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MOMENT PROBLEM IN INFINITELY MANY VARIABLES

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ABSTRACT

The multivariate moment problem is investigated in the general context of the polynomial algebra $\mathbb{R}[x_i \mid i \in \Omega]$ in an arbitrary number of variables x_i, $i \in \Omega$. The results obtained are sharpest when the index set Ω is countable. Extensions of Haviland's theorem [17] and Nussbaum's theorem [34] are proved. Lasserre's description of the support of the measure in terms of the non-negativity of the linear functional on a quadratic module of $\mathbb{R}[x_i \mid i \in \Omega]$ in [27] is shown to remain valid in this more general situation. The main tool used in the paper is an extension of the localization method developed by the third author in [30], [32] and [33]. Various results proved in [30], [32] and [33] are shown to continue to hold in this more general setting.

* Murray Marshall passed away on May 1, 2015. We lost a wonderful collaborator and a dear friend. We miss him sorely.

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1. Introduction

The univariate moment problem is an old problem with origins tracing back to work of Stieltjes [45]. Given a sequence $(s_k)_{k\geq 0}$ of real numbers one wants to know when there exists a Radon measure μ on $\mathbb{R}$ such that

$$s_k = \int x^k d\mu \;\forall\; k \geq 0.^1$$

Since the monomials x^k, $k \geq 0$ form a basis for the polynomial algebra $\mathbb{R}[x]$, this problem is equivalent to the following one: Given a linear functional $L:\mathbb{R}[x]\to\mathbb{R}$, when does there exist a Radon measure μ on $\mathbb{R}$ such that

$$L(f) = \int f d\mu \quad \forall\, f \in \mathbb{R}[x].$$

One also wants to know to what extent the measure is unique, assuming it exists; [1] and [43] are standard references.

Work on the multivariate moment problem is more recent. For $n \geq 1$,

$$\mathbb{R}[\underline{x}] := \mathbb{R}[x_1, \dots, x_n]$$

denotes the polynomial ring in n variables $x_1, \dots, x_n$. Given a linear functional $L : \mathbb{R}[\underline{x}] \to \mathbb{R}$ one wants to know when there exists a Radon measure μ on $\mathbb{R}^n$ such that $L(f) = \int f d\mu\ \forall f \in \mathbb{R}[\underline{x}]$. Again, one also wants to know to what extent the measure is unique, assuming it exists; [7], [12], [25], [31], [37] are general references. A major motivation here is the close connection between the multivariate moment problem and polynomial optimization using semidefinite programming; see [25], [28], [31] and the references therein.

There are also papers dealing with the moment problem in infinitely many variables. These deal with cases where the linear functional in question is continuous for a certain topology: [2] applies Schmüdgen's solution of the moment problem in [42] to represent L^1-continuous linear functionals on the vector space of polynomials of Brownian motion as integration with respect to probability measures on the Wiener space of $\mathbb{R}$; [5, Theorem 2.1], [6], [11], [18], [19], [20] and [40, Theorem 12.5.2] consider continuous linear functionals on the symmetric algebra of a nuclear space; [13] deals with linear functionals on the symmetric algebra of a locally convex space (V, τ) which are continuous with respect to the finest locally multiplicatively convex topology extending τ; [10], [14] and

[1] All Radon measures considered are assumed to be positive.

[15] are precursors of [13]. The present paper seems to be the first to deal with the general case.

The method used in the present paper is an extension of the localization method in [30], [32] and [33], the latter method being motivated in turn by results in [24], [29] and [38]. It is worth noting that, although some of the results in [30], [32] and [33] are similar in nature to those in [38], the arguments are completely different.

The paper was written with no particular application in mind. At the same time, it seems reasonable to expect that applications do exist. E.g., there may be connections to some variant of the semi-infinite polynomial optimization problem considered in [26], [46].

Section 2 introduces terminology and notation. Two important new concepts, constructibly Borel sets and constructibly Radon measures, are defined in this section. In Section 3 we introduce three algebras $A = A_\Omega := \mathbb{R}[x_i \mid i \in \Omega]$, $B = B_\Omega := \mathbb{R}[x_i, \frac{1}{1+x_i^2} \mid i \in \Omega]$, and $C = C_\Omega := \mathbb{R}[\frac{1}{1+x_i^2}, \frac{x_i}{1+x_i^2} \mid i \in \Omega]$, show how the moment problem for A_Ω reduces to understanding the extensions of a linear functional $L : A_\Omega \to \mathbb{R}$ to a positive linear functional on B_Ω, see Corollary 3.4, and prove that positive linear functionals $L : B_\Omega \to \mathbb{R}$ correspond bijectively to constructibly Radon measures on $\mathbb{R}^\Omega$, see Theorem 3.8. We also consider the important question of when the constructibly Radon measures thus obtained are actually Radon; see Lemma 3.6, Remark 3.7 and Theorem 5.4. In Section 4 we explain how results in [32] and [33] carry over, more-or-less word-for-word, to the case of infinitely many variables. In particular, we extend results of Fuglede [12] and Petersen [35], see Corollary 4.3 and Corollary 4.5, and we establish extensions of Nussbaum's well-known sufficient condition for a linear functional $L : A_\Omega \to \mathbb{R}$ to correspond to a measure [34]; see Theorem 4.6, Corollary 4.8 and Theorem 4.10. Section 5 deals with the problem of describing supporting sets for the measure. There are a number of important results in Section 5; see for example Theorem 5.1, which is an extension of Haviland's theorem [17], and Theorem 5.2, which is an extension of a result of Lasserre [27]. In Section 6 we explain how the cylinder results in [30], [32] and [33] extend to infinitely many variables.

The reader will notice that everything works more-or-less perfectly in case Ω is countable. If Ω is uncountable everything still works, but one typically only knows that the measures obtained are constructibly Radon (as opposed to

Radon) and the results obtained concerning the support of the measure are a bit more restrictive than one might like.

2. Terminology and notation

All rings considered are commutative with 1. All ring homomorphisms considered send 1 to 1. All rings we are interested in are $\mathbb{R}$-algebras. For $n \geq 1$, $\mathbb{R}[\underline{x}] := \mathbb{R}[x_1, \ldots, x_n]$. For a topological space X, $C(X)$ denotes the ring of all continuous functions from X to $\mathbb{R}$.

Let A be a commutative ring; $X(A)$ denotes the **character space** of A, i.e., the set of all ring homomorphisms $\alpha : A \to \mathbb{R}$. For $a \in A$, $\hat{a} = \hat{a}_A : X(A) \to \mathbb{R}$ is defined by $\hat{a}_A(\alpha) = \alpha(a)$; $X(A)$ is given the weakest topology such that the functions $\hat{a}_A$, $a \in A$ are continuous. The mapping $a \mapsto \hat{a}_A$ defines a ring homomorphism from A into $\mathrm{C}(X(A))$. The only ring homomorphism from $\mathbb{R}$ to itself is the identity. Ring homomorphisms from $\mathbb{R}[\underline{x}]$ to $\mathbb{R}$ correspond to point evaluations $f \mapsto f(\alpha)$, $\alpha \in \mathbb{R}^n$; $X(\mathbb{R}[\underline{x}])$ is identified (as a topological space) with $\mathbb{R}^n$. By a **quadratic module** of A we mean a subset M of A satisfying

$$1 \in M, \quad M + M \subseteq M \quad \text{and} \quad a^2 M \subseteq M \quad \text{for each } a \in A.$$

A **preordering** of A is a quadratic module of A which is also closed under multiplication. For a subset X of $X(A)$,

$$\mathrm{Pos}_A(X) := \{a \in A \mid \hat{a}_A \geq 0 \text{ on } X\}$$

is a preordering of A. We denote by $\sum A^2$ the set of all finite sums $\sum a_i^2$, $a_i \in A$; $\sum A^2$ is the unique smallest quadratic module of A; $\sum A^2$ is closed under multiplication, so $\sum A^2$ is also the unique smallest quadratic preordering of A. For a subset $S \subseteq A$, the quadratic module of A generated by S consists of all finite sums $s_0 + s_1 g_1 + \cdots + s_k g_k$, $g_1, \ldots, g_k \in S$, $s_0, \ldots, s_k \in \sum A^2$. Also,

$$X_S := \{\alpha \in X(A) \mid \hat{a}_A(\alpha) \geq 0 \ \forall a \in S\}.$$

If $M = \sum A^2$ then $X_M = X(A)$. If M is the quadratic module of A generated by S then $X_M := X_S$. A quadratic module M in A is said to be **archimedean** if for each $a \in A$ there exists an integer k such that $k \pm a \in M$. If M is a quadratic module of A which is archimedean, then X_M is compact. The converse is false in general [22].

For simplicity, we assume from now on that A is an $\mathbb{R}$-algebra. We record the following special case of the representation theorem of T. Jacobi [21].

THEOREM 2.1: *Suppose M is an archimedean quadratic module of A. Then, for any $a \in A$, the following are equivalent:*

(1) $\hat{a}_A \geq 0$ *on* X_M.
(2) $a + \epsilon \in M$ *for all real* $\epsilon > 0$.

Note: The implication (2) ⇒ (1) is trivial. The implication (1) ⇒ (2) is non-trivial. See [4], [23] and [36] for early versions of Jacobi's theorem. See [31] for a simple proof.

The open sets

$$U_A(a) := \{\alpha \in X(A) \mid \hat{a}_A(\alpha) > 0\}, \ a \in A$$

form a subbasis for the topology on $X(A)$ (even a basis). Suppose A is generated as an $\mathbb{R}$-algebra by x_i, $i \in \Omega$. The embedding $X(A) \hookrightarrow \mathbb{R}^\Omega$ defined by $\alpha \mapsto (\alpha(x_i))_{i\in\Omega}$ identifies $X(A)$ with a subspace of $\mathbb{R}^\Omega$. Sets of the form

$$\Big\{b \in \mathbb{R}^\Omega \mid \sum_{i\in I}(b_i - p_i)^2 < r\Big\},$$

where $r, p_i \in \mathbb{Q}$ and I is a finite subset of Ω, form a basis for the product topology on $\mathbb{R}^\Omega$. It follows that sets of the form

$$U_A\Big(r - \sum_{i\in I}(x_i - p_i)^2\Big), \quad r, p_i \in \mathbb{Q}, \quad I \text{ a finite subset of } \Omega, \tag{2.1}$$

form a basis for $X(A)$.

A subset E of $X(A)$ is called **Borel** if E is an element of the σ-algebra of subsets of $X(A)$ generated by the open sets. A subset E of $X(A)$ is said to be **constructible** (resp., **constructibly Borel**) if E is an element of the algebra (resp., σ-algebra) of subsets of $X(A)$ generated by the $U_A(a)$, $a \in A$.[2]

Clearly constructible ⇒ constructibly Borel ⇒ Borel.

PROPOSITION 2.2: *If A is generated as an $\mathbb{R}$-algebra by a countable set $\{x_i \mid i \in \Omega\}$, then sets of the form* (2.1) *form a countable basis for the topology on $X(A)$ and every Borel set of $X(A)$ is constructibly Borel.*

Proof. This is clear. ∎

[2] The descriptor 'constructible' is borrowed from standard terminology in real algebraic geometry; e.g., see [3]. Constructible sets in $\mathbb{R}^n$ are also called semialgebraic sets.

Proposition 2.3: *A subset E of $X(A)$ is constructibly Borel iff $E = \pi^{-1}(E')$ for some Borel set E' of $X(A')$, where A' is a countably generated subalgebra of A and $\pi : X(A) \to X(A')$ is the canonical restriction map, i.e., $\pi(\alpha) = \alpha|_{A'}$.*

Proof. Clearly $U_A(a) = \pi^{-1}(U_{A'}(a))$ for any $a \in A'$. Coupled with Proposition 2.2 this implies that, for each Borel set E' of $X(A')$, $\pi^{-1}(E')$ is an element of the σ-algebra of subsets of $X(A)$ generated by the $U_A(a)$, $a \in A'$ (and conversely). Denote this σ-algebra by $\Sigma_{A'}$. It remains now to show that the union of the σ-algebras $\Sigma_{A'}$, A' running through the countably generated subalgebras of A, is itself a σ-algebra. This follows from the well-known fact that a countable union of countable sets is countable (so the subalgebra of A generated by countably many countably generated subalgebras of A is itself countably generated). ∎

The support of a measure is not defined in general. For a measure space (X, Σ, μ) and a subset Y of X, we say μ is **supported** by Y if

$$E \cap Y = \emptyset \Rightarrow \mu(E) = 0 \quad \forall\ E \in \Sigma.$$

In this situation, if $\Sigma' := \{E \cap Y \mid E \in \Sigma\}$, and $\mu'(E \cap Y) := \mu(E)\ \forall E \in \Sigma$, then Σ' is a σ-algebra of subsets of Y, μ' is a well-defined measure on (Y, Σ'), the inclusion map $i : Y \to X$ is a measurable function, and μ is the pushforward of μ' to X.

Recall that if (Y, Σ', μ') is a measure space, (X, Σ) is a σ-algebra, $i : Y \to X$ is any measurable function, and μ is the pushforward of μ' to (X, Σ), then for each measurable function $f : X \to \mathbb{R}$, $\int f d\mu = \int (f \circ i) d\mu'$ [16, Theorem 39C]. This is the well-known change in variables theorem.

A **Radon measure** on $X(A)$ is a positive measure μ on the σ-algebra of Borel sets of $X(A)$ which is locally finite and inner regular. Locally finite means that every point has a neighbourhood of finite measure. Inner regular means each Borel set can be approximated from within using a compact set.

Definition 2.4: A **constructibly Radon measure** on $X(A)$ is a positive measure μ on the σ-algebra of constructibly Borel sets of $X(A)$ such that, for each countably generated subalgebra A' of A, the pushforward of μ to $X(A')$ via the restriction map $\alpha \mapsto \alpha|_{A'}$ is a Radon measure on $X(A')$.

We are interested here in Radon and constructibly Radon measures having the additional property that $\hat{a}_A$ is μ-integrable (i.e., $\int \hat{a}_A d\mu$ is well-defined and finite) for all $a \in A$. For a linear functional $L : A \to \mathbb{R}$, one can consider the set of Radon or constructibly Radon measures μ on $X(A)$ such that $L(a) = \int \hat{a}_A d\mu$ $\forall a \in A$. The **moment problem** is to understand this set of measures, for a given linear functional $L : A \to \mathbb{R}$. In particular, one wants to know: (i) When is this set non-empty? (ii) In case it is non-empty, when is it a singleton set?

A linear functional $L : A \to \mathbb{R}$ is said to be **positive** if

$$L\Big(\sum A^2\Big) \subseteq [0, \infty)$$

and M**-positive**, for some quadratic module M of A, if

$$L(M) \subseteq [0, \infty).$$

3. Three special $\mathbb{R}$-algebras

Let

$$A = A_\Omega := \mathbb{R}[x_i \mid i \in \Omega],$$

the ring of polynomials in the variables x_i, $i \in \Omega$ with coefficients in $\mathbb{R}$,

$$B = B_\Omega := \mathbb{R}\Big[x_i, \frac{1}{1+x_i^2} \mid i \in \Omega\Big],$$

the localization of A at the multiplicative set generated by the $1 + x_i^2$, $i \in \Omega$, and

$$C = C_\Omega := \mathbb{R}\Big[\frac{1}{1+x_i^2}, \frac{x_i}{1+x_i^2} \mid i \in \Omega\Big],$$

the $\mathbb{R}$-subalgebra of B generated by the elements $\frac{1}{1+x_i^2}$, $\frac{x_i}{1+x_i^2}$, $i \in \Omega$. Here, Ω is an arbitrary index set.

By definition, A (resp., B, resp., C) is the direct limit of the $\mathbb{R}$-algebras A_I (resp., B_I, resp., C_I), I running through all finite subsets of Ω. Because of this, many questions about A, B and C reduce immediately to the case where Ω is finite. Observe also that if Ω is finite, then B is equal to the localization of A at

$$p := \prod_{i \in \Omega} (1 + x_i^2)$$

considered in [32]. This is clear.

Elements of $X(A)$ and $X(B)$ are naturally identified with point evaluations $f \mapsto f(\alpha)$, $\alpha \in \mathbb{R}^\Omega$. Note that $X(A) = X(B) = \mathbb{R}^\Omega$, not just as sets, but also as topological spaces, giving $\mathbb{R}^\Omega$ the product topology.

Theorem 3.1:

(1) *For $f \in B$, the following are equivalent:*
   (i) $f \in C$;
   (ii) *f is **geometrically bounded**, i.e., $\exists\, k \in \mathbb{N}$ such that $|f(\alpha)| \le k$ $\forall \alpha \in \mathbb{R}^\Omega$;*
   (iii) *f is **algebraically bounded**, i.e., $\exists\, k \in \mathbb{N}$ such that $k \pm f \in \sum B^2$.*

(2) *$\sum B^2 \cap C = \sum C^2$. In particular, $\sum C^2$ is archimedean.*

(3) *C is naturally identified (via $y_i \leftrightarrow \frac{1}{1+x_i^2}$ and $z_i \leftrightarrow \frac{x_i}{1+x_i^2}$) with the polynomial algebra $\mathbb{R}[y_i, z_i \mid i \in \Omega]$ factored by the ideal generated by the polynomials $y_i^2 + z_i^2 - y_i = (y_i - \frac{1}{2})^2 + z_i^2 - \frac{1}{4}$, $i \in \Omega$. Consequently, $X(C)$ is identified naturally with $\mathbb{S}^\Omega$, where*

$$\mathbb{S} := \Big\{(y,z) \in \mathbb{R}^2 \mid \Big(y - \frac{1}{2}\Big)^2 + z^2 = \frac{1}{4}\Big\}.$$

(4) *The restriction map $\alpha \mapsto \alpha|_C$ identifies $X(B)$ with a subspace of $X(C)$. In terms of coordinates, this map is given by*

$$\alpha = (x_i)_{i\in\Omega} \mapsto \beta = (y_i, z_i)_{i\in\Omega},$$

where $y_i := \frac{1}{1+x_i^2}$, $z_i := \frac{x_i}{1+x_i^2}$. In particular, the image of $X(B)$ is dense in $X(C)$.

Proof. See [32, Remark 2.2]. ∎

Corollary 3.2:

(1) *For $f \in C$, $\hat{f}_C \ge 0$ on $X(C)$ iff $f + \epsilon \in \sum C^2$ $\forall$ real $\epsilon > 0$.*

(2) *For $f \in B$, $\hat{f}_B \ge 0$ on $X(B)$ iff $\exists\, q$ of the form $q = \prod_{k=1}^n (1 + x_{i_k}^2)^{\ell_k}$, where $x_{i_1}, \dots, x_{i_n}$ are the variables appearing in f, such that*

$$f + \epsilon q \in \sum B^2,$$

for all real $\epsilon > 0$.

Proof. (1) Since $\sum C^2$ is archimedean, this is immediate from Theorem 2.1. (2) If $f \in B$, say $f \in B_{\{i_1,\dots,i_n\}}$, there exists an element q of the form

$$q = \prod_{k=1}^n (1 + x_{i_k}^2)^{\ell_k}$$

such that $\frac{f}{q} \in C$. Thus, if $f \ge 0$ on $X(B)$ then $\frac{f}{q} \ge 0$ on $X(C)$ so $\frac{f}{q} + \epsilon \in \sum C^2$ and, consequently, $f + \epsilon q \in \sum B^2$, $\forall$ real $\epsilon > 0$. ∎

It follows from Corollary 3.2 that a linear functional $L : C \to \mathbb{R}$ is positive iff it is $\mathrm{Pos}_C(X(C))$-positive, and a linear functional $L : B \to \mathbb{R}$ is positive iff it is $\mathrm{Pos}_B(X(B))$-positive. For linear functionals $L : A \to \mathbb{R}$ this is never the case, except when $|\Omega| \leq 1$; see [9], [39].

Lemma 3.3: *Suppose $L : A \to \mathbb{R}$ is a linear functional and $L(\mathrm{Pos}_A(Y)) \subseteq [0, \infty)$ for some closed set $Y \subseteq \mathbb{R}^\Omega$. Then L extends to a linear functional $L : B \to \mathbb{R}$ such that $L(\mathrm{Pos}_B(Y)) \subseteq [0, \infty)$.*

Note: The extension is not unique, in general.

Proof. The proof is a simple modification of the Zorn's lemma argument in [30, Theorem 3.1]. Denote by $C'(Y)$ the $\mathbb{R}$-algebra of all continuous functions $f : Y \to \mathbb{R}$ which are bounded by some $\hat{a}$, $a \in A$ in the sense that $|f| \leq |\hat{a}|$ on Y. As in the proof of [30, Theorem 3.1] $\exists$ a linear functional $\overline{L} : C'(Y) \to \mathbb{R}$ which is $\mathrm{Pos}_{C'(Y)}(Y)$-positive (i.e., $\forall f \in C'(Y)$, $f \geq 0$ on $Y \Rightarrow \overline{L}(f) \geq 0$) such that $L(a) = \overline{L}(\hat{a}|_Y)$ $\forall a \in A$. If $b \in B$ then

$$b = \frac{a}{(1 + x_{i_1}^2)^{\ell_1} \cdots (1 + x_{i_n}^2)^{\ell_n}}$$

for some $a \in A$, $n \geq 1$, $i_k \in \Omega$, $\ell_k \geq 0$, $k = 1, \dots, n$. Since $1 + \alpha^2 \geq 1$ $\forall \alpha \in \mathbb{R}$ it follows that $|\hat{b}| \leq |\hat{a}|$ on Y, i.e., $\hat{b}|_Y \in C'(Y)$ $\forall b \in B$. Define $L : B \to \mathbb{R}$ by $L(b) = \overline{L}(\hat{b}|_Y)$. ∎

Corollary 3.4: *For a linear functional $L : A \to \mathbb{R}$ the following are equivalent:*

(1) *L is $\mathrm{Pos}_A(X(A))$-positive.*
(2) *L extends to a positive linear functional $L : B \to \mathbb{R}$.*
(3) *$\forall f \in A$ and $\forall p$ of the form $p = \prod_{k=1}^n (1 + x_{i_k}^2)^{\ell_k}$, where $x_{i_1}, \dots, x_{i_n}$ are the variables appearing in f and $\ell_k \geq 0$, $k = 1, \dots, n$,*

$$pf \in \sum A^2 \Rightarrow L(f) \geq 0.$$

Proof. (1) $\Rightarrow$ (2). Apply Lemma 3.3 with $Y = \mathbb{R}^\Omega$.

(2) $\Rightarrow$ (3). Since $pf \in \sum A^2$, it follows that $p^2 f \in \sum A^2$, so $f \in \sum B^2$. Since the extension of L to B is positive this implies $L(f) \geq 0$.

(3) $\Rightarrow$ (1). Suppose $f \in A$, $\hat{f} \geq 0$. By Corollary 3.2 (2) $\exists$ $q = \prod_{k=1}^n (1+x_{i_k}^2)^{\ell_k}$, where $x_{i_1}, \dots, x_{i_n}$ are the variables appearing in f, such that $f + \epsilon q \in \sum B^2$ $\forall \epsilon > 0$. Clearing denominators, $p^2(f + \epsilon q) \in \sum A^2$ for some p (depending on ϵ)

of the form $p = \prod_{k=1}^{n}(1+x_{i_k}^2)^{m_k}$. By (3), $L(f)+\epsilon L(q) = L(f+\epsilon q) \geq 0$. Since $\epsilon > 0$ is arbitrary, this implies $L(f) \geq 0$. ∎

Positive linear functionals $L : B \to \mathbb{R}$ restrict to positive linear functionals on C. Positive linear functionals $L : C \to \mathbb{R}$ are in natural one-to-one correspondence with Radon measures μ on the compact space $X(C)$ via $L \leftrightarrow \mu$ iff $L(f) = \int \hat{f}_C d\mu \ \forall f \in C$. This is well-known, e.g., see [30, Corollary 3.3 and Remark 3.5].

For $i \in \Omega$, let $\Delta_i := \{\beta \in X(C) \mid \beta(\frac{1}{1+x_i^2}) = 0\}$. Because of the way $X(C)$ is being identified with $\mathbb{S}^\Omega$, Δ_i is identified with the set

$$\{(y_j, z_j)_{j\in\Omega} \in \mathbb{S}^\Omega \mid y_i = z_i = 0\}.$$

It is clear that $X(C)\backslash X(B)$ is the union of the sets Δ_i, $i \in \Omega$. For each $f \in B$ one can associate a continuous function

$$\tilde{f} : X(C)\backslash(\Delta_{i_1} \cup \cdots \cup \Delta_{i_n}) \to \mathbb{R},$$

where $x_{i_1}, \dots, x_{i_n}$ are the variables appearing in f. Observe that $f \in B_{\{i_1,\dots,i_n\}}$. Define

$$\tilde{f} = \hat{f}_{B_{\{i_1,\dots,i_n\}}} \circ \pi$$

where $\pi : X(C) \to X(C_{\{i_1,\dots,i_n\}})$ is the restriction map. Observe that the inverse image under π of the set $X(C_{\{i_1,\dots,i_n\}})\backslash X(B_{\{i_1,\dots,i_n\}})$ is precisely the set $\Delta_{i_1} \cup \cdots \cup \Delta_{i_n}$. Note also that $\tilde{f}|_{X(B)} = \hat{f}_B$.

Lemma 3.5: *For each positive linear functional $L : B \to \mathbb{R}$ there exists a unique Radon measure μ on $X(C)$ such that $L(f) = \int \hat{f}_C d\mu \ \forall f \in C$. This satisfies $\mu(\Delta_i) = 0 \ \forall i \in \Omega$ and $L(f) = \int \tilde{f} d\mu \ \forall f \in B$.*

Proof. Fix a finite subset $I = \{i_1, \dots, i_n\}$ of Ω. By [30, Corollary 3.3] there exists a Radon measure μ on $X(C)$ and a Radon measure μ_I on $X(C_I)$ such that $L(f) = \int \hat{f}_C d\mu \ \forall f \in C$ and $L(f) = \int \hat{f}_{C_I} d\mu_I \ \forall f \in C_I$. Applying [30, Corollary 3.4] with $p = (1+x_{i_1}^2)\cdots(1+x_{i_n}^2)$, there exists a Radon measure ν_I on $X(B_I)$ such that $L(f) = \int \hat{f}_{B_I} d\nu_I \ \forall f \in B_I$. By [30, Remark 3.5] the measures μ, μ_I, ν_I are unique. Denote by μ_I' the pushforward of μ to $X(C_I)$ by the restriction map $\pi : X(C) \to X(C_I)$. Since $\hat{f}_C = \hat{f}_{C_I} \circ \pi \ \forall f \in C_I$, it follows that

$$\int \hat{f}_{C_I} d\mu_I' = \int \hat{f}_C d\mu = L(f) \quad \forall f \in C_I,$$

so uniqueness of μ_I implies $\mu'_I = \mu_I$. A similar argument shows that μ_I is the pushforward of ν_I via the natural embedding $X(B_I) \hookrightarrow X(C_I)$. It follows that $\mu(\Delta_{i_1} \cup \cdots \cup \Delta_{i_n}) = \mu_I(X(C_I)\backslash X(B_I)) = 0$. Since I is an arbitrary finite subset of Ω, this implies $\mu(\Delta_i) = 0\ \forall i \in \Omega$. Suppose now that $f \in B_I$. Since $\tilde{f} = \hat{f}_{B_I} \circ \pi$, $\int \tilde{f} d\mu = \int \hat{f}_{B_I} d\nu_I = L(f)$ as required. ∎

One would like to know when there exists a Radon measure ν on $X(B)$ such that $L(f) = \int \hat{f}_B d\nu\ \forall f \in B$.

Lemma 3.6: *Let L be a positive linear functional on B and let μ be the Radon measure on $X(C)$ associated to L. The following are equivalent:*

(1) *$\exists$ a Radon measure ν on $X(B)$ such that $L(f) = \int \hat{f}_B d\nu\ \forall f \in B$.*
(2) *$\forall$ Borel sets E of $X(C)$,*

$$\mu(E) = \sup\{\mu(K) \mid K \text{ compact, } K \subseteq X(B) \cap E\}.$$

(3) *$\mu(X(C)) = \sup\{\mu(K) \mid K$ compact, $K \subseteq X(B)\}$.*
(4) *μ is supported by a Borel set E of $X(C)$ such that $E \subseteq X(B)$.*

Moreover, if this is the case, then $\nu(E) = \sup\{\mu(K) \mid K$ compact, $K \subseteq E\}$ for all Borel sets E of $X(B)$. In particular, ν is uniquely determined by μ.

Proof. Assume (1). Denote by μ' the pushforward of ν to $X(C)$. Then, $\forall f \in C$, $\int \hat{f}_C d\mu' = \int \hat{f}_B d\nu = L(f)$. Uniqueness of μ implies $\mu' = \mu$. Since ν is Radon, (2) is now clear. (2) $\Rightarrow$ (3) is obvious. Assume (3). Define $E = \bigcup_{n\geq 1} K_n$ where K_n is a compact subset of $X(B)$ such that $\mu(X(C)\backslash K_n) \leq \frac{1}{n}$. Clearly $E \subseteq X(B)$, E is a Borel set of $X(C)$ and μ is supported by E. This proves (4). Assume (4). Then ν defined by $\nu(E' \cap X(B)) = \mu(E')$ $\forall$ Borel sets E' of $X(C)$ is a Radon measure on $X(B)$. Since μ is the pushforward of ν to $X(C)$, $\int \hat{f}_B d\nu = \int \tilde{f} d\mu = L(f)$, so (1) is clear. The last assertion is clear. ∎

Remark 3.7: If Ω is countable, then $X(C)\backslash X(B) = \bigcup_{i\in\Omega} \Delta_i$ is a Borel set of measure zero, so the equivalent conditions of Lemma 3.6 hold in this case. We know of no example where the conditions of Lemma 3.6 fail. It would be nice to have an example.[3]

[3] If we assume Ω is uncountable, then it is easy enough to construct a Radon measure μ on $X(C)$ so that the equivalent conditions (2) and (3) fail. This is not a problem. The problem is to choose such a μ so that, in addition, $\int \tilde{f} d\mu$ is well-defined and finite for all $f \in B$.

It seems probable that, to handle the most general case, one has to relax the requirement that ν be Radon, requiring only that ν be constructibly Radon.

THEOREM 3.8: *There is a canonical one-to-one correspondence $L \leftrightarrow \nu$ given by $L(f) = \int \hat{f}_B d\nu$ $\forall f \in B$ between positive linear functionals L on B and constructibly Radon measures ν on $X(B)$ with the property that $\hat{f}_B$ is ν-integrable $\forall f \in B$.*

Proof. If ν is a constructibly Radon measure on $X(B)$ and $\hat{f}_B$ is ν-integrable $\forall f \in B$, then it is clear that the map $f \mapsto \int \hat{f}_B d\nu$, $f \in B$ is a positive linear functional on B. Conversely, suppose L is a positive linear functional on B. Let μ be the measure defined in Lemma 3.5. For each subset I of Ω, consider the subalgebra B_I of B and the subalgebra C_I of C. Denote by μ_I the pushforward of μ via the canonical restriction map $\pi : X(C) \to X(C_I)$. One checks that μ_I is the Radon measure on $X(C_I)$ corresponding to the positive linear map $L|_{B_I}$. In particular, if I is countable then $\mu_I(X(C_I)\backslash X(B_I)) = 0$.

CLAIM 1: *If E in $X(C)$ is constructibly Borel and $X(B)\cap E = \emptyset$, then $\mu(E) = 0$. Say $E = \pi^{-1}(E')$, E' a Borel set in $X(C_I)$, $I \subseteq \Omega$ countable. Since the restriction map $X(B) \to X(B_I)$ is surjective, our hypothesis implies that $X(B_I) \cap E' = \emptyset$. It follows that $\mu(E) = \mu_I(E') = 0$ as required.*

CLAIM 2: *The constructibly Borel sets in $X(B)$ are precisely the sets of the form $X(B) \cap E$ where E is constructibly Borel in $X(C)$. This is more or less clear. If $f \in C$ then $U_B(f) = X(B) \cap U_C(f)$. If $f \in B$ then there exists p of the form $p = \prod_{k=1}^{n}(1 + x_{i_k}^2)^{\ell_k}$ where $x_{i_1}, \dots, x_{i_n}$ are the variables appearing in f such that $\frac{f}{p} \in C$. Also, $U_B(f) = U_B(\frac{f}{p})$ for any such p.*

Define a measure ν on the σ-algebra of constructibly Borel subsets of $X(B)$ by $\nu(X(B)\cap E) := \mu(E)$ $\forall$ constructibly Borel subsets E of $X(C)$. By Claim 1, ν is well-defined. By construction, μ is the pushforward of ν to $X(C)$. Also, $\hat{f}_B = \tilde{f}|_{X(B)}$ $\forall f \in B$. It follows that $\int \hat{f}_B d\nu = \int \tilde{f} d\mu = L(f)$ $\forall f \in B$. For each countable $I \subseteq \Omega$, the pushforward of ν to $X(B_I)$ is the Radon measure ν_I on $X(B_I)$ induced by μ_I using Lemma 3.6 and Remark 3.7. It follows that ν is constructibly Radon.

It remains to show that ν is unique. Let ν' be any constructibly Radon measure on $X(B)$ such that $\int \hat{f}_B d\nu' = L(f)$ $\forall f \in B$. For $I \subseteq \Omega$ countable, let ν'_I be the pushforward of ν' to $X(B_I)$ and let μ'_I be the pushforward of

ν'_I to $X(C_I)$. Then $L(f) = \int \hat{f}_B d\nu' = \int \hat{f}_{C_I} d\mu'_I$ $\forall f \in C_I$. Since we also have $L(f) = \int \hat{f}_B d\nu = \int \hat{f}_{C_I} d\mu_I$ $\forall f \in C_I$, this implies $\int \hat{f}_{C_I} d\mu'_I = \int \hat{f}_{C_I} d\mu_I$ $\forall f \in C_I$. Thus by uniqueness of μ_I, $\mu'_I = \mu_I$ $\forall$ countable $I \subseteq \Omega$. This in turn implies that $\nu'_I = \nu_I$ $\forall$ countable $I \subseteq \Omega$, so $\nu' = \nu$. ∎

Remark 3.9: If μ is supported by a constructibly Borel set K in $X(C)$, then ν is supported by $K \cap X(B)$. This follows from Claim 1. If E is a constructibly Borel set in $X(C)$ and $E \cap K \cap X(B) = \emptyset$, then $\mu(E \cap K) = 0$. Since μ is supported by K this implies in turn that $\nu(E \cap X(B)) = \mu(E) = 0$. Unfortunately, we are unable to prove this in the more general setting where K is only assumed to be a Borel set of $X(C)$. Of course, if μ happens to be the pushforward of a Radon measure ν on $X(B)$ (the case considered in Lemma 3.6), then μ supported by $K \Rightarrow \nu$ supported by $K \cap X(B)$ for any Borel set K of $X(C)$.

4. Moment problem

We fix an index set Ω and define $A = A_\Omega$, $B = B_\Omega$ and $C = C_\Omega$ as in the previous section. We identify $X(A) = X(B) = \mathbb{R}^\Omega$. The measures ν arising in Theorem 3.8 have **finite moments**, i.e., $\int \hat{x^\alpha} d\nu$ is well-defined and finite for all monomials $x^\alpha := x_{i_1}^{\alpha_1} \cdots x_{i_n}^{\alpha_n}$, $\{i_1, \dots, i_n\} \subseteq \Omega$, $\alpha_k \geq 0$, $k = 1, \dots, n$. Conversely, if ν is a constructibly Borel measure on $\mathbb{R}^\Omega$ having finite moments, then $L : B \to \mathbb{R}$ defined by $L(f) := \int \hat{f} d\nu$ is a well-defined positive linear functional on B. This is clear.

Much of what was done in [32] and [33] in the finite-dimensional case carries over, more or less word for word, to the infinite-dimensional case. Recall if (X, Σ, μ) is a measure space and $f : X \to \mathbb{C}$ is a measurable function, then

$$\|f\|_{s,\mu} := \left[\int |f|^s d\mu \right]^{1/s}, \ s \in [1, \infty).$$

The **Lebesgue space** $\mathcal{L}^s(\mu)$, by definition, is the $\mathbb{C}$-vector space

$$\mathcal{L}^s(\mu) := \{f : X \to \mathbb{C} \mid f \text{ is measurable and } \|f\|_{s,\mu} < \infty\}$$

equipped with the norm $\| \cdot \|_{s,\mu}$.

THEOREM 4.1: *Suppose ν is a constructibly Radon measure on $\mathbb{R}^\Omega$ having finite moments. Then for any $s \in [1, \infty)$ the obvious $\mathbb{C}$-linear map $B \otimes_{\mathbb{R}} \mathbb{C} \to \mathcal{L}^s(\nu)$, $f \otimes r \mapsto r\hat{f}$ has dense image, equivalently, the image of B under the $\mathbb{R}$-linear map $f \mapsto \hat{f}$ is dense in the real part of $\mathcal{L}^s(\nu)$.*

Note that

$$A \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}[x_i \mid i \in \Omega], \quad B \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}\Big[x_i, \frac{1}{1+x_i^2} \mid i \in \Omega\Big],$$

and

$$C \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}\Big[\frac{1}{1+x_i^2}, \frac{x_i}{1+x_i^2} \mid i \in \Omega\Big].$$

Proof. It suffices to show that the step functions $\sum_{j=1}^m r_j \chi_{E_j}$, $r_j \in \mathbb{C}$, $E_j \subseteq X(B)$ a constructibly Borel set, belong to the closure of the image of $B \otimes_{\mathbb{R}} \mathbb{C}$. Using the triangle inequality we are reduced further to the case $m = 1$, $r_1 = 1$. Let $E \subseteq X(B)$ be a constructibly Borel set. Writing $E = \pi^{-1}(E')$ where E' is a Borel set in $X(B_I)$, for some appropriate countable $I \subseteq \Omega$, and applying the change of variable theorem, we see that $\|\chi_E - \hat{f}_B\|_{s,\nu} = \|\chi_{E'} - \hat{f}_{B_I}\|_{s,\nu_I}$ $\forall f \in B_I$. In this way we are reduced to the case where Ω is countable. Choose K compact, U open in $X(C)$ such that $K \subseteq E \subseteq U$, $\mu(U \backslash K) < \epsilon$. By Urysohn's lemma there exists a continuous function $\phi : X(C) \to \mathbb{R}$ such that $0 \leq \phi \leq 1$ on $X(C)$, $\phi = 1$ on K, $\phi = 0$ on $X(C)\backslash U$. Then $\|\chi_E - \phi\|_{s,\mu} \leq \epsilon^{1/s}$. Use the Stone–Weierstrass approximation theorem to get $f \in C$ such that $\|\phi - \hat{f}_C\|_\infty < \epsilon$, where $\|\cdot\|_\infty$ denotes the sup-norm. Then $\|\phi - \hat{f}_C\|_{s,\mu} \leq \epsilon\mu(X(C))^{1/s}$. Putting these things together yields

$$\begin{aligned} \|\chi_E - \hat{f}_B\|_{s,\nu} =& \|\chi_E - \hat{f}_C\|_{s,\mu} \\ \leq& \|\chi_E - \phi\|_{s,\mu} + \|\phi - \hat{f}_C\|_{s,\mu} \\ \leq& \epsilon^{1/s} + \epsilon\mu(X(C))^{1/s}. \qquad \blacksquare \end{aligned}$$

From now on, by a constructibly Radon measure on $\mathbb{R}^\Omega$ we will mean a constructibly Radon measure on $\mathbb{R}^\Omega$ having finite moments.

Corollary 4.2: *For any constructibly Radon measure ν on $\mathbb{R}^\Omega$ and any $s \in [1, \infty)$, $A_\Omega \otimes_{\mathbb{R}} \mathbb{C}$ is dense in $\mathcal{L}^s(\nu)$ iff $A_\Omega \otimes_{\mathbb{R}} \mathbb{C}$ is dense in $B_\Omega \otimes_{\mathbb{R}} \mathbb{C}$ in the $\|\cdot\|_{s,\nu}$-norm.*

Proof. Since the density property in question is transitive, this is immediate from Theorem 4.1. ∎

Corollary 4.3: *Suppose ν is a constructibly Radon measure on $\mathbb{R}^\Omega$ and $s \in (1, \infty)$. Suppose for each $j \in \Omega$ $\exists\, q_{jk} \in A_\Omega \otimes_{\mathbb{R}} \mathbb{C}$ such that $\|q_{jk} - \frac{1}{x_j - \mathrm{i}}\|_{s,\nu} \to 0$ as $k \to \infty$. Then $A_\Omega \otimes_{\mathbb{R}} \mathbb{C}$ is dense in $\mathcal{L}^{s'}(\nu)$ for each $s' \in [1, s)$.*

Proof. Denote by $\overline{A_\Omega \otimes_{\mathbb{R}} \mathbb{C}}$ the closure of $A_\Omega \otimes_{\mathbb{R}} \mathbb{C}$ with respect to topology induced by the norm $\|\cdot\|_{s',\nu}$. It suffices to show that each $f \in B_\Omega \otimes_{\mathbb{R}} \mathbb{C}$ belongs to $\overline{A_\Omega \otimes_{\mathbb{R}} \mathbb{C}}$. The proof is by induction on the number of factors of the form $x_j \pm \mathrm{i}$ appearing in the denominator of f. Suppose $x_j - \mathrm{i}$ appears in the denominator of f. By induction, $fq_{jk}(x_j - \mathrm{i})$ belongs to $\overline{A_\Omega \otimes_{\mathbb{R}} \mathbb{C}}$, for each $k \geq 1$. Applying Hölder's inequality

$$\int |PQ| d\nu \leq \left[\int |P|^a d\nu\right]^{\frac{1}{a}} \cdot \left[\int |Q|^b d\nu\right]^{\frac{1}{b}}, \quad \frac{1}{a} + \frac{1}{b} = 1$$

with $P = |q_{jk} - \frac{1}{x_j - \mathrm{i}}|^{s'}$, $Q = |f(x_j - \mathrm{i})|^{s'}$, $a = \frac{s}{s'}$, $b = \frac{s}{s-s'}$, we see that

$$\begin{aligned}\|fq_{jk}(x - \mathrm{i}) - f\|_{s',\nu} =& \left\|\left(q_{jk} - \frac{1}{x_j - \mathrm{i}}\right) f(x_j - \mathrm{i})\right\|_{s',\nu} \\ \leq& \left\|q_{jk} - \frac{1}{x_j - \mathrm{i}}\right\|_{s,\nu} \cdot \|f(x_j - \mathrm{i})\|_{\frac{ss'}{s-s'},\nu}.\end{aligned}$$

It follows that f belongs to $\overline{A_\Omega \otimes_{\mathbb{R}} \mathbb{C}}$. The case where $x_j + \mathrm{i}$ appears in the denominator of f is dealt with similarly, replacing q_{jk} by $\overline{q_{jk}}$. ∎

Corollary 4.3 extends Petersen's result in [35, Proposition]. In the one variable case, i.e., when $|\Omega| = 1$, one can conclude also that $A_\Omega \otimes_{\mathbb{R}} \mathbb{C}$ is dense in $\mathcal{L}^s(\nu)$; see [32, Corollary 3.3].

Caution: The proof given in [32, Corollary 3.6] is not correct. The proof in [32, Corollary 3.6] is correct when $q_{jk} \in \mathbb{C}[x_j]$ for each j and k.

The following result extends [32, Corollary 2.5] to the case where Ω is infinite.

Corollary 4.4: *For any linear functional $L : A_\Omega \to \mathbb{R}$, the set of constructibly Radon measures ν on $\mathbb{R}^\Omega$ satisfying $L(f) = \int \hat{f} d\nu$ $\forall f \in A_\Omega$ is in natural one-to-one correspondence with the set of positive linear functionals $L' : B_\Omega \to \mathbb{R}$ extending L.*

Proof. If ν is a constructibly Radon measure on $\mathbb{R}^\Omega$ such that $L(f) = \int \hat{f} d\nu$ $\forall f \in A_\Omega$, the corresponding extension of L to a positive linear functional $L' : B_\Omega \to \mathbb{R}$ is defined by $L'(f) = \int \hat{f} d\nu$ $\forall f \in B_\Omega$. The correspondence $\nu \mapsto L'$ has the desired properties by Theorem 3.8. ∎

For ν any constructibly Radon measure on $\mathbb{R}^\Omega$, define $L_\nu : A_\Omega \to \mathbb{R}$ by $L_\nu(f) = \int \hat{f} d\nu$ $\forall f \in A_\Omega$. If ν' is another constructibly Radon measure on $\mathbb{R}^\Omega$, we write $\nu \sim \nu'$ to indicate that ν and ν' have the same moments, i.e., $L_\nu = L_{\nu'}$.

We say ν is **determinate** if $\nu \sim \nu' \Rightarrow \nu = \nu'$ and **indeterminate** if this is not the case.

Corollary 4.5: *Suppose $L : A_\Omega \to \mathbb{R}$ is linear and, for each $j \in \Omega$,*

(4.1) $\exists$ *a sequence* $\{p_{jk}\}_{k=1}^\infty$ *in* $A_\Omega \otimes \mathbb{C}$ *such that* $\lim_{k\to\infty} L(|1-(x_j-\mathrm{i})p_{jk}|^2) = 0.$

Then there is at most one constructibly Radon measure ν on $\mathbb{R}^\Omega$ such that $L = L_\nu$.

Proof. Argue as in [32, Corollary 2.7]. ∎

Corollary 4.5 extends Fuglede's result in [12, Section 7] and Petersen's result in [35, Theorem 3].

Theorem 4.6: *Suppose $L : A_\Omega \to \mathbb{R}$ is linear and positive and, for each $j \in \Omega$,*

$$\begin{aligned} &\exists \text{ a sequence } \{q_{jk}\}_{k=1}^\infty \text{ in } A_\Omega \otimes \mathbb{C} \\ &\text{such that } \lim_{k\to\infty} L(|1-(1+x_j^2)q_{jk}\overline{q_{jk}}|^2) = 0. \end{aligned} \tag{4.2}$$

Then there exists a unique constructibly Radon measure ν on $\mathbb{R}^\Omega$ such that $L = L_\nu$.

Proof. Argue as in [32, Corollaries 4.7 and 4.8] and [33, Theorem 0.1]. ∎

Remark 4.7:

(i) For each $j \in \Omega$, condition (4.1) is implied by condition (4.2). This is clear. Just take $p_{jk} = (x_j + \mathrm{i})q_{jk}\overline{q_{jk}}$.

(ii) For each $j \in \Omega$, condition (4.2) is implied by the Carleman condition

$$\sum_{k=1}^\infty \frac{1}{\sqrt[2k]{L(x_j^{2k})}} = \infty. \tag{4.3}$$

See [8, Théorème 3] and [33, Lemma 0.2 and Theorem 0.3] for the proof.

(iii) The example in [44] shows that (4.2) is strictly weaker than (4.3).

Combining Theorem 4.6 and Remark 4.7 (ii) yields the following result, which is an extension of Nussbaum's result in [34].

Corollary 4.8: *Suppose $L : A_\Omega \to \mathbb{R}$ is linear and positive and, for each $j \in \Omega$, the Carleman condition* (4.3) *holds. Then there exists a unique constructibly Radon measure ν on $\mathbb{R}^\Omega$ such that $L = L_\nu$.*

Remark 4.9: Condition (4.3) holds in a large number of cases. It holds, for example, if there exists a constant $C_j > 0$ such that $L(x_j^{2k}) \leq C_j(2k)!$ for all $k \geq 1$. It holds, in particular, if L is continuous with respect to the vector space norm $\rho_w : A_\Omega \to [0,\infty)$ defined by $\rho_w(\sum a_\alpha x^\alpha) := \sum_\alpha |a_\alpha| w_\alpha$ where $w_\alpha := (2\lceil|\alpha|/2\rceil)!$, see [27] for the definition of ρ_w in case $|\Omega| < \infty$, or if L is continuous with respect to the finest locally multiplicatively convex topology on A_Ω, see [13] and [14].

We mention another result of the same flavour which, in case $|\Omega| < \infty$, is due to Schmüdgen; see [32, Theorem 4.11], [41, Proposition 1].

THEOREM 4.10: *Suppose $L : A_\Omega \to \mathbb{R}$ is linear and positive. For each $j \in \Omega$ fix a Radon measure μ_j on $\mathbb{R}$ such that $L|_{\mathbb{R}[x_j]} = L_{\mu_j}$ and suppose, for each $j \in \Omega$, that $\mathbb{C}[x_j]$ is dense in $\mathcal{L}^4(\mu_j)$, i.e.,*

$$(4.4) \quad \exists \text{ a sequence } \{q_{jk}\}_{k=1}^\infty \text{ in } \mathbb{C}[x_j] \text{ such that } \lim_{k\to\infty} \|q_{jk} - \frac{1}{x_j - \mathrm{i}}\|_{4,\mu_j} = 0.$$

Then there exists a unique constructibly Radon measure μ on $\mathbb{R}^\Omega$ such that $L = L_\mu$.

Proof. Argue as in [32, Theorem 4.11]. ∎

One knows that (4.4) is also strictly weaker than (4.3). The exact relationship between (4.2) and (4.4) remains unclear.

5. The support of the measure

We turn now to the problem of describing the support of the measure. As one might expect, the results we obtain are sharpest when Ω is countable.

We begin with an extension of Haviland's theorem [17], [31, Theorem 3.1.2]. Note that for a closed set $Y \subseteq \mathbb{R}^\Omega$ the following are equivalent:

(i) Y is described by countably many inequalities of the form $\hat{g} \geq 0$, $g \in A_\Omega$, i.e., $\exists$ a countable subset S of A_Ω such that

$$Y = X_S = \{\alpha \in \mathbb{R}^\Omega \mid \hat{g}(\alpha) \geq 0 \ \forall\, g \in S\}.$$

(ii) $\exists$ a countable subset $I \subseteq \Omega$ and a closed subset Y' of $\mathbb{R}^I$ such that $Y = \pi^{-1}(Y')$, where $\pi : \mathbb{R}^\Omega \to \mathbb{R}^I$ is the projection.

The equivalence of (i) and (ii) is a consequence of Proposition 2.2. If Ω is countable, then any closed subset Y of $\mathbb{R}^\Omega$ satisfies these conditions.

Theorem 5.1: *Suppose $L : A_\Omega \to \mathbb{R}$ is linear and $L(\mathrm{Pos}_{A_\Omega}(Y)) \subseteq [0, \infty)$ where Y is a closed subset of $\mathbb{R}^\Omega$ satisfying either of the equivalent conditions (i), (ii). Then there exists a constructibly Radon measure ν on $\mathbb{R}^\Omega$ supported by Y such that $L = L_\nu$.*[4]

Proof. By Lemma 3.3 there exists an extension of L to a linear functional L on B_Ω such that $L(\mathrm{Pos}_{B_\Omega}(Y)) \subseteq [0, \infty)$. Denote by ν the constructibly Radon measure on $\mathbb{R}^\Omega$ corresponding to this extension. Fix a countable set S in A_Ω such that $Y = X_S$. For each $g \in S$, choose $g' \in C_\Omega$ of the form $g' = g/p_g$ for some suitably chosen element $p_g = (1 + x_{j_1}^2)^{e_1} \cdots (1 + x_{j_k}^2)^{e_k}$. Let $S' = \{g' \mid g \in S\}$. Let Q' be the quadratic module of C_Ω generated by S', and Q the quadratic module of B_Ω generated by S. Note that Q is also the quadratic module in B_Ω generated by S', and $Q' \subseteq Q \subseteq \mathrm{Pos}_{B_\Omega}(Y)$, so $L'(Q') \subseteq [0, \infty)$ where $L' := L|_{C_\Omega}$. By [30, Corollary 3.4] there exists a Radon measure μ on $X(C_\Omega)$ supported by $X_{Q'}$ such that $L'(f) = \int \hat{f} d\mu \ \forall f \in C_\Omega$. Uniqueness implies that μ is the Radon measure on $X(C_\Omega)$ defined in Lemma 3.5. Remark 3.9 implies that ν is supported by $X_{Q'} \cap X(B_\Omega) = X_Q = X_S = Y$. ∎

Our next results extend [33, Corollary 0.6] and [33, Remark 0.7].

Theorem 5.2: *Suppose $L : A_\Omega \to \mathbb{R}$ is a positive linear map,* (4.2) *holds for each $j \in \Omega$, and $L(M) \subseteq [0, \infty)$ for some quadratic module M of A_Ω which is the extension of a quadratic module of A_I for some countable $I \subseteq \Omega$. Then the associated constructibly Radon measure ν on $\mathbb{R}^\Omega$ is supported by X_M.*

An earlier version of Theorem 5.2 is proved already in [27, Theorem 2.2].

Proof. Denote by $L : B_\Omega \to \mathbb{R}$ the positive linear extension of L defined by

$$L(f) := \int \hat{f} d\nu \quad \forall f \in B_\Omega.$$

Arguing as in [33, Theorem 0.5] one sees that $L(gh\overline{h}) \geq 0 \ \forall h \in B_\Omega \otimes \mathbb{C}$ (so, in particular, $L(gh^2) \geq 0 \ \forall h \in B_\Omega$) $\forall g \in M$. Denote by Q the extension of M to B_Ω. It follows that $L(Q) \subseteq [0, \infty)$. By hypothesis, Q is the extension of a quadratic module Q_0 of B_I, so $Q' := Q \cap C_\Omega$ is the extension of $Q'_0 := Q_0 \cap C_I$,

[4] As was brought to our attention by the referee, for Ω countable this can also be deduced as a corollary of [11, Theorem 4.2] (using the fact that, if Ω is countable, then the finest locally convex topology on A_Ω is nuclear). See also [18, Remark] and [40, Theorem 12.5.2].

for some countable $I \subseteq \Omega$. Then $X_{Q'} = \pi^{-1}(X_{Q'_0})$, where $\pi : X(C_\Omega) \to X(C_I)$ denotes the restriction, so $X_{Q'}$ is constructibly Borel. By [30, Corollary 3.4] the Radon measure μ on $X(C_\Omega)$ associated to $L' = L|_{C_\Omega}$ is supported by $X_{Q'}$, so, by Remark 3.9, ν is supported by $X_M = X_Q = X_{Q'} \cap X(B_\Omega)$. ∎

For a quadratic module of the form $M = \sum A_\Omega^2 + J$, J an ideal of A_Ω, one can weaken the hypothesis a bit.

Theorem 5.3: *Suppose $L = L_\nu$ for some constructibly Radon measure ν on $\mathbb{R}^\Omega$ and $L(J) = \{0\}$ for some countably generated ideal J of A_Ω. Then ν is supported by $Z(J)$. Here, $Z(J) := \{\alpha \in \mathbb{R}^\Omega \mid \hat{g}(\alpha) = 0 \ \forall g \in J\}$.*

Proof. Let $M = \sum A_\Omega^2 + J$. Since L is positive the hypothesis on J is equivalent to $L(M) \subseteq [0, \infty)$. The extension of M to B_Ω is $Q = \sum B_\Omega^2 + JB_\Omega$, where JB_Ω denotes the extension of J to B_Ω. Extend L to B_Ω in the obvious way, i.e., $L(f) = \int \hat{f} d\mu \ \forall f \in B_\Omega$. By the Cauchy–Schwartz inequality, for $g \in A_\Omega$,

$$L(gh) = 0 \ \forall \ h \in A_\Omega \Leftrightarrow L(g^2) = 0 \Leftrightarrow L(gh) = 0 \ \forall \ h \in B_\Omega.$$

This implies $L(JB_\Omega) = \{0\}$, i.e., $L(Q) \subseteq [0, \infty)$. At this point everything is clear. ∎

A special feature of the following result is that the measure ν obtained is Radon (not just constructibly Radon).

Theorem 5.4: *Suppose M is a quadratic module of A_Ω and there exists a countable subset I of Ω such that the quadratic module $M \cap A_{\Omega \setminus I}$ of $A_{\Omega \setminus I}$ is archimedean. Suppose $L : A_\Omega \to \mathbb{R}$ is linear, $L(M) \subseteq [0, \infty)$, and (4.2) holds for each $j \in I$. Then there exists a unique Radon measure ν on $\mathbb{R}^\Omega$ such that $L = L_\nu$. Moreover, ν is supported by X_M.*

Special cases:

(i) If M is an archimedean quadratic module of A_Ω then Theorem 5.4 applies, taking $I = \emptyset$.

(ii) If Ω is countable then Theorem 5.4 applies to any quadratic module M of A_Ω, taking $I = \Omega$ (so $M \cap A_{\Omega \setminus I} = M \cap A_\emptyset = M \cap \mathbb{R} = \{r \in \mathbb{R} \mid r \geq 0\}$, a quadratic module of $\mathbb{R}$ which is obviously archimedean).

Proof. By hypothesis, there exists $N_j > 0$ such that $N_j^2 - x_j^2 \in M$ for each $j \in \Omega \setminus I$. It follows, e.g., using Theorem 2.1, that $L(x_j^{2k}) \leq N_j^{2k} L(1)$, so (4.3) holds for each $j \in \Omega\backslash I$. By Theorem 4.6 there exists a unique constructibly Radon measure ν on $\mathbb{R}^\Omega$ such that $L = L_\nu$. Extending L to B_Ω in the obvious way and arguing as in Theorem 5.2 we see that $L(Q) \subseteq [0,\infty)$ where Q is the extension of M to B_Ω. By [30, Corollary 3.4] there exists a Radon measure μ on $X(C_\Omega)$ supported by $X_{Q\cap C_\Omega}$ such that $L(f) = \int \hat{f}_{C_\Omega} d\mu$ $\forall f \in C_\Omega$. By Lemma 3.5, μ is supported by the Borel set $E := X_{Q\cap C_\Omega} \backslash \bigcup_{j\in I} \Delta_j$. According to Lemma 3.6 it suffices to show $E \subseteq X(B_\Omega)$. But this is clear. Let $\alpha \in E$. If $j \in \Omega\backslash I$ then $N_j^2 - x_j^2 \in Q$, so

$$\frac{1}{1+x_j^2} - \frac{1}{1+N_j^2} = \frac{N_j^2 - x_j^2}{(1+x_j^2)(1+N_j^2)} \in Q \cap C_\Omega.$$

Thus $\alpha(\frac{1}{1+x_j^2}) \geq \frac{1}{1+N_j^2}$. If $j \in I$ then $\alpha \notin \Delta_j$, so $\alpha(\frac{1}{1+x_j^2}) > 0$. ∎

6. Cylinder results

Fix $i_0 \in \Omega$ and let $\Omega' := \Omega\backslash\{i_0\}$. Consider the subalgebras $A_\Omega \subseteq B_{\Omega'}[x_{i_0}] \subseteq B_\Omega$ and $C_{\Omega'}[x_{i_0}] \subseteq C_\Omega$. Observe that $X(B_{\Omega'}[x_{i_0}])$ is naturally identified with $\mathbb{R}^\Omega$ and $X(C_{\Omega'}[x_{i_0}])$ is naturally identified with $\mathbb{S}^{\Omega'} \times \mathbb{R}$.

The cylinder results in [32, Section 4] and [33] extend in a straightforward way to the case where Ω is infinite. As a consequence, we are able to strengthen slightly the statement of Theorem 4.6, Corollary 4.8 and Theorem 4.10.

Theorem 6.1:

(1) *For $f \in C_{\Omega'}[x_{i_0}]$, $\hat{f} \geq 0$ on $\mathbb{S}^{\Omega'} \times \mathbb{R}$ iff $\exists$ $k \geq 0$ such that*

$$f + \epsilon(1+x_{j_0}^2)^k \in \sum C_{\Omega'}[x_{j_0}]^2 \quad \forall \text{ real } \epsilon > 0.$$

(2) *For $f \in B_{\Omega'}[x_{i_0}]$, $\hat{f} \geq 0$ on $\mathbb{R}^\Omega$ iff $\exists$ q of the form $q = \prod_{k=1}^n (1+x_{i_k}^2)^{\ell_k}$, where $x_{i_1}, \ldots, x_{i_n}$ are the variables appearing in the coefficients of f and $k \geq 0$ such that $f + \epsilon q(1+x_{i_0}^2)^k \in \sum B_{\Omega'}[x_{i_0}]^2$ $\forall$ real $\epsilon > 0$.*

Proof. (1) Since the quadratic module $\sum C_{\Omega'}^2$ of $C_{\Omega'}$ is archimedean, this is an immediate consequence of [30, Theorem 5.1]. (2) If $f \in B_{\Omega'}[x_{i_0}]$, say $f \in B_I[x_{i_0}]$ where $I \subseteq \Omega'$ is finite, there exists an element q of the form $q = \prod_{j\in I}(1+x_j^2)^{\ell_j}$ such that $\frac{f}{q} \in C_{\Omega'}[x_{i_0}]$. Thus, if $f \geq 0$ on $\mathbb{R}^\Omega$ then $\frac{f}{q} \geq 0$ on $\mathbb{S}^{\Omega'} \times \mathbb{R}$, so

$$\frac{f}{q} + \epsilon(1+x_{i_0}^2)^k \in \sum C_{\Omega'}[x_{i_0}]^2$$

for some $k \geq 0$ and, consequently, $f + \epsilon q(1+x_{i_0}^2)^k \in \sum B_{\Omega'}[x_{i_0}]^2$ $\forall$ real $\epsilon > 0$. ∎

Corollary 6.2: *For a linear functional $L : A_\Omega \to \mathbb{R}$ the following are equivalent:*

(1) *L is $\mathrm{Pos}_A(X(A))$-positive.*
(2) *L extends to a positive linear functional $L : B_\Omega \to \mathbb{R}$.*
(3) *L extends to a positive linear functional $L : B_{\Omega'}[x_{i_0}] \to \mathbb{R}$.*
(4) *$\forall f \in A_\Omega$ and $\forall p$ of the form $p = \prod_{j=1}^n (1+x_{i_j}^2)^{\ell_j}$, where $x_{i_1}, \dots, x_{i_n}$ are the variables appearing in the coefficients of f (viewing f as a polynomial in x_{i_0} with coefficients in $A_{\Omega'}$) and $\ell_j \geq 0$, $j = 1, \dots, n$,*

$$pf \in \sum A_\Omega^2 \Rightarrow L(f) \geq 0.$$

Proof. (1) ⇒ (2). By Corollary 3.4. (2) ⇒ (3). Immediate. (3) ⇒ (4). Since $pf \in \sum A_\Omega^2$, it follows that $p^2 f \in \sum A_\Omega^2$, so $f \in \sum B_{\Omega'}[x_{i_0}]^2$. Since the extension of L to $B_{\Omega'}[x_{i_0}]$ is positive this implies $L(f) \geq 0$. (4) ⇒ (1). Suppose $f \in A_\Omega$, $\hat{f} \geq 0$ on $\mathbb{R}^\Omega$. By Theorem 6.1 (2) $\exists\, q = \prod_{j=1}^n (1+x_{i_j}^2)^{\ell_j}$, where $x_{i_1}, \dots, x_{i_n}$ are the variables appearing in the coefficients of f and $k \geq 0$ such that $f + \epsilon q (1+x_{i_0}^2)^k \in \sum B_{\Omega'}[x_{i_0}]^2$ $\forall \epsilon > 0$. Clearing denominators, $p^2(f + \epsilon q(1+x_{i_0}^2)^k) \in \sum A_\Omega^2$ for some p (depending on ϵ) of the form

$$p = \prod_{j=1}^n (1+x_{i_j}^2)^{m_j}.$$

By (4),

$$L(f) + \epsilon L(q(1+x_{i_0}^2)^k) = L(f + \epsilon q(1+x_{i_0}^2)^k) \geq 0.$$

Since $\epsilon > 0$ is arbitrary, this implies $L(f) \geq 0$. ∎

Theorem 6.3: *Suppose $L : A_\Omega \to \mathbb{R}$ is linear and positive and condition* (4.2) *holds, for each $j \in \Omega$, $j \neq i_0$. Then there exists a constructibly Radon measure ν on $\mathbb{R}^\Omega$ such that $L = L_\nu$. If condition* (4.2) *also holds for $j = i_0$ then ν is unique.*

Proof. Argue as in [32, Corollary 4.7 and 4.8] and [33, Theorem 0.1]. ∎

Combining Theorem 6.3 and Remark 4.7 yields the following result which is due to Nussbaum [34] in case $|\Omega| < \infty$.

Corollary 6.4: *Suppose $L : A_\Omega \to \mathbb{R}$ is linear and positive and, for each $j \in \Omega$, $j \neq i_0$ the Carleman condition* (4.3) *holds. Then there exists a constructibly Radon measure ν on $\mathbb{R}^\Omega$ such that $L = L_\nu$. If condition* (4.3) *also holds for $j = i_0$ then ν is unique.*

Theorem 4.10 extends in a similar way.

Theorem 6.5: *Suppose $L : A_\Omega \to \mathbb{R}$ is linear and positive. For each $j \in \Omega$ fix a Radon measure μ_j on $\mathbb{R}$ such that $L|_{\mathbb{R}[x_j]} = L_{\mu_j}$ and suppose, for each $j \in \Omega$, $j \neq i_0$, condition* (4.4) *holds. Then there exists a constructibly Radon measure μ on $\mathbb{R}^\Omega$ such that $L = L_\mu$. If condition* (4.4) *also holds for $j = i_0$ then ν is unique.*

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