frac{\mu}{n}(at+\tau)^{-\alpha\kappa}\\ &\ge \frac12 A(t)+\frac12\underline{A}-\frac{\mu}{n}\tau^{-\alpha\kappa}\\ &\ge \frac12 A(t) \quad \text{for all } t>0 \text{ and } s\in((at+\tau)^{-\alpha},(at+\tau)^{-\alpha\kappa}).\end{aligned}$$

In view of (3.19), (1.8) and (2.8), we therefore obtain that since $A'(t) \ge 0$ for all $t > 0$ by (3.7),

$$
\begin{aligned}
(\mathcal{P}\underline{w}_{\rm mid})(s,t) \le& \frac{\gamma}{\gamma-1}A'(t) \\
& - \frac{1}{\gamma-1}A'(t)\cdot(at+\tau)^{-\alpha(\gamma-1)}s^{1-\gamma} \\
& + \alpha a A(t)\cdot(at+\tau)^{-\alpha(\gamma-1)-1}s^{1-\gamma} \\
& - n^2 s^{2-\frac{2}{n}}\cdot K_D\{nA(t)\cdot(at+\tau)^{-\alpha(\gamma-1)}s^{-\gamma}\}^{m-1} \\
& \quad \times\{-\gamma A(t)\cdot(at+\tau)^{-\alpha(\gamma-1)}s^{-\gamma-1}\} \\
& - \{\underline{w}_{\rm mid}(s,t) - \frac{\mu}{n}s\}\cdot S(n\partial_s\underline{w}_{\rm mid}(s,t))\cdot n\partial_s\underline{w}_{\rm mid}(s,t) \\
\le& \frac{n\gamma\alpha\cdot[(1-\kappa)\gamma-1]\cdot A^2(t)}{(\gamma-1)\mu}\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]-1} \\
& + \alpha a A(t)\cdot(at+\tau)^{-\alpha(\gamma-1)-1}s^{1-\gamma} \\
& + n^{m+1}\gamma K_D\,A^m(t)\cdot(at+\tau)^{-m\alpha(\gamma-1)}s^{-m\gamma+1-\frac{2}{n}} \\
& - \frac{1}{2}A(t)\cdot S(n\partial_s\underline{w}_{\rm mid}(s,t))\cdot n\partial_s\underline{w}_{\rm mid}(s,t) \\
& \text{for all } t>0 \text{ and } s\in((at+\tau)^{-\alpha},(at+\tau)^{-\alpha\kappa}).
\end{aligned} \tag{3.42}
$$

To proceed from this, let us split

$$Q := \{(s,t)\in(0,R^n)\times(0,\infty) \mid (at+\tau)^{-\alpha} < s < (at+\tau)^{-\alpha\kappa}\}$$

according to $Q = Q_1 \cup Q_2$, where

$$Q_1 := \{(s,t)\in Q \mid n\partial_s\underline{w}_{\rm mid}(s,t) \le 1\}$$

and

$$Q_2 := \{(s,t)\in Q \mid n\partial_s\underline{w}_{\rm mid}(s,t) > 1\}.$$

Then using (1.9) and again (3.19), we see that in the latter of these regions,

$$
\begin{aligned}
& S(n\partial_s\underline{w}_{\rm mid}(s,t))\cdot n\partial_s\underline{w}_{\rm mid}(s,t) \\
&\qquad \ge K_S\cdot\{n\partial_s\underline{w}_{\rm mid}(s,t)+1\}^{-\lambda-1}\cdot n\partial_s\underline{w}_{\rm mid}(s,t) \\
&\qquad \ge 2^{-\lambda-1}K_S\cdot\{n\partial_s\underline{w}_{\rm mid}(s,t)\}^{-\lambda} \\
&\qquad = 2^{-\lambda-1}n^{-\lambda}K_S\cdot A^{-\lambda}(t)\cdot(at+\tau)^{\alpha\lambda(\gamma-1)}s^{\lambda\gamma} \\
&\qquad = 2^{-\lambda-1}n^{-\lambda}K_S\cdot A^{-\lambda}(t)\cdot(at+\tau)^{\gamma-1}s^{\lambda\gamma} \quad \text{for all } (s,t)\in Q_2,
\end{aligned} \tag{3.43}
$$

because $\alpha\lambda = 1$ by (3.4). On the other hand, again drawing on (3.8), (3.4) and the positivity of $(1-\kappa)\gamma - 1$ we may utilize (3.36) to infer that within this subset of Q we can control the first summand on the right of (3.42) according to

$$\begin{aligned}
&\frac{\frac{n\gamma\alpha\cdot[(1-\kappa)\gamma-1]\cdot A^2(t)}{(\gamma-1)\mu}\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]-1}}{\frac{1}{3}\cdot\frac{1}{2}A(t)\cdot 2^{-\lambda-1}n^{-\lambda}K_S\,A^{-\lambda}(t)\cdot(at+\tau)^{\gamma-1}s^{\lambda\gamma}} \\
&=\frac{3\cdot 2^{\lambda+2}n^{\lambda+1}\gamma\alpha\cdot[(1-\kappa)\gamma-1]\cdot A^{\lambda+1}(t)}{(\gamma-1)K_S\,\mu}\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]-\gamma}s^{-\lambda\gamma} \\
&\le\frac{3\cdot 2^{\lambda+2}n^{\lambda+1}\gamma\alpha\cdot[(1-\kappa)\gamma-1]\cdot\overline{A}^{\lambda+1}}{(\gamma-1)K_S\,\mu}\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]-\gamma}\cdot(at+\tau)^{\alpha\lambda\gamma} \\
&=\frac{3\cdot 2^{\lambda+2}n^{\lambda+1}\gamma\alpha\cdot[(1-\kappa)\gamma-1]\cdot\overline{A}^{\lambda+1}}{(\gamma-1)K_S\,\mu}\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]} \\
&\le\frac{3\cdot 2^{\lambda+2}n^{\lambda+1}\gamma\alpha\cdot[(1-\kappa)\gamma-1]\cdot\overline{A}^{\lambda+1}}{(\gamma-1)K_S\,\mu}\cdot\tau^{-\alpha[(1-\kappa)\gamma-1]} \\
&\le 1 \qquad\qquad \text{for all } (s,t)\in Q_2,
\end{aligned}\tag{3.44}$$

while (3.34) together with our smallness assumption on $a_{\rm mid}$ in (3.8) as well as the evident fact that $1-\gamma-\lambda\gamma<0$ ensures that

$$\begin{aligned}
&\frac{\alpha a A(t)\cdot(at+\tau)^{-\alpha(\gamma-1)-1}s^{1-\gamma}}{\frac{1}{3}\cdot\frac{1}{2}A(t)\cdot 2^{-\lambda-1}n^{-\lambda}K_S\,A^{-\lambda}(t)\cdot(at+\tau)^{\gamma-1}s^{\lambda\gamma}} \\
&=\frac{3\cdot 2^{\lambda+2}n^{\lambda}\alpha A^{\lambda}(t)}{K_S}\cdot a\cdot(at+\tau)^{-\alpha(\gamma-1)-\gamma}s^{1-\gamma-\lambda\gamma} \\
&\le\frac{3\cdot 2^{\lambda+2}n^{\lambda}\alpha\overline{A}^{\lambda}}{K_S}\cdot a\cdot(at+\tau)^{-\alpha(\gamma-1)-\gamma}\cdot(at+\tau)^{-\alpha(1-\gamma-\lambda\gamma)} \\
&=\frac{3\cdot 2^{\lambda+2}n^{\lambda}\alpha\overline{A}^{\lambda}}{K_S}\cdot a \\
&\le 1 \qquad\qquad \text{for all } (s,t)\in Q_2.
\end{aligned}\tag{3.45}$$

Furthermore, thanks to the positivity of

$$1-\frac{2}{n}-(m+\lambda)\gamma$$

asserted by (3.1), and to the inequality $\delta_2 > 0$ satisfied by the number δ_2 defined in (3.40), we may use (3.37) to find that since

$$A^{m+\lambda-1} \le \underline{A}^{m+\lambda-1}$$

in $(0,\infty)$ by (3.8) and the inequality $m+\lambda-1<0$ clearly implied by (1.10),

$$\begin{aligned}
&\frac{n^{m+1}\gamma K_D\, A^m(t)\cdot(at+\tau)^{-m\alpha(\gamma-1)}s^{-m\gamma+1-\frac{2}{n}}}{\frac{1}{3}\cdot\frac{1}{2}A(t)\cdot 2^{-\lambda-1}n^{-\lambda}K_S\,A^{-\lambda}(t)\cdot(at+\tau)^{\gamma-1}s^{\lambda\gamma}} \\
&\quad = \frac{3\cdot 2^{\lambda+2}n^{m+\lambda+1}\gamma K_D\,A^{m+\lambda-1}(t)}{K_S} \\
&\qquad \times (at+\tau)^{-m\alpha(\gamma-1)-\gamma+1}s^{1-\frac{2}{n}-(m+\lambda)\gamma} \\
&\quad \le \frac{3\cdot 2^{\lambda+2}n^{m+\lambda+1}\gamma K_D\underline{A}^{m+\lambda-1}}{K_S} \\
&\qquad \times (at+\tau)^{-m\alpha(\gamma-1)-\gamma+1}\cdot(at+\tau)^{-\alpha\kappa\cdot[1-\frac{2}{n}-(m+\lambda)\gamma]} \\
&\quad = \frac{3\cdot 2^{\lambda+2}n^{m+\lambda+1}\gamma K_D\underline{A}^{m+\lambda-1}}{K_S}\cdot(at+\tau)^{-\delta_2} \\
&\quad \le \frac{3\cdot 2^{\lambda+2}n^{m+\lambda+1}\gamma K_D\underline{A}^{m+\lambda-1}}{K_S}\cdot\tau^{-\delta_2} \\
&\quad \le 1 \quad \text{for all } (s,t)\in Q_2,
\end{aligned} \tag{3.46}$$

so that combining (3.42)–(3.46) shows that, indeed,

$$(\mathcal{P}\underline{w}_{\rm mid})(s,t) \le 0 \quad \text{for all } (s,t)\in Q_2. \tag{3.47}$$

In the corresponding complementary region, however, (1.9) along with (3.19) implies that

$$\begin{aligned}
&S(n\partial_s\underline{w}_{\rm mid}(s,t))\cdot n\partial_s\underline{w}_{\rm mid}(s,t) \\
&\qquad \ge nK_S\cdot\{n\partial_s\underline{w}_{\rm mid}(s,t)+1\}^{-\lambda-1}\partial_s\underline{w}_{\rm mid}(s,t) \\
&\qquad \ge 2^{-\lambda-1}nK_S\cdot\partial_s\underline{w}_{\rm mid}(s,t) \\
&\qquad \ge 2^{-\lambda-1}nK_S\,A(t)\cdot(at+\tau)^{-\alpha(\gamma-1)}s^{-\gamma} \\
&\qquad\qquad \text{for all } (s,t)\in Q_1,
\end{aligned} \tag{3.48}$$

whereas

$$
\begin{aligned}
&\frac{\frac{n\gamma\alpha\cdot[(1-\kappa)\gamma-1]\cdot A^2(t)}{(\gamma-1)\mu}\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]-1}}{\frac{1}{3}\cdot\frac{1}{2}A(t)\cdot 2^{-\lambda-1}nK_S\,A(t)\cdot(at+\tau)^{-\alpha(\gamma-1)}s^{-\gamma}} \\
&\quad=\frac{3\cdot 2^{\lambda+2}\gamma\alpha\cdot[(1-\kappa)\gamma-1]}{(\gamma-1)K_S\,\mu}\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]-1+\alpha(\gamma-1)}s^{\gamma} \\
&\quad\le\frac{3\cdot 2^{\lambda+2}\gamma\alpha\cdot[(1-\kappa)\gamma-1]}{(\gamma-1)K_S\,\mu} \\
&\qquad\times(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]-1+\alpha(\gamma-1)}\cdot(at+\tau)^{-\alpha\kappa\gamma} \\
&\quad=\frac{3\cdot 2^{\lambda+2}\gamma\alpha\cdot[(1-\kappa)\gamma-1]}{(\gamma-1)K_S\,\mu}\cdot\frac{1}{at+\tau} \\
&\quad\le\frac{3\cdot 2^{\lambda+2}\gamma\alpha\cdot[(1-\kappa)\gamma-1]}{(\gamma-1)K_S\,\mu}\cdot\frac{1}{\tau} \\
&\quad\le 1 \qquad\qquad \text{for all } (s,t)\in Q_1
\end{aligned}
\tag{3.49}
$$

by (3.38), and

$$
\begin{aligned}
&\frac{\alpha aA(t)\cdot(at+\tau)^{-\alpha(\gamma-1)-1}s^{1-\gamma}}{\frac{1}{3}\cdot\frac{1}{2}A(t)\cdot 2^{-\lambda-1}nK_S\,A(t)\cdot(at+\tau)^{-\alpha(\gamma-1)}s^{-\gamma}} \\
&\quad=\frac{3\cdot 2^{\lambda+2}\alpha a}{nK_SA(t)}\cdot\frac{s}{at+\tau} \\
&\quad\le\frac{3\cdot 2^{\lambda+2}\alpha}{nK_S\underline{A}}\cdot(at+\tau)^{-\alpha\kappa-1} \\
&\quad\le\frac{3\cdot 2^{\lambda+2}\alpha}{nK_S\underline{A}}\cdot\tau^{-\alpha\kappa-1} \\
&\quad\le 1 \qquad\qquad \text{for all } (s,t)\in Q_1
\end{aligned}
\tag{3.50}
$$

due to (3.39) and the inequality $a\le 1$. Apart from that, the positivity not only of the number δ_1 from (3.41) but also of

$$
1-\frac{2}{n}-(m-1)\gamma=\Big\{1-\frac{2}{n}-(m+\lambda)\gamma\Big\}+(\lambda+1)\gamma,
$$

the latter being implied by (3.1), enables us to moreover use (3.41) and (3.8) along with the inequality $m-2<0$, as particularly guaranteed by (1.10), in estimating $A^{m-2} \le \underline{A}^{m-2}$ and therefore

$$\begin{aligned}
&\frac{n^{m+1}\gamma K_D\, A^m(t)\cdot(at+\tau)^{-m\alpha(\gamma-1)}s^{-m\gamma+1-\frac{2}{n}}}{\frac{1}{3}\cdot\frac{1}{2}A(t)\cdot 2^{-\lambda-1}nK_S\, A(t)\cdot(at+\tau)^{-\alpha(\gamma-1)}s^{-\gamma}} \\
&\quad = \frac{3\cdot 2^{\lambda+2}n^m\gamma K_D\, A^{m-2}(t)}{K_S}\cdot(at+\tau)^{-(m-1)\alpha(\gamma-1)}\cdot s^{1-\frac{2}{n}-(m-1)\gamma} \\
&\quad \le \frac{3\cdot 2^{\lambda+2}n^m\gamma K_D\underline{A}^{m-2}}{K_S}\cdot(at+\tau)^{-(m-1)\alpha(\gamma-1)}\cdot(at+\tau)^{-\alpha\kappa\cdot[1-\frac{2}{n}-(m-1)\gamma]} \\
&\quad = \frac{3\cdot 2^{\lambda+2}n^m\gamma K_D\underline{A}^{m-2}}{K_S}\cdot(at+\tau)^{-\alpha\delta_1} \\
&\quad \le \frac{3\cdot 2^{\lambda+2}n^m\gamma K_D\underline{A}^{m-2}}{K_S}\cdot\tau^{-\alpha\delta_1} \\
&\quad \le 1 \qquad\qquad \text{for all } (s,t)\in Q_1.
\end{aligned}$$

From (3.42) and (3.48)-(3.50) we thus infer that

$$(\mathcal{P}\underline{w}_{\rm mid})(s,t) \le 0 \quad \text{for all } (s,t)\in Q_1,$$

and hence obtain (3.33) upon recalling (3.47). ∎

4. Proofs of Theorem 1.1 and Proposition 1.2

The derivation of our main result on complete infinite-time aggregation, occurring at a grow-up rate satisfying (1.14), can now be accomplished by an application of the comparison principle from Lemma 2.3 to suitable members of the subsolution family introduced in (3.14).

Proof of Theorem 1.1. Given $\mu > 0$, we let $a_{\rm in} = a_{\rm in}^{(\mu)}$, $a_{\rm mid} = a_{\rm mid}^{(\mu)}$ and $a_{\rm out} = a_{\rm out}^{(\mu)}$ as well as $\tau_{\rm in} = \tau_{\rm in}^{(\mu)}$, $\tau_{\rm mid} = \tau_{\rm mid}^{(\mu)}$ and $\tau_{\rm out} = \tau_{\rm out}^{(\mu)}$ be as provided by Lemma 3.4, Lemma 3.6 and Lemma 3.5, and choosing any

$$a \in (0, \min\{a_{\rm in}, a_{\rm mid}, a_{\rm out}\}) \quad \text{and} \quad \tau > \max\{\tau_{\rm in}, \tau_{\rm mid}, \tau_{\rm out}\}$$

we let $\underline{w} = \underline{w}^{(\mu,a,\tau)}$ be as defined in (3.14), with $A = A^{(\mu,a,\tau)}$ as correspondingly introduced in (3.5), and with γ and κ taken from Lemma 3.1. Then from Lemma 3.3 it follows that

$$M(r) \equiv M^{(\mu)}(r) := n|B_1(0)|\underline{w}(r^n, 0), \quad r \in [0, R],$$

defines a nondecreasing function $M \in C^0([0,R^n])$ about which we know that

$$M(R) = n|B_1(0)| \cdot \frac{\mu R^n}{n} = \mu|\Omega| \quad \text{and} \quad \frac{M(r)}{r^n} \le n|B_1(0)| \cdot A(0)\tau^\alpha$$

for all $r \in (0,R)$, because $\underline{w}(\cdot,0)$ is concave on $[0,R^n]$ by (3.18), (3.19) and (3.20), and because $\underline{w}_s(0,0) = A(0)\tau^\alpha$ by (3.14) and (3.18). Moreover, if u_0 is any function fulfilling (1.11) as well as (1.12), then the function w accordingly given by (2.4) satisfies

$$\begin{aligned} w(s,0) &= \frac{1}{n|B_1(0)|} \int_{B_{s^{\frac{1}{n}}}(0)} u_0 dx \\ &\ge \frac{1}{n|B_1(0)|} \cdot M(s^{\frac{1}{n}}) = \underline{w}(s,0) \quad \text{for all } s \in [0,R^n], \end{aligned}$$

so that since our choices of a and τ ensure that

$$(\mathcal{P}\underline{w})(s,t) \le 0 \quad \text{for all } t>0 \text{ and } s \in (0,R^n) \setminus \{(at+\tau)^{-\alpha}, (at+\tau)^{-\alpha\kappa}\}$$

thanks to Lemma 3.4, Lemma 3.5 and Lemma 3.6, by means of the comparison principle from Lemma 2.3 we conclude that

$$w(s,t) \ge \underline{w}(s,t) \quad \text{for all } s \ge 0 \text{ and } t \ge 0. \tag{4.1}$$

Since

$$\begin{aligned} (at+\tau)^\alpha \cdot w((at+\tau)^{-\alpha},t) &= \frac{w((at+\tau)^{-\alpha},t) - w(0,t)}{(at+\tau)^{-\alpha}} \\ &\le \sup_{s \in (0,R^n)} w_s(s,t) \qquad \text{for all } t>0 \end{aligned}$$

due to the mean value theorem, in view of (2.5) and (3.8) this firstly implies that

$$\begin{aligned} \|u(\cdot,t)\|_{L^\infty(\Omega)} &= n\|w_s(\cdot,t)\|_{L^\infty((0,R^n))} \\ &\ge n(at+\tau)^\alpha \cdot w((at+\tau)^{-\alpha},t) \\ &\ge n(at+\tau)^\alpha \cdot \underline{w}((at+\tau)^{-\alpha},t) \\ &= nA(t) \cdot (at+\tau)^\alpha \\ &\ge n\underline{A} \cdot (at+\tau)^\alpha \qquad \text{for all } t>0 \end{aligned}$$

with $\underline{A} = \underline{A}^{(\mu)}$ as in (3.9), so that recalling (3.4) we obtain (1.14) with some suitably small $C>0$.

Apart from that, however, (4.1) may also be used to derive (1.15), because for any fixed $r \in (0, R)$ we see that if we let $t_0(r) := (ar^{\frac{n}{\alpha\kappa}})^{-1}$, then

$$(at+\tau)^{-\alpha\kappa} \le (at)^{-\alpha\kappa} \le r^n \quad \text{for all } t > t_0(r),$$

whence from (3.14) and (3.8) we obtain that

$$\begin{aligned} \underline{w}(r^n,t) &= \frac{\mu R^n}{n} - A(t)\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]}\cdot(R^n-r^n) \\ &\ge \frac{\mu R^n}{n} - \overline{A}R^n\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]} \qquad \text{for all } t > t_0(r), \end{aligned}$$

where $\overline{A} = \overline{A}^{(\mu)}$ is taken from (3.9). In line with (2.4), when combined with (4.1) this entails that

$$\begin{aligned} \|u(\cdot,t)\|_{L^1(\Omega\setminus B_r(0))} &= n|B_1(0)|\cdot\{w(R^n,t)-w(r^n,t)\} \\ &= n|B_1(0)|\cdot\Big\{\frac{\mu R^n}{n} - w(r^n,t)\Big\} \\ &\le n|B_1(0)|\cdot\Big\{\frac{\mu R^n}{n} - \Big\{\frac{\mu R^n}{n} - \overline{A}R^n\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]}\Big\}\Big\} \\ &= n|B_1(0)|\cdot\overline{A}R^n\cdot(at+\tau)^{-\alpha[(1-\kappa)\gamma-1]} \qquad \text{for all } t>t_0(r), \end{aligned}$$

and thereby establishes (1.15) because of the fact that the inequality $\kappa < \frac{\gamma-1}{\gamma}$ warrants positivity of $(1-\kappa)\gamma - 1$. ∎

Our verification of the statement on upper grow-up bounds made in (1.17) can be obtained independently of the above by once again returning to the original variables, and refining the comparison argument from Lemma 2.1 so as to appropriately exploit the stronger hypothesis on asymptotics of S in (1.16):

Proof of Proposition 1.2. According to Lemma 2.1, our assumptions ensure the existence of a unique global classical solution with the regularity features in (1.13), fulfilling $u > 0$ in $\overline{\Omega} \times [0, \infty)$ and being such that $(u, v)(\cdot, t)$ is radially symmetric for all $t > 0$, so that we only need to derive (1.17). To this end, we fix $B > 0$ large enough such that

$$B \ge \|u_0\|_{L^\infty(\Omega)} \tag{4.2}$$

and

$$\frac{\widehat{K}_S}{B^\lambda} \le \frac{1}{\lambda}, \tag{4.3}$$

and let

$$\overline{u}(x,t) := B\cdot(t+1)^{\frac{1}{\lambda}}, \quad x \in \overline{\Omega},\ t \ge 0,$$

so we obtain a function $\overline{u} \in C^\infty(\overline{\Omega} \times [0,\infty))$ fulfilling $\overline{u}(x,0) \ge B \ge u_0(x)$ for all $x \in \overline{\Omega}$ by (4.2). As (4.2) furthermore ensures that

$$\mu - \overline{u}(x,t) \le \|u_0\|_{L^\infty(\Omega)} - B \le 0 \quad \text{for all } (x,t) \in \Omega \times (0,\infty),$$

we can utilize (1.16) and (4.3) in estimating

$$\begin{aligned}
\overline{u}_t(x,t) &- \nabla\cdot(D(u(x,t))\nabla\overline{u}(x,t)) \\
&\quad + (S(u(x,t)) + u(x,t)S'(u(x,t)))\nabla v(x,t)\cdot\nabla\overline{u}(x,t) \\
&\quad + (\mu - \overline{u}(x,t))\overline{u}(x,t)S(\overline{u}(x,t)) \\
&= \frac{B}{\lambda}\cdot(t+1)^{\frac{1}{\lambda}-1} + (\mu - \overline{u}(x,t))\cdot\overline{u}(x,t)S(\overline{u}(x,t)) \\
&\ge \frac{B}{\lambda}\cdot(t+1)^{\frac{1}{\lambda}-1} + (\mu - \overline{u}(x,t))\cdot\widehat{K}_S\overline{u}^{-\lambda}(x,t) \\
&\ge \frac{B}{\lambda}\cdot(t+1)^{\frac{1}{\lambda}-1} - \widehat{K}_S\overline{u}^{1-\lambda}(x,t) \\
&= B\cdot\Big\{\frac{1}{\lambda} - \widehat{K}_S B^{-\lambda}\Big\}\cdot(t+1)^{\frac{1}{\lambda}-1} \\
&\ge 0 \quad \text{for all } x\in\Omega \text{ and } t>0.
\end{aligned}$$

In view of (1.2), a comparison argument thus shows that $\overline{u}(x,t) \ge u(x,t)$ for all $(x,t) \in \Omega \times (0,\infty)$, and that hence (1.17) results if we let $C := B$, for instance. ∎

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