SCATTERING THEORY WITH UNITARY TWISTS

By

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Abstract. We study the spectral properties of the Laplace operator associated to a hyperbolic surface in the presence of a unitary representation of the fundamental group. Following the approach by Guillopé and Zworski, we establish a factorization formula for the twisted scattering determinant and describe the behavior of the scattering matrix in a neighborhood of $1/2$.

Contents

1 Introduction

We consider a finitely generated Fuchsian group $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ and denote the associated hyperbolic surface by X. Thus $X = \Gamma\backslash\mathbb{H}$, where $\mathbb{H}$ denotes the hyperbolic upper half-plane and $\mathrm{PSL}(2, \mathbb{R})$ acts via Möbius transformations on $\mathbb{H}$. Throughout this article, we will suppose that X is non-elementary, geometrically finite and of infinite volume. However, we allow that X has orbifold singularities or, equivalently, that Γ has torsion. We note that the condition of being geometrically finite is equivalent to the group Γ being finitely generated [Kat92, Theorem 3.5.4 and

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Theorem 4.6.1]. For the analysis of the twisted Laplacian on hyperbolic surfaces of finite volume, we refer to Venkov [Ven82] and Phillips [Phi97, Phi98].

We further consider a finite-dimensional unitary representation

$$\chi\colon \Gamma \to \mathrm{U}(V)$$

on a Hermitian vector space V. The representation χ induces a Hermitian vector orbibundle

$$E_\chi \coloneqq \Gamma\backslash(\mathbb{H} \times V) \to X$$

with typical fiber V. The action of Γ on $\mathbb{H} \times V$ above is diagonal:

$$g.(z, v) := (g.z, \chi(g)v), \quad g \in \Gamma, z \in \mathbb{H}, v \in V.$$

It is well-known that the (smooth) sections of E_χ are in bijection with the smooth functions $f\colon \mathbb{H} \to V$ that obey the **twisting** equivariance

$$f(g.z) = \chi(g)f(z), \quad z \in \mathbb{H},\ g \in \Gamma. \tag{1}$$

See, for example, [DFP, Lemma 3.3] for details. On smooth maps $f\colon \mathbb{H} \to V$, the hyperbolic Laplacian is given by

$$\Delta_{\mathbb{H}} f(z) = -y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)f(z),$$

where $z = x + iy \in \mathbb{H}$. We note that if the function f satisfied (1), then $\Delta_{\mathbb{H}} f$ would as well. Using the identification of twisted functions (see (1)) and sections of E_χ and the fact that χ is unitary, we notice that the Laplacian $\Delta_{\mathbb{H}}$ gives rise to a non-negative self-adjoint operator

$$\Delta_{X,\chi}\colon L^2(X, E_\chi) \to L^2(X, E_\chi).$$

For $\mathrm{Re}\, s > 1/2$ and $s \notin [1/2, 1]$, the resolvent of $\Delta_{X,\chi}$ is defined by

$$R_{X,\chi}(s) \coloneqq (\Delta_{X,\chi} - s(1-s))^{-1}\colon L^2(X, E_\chi) \to L^2(X, E_\chi).$$

As shown in [DFP, Theorem A], the resolvent $R_{X,\chi}$ admits a meromorphic continuation to $s \in \mathbb{C}$ as an operator

$$R_{X,\chi}(s)\colon L^2_{\mathrm{cpt}}(X, E_\chi) \to L^2_{\mathrm{loc}}(X, E_\chi).$$

The poles of $R_{X,\chi}(s)$ are the **resonances** of $\Delta_{X,\chi}$. The **multiplicity** of the pole $s \in \mathbb{C}$ is the rank of the residue at s and is denoted by $m_{X,\chi}(s)$.

In [DFP, Theorem B], we showed that the resonance counting function grows at most quadratically, i.e.,

$$\sum_{\substack{s\in\mathcal{R}_{X,\chi}\\ |s|\le r}} m_{X,\chi}(s) = O(r^2) \quad \text{as } r\to\infty,$$

where $\mathcal{R}_{X,\chi}$ denotes the set of resonances and $m_{X,\chi}(s)$ the multiplicity of $s\in\mathcal{R}_{X,\chi}$. Hence, by the Weierstrass factorization theorem, there exists an entire function, $\mathcal{P}_{X,\chi}$, such that its zeros coincide with the resonances, and the multiplicity of a zero s of $\mathcal{P}_{X,\chi}$ is equal to $m_{X,\chi}(s)$. We also define the Weierstrass product $\mathcal{P}_{X_f,\chi}(s)$ associated to the resonances of the disjoint union of funnel ends X_f (see Section 5.5 for details).

We consider the **scattering matrix**, which is a certain operator

$$S_{X,\chi}(s)\colon \mathcal{C}^\infty(\partial_\infty X, E_\chi|_{\partial_\infty X}) \to \mathcal{C}^\infty(\partial_\infty X, E_\chi|_{\partial_\infty X}), \qquad s\notin \mathcal{R}_{X,\chi}\cup \mathbb{Z}/2,$$

defined on the boundary $\partial_\infty X$ of a suitable compactification of X (see Sections 3 and 5). For each $\psi\in\mathcal{C}^\infty(\partial_\infty X, E_\chi)$ there exists $u\in\mathcal{C}^\infty(X,E_\chi)$ such that

$$(\Delta_{X,\chi} - s(1-s))u = 0$$

and

$$(2s-1)u \sim \rho_f^{1-s}\rho_c^{-s}\psi + \rho_f^{s}\rho_c^{s-1}S_{X,\chi}(s)\psi \quad \text{as } \rho_f\rho_c\to 0,$$

where ρ_f and ρ_c are the boundary defining functions in the funnel and cusp ends, respectively. Even though the scattering matrix is not trace class, we can define a regularized determinant of $S_{X,\chi}(s)$, that we will call the **relative scattering determinant**, $\tau_{X,\chi}(s)$.

As the first main result of this article, we prove a factorization of the relative scattering determinant in terms of the Weierstrass product over the resonances.

Theorem A. *The scattering determinant admits the factorization*

$$\tau_{X,\chi}(s) = e^{q(s)}\frac{\mathcal{P}_{X,\chi}(1-s)}{\mathcal{P}_{X,\chi}(s)}\frac{\mathcal{P}_{X_f,\chi}(s)}{\mathcal{P}_{X_f,\chi}(1-s)},$$

where $q\colon\mathbb{C}\to\mathbb{C}$ *is a polynomial of degree at most* 4.

For $\dim V = 1$ and $\chi = \mathrm{id}$, Theorem A reduces to [GZ97, Proposition 3.7]. In contrast to Guillopé–Zworski in our setting the Laplacian is vector-valued. This does not impact the functional-analytic parts of the proofs, but one has to be very careful when dealing with the compactifications, especially at the cusps. There, the choice of representation affects the compactification of the bundle.

The results by Guillopé–Zworski play a crucial role in the proof of the factorization of the Selberg zeta function by Borthwick–Judge–Perry [BJP05]. Theorem A will be used in future work to prove an extension of the theorem by Borthwick–Judge–Perry to arbitrary geometrically finite hyperbolic surfaces with unitary twists.

We remark that Theorem A implies that the scattering determinant has no pole or zero at $s = 1/2$. However, $s = 1/2$ might be a resonance. The second main result of this article shows that we are able to describe the behavior of the scattering matrix $S_{X,\chi}(s)$ in some (small) neighborhood of $1/2$. For this, we set

$$P \coloneqq \frac{1}{2}\left(S_{X,\chi}\left(\frac{1}{2}\right) + \mathrm{id}\right). \tag{2}$$

Then

$$S_{X,\chi}(s) = -\mathrm{id} + 2P + (2s-1)T_{X,\chi}(s)$$

with $T_{X,\chi}$ being an operator family that is holomorphic in a small neighborhood of $s = 1/2$.

Theorem B. *The operator P is an orthogonal projection of rank $m_{X,\chi}(1/2)$ onto the space of elements in $\mathcal{C}^\infty(\partial_\infty X, E_\chi|_{\partial_\infty X})$ that are invariant under the map $S_{X,\chi}(1/2)$.*

Structure of this article. In Section 3, we discuss the scattering matrices for the model funnel and the parabolic cylinder. In Section 4, we obtain a decomposition of the resolvent, study the structure of the resolvent close to a resonance and obtain that there are no resonances on the line $\mathrm{Re}(s) = 1/2$ except for, maybe, $s = 1/2$. In Section 5, we introduce the scattering matrix, the relative scattering determinant and prove Theorems A and B.

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2 Preliminaries and notation

We let X and E_χ be as above. We denote by $(\cdot,\cdot)_{E_\chi}$ the Hermitian bundle metric on E_χ that is induced from the sesquilinear inner product $(\cdot,\cdot)_V$ on V. We denote by $\langle\cdot,\cdot\rangle_{E_\chi}$ the bilinear metric on E_χ given by

$$\langle u, v\rangle_{E_\chi} := \int_X (u(x), \overline{v(x)})_V \, d\mu.$$

We abbreviate the norm $|v|_{E_\chi} = \sqrt{(v,v)_{E_\chi}}$ of any $v \in E_\chi$ by $|v|$.

By Selberg's Lemma [Sel60, Lemma 8], there is a finite cover

$$\widetilde{X} = \tilde{\Gamma}\backslash\mathbb{H}$$

of X such that the Fuchsian group $\tilde{\Gamma}$ is a torsion-free subgroup of Γ. We denote the pull-back of E_χ under the covering map $\widetilde{X} \to X$ by $\widetilde{E}$, which becomes a vector bundle over $\widetilde{X}$. We call an operator A acting on the sections of E_χ a **pseudodifferential operator of order** $m \in \mathbb{R}$ if its pull-back, $\widetilde{A}$, under the map $\widetilde{X} \to X$ is a pseudodifferential operator of order m, acting on the sections of $\widetilde{E}$.

In the case of the 1-sphere $\mathbb{S}^1$, pseudodifferential operators have a very simple characterization using Fourier series, which we recall now. To that end let $A\colon \mathcal{C}^\infty(\mathbb{S}^1) \to \mathcal{C}^\infty(\mathbb{S}^1)$ be a continuous linear operator. As proven by McLean [McL91, Theorem 4.4], A is a pseudodifferential operator of order $m \in \mathbb{R}$ if and only if

$$a(x,\xi) := e^{-2\pi i\langle x,\xi\rangle} A\left(e^{2\pi i\langle\cdot,\xi\rangle}\right)(x), \quad (x,\xi) \in \mathbb{S}^1 \times \mathbb{Z},$$

is a periodic symbol of order m. This means that $a \in \mathcal{C}^\infty(\mathbb{S}^1 \times \mathbb{Z})$, and for all $b, c \in \mathbb{N}_0$, we have

$$|\partial_x^b \triangle_\xi^c a(x,\xi)| \lesssim_{b,c} \langle\xi\rangle^{m-c}, \tag{3}$$

where $\langle\cdot\rangle$ is the **Japanese bracket** $\langle z\rangle := (1+|z|^2)^{1/2}$ for $z \in \mathbb{C}$ and $\triangle_\xi$ denotes the discrete derivative, i.e.,

$$\triangle_\xi a(x,\xi) := a(x,\xi+1) - a(x,\xi).$$

Further, $\lesssim$ indicates an upper bound with implied constants. More precisely, for any set Y and any functions $a, b\colon Y \to \mathbb{R}$, we write

$$a \lesssim b \quad \text{or} \quad a(y) \lesssim b(y)$$

if there exists a constant $C > 0$ such that for all $y \in Y$ we have

$$|a(y)| \leq C|b(y)|.$$

If the constant, C, depends on additional parameters, we indicate the dependence in the subscript.

Let H be a Hilbert space and let $B\colon H \to H$ be a bounded operator. The non-zero eigenvalues of $(B^*B)^{1/2}$ are called the **singular values** of the operator B. We denote these singular values by $\mu_k(B)$, $k \in \mathbb{N}$, listed in decreasing order.

We use the convention to call a function, f, **meromorphic on** an open set $U \subseteq \mathbb{C}$ if there exists a discrete subset, P, of U such that f, considered as a function, is defined on $U \setminus P$ only, and f is holomorphic on $U \setminus P$ and has poles (of finite order, which might be zero) at the points in P.

Let X and Y be smooth manifolds and $E \to X$ and $F \to Y$ be smooth vector bundles. We denote by $E \boxtimes F$ the **exterior tensor product**, which is given by

$$(E \boxtimes F)_{(x,y)} = E_x \otimes F_y. \tag{4}$$

If we choose coordinate neighborhoods $U_X \subset X$ and $U_Y \subset Y$, then $E|_{U_X} \cong U_X \times V_X$ and $F|_{U_Y} \cong U_Y \times V_Y$ for some fixed vector spaces V_X and V_Y. The exterior tensor product has the trivialization

$$(E \boxtimes F)|_{U_X \times U_Y} \cong U_X \times U_Y \times V_X \times V_Y.$$

In particular, if $V := V_X = V_Y$ and $F = E'$, then we have that

$$(E \boxtimes E')|_{U_X \times U_X} \cong U_X \times U_X \times \mathrm{End}(V). \tag{5}$$

3 The scattering matrix for the model cylinders

In this section we present the structure of the twisted scattering matrix for the model ends. We discuss the model funnel in Section 3.1 and the model cusp in Section 3.2. The analysis was originally done in [DFP, Section 4]. Here we restrict to presenting the main results only.

3.1 Model funnel. Let $\ell \in (0, \infty)$ and set $\omega := 2\pi/\ell$. We define the **hyperbolic cylinder** as the quotient $C_\ell := \langle h_\ell \rangle \backslash \mathbb{H}$, where $h_\ell.z = e^\ell z$. We may change coordinates via

$$z = e^{\omega^{-1}\phi} \frac{e^r + i}{e^r - i}$$

to $(r, \phi) \in \mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z} \cong C_\ell$ in such a way that the induced metric from the hyperbolic plane becomes

$$g_{C_\ell}(r, \phi) := dr^2 + \frac{\ell^2}{4\pi^2} \cosh^2 r \, d\phi^2.$$

We define the **model funnel** as

$$F_\ell := \{(r, \phi) \in C_\ell \colon r > 0\}$$

with the metric $g_{F_\ell} \coloneqq g_{C_\ell}|_{F_\ell}$. The canonical boundary defining function is

$$\rho_f(r,\phi) = \cosh(r)^{-1}.$$

Taking the boundary defining function, ρ, as a coordinate function (for more details, see [DFP, Section 3.2.4]), we have that $F_\ell \cong (0,1) \times \mathbb{R}/2\pi\mathbb{Z}$. Its compactification, $\overline{F_\ell}$, is given in these coordinates by $[0,1] \times \mathbb{R}/2\pi\mathbb{Z}$. This means we compactify at both $r = 0$ and $r = \infty$ in the (r,ϕ) coordinates. In the (ρ,ϕ) coordinates, the Riemannian metric becomes

$$g_{F_\ell}(\rho,\phi) = \rho^{-2}\Big(\frac{\ell^2}{4\pi^2}d\phi^2 + \frac{d\rho^2}{1-\rho^2}\Big). \tag{6}$$

The volume form is

$$d\mu_{F_\ell} = \frac{\ell}{2\pi}\frac{d\rho\, d\phi}{\rho^2\sqrt{1-\rho^2}}. \tag{7}$$

We also define the metric restricted to the boundary at infinity

$$\begin{aligned} g_{\partial_\infty F_\ell}(\phi, \partial_\phi) &\coloneqq \rho^2 g_{F_\ell}(\rho,\phi,0,\partial_\phi)|_{\rho=0} \\ &= \frac{\ell^2}{4\pi^2}\, d\phi^2 \end{aligned} \tag{8}$$

and denote the corresponding measure by

$$d\sigma_{\partial_\infty F_\ell} \coloneqq \frac{\ell}{2\pi}\, d\phi.$$

The Laplacian acting on functions $F_\ell \to \mathbb{C}$ takes the form

$$\Delta_{F_\ell} = -\rho^2(1-\rho^2)\partial_\rho^2 + \rho^3\partial_\rho - \frac{4\pi^2}{\ell^2}\rho^2\partial_\phi^2. \tag{9}$$

Let $\chi\colon \langle h_\ell\rangle \to \mathrm{U}(V)$ be a finite-dimensional unitary representation. As above, we denote by $\Delta_{C_\ell,\chi}$ the Laplacian acting on sections of the vector bundle $E_\chi := \langle h_\ell\rangle\backslash(\mathbb{H}\times V)$ over C_ℓ. The Laplacian $\Delta_{F_\ell,\chi}$ is the restriction of the Laplacian $\Delta_{C_\ell,\chi}$ to F_ℓ with Dirichlet boundary conditions at $r = 0$. We will also denote by E_χ the restriction of E_χ to F_ℓ. It was shown in [DFP, Proposition 4.12] that the resolvent of the model funnel, $(\Delta_{F_\ell,\chi} - s(1-s))^{-1}$, which is initially defined for $\operatorname{Re} s > 1/2$ and $s \notin [1/2, 1]$ as a bounded operator on $L^2(F_\ell, E_\chi)$, admits a meromorphic continuation to $\mathbb{C}$ as an operator

$$R_{F_\ell,\chi}(s)\colon L^2_{\mathrm{cpt}}(F_\ell, E_\chi) \to L^2_{\mathrm{loc}}(F_\ell, E_\chi).$$

The multiset of its resonances, $\mathcal{R}_{F_\ell,\chi}$, is given by

$$\mathcal{R}_{F_\ell,\chi} \coloneqq \bigcup_{\lambda\in \mathrm{EV}(\chi(h_\ell))}\ \bigcup_{p\in\{\pm1\}} (-(1+2\mathbb{N}_0) + p\ell^{-1}(\log\lambda + 2\pi i\mathbb{Z})), \tag{10}$$

where $\mathrm{EV}(\chi(h_\ell))$ denotes the multiset of eigenvalues of $\chi(h_\ell)$. Since χ is unitary, all eigenvalues have modulus 1, therefore $\log\lambda + 2\pi\mathbb{Z}$ is well-defined. In the case of the trivial representation, we have that $\mathcal{R}_{F_\ell}$ is equal as a set to $-(1+2\mathbb{N}_0)+2\pi i\ell^{-1}\mathbb{Z}$, but every resonance has multiplicity 2, see [DFP, Proposition 4.12] more details.

For the definitions of the restriction of vector bundles to the boundary at infinity, we refer to [DFP, Section 3.7].

Let $\psi \in \mathcal{C}^\infty(\overline{F_\ell} \times \overline{F_\ell})$ such that ψ is supported away from the diagonal and $s \in \mathbb{C}$ is not a pole of the resolvent $R_{F_\ell,\chi}$. By [DFP, Proposition 4.12], we have

$$\psi R_{F_\ell,\chi}(s;\cdot,\cdot) \in (\rho_f\rho_f')^s\mathcal{C}^\infty(\overline{F_\ell} \times \overline{F_\ell}, E_\chi \boxtimes E_\chi'), \tag{11}$$

where $E_\chi \boxtimes E_\chi'$ is the exterior tensor product of E_χ and its dual E_χ' as defined in (4). By (11), the function

$$E_{F_\ell,\chi}(s;r,\phi,\phi') := \lim_{r'\to\infty}(\rho_f(r'))^{-s}R_{F_\ell,\chi}(s;r,\phi,r',\phi') \tag{12}$$

is well-defined. This allows us to introduce the *Poisson operator*

$$\begin{aligned} E_{F_\ell,\chi}(s)\colon \mathcal{C}^\infty(\partial_\infty F_\ell, E_\chi|_{\partial_\infty F_\ell}) &\to \mathcal{C}^\infty(F_\ell, E_\chi), \\ (E_{F_\ell,\chi}(s)f)(r,\phi) &:= \frac{\ell}{2\pi}\int_0^{2\pi} E_{F_\ell,\chi}(s;r,\phi,\phi')f(\phi')\,d\phi'. \end{aligned}$$

We now recall the Fourier expansion of the Poisson operator. Let $(\psi_j)_{j=1}^{\dim V} \subset V$ be an eigenbasis of $\chi(h_\ell)$ with eigenvalues $\lambda_j = e^{2\pi i\vartheta_j}$, $j = 1,\ldots,\dim V$. For $\kappa \in \mathbb{R}$ and $s \in \mathbb{C} \setminus (-1-2\mathbb{N}_0 \pm i\omega\kappa)$ we define,

$$\beta_\kappa(s) := \frac{1}{2}\Gamma\Big(\frac{s+i\omega\kappa+1}{2}\Big)\Gamma\Big(\frac{s-i\omega\kappa+1}{2}\Big). \tag{13}$$

We recall that the **regularized hypergeometric function** $\mathbf{F}(a,b;c;z)$ is defined for $a,b,c \in \mathbb{C}$ and $z \in \mathbb{C}$, $|z| < 1$, by the power series (see [Olv97, Theorem 9.1])

$$\mathbf{F}(a,b;c;z) := \sum_{n=0}^\infty \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(a)\Gamma(b)}\frac{1}{\Gamma(c+n)}\cdot\frac{z^n}{n!}.$$

For arbitrary $\kappa \in \mathbb{R}$, $s \in \mathbb{C}$ and $r \geq 0$, we define

$$v_\kappa^0(s;r) := \tanh(r)(\cosh(r))^{-s}\mathbf{F}\Big(\frac{s+i\omega\kappa+1}{2}, \frac{s-i\omega\kappa+1}{2}; \frac{3}{2}; \tanh(r)^2\Big).$$

It was shown in [DFP, Remark 4.13] that using the identification (5), we have that

$$E_{F_\ell,\chi}(s;r,\phi,\phi')\psi_j = \frac{1}{\ell}\sum_{k\in\mathbb{Z}} e^{i(k+\vartheta_j)(\phi-\phi')}E_{F_\ell,\chi}(s;r)_k^j\psi_j,$$

where

$$E_{F_\ell,\chi}(s;r)^j_k := \frac{\beta_{k+\vartheta_j}(s)v^0_{k+\vartheta_j}(s;r)}{\Gamma(s+\frac{1}{2})}.$$

By the proof of [DFP, Lemma 6.15], we have for $\varepsilon \in (0, 1/2)$, $A \in \mathrm{Diff}^1(F_\ell)$ compactly supported, there exist $C, c > 0$ such that for all $s \in \mathbb{C}$ with $\operatorname{Re} s > \varepsilon$ we have the estimate

$$\mu_j(AE_{F_\ell,\chi}(s)) \leq e^{C\langle s\rangle - cj}. \tag{14}$$

Above, in order to define $\mu_j(AE_{F_\ell,\chi}(s))$, we consider $AE_{F_\ell,\chi}(s)$ as an operator mapping $L^2(\partial_\infty F_\ell, E_\chi|_{\partial_\infty F_\ell})$ to $L^2(F_\ell, E_\chi)$. Moreover, the scattering matrix

$$S_{F_\ell,\chi}(s)\colon \mathcal{C}^\infty(\partial_\infty F_\ell, E_\chi|_{\partial_\infty F_\ell}) \to \mathcal{C}^\infty(\partial_\infty F_\ell, E_\chi|_{\partial_\infty F_\ell})$$

was defined in [DFP, (66)] via the Fourier coefficients of its Schwartz kernel,

$$S_{F_\ell,\chi}(s;\phi,\phi')\psi_j = \frac{1}{\ell}\sum_{k\in\mathbb{Z}} e^{i(k+\vartheta_j)(\phi-\phi')} S_{F_\ell,\chi}(s)^j_k \psi_j,$$

where

$$S_{F_\ell,\chi}(s)^j_k := \frac{\Gamma(\frac{1}{2}-s)\beta_{k+\vartheta_j}(s)}{\Gamma(s-\frac{1}{2})\beta_{k+\vartheta_j}(1-s)}. \tag{15}$$

From this we directly obtain that

$$S_{F_\ell,\chi}(1-s)S_{F_\ell,\chi}(s) = \mathrm{id}\,. \tag{16}$$

From the Fourier expansion, we obtain that

$$\begin{aligned}
[E_{F_\ell,\chi}(1-s)S_{F_\ell,\chi}(s)]^j_k &= \frac{\beta_{k+\vartheta_j}(1-s)v^0_{k+\vartheta_j}(1-s;r)}{\Gamma(1-s+\frac{1}{2})} \cdot \frac{\Gamma(\frac{1}{2}-s)\beta_{k+\vartheta_j}(s)}{\Gamma(s-\frac{1}{2})\beta_{k+\vartheta_j}(1-s)} \\
&= \frac{v^0_{k+\vartheta_j}(1-s;r)}{\Gamma(1-s+\frac{1}{2})} \cdot \frac{\Gamma(\frac{1}{2}-s)\beta_{k+\vartheta_j}(s)}{\Gamma(s-\frac{1}{2})} \\
&= -\frac{\beta_{k+\vartheta_j}(s)v^0_{k+\vartheta_j}(1-s;r)}{(s-\frac{1}{2})\Gamma(s-\frac{1}{2})} \\
&= -\frac{\beta_{k+\vartheta_j}(s)v^0_{k+\vartheta_j}(1-s;r)}{\Gamma(s+\frac{1}{2})},
\end{aligned}$$

where $[E_{F_\ell,\chi}(1-s)S_{F_\ell,\chi}(s)]^j_k$ are the Fourier coefficients of the Schwartz kernel of $E_{F_\ell,\chi}(1-s)S_{F_\ell,\chi}(s)$, see [DFP, Section 4.4]. Therefore

$$E_{F_\ell,\chi}(1-s)S_{F_\ell,\chi}(s) = -E_{F_\ell,\chi}(s) \tag{17}$$

since $v_\kappa^0(s,r) = -v_\kappa^0(1-s,r)$ for all $r \geq 0$ by a connection formula (see [Bor16, p. 93]). We note that by [DFP, (67)] we have for $f \in \mathcal{C}^\infty(\partial_\infty F_\ell, E_\chi|_{\partial_\infty F_\ell})$ that

$$(2s-1)E_{F_\ell,\chi}(s;r)f \sim \sum_{m=0}^\infty \rho_f^{1-s+2m} a_m(s) + \sum_{m=0}^\infty \rho_f^{s+2m} b_m(s) \tag{18}$$

as $r \to \infty$, where the coefficient functions a_m, b_m for $m \in \mathbb{N}_0$ are meromorphic, with the leading coefficient functions being

$$a_0(s) = f \quad \text{and} \quad b_0(s) = S_{F_\ell,\chi}(s)f.$$

From this, we obtain for $\operatorname{Re} s < 1/2$ that

$$S_{F_\ell,\chi}(s;\phi,\phi') = (2s-1)(\rho_f\rho_f')^{-s} R_{F_\ell,\chi}(s;\rho_f,\phi,\rho_f',\phi')|_{\rho_f=\rho_f'=0}.$$

In what follows we will argue that the scattering matrix is a pseudodifferential operator on $\partial_\infty F_\ell$ and calculate its principal symbol. From [DLMF, Eq. 5.11.13], we have that for $a, b \in \mathbb{C}$ and $|\arg(z)| < \pi - \varepsilon$ for fixed $\varepsilon > 0$,

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b} \sum_{k=0}^\infty \frac{G_k(a,b)}{z^k} \quad \text{as } z \to \infty,$$

for some $G_k(a,b) \in \mathbb{C}$. This immediately implies that for $y \to \infty$,

$$\begin{aligned}\frac{\Gamma(iy+a)}{\Gamma(iy+b)}\frac{\Gamma(-iy+a)}{\Gamma(-iy+b)} &\sim \left((iy)^{a-b}\sum_{k=0}^\infty \frac{G_k(a,b)}{(iy)^k}\right)\left((-iy)^{a-b}\sum_{k=0}^\infty \frac{G_k(a,b)}{(-iy)^k}\right) \\ &= |y|^{2(a-b)} \sum_{k=0}^\infty \frac{1}{y^k} \sum_{n=0}^k e^{\frac{\pi i}{2}(k-2n)} G_n(a,b) G_{k-n}(a,b).\end{aligned}$$

Taking $y = \omega\kappa/2$, $a = 1/2 + s/2$ and $b = 1 - s/2$, we obtain

$$\begin{aligned}&\frac{\Gamma(\frac{s+i\omega\kappa+1}{2})\Gamma(\frac{s-i\omega\kappa+1}{2})}{\Gamma(\frac{2-s+i\omega\kappa}{2})\Gamma(\frac{2-s-i\omega\kappa}{2})} \\ &\qquad \sim \left|\frac{\omega\kappa}{2}\right|^{2s-1} \sum_{k=0}^\infty \left(\frac{2}{\omega\kappa}\right)^k \sum_{n=0}^k e^{\frac{\pi i}{2}(k-2n)} G_n(a,b) G_{k-n}(a,b).\end{aligned} \tag{19}$$

We note that the terms in (19) with odd k vanish, since the left hand side is even as a function of $\omega\kappa$. The definition in (13) combined with (19) shows that

$$\begin{aligned}\frac{\beta_\kappa(s)}{\beta_\kappa(1-s)} &= \frac{\Gamma(\frac{s+i\omega\kappa+1}{2})\Gamma(\frac{s-i\omega\kappa+1}{2})}{\Gamma(\frac{2-s+i\omega\kappa}{2})\Gamma(\frac{2-s-i\omega\kappa}{2})} \\ &\sim \sum_{k=0}^\infty \left|\frac{\omega\kappa}{2}\right|^{2s-1-2k} \sum_{n=0}^{2k} e^{\pi i(k-n)} G_n(a,b) G_{2k-n}(a,b).\end{aligned}$$

By [DLMF, Eq. 5.11.15], the leading coefficient is given by $G_0(a,b)^2 = 1$. Combining this with (15), we obtain that

$$S_{F_\ell,\chi}(s)^j_k \sim 2^{1-2s}\frac{\Gamma(\frac{1}{2}-s)}{\Gamma(s-\frac{1}{2})}|(k+\vartheta_j)\omega|^{2s-1}.$$

The full asymptotic expansion now implies that $S_{F_\ell,\chi}(s)^j_k$ satisfies the symbol estimates for global pseudodifferential operators on the torus, as stated in (3). Hence,

$$S_{F_\ell,\chi}(s) \in \Psi^{2\operatorname{Re}s-1}(\partial_\infty F_\ell, E_\chi|_{\partial_\infty F_\ell}), \quad s \notin \mathcal{R}_{F_\ell,\chi} \cup \left(\mathbb{N}_0 + \frac{1}{2}\right). \tag{20}$$

We define the **reduced scattering matrix** $\tilde{S}_{F_\ell,\chi}(s)$ as follows: we consider the invertible elliptic pseudodifferential operator $\Lambda(s) \in \Psi^{\operatorname{Re}s}(\partial_\infty F_\ell, E_\chi|_{\partial_\infty F_\ell})$ defined by

$$\Lambda(s)\psi_j = \sum_{k\in\mathbb{Z}} e^{i(k+\vartheta_j)(\phi-\phi')}\langle k\rangle^s\psi_j,$$

set

$$G(s) := \Gamma\left(s+\frac{1}{2}\right)\operatorname{id}_{\mathcal{C}^\infty(\partial_\infty F_\ell, E_\chi|_{\partial_\infty F_\ell})}$$

and define

$$\tilde{S}_{F_\ell,\chi}(s) := G(s)\Lambda(s)S_{F_\ell,\chi}(s)\Lambda(1-s)^{-1}G(1-s)^{-1}, \tag{21}$$

for $s \notin \mathcal{R}_{F_\ell,\chi} \cup (\mathbb{N}_0 + 1/2)$. A straightforward calculation shows that the Fourier coefficients of $\tilde{S}_{F_\ell,\chi}(s)$ are

$$\begin{aligned}\tilde{S}_{F_\ell,\chi}(s)^j_k &= \frac{\Gamma(s+\frac{1}{2})}{\Gamma(\frac{1}{2}-s)}\langle k\rangle^{-2s+1}S_{F_\ell,\chi}(s)^j_k \\ &= \langle k\rangle^{-2s+1}\frac{(s-\frac{1}{2})\beta_{k+\vartheta_j}(s)}{\beta_{k+\vartheta_j}(1-s)}.\end{aligned} \tag{22}$$

Since the right-hand side of the last equation is defined for all $s \notin \mathcal{R}_{F_\ell,\chi}$, the scattering matrix is defined as an operator $\tilde{S}_{F_\ell,\chi}(s) \in \Psi^0(\partial_\infty F_\ell, E_\chi|_{\partial_\infty F_\ell})$ for $s \notin \mathcal{R}_{F_\ell,\chi}$. Taking advantage of this property, we can characterize the resonances in terms of the scattering matrix.

Proposition 3.1. *Let $s \in \mathbb{C}$, $\operatorname{Re}s < 1/2$ and let $m \in \mathbb{N}$. Then the reduced scattering matrix $\tilde{S}_{F_\ell,\chi}$ has a pole of rank m at s if and only if s is a resonance of multiplicity m of $\Delta_{F_\ell,\chi}$. In this case, $m = m_{X,\chi}(s)$.*

Proof. By definition of β_κ, we have that

$$\langle k\rangle^{-2s+1}\frac{(s-\frac{1}{2})}{\beta_{k+\vartheta_j}(1-s)}$$

is holomorphic and non-zero for $\operatorname{Re} s < 1/2$. Therefore, the poles counted with multiplicities of $\tilde{S}_{F_\ell,\chi}(s)^j_k$ coincide with $\beta_{k+\vartheta_j}(s)$ and are given by the multiset

$$\bigcup_{p\in\{\pm1\}}(-(1+2\mathbb{N}_0)+2\pi p\ell^{-1}(\vartheta_j+k)).$$

Since we have the Fourier type expansion,

$$\tilde{S}_{F_\ell,\chi}(s;\phi,\phi')\psi_j=\frac{1}{\ell}\sum_{k\in\mathbb{Z}}e^{i(k+\vartheta_j)(\phi-\phi')}\tilde{S}_{F_\ell,\chi}(s)^j_k\psi_j,$$

the poles of $\tilde{S}_{F_\ell,\chi}(s)$ are given by the multiset (10) with correct multiplicities. □

Finally, we recall the singular value estimate for the scattering matrix from [DFP, Lemma 6.14]. For this, we will define functions $d_k\colon\mathbb{C}\to\mathbb{C}$ for $k\in\mathbb{N}$, which have poles contained in the set of resonances of $\Delta_{F_\ell,\chi}$. We set

$$\begin{aligned}
\tilde{\mathcal{R}}_0 &:= 1-2\mathbb{N}_0,\\
\mathcal{R}_0 &:= 1-2\mathbb{N}_0+i\omega\mathbb{Z}\setminus\{0\},\\
\mathcal{R}_{1/2} &:= 1-2\mathbb{N}_0+i\omega\Big(\frac{1}{2}+\mathbb{Z}\Big),\\
\mathcal{R}_\vartheta &:= \bigcup_{p\in\{\pm1\}}(1-2\mathbb{N}_0+ip\omega(\vartheta+\mathbb{Z})),\qquad \vartheta\notin\Big\{0,\frac{1}{2}\Big\},
\end{aligned}$$

where we denote by m_ϑ the multiplicity of the eigenvalue $\lambda=e^{2\pi i\vartheta}$ of $\chi(h_\ell)$. We can assume without loss of generality that $\vartheta\in[0,1)$. Denote by $d_{\mathbb{C}}$ the Euclidean distance on $\mathbb{C}$. For $k\in\mathbb{N}$ we define $d_{k,\vartheta}(s)$ as follows: for $\vartheta\in(0,1)\setminus\{1/2\}$, we set

$$d_{k,\vartheta}(s):=\begin{cases}d_{\mathbb{C}}(s,\mathcal{R}_\vartheta)^{-1}, & k\le m_\vartheta,\\ 1, & k>m_\vartheta,\end{cases}$$

and for $\vartheta\in\{0,1/2\}$, we set

$$d_{k,\vartheta}(s):=\begin{cases}d_{\mathbb{C}}(s,\mathcal{R}_\vartheta)^{-1}, & k\le 2m_\vartheta,\\ 1, & k>2m_\vartheta.\end{cases}$$

Moreover, we define the function $\tilde{d}_{k,0}$ by

$$\tilde{d}_{k,0}(s):=\begin{cases}d_{\mathbb{C}}(s,\tilde{\mathcal{R}}_0)^{-2}, & k\le m_0,\\ 1, & k>m_0.\end{cases}$$

Finally, we set

$$d_k(s) := \tilde{d}_{k,0}(s) \cdot \prod_{\vartheta} d_{k,\vartheta}(s). \tag{23}$$

Let

$$m_{\max} = \max\Big\{m_0,\, 2 \cdot \max_{j=1}^{\dim V}\{m_{\vartheta_j} :\, \vartheta_j \neq 0\}\Big\}. \tag{24}$$

It is shown in [DFP, Lemma 6.14] that for any $\varepsilon \in (0, 1/2)$ there exists $C > 0$ such that for $s \in \mathbb{C}$ with $\operatorname{Re} s < 1/2 - \varepsilon$,

$$\mu_k(S_{F_f,\chi}(s)) \leq e^{C\langle s\rangle} \langle s\rangle^{1-2\operatorname{Re} s} \times \begin{cases} d_k(s), & k \leq m_{\max}, \\ k^{2\operatorname{Re} s-1}, & k > m_{\max}. \end{cases} \tag{25}$$

3.2 Parabolic cylinders. We now turn to the parabolic cylinder, where the structure of the resolvent is slightly simpler than for the hyperbolic cylinder.

The **parabolic cylinder** is given by $C_\infty := \langle T\rangle\backslash\mathbb{H}$, where $T.z := z+1$. We can choose as fundamental domain the set

$$\mathcal{F} := \{x + iy \in \mathbb{H} :\, x \in (0, 1)\}.$$

With the coordinates $(\rho, \phi) = (y^{-1}, (2\pi)^{-1}x)$, the induced Riemannian metric reads

$$g_{C_\infty} = \frac{d\rho^2}{\rho^2} + \rho^2 \frac{d\phi^2}{4\pi^2} \tag{26}$$

and $\rho_c(\rho, \phi) = \rho$, where ρ_c is the canonical boundary defining function. In the (x, y)-coordinates the Laplacian is given by

$$\Delta_{C_\infty} = -y^2(\partial_x^2 + \partial_y^2).$$

Let $\chi\colon \langle T\rangle \to \mathrm{U}(V)$ be a finite-dimensional unitary representation. We denote by $E_1(\chi(T))$ the eigenspace of $\chi(T)$ for eigenvalue 1, and we set $n_c^\chi := \dim E_1(\chi(T))$.

Let $\overline{C_\infty}$ be the compactification of C_∞ as described in [DFP, Section 3.3]. We also define $\mathcal{C}^\infty(\overline{C_\infty})$ as on [DFP, p. 9]. The meromorphically continued resolvent $R_{C_\infty,\chi}(s)$ defines a continuous map

$$\psi R_{C_\infty,\chi}(s) \colon \mathcal{C}_c^\infty(C_\infty, E_\chi) \to \rho_c^{s-1}\mathcal{C}^\infty(\overline{C_\infty}, E_\chi) \tag{27}$$

provided that $s \neq 1/2$, where ψ is any element of $\mathcal{C}^\infty(\overline{C_\infty})$ that is supported away from $\{y = 0\}$. The only pole of $R_{C_\infty,\chi}(s)$ is at the point $s = 1/2$ and its multiplicity is equal to n_c^χ.

For $\kappa \in \mathbb{R}$, we define the function u_κ as follows: for $\kappa \in \mathbb{R} \setminus \{0\}$, we set

$$u_\kappa(s; y, y') := \begin{cases} \sqrt{yy'} I_{s-1/2}(|\kappa| y) K_{s-1/2}(|\kappa| y'), & y \le y', \\ \sqrt{yy'} K_{s-1/2}(|\kappa| y) I_{s-1/2}(|\kappa| y'), & y > y', \end{cases}$$

where $I_{s-1/2}$ and $K_{s-1/2}$ is the modified Bessel function of the first and the second kind, respectively (see [Wat66, § 3.7]). For $\kappa = 0$ and $s \neq 1/2$ we set

$$u_0(s; y, y') := \frac{1}{2s-1} \begin{cases} y^s (y')^{1-s}, & y \le y', \\ y^{1-s} (y')^s, & y > y'. \end{cases}$$

The integral kernel of the resolvent $R_{C_\infty,\chi}(s)$ admits a Fourier decomposition. For any $j \in \{1, \ldots, \dim V\}$, the Fourier decomposition of the non-vanishing matrix coefficients $R_{C_\infty,\chi}(s; z, z')^j$ is given by

$$R_{C_\infty,\chi}(s; z, z')^j = \sum_{k \in \mathbb{Z}} e^{2\pi i (k+\vartheta_j)(x-x')} u_{2\pi(k+\vartheta_j)}(s; y, y'). \tag{28}$$

The **Poisson operator** $E_{C_\infty,\chi}(s)$ is given by

$$\begin{aligned} E_{C_\infty,\chi}(s) &: \mathcal{C}^\infty(\partial_c C_\infty, E_\chi) \to \mathcal{C}^\infty(C_\infty, E_\chi) \\ (E_{C_\infty,\chi}(s) u)(x, y) &:= \frac{y^s}{2s-1} u(x), \end{aligned}$$

where by [DFP, (25)], $u \in \mathcal{C}^\infty(\partial_c C_\infty) \cong \mathbb{C}^{n_c^\chi}$. The Schwartz kernel of $E_{C_\infty,\chi}(s)$ is given by

$$\begin{aligned} E_{C_\infty,\chi}(s; x, y, x') &= \frac{y^s}{2s-1} \operatorname{id}_{E_1(\chi(T))} \\ &= \lim_{y' \to \infty} \rho_c(y')^{1-s} R_{C_\infty,\chi}(s; x, y, x', y') \end{aligned} \tag{29}$$

by the Fourier decomposition in (28). In particular,

$$E_{C_\infty,\chi}(s; x, y, x') = E_{C_\infty,\chi}(s; y)$$

is independent of x, x'.

4 Analysis of the resolvent

In this section, we discuss fine-structure properties of the resolvent of $\Delta_{X,\chi}$. We start, in Theorem 4.1, with a decomposition of its resolvent into interior and residual terms, which are then discussed separately in more detail. In Section 4.1, we give a description of the resolvent near a resonance. In Section 4.2, we prove that on the line $\mathrm{Re}(s) = 1/2$ there are no resonances except for potentially $s = 1/2$. Moreover, in Proposition 4.7, we prove that if the hyperbolic surface X has infinite volume, then $\Delta_{X,\chi}$ has no eigenvalues larger than $1/4$.

As in [DFP, Section 3.2.3], we take advantage of the decomposition

$$X = K \sqcup X_f \sqcup X_c,$$

where K is compact and X_f and X_c are finite collections of funnels and cusps, respectively. For $\bullet \in \{f, c\}$ and $r \in [0, \infty)$, we choose a cutoff function $\eta_{\bullet,r} \in \mathcal{C}^\infty(X)$ such that

$$\eta_{\bullet,r}(x) = \begin{cases} 1, & \text{if } d(X \setminus X_\bullet, x) < r, \\ 0, & \text{if } d(X \setminus X_\bullet, x) > r + \frac{1}{2}. \end{cases}$$

We fix $s_0 \in \mathbb{C}$ with sufficiently large real part (such that $s(1-s)$ is sufficiently far away from the spectrum of $\Delta_{X,\chi}$) and denote by n_f and n_c the number of connected components of X_f and X_c, respectively:

$$X_\bullet = \bigsqcup_{j=1}^{n_\bullet} X_{\bullet,j}, \qquad \bullet \in \{f, c\}.$$

For each $X_{f,j}$ ($X_{c,j}$) there exists a hyperbolic (parabolic) element $\gamma_j \in \Gamma$ such that

$$E_\chi|_{X_{\bullet,j}} = (\mathbb{H} \times_{\chi_j} V)|_{X_{\bullet,j}},$$

where $\chi_j \coloneqq \chi|_{\langle\gamma_j\rangle} : \langle\gamma_j\rangle \to \mathrm{U}(V)$. As in [DFP, Section 5], we set

$$M_i \coloneqq \eta_{f,2}\eta_{c,2}R_{X,\chi}(s_0)\eta_{f,1}\eta_{c,1}, \tag{30}$$

$$M_f(s) \coloneqq (1 - \eta_{f,0})R_{X_f,\chi}(s)(1 - \eta_{f,1}), \tag{31}$$

$$M_c(s) \coloneqq (1 - \eta_{c,0})R_{X_c,\chi}(s)(1 - \eta_{c,1}), \tag{32}$$

where

$$R_{X_f,\chi}(s) \colon L^2(X_f, E_\chi) \to L^2(X_f, E_\chi),$$
$$R_{X_f,\chi}(s) \coloneqq R_{X_{f,1},\chi_1}(s) \oplus \cdots \oplus R_{X_{f,n_f},\chi_{n_f}}(s)$$

and

$$R_{X_c,\chi}(s) \colon L^2(X_c, E_\chi) \to L^2(X_c, E_\chi),$$
$$R_{X_c,\chi}(s) \coloneqq R_{X_{c,1},\chi_1}(s) \oplus \cdots \oplus R_{X_{c,n_c},\chi_{n_c}}(s).$$

Further, we set

$$M(s) \coloneqq M_i + M_f(s) + M_c(s)$$

and, as in [DFP, (85)], we define

$$L(s) \coloneqq L_i(s) + L_f(s) + L_c(s), \tag{33}$$

where

$$L_i(s) := -[\Delta_{X,\chi}, \eta_{f,2}\eta_{c,2}]R_{X,\chi}(s_0)\eta_{f,1}\eta_{c,1} + (s(1-s) - s_0(1-s_0))M_i(s_0),$$

$$L_f(s) := [\Delta_{X,\chi}, \eta_{f,0}]R_{X_f,\chi}(s)(1-\eta_{f,1}), \tag{34}$$

and

$$L_c(s) := [\Delta_{X,\chi}, \eta_{c,0}]R_{X_c,\chi}(s)(1-\eta_{c,1}). \tag{35}$$

It follows that

$$(\Delta_{X,\chi} - s(1-s))M(s) = \mathrm{id} - L(s). \tag{36}$$

It was proven in [DFP, Section 5] that for $\operatorname{Re} s$ sufficiently large, the operator

$$\mathrm{id} - L(s) : L^2(X, E_\chi) \to L^2(X, E_\chi)$$

is invertible, and its inverse admits a meromorphic continuation to $s \in \mathbb{C}$ as an operator

$$(\mathrm{id} - L(s))^{-1} : L^2_{\mathrm{cpt}}(X, E_\chi) \to L^2_{\mathrm{loc}}(X, E_\chi).$$

Theorem 4.1. *Let $X = \Gamma\backslash\mathbb{H}$ be geometrically finite and let $\chi\colon \Gamma \to \mathrm{U}(V)$ be a finite-dimensional unitary representation of Γ. For $s \in \mathbb{C}$ not a pole of neither $R_{X,\chi}(s)$ nor $M_f(s)$ nor $M_c(s)$, the resolvent admits a decomposition*

$$R_{X,\chi}(s) = \tilde{M}_i(s) + M_f(s) + M_c(s) + Q(s),$$

where

- *$\tilde{M}_i(s)$ is a compactly supported pseudodifferential operator of order -2,*
- *$M_f(s)$ and $M_c(s)$ are as in* (31) *and* (32)*, respectively, and*
- *$Q(s)$ is an integral operator with the integral kernel $Q(s;\cdot,\cdot)$ satisfying*

$$Q(s;\cdot,\cdot) \in (\rho_f\rho_f')^s(\rho_c\rho_c')^{s-1}\mathcal{C}^\infty(\overline{X}\times\overline{X}, E_\chi \boxtimes E_\chi').$$

Remark 4.2. The product $\overline{X}\times\overline{X}$ is not a smooth manifold (even in the absence of orbifold points). The reason is that the geodesic boundary at infinity of a cusp end is a single point. Blowing up each parabolic fixed point to a 1-sphere, we obtain an orbifold with boundary $\overline{X}'$. We define smooth functions on $\overline{X}\times\overline{X}$ as the set of function that pullback to smooth functions on $\overline{X}'\times\overline{X}'$.

Note that blowing up a parabolic fixed point amounts to introducing coordinates (ρ, ϕ) as in Section 3.2, where $\{\rho = 0\} \cong \mathbb{S}^1$ is the blowup of the parabolic fixed point.

Proof of Theorem 4.1. We set

$$K(s) := (\mathrm{id} - L(s))^{-1} L(s). \tag{37}$$

Note that

$$\mathrm{id} + K(s) = (\mathrm{id} - L(s))^{-1}. \tag{38}$$

Then (36) implies

$$R_{X,\chi}(s) = M(s)(\mathrm{id} + K(s)).$$

For notational simplicity, define $\eta_3 := \eta_{f,3}\eta_{c,3}$. We now split $M(s)K(s)$ as

$$M(s)K(s) = \eta_3 M(s)K(s)\eta_3 + Q(s),$$

where

$$Q(s) := (1 - \eta_3)M(s)K(s)\eta_3 + M(s)K(s)(1 - \eta_3).$$

Moreover, we define $\tilde{M}_i(s) := M_i + \eta_3 M(s)K(s)\eta_3$ and note that

$$R_{X,\chi}(s) = \tilde{M}_i(s) + M_f(s) + M_c(s) + Q(s).$$

We now have to show that $\tilde{M}_i(s)$ and $Q(s)$ have the claimed properties.

Interior term. The operator M_i is a compactly supported pseudodifferential operator by definition, so it suffices to show that $\eta_3 M(s)K(s)\eta_3$ is a pseudodifferential operator of order at most -2. By (33) we have

$$\eta_3 L(s) = L(s).$$

Now equation (38) directly implies that

$$K(s)\eta_3 = (\mathrm{id} - L(s))^{-1}\eta_3 - \eta_3.$$

From $\eta_3 L(s) = L(s)$, we obtain

$$(\mathrm{id} + K(s)\eta_3)(\mathrm{id} - L(s)\eta_3) = \mathrm{id}.$$

Consequently,

$$\mathrm{id} + K(s)\eta_3 = (\mathrm{id} - L(s)\eta_3)^{-1} \tag{39}$$

is meromorphic for s close to s_0. By the identity theorem for holomorphic functions, the equality in (39) is valid for all $s \in \mathbb{C}$. Formally, we can also obtain this from the geometric series,

$$\mathrm{id} + K(s)\eta_3 = \mathrm{id} + \sum_{k>0} L(s)^k \eta_3 = (\mathrm{id} - L(s)\eta_3)^{-1}.$$

We have that

$$L(s)\eta_3 = (s(1-s) - s_0(1-s_0))M_i + \tilde{Q}(s),$$

where $\tilde{Q}(s)$ is compactly supported and smoothing. Thus, $L(s)\eta_3$ is a pseudodifferential operator of order -2 and therefore

$$K(s)\eta_3 = (\mathrm{id} - L(s)\eta_3)^{-1}L(s)\eta_3$$

is also a pseudodifferential operator of order -2. By the definition of $M(s)$, the operator $\eta_3 M(s)$ is a pseudodifferential operator of order -2 and hence $\eta_3 M(s)K(s)\eta_3$ is a pseudodifferential operator of order -4.

Residual term. To study the operator $Q(s)$, we start by considering the operator $M(s)K(s)(1-\eta_3)$. We use (37) to show that

$$K(s) = L(s)(\mathrm{id} + K(s)).$$

Since $L(s)$ maps to compactly supported smooth sections, we use the explicit calculations for the model resolvents to obtain that, for any $\varphi \in \mathcal{C}_c^\infty(X, E_\chi)$, we have

$$L(s)^T\varphi \in (\rho_f)^s(\rho_c)^{s-1}\mathcal{C}^\infty(\overline{X}, E_\chi).$$

Moreover, the property

$$L(s)\varphi \in (\rho_f)^s(\rho_c)^{s-1}\mathcal{C}^\infty(\overline{X}, E_\chi)$$

implies that the integral kernel $K(s)(1-\eta_3)(\cdot,\cdot)$ of the operator $K(s)(1-\eta_3)$ satisfies

$$K(s)(1-\eta_3)(\cdot,\cdot) \in (\rho_f\rho_c)^\infty(\rho_f')^s(\rho_c')^{s-1}\mathcal{C}^\infty(\overline{X}\times\overline{X}, E_\chi \boxtimes E_\chi')$$

and is compactly supported in the left-most variable. Using that M_i is a compactly supported pseudodifferential operator and $M_f(s)$ and $M_c(s)$ are given by the model resolvents, we conclude that the integral kernel of the operator $M(s)K(s)(1-\eta_3)$ satisfies

$$M(s)K(s)(1-\eta_3)(\cdot,\cdot) \in (\rho_f\rho_f')^s(\rho_c\rho_c')^{s-1}\mathcal{C}^\infty(\overline{X}\times\overline{X}, E_\chi \boxtimes E_\chi').$$

For $(1-\eta_3)M(s)K(s)$, we use that $K(s)\eta_3$ is compactly supported and that the integral kernel of the operator $(1-\eta_3)M(s)K(s)\eta_3$ satisfies

$$(1-\eta_3)M(s)K(s)\eta_3(\cdot,\cdot) \in \rho_f^s\rho_c^{s-1}(\rho_f'\rho_c')^\infty\mathcal{C}^\infty(\overline{X}\times\overline{X}, E_\chi \boxtimes E_\chi').$$

This proves the theorem. □

We will now provide a formula for $Q(s)$ restricted to the boundary that will be useful later on. Let $\varphi \in \mathcal{C}^\infty(\overline{X}, E_\chi)$ such that $\eta_3\varphi = 0$. In this case $Q(s)\varphi$ simplifies to

$$\begin{aligned} Q(s)\varphi &= M(s)K(s)\varphi \\ &= M(s)(\mathrm{id} - L(s))^{-1}L(s)\varphi. \end{aligned}$$

Using that $L(s) = \eta_3 L(s)$, we obtain

$$\begin{aligned} L(s) &= \eta_3(\mathrm{id} - L(s)\eta_3)(\mathrm{id} - L(s)\eta_3)^{-1}L(s) \\ &= (\mathrm{id} - L(s))\eta_3(\mathrm{id} - L(s)\eta_3)^{-1}L(s). \end{aligned}$$

Hence, we have

$$Q(s)\varphi = M(s)\eta_3(\mathrm{id} - L(s)\eta_3)^{-1}L(s)\varphi. \tag{40}$$

4.1 Resolvent at a resonance. Let $s_0 \in \mathbb{C}$ be a resonance of $\Delta_{X,\chi}$. As in [DFP, Section 6], we define the **multiplicity** of the resonance s_0 as the number

$$m_{X,\chi}(s_0) := \operatorname{rank} \int_{\gamma_{\varepsilon,s_0}} R_{X,\chi}(s)\, ds,$$

where $\varepsilon > 0$ is chosen such that the path $\gamma_{\varepsilon,s_0}\colon [0,1] \to \mathbb{C}$ with

$$\gamma_{\varepsilon,s_0}(t) := s_0 + \varepsilon e^{2\pi i t}$$

encloses exactly one resonance (namely s_0). We denote the multiset of resonances by

$$\mathcal{R}_{X,\chi} := \{(s_0, m) \in \mathbb{C} \times \mathbb{N} \colon s_0 \text{ is a resonance}, m = m_{X,\chi}(s_0)\}$$

and the multiset of resonances of the model funnel ends by

$$\mathcal{R}_{X_f,\chi} := \bigcup_{j=1}^{n_f} \mathcal{R}_{X_{f,j},\chi_j}, \tag{41}$$

where the multiset $\mathcal{R}_{X_{f,j},\chi_j}$ is given as in (10).

In a small neighborhood of the resonance s_0, the resolvent admits an expansion

$$R_{X,\chi}(s) = \sum_{j=1}^{p} \frac{A_j(s_0)}{(s(1-s) - s_0(1-s_0))^j} + H(s, s_0) \tag{42}$$

for some $p \in \mathbb{N}$, further referred to as the **order** of the resonance, where, for $j = 1, \ldots, p$, the coefficient $A_j(s_0)$ is a finite rank operator, $A_p(s_0) \neq 0$, and the map $s \mapsto H(s, s_0)$ is holomorphic in a small neighborhood of s_0.

Now let $s_0 \neq 1/2$ and fix $j = 1, \ldots, p$. We multiply (42) by

$$(s(1-s) - s_0(1-s_0))^{j-1}$$

and integrate both sides along the path γ_{ε,s_0}. We substitute $\lambda = s(1-s)$ and use $d\lambda = (1-2s)ds$. The path of the integration changes to

$$\tilde{\gamma}_{\varepsilon,s_0}(t) = s_0(1-s_0) + (1-2s_0)\varepsilon e^{2\pi i t} + \varepsilon^2 e^{4\pi i t}.$$

For $s_0 \neq 1/2$ and ε small enough, $\tilde{\gamma}_{\varepsilon,s_0}(t)$ winds around $s_0(1-s_0)$ once. Applying the Cauchy integration formula, we obtain

$$A_j(s_0) = \frac{1}{2\pi i} \int_{\gamma_{\varepsilon,s_0}} (1-2s)(s(1-s) - s_0(1-s_0))^{j-1} R_{X,\chi}(s)\, ds \tag{43}$$

for any $j = 1, \ldots, p$. It is also convenient to define

$$A_{p+1}(s_0) \coloneqq 0.$$

We note that for $j = 1$, the equality (43) was obtained in the proof of [GZ97, Lemma 2.4].

Note that (43) implies that $A_j(s_0)^* = A_j(\overline{s_0})$. The proof of [DFP, Theorem A] implies that

$$R(s) : \rho^N L^2(X, \chi) \mapsto \rho^{-N} L^2(X, \chi).$$

Together with $m_{X,\chi}(s_0) = \operatorname{rank} A_1(s_0)$, this yields

$$A_1(s_0) = \sum_{\ell,m=1}^{m_{X,\chi}(s_0)} a_1^{\ell,m}(s_0) \phi_\ell \langle \phi_m, \cdot \rangle, \tag{44}$$

where $a_1(s_0) = (a_1^{\ell,m}(s_0))_{\ell,m=1}^{m_{X,\chi}(s_0)}$ is a symmetric invertible matrix and

$$\phi_j \in \rho^{-N} L^2(X, E_\chi)$$

for $j = 1, \ldots, m_{X,\chi}(s_0)$ and $\operatorname{Re}(s_0) > 1/2 - N$ for any $N \in \mathbb{N}$. The definition of the resolvent implies that for any $j = 1, \ldots, p$,

$$\begin{aligned} A_{j+1}(s_0) &= A_j(s_0)(\Delta_{X,\chi} - s_0(1-s_0)) \\ &= (\Delta_{X,\chi} - s_0(1-s_0)) A_j(s_0). \end{aligned}$$

We define the matrix $d(s_0)$ in such a way that

$$\sum_{k=1}^{m_{X,\chi}(s_0)} d^{m,k}(s_0) \phi_k = (\Delta_{X,\chi} - s_0(1-s_0)) \phi_m.$$

Note that the matrix $d(s_0)$ is nilpotent with $d(s_0)^p = 0$. We obtain that

$$A_k(s_0) = \sum_{\ell,m=1}^{m_{X,\chi}(s_0)} a_k^{\ell,m}(s_0)\phi_\ell\langle\phi_m,\cdot\rangle, \tag{45}$$

where $a_k(s_0) = a_1(s_0)d(s_0)^{k-1}$.

4.2 Absence of poles with $\operatorname{Re} s = 1/2$**.** In this section, we will show that for $s \in \mathbb{C}$ with $\operatorname{Re} s = 1/2$ there is at most one resonance at $s = 1/2$. This will imply that there are no eigenvalues larger than $1/4$.

The Carleman estimate [Maz91, Theorem (7)] reads in our setting as follows (cf. Borthwick [Bor16, Lemma 7.6]).

Proposition 4.3. *Let $F_\ell \subset C_\ell = \langle h_\ell\rangle\backslash\mathbb{H}$ be a hyperbolic funnel and let $\chi\colon \langle h_\ell\rangle \to \mathrm{U}(V)$ be a unitary finite-dimensional representation. Denote by ρ_f the boundary defining function of $\partial_\infty F_\ell$. Let $r_0, k \geq 0$ and suppose that $u \in \mathcal{C}^\infty(\overline{F_\ell}, E_\chi)$ satisfies $u = O(\rho_f^\infty)$ and is supported in $\{ r \geq r_0\}$, where r denotes the distance to the geodesic boundary. For r_0 and k sufficiently large there exists $C > 0$ independent of k such that*

$$k^3 \int_{F_\ell} e^{2kr}|u|^2 d\mu_{F_\ell} + k\int_{F_\ell} e^{2kr}|\nabla_\chi u|^2\, d\mu_{F_\ell} \leq C\int_{F_\ell} e^{2kr}|\Delta_{F_\ell,\chi}u|^2\, d\mu_{F_\ell}. \tag{46}$$

The Carleman estimate implies the following result on unique continuations (see [Bor16, Proposition 7.4] for the untwisted case).

Proposition 4.4. *Let $X = \Gamma\backslash\mathbb{H}$ be an infinite-volume hyperbolic surface and $\chi\colon \Gamma \to \mathrm{U}(V)$ be a unitary finite-dimensional representation. Suppose that $u \in \mathcal{C}^\infty(X, E_\chi)$ is a solution of $(\Delta_{X,\chi} - s(1-s))u = 0$ for some $s \notin -\mathbb{N}_0/2$. If*

$$u|_{X_{f,j}} \in \rho_f^{s+1}\mathcal{C}^\infty(\overline{X_{f,j}}, E_\chi) \tag{47}$$

for some $j = 1, \ldots, n_f$, then $u \equiv 0$.

We adapt the proof of [Bor16, Proposition 7.4] to the twisted case.

Proof of Proposition 4.4. Without loss of generality, we assume that X has only one funnel end, that is $n_f = 1$ and $X_f = X_{f,1}$. We prove the proposition in two steps.

Step 1. We want to show by induction that

$$u|_{X_f} \in \rho_f^{s+n}\mathcal{C}^\infty(\overline{X_f}, E_\chi), \quad \forall n \in \mathbb{N}.$$

The base case is true by (47). Suppose that $u|_{X_f} \in \rho_f^{s+n}\mathcal{C}^\infty(\overline{X_f}, E_\chi)$ for some $n \in \mathbb{N}$. Write $u|_{X_f} = \rho_f^{s+n} v$, where $v \in \mathcal{C}^\infty(\overline{X_f}, E_\chi)$. Using (9), we obtain

$$(\Delta_{X,\chi} - s(1-s))\rho_f^{s+n} v = n(1-2s-n)\rho_f^{s+n} v + O(\rho_f^{s+n+1}).$$

Since u solves $(\Delta_{X,\chi} - s(1-s))u = 0$, it follows that $v = O(\rho_f)$ under the assumption that $s \notin -\mathbb{N}_0/2$. Therefore,

$$u|_{X_f} \in \rho_f^{s+n+1}\mathcal{C}^\infty(\overline{X_f}, E_\chi).$$

By induction, we obtain that $u|_{X_f} \in \rho_f^{\infty}\mathcal{C}^\infty(\overline{X_f}, E_\chi)$.

Step 2. We want to show that $u \equiv 0$. Choose $r_0, r_1 \in (0,1)$ with $r_1 > r_0$ and choose $\eta \in \mathcal{C}^\infty([0,1])$ such that $\eta(r) = 1$ for $r \le r_0$ and $\eta(r) = 0$ for $r \ge r_1$. The function $\eta(\rho_f)u|_{X_f}$ satisfies the assumptions of Proposition 4.3. We note that the second summand in the left-hand side of (46) is positive, that implies

$$k^3 \int_{X_f} \rho_f^{-2k} |\eta(\rho_f)|^2 |u|^2 \, d\mu_X \le C \int_{X_f} \rho_f^{-2k} |\,\Delta_{X,\chi}\eta(\rho_f)u|^2 \, d\mu_X$$

for $k > 0$ large enough. Denote

$$I_1 \coloneqq (1 + |s(1-s)|^2) \int_{X_f \cap \{\rho_f \le r_0\}} \rho_f^{-2k} |u|^2 \, d\mu_X.$$

Using the equation $(\Delta_{X,\chi} - s(1-s))u = 0$ and the fact that $\eta(r) = 1$ for $r \le r_0$, we obtain

$$\begin{aligned}
\frac{I_1 \cdot k^3}{1 + |s(1-s)|^2} &= k^3 \int_{X_f \cap \{\rho_f \le r_0\}} \rho_f^{-2k} |u|^2 \, d\mu_X \\
&\le k^3 \int_{X_f} \rho_f^{-2k} |\eta(\rho_f)|^2 |u|^2 \, d\mu_X \\
&\le C \int_{X_f} \rho_f^{-2k} |\,\Delta_{X,\chi}\eta(\rho_f)u|^2 \, d\mu_X \\
&= C \int_{X_f \cap \{r_0 \le \rho_f \le r_1\}} \rho_f^{-2k} |\,\Delta_{X,\chi}\eta(\rho_f)u|^2 \, d\mu_X \\
&\quad + C \int_{X_f \cap \{\rho_f \le r_0\}} \rho_f^{-2k} |\,\Delta_{X,\chi}\eta(\rho_f)u|^2 \, d\mu_X \\
&\le C(I_2 + I_3 + I_1),
\end{aligned}$$

where $C = C(r_0, r_1, \eta) > 0$ and

$$\begin{aligned}
I_2 &\coloneqq (1 + |s(1-s)|^2) \int_{X_f \cap \{r_0 \le \rho_f \le r_1\}} \rho_f^{-2k} |u|^2 \, d\mu_X, \\
I_3 &\coloneqq \int_{X_f \cap \{r_0 \le \rho_f \le r_1\}} \rho_f^{-2k} |\nabla_{X,\chi} u|^2 \, d\mu_X.
\end{aligned}$$

Setting $C' = (1 + |s(1-s)|^2)^{-1}C^{-1}$, we rewrite the above estimate as

$$I_1 \le (C'k^3 - 1)^{-1}(I_2 + I_3).$$

Using Step 1 and (7), we estimate I_2 and I_3 by

$$\begin{aligned} I_2 + I_3 &\le C'' \int_{r_0}^{r_1} \rho^{-2k-1}\, d\rho \\ &= \frac{C''}{2kr_0^{2k}}\Big(1 - \Big(\frac{r_1}{r_0}\Big)^{-2k}\Big), \end{aligned}$$

for some $C'' > 0$, which depends on r_0, r_1, s, and u, but is independent of k. Note that this estimate is far from optimal, but it suffices for our purposes. Therefore we arrive at

$$\begin{aligned} \int_{X_f \cap \{\rho_f \le r_0\}} |u|^2\, d\mu_X &\le \frac{r_0^{2k}}{1 + |s(1-s)|^2} I_1 \\ &\le \frac{C''(1 - (\frac{r_1}{r_0})^{-2k})}{2k(1 + |s(1-s)|^2)(C'k^3 - 1)}. \end{aligned}$$

Letting $k \to \infty$, we obtain that

$$\|u\|_{L^2(X_f \cap \{\rho_f \le r_0\}, E_\chi)} = 0$$

and consequently $u = 0$ on $X_f \cap \{\rho_f \le r_0\}$. By standard uniqueness results of elliptic differential operators, we conclude that $u = 0$ everywhere. □

In the case $\operatorname{Re} s = 1/2$ and $s \ne 1/2$, we can prove a better result following [Bor16, Lemma 7.7].

Proposition 4.5. *Let X and χ be as above, and let* $\operatorname{Re} s = 1/2$ *with* $s \ne 1/2$. *If $u \in \mathcal{C}^\infty(X, E_\chi)$ satisfies $u|_{X_{f,j}} \in \rho_f^s \mathcal{C}^\infty(\overline{X_{f,j}}, E_\chi)$ for some $j \in \{1, \dots, n_f\}$ and*

$$(\Delta_{X,\chi} - s(1-s))u = 0,$$

then $u \equiv 0$.

Proof. Without loss of generality, we may suppose that $n_f = 1$ and $X_f = X_{f,1}$. We take local coordinates $(\rho, \phi) \in \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z} \cong X_f$, where ρ is the boundary defining function. We have that

$$u(\rho, \phi + 2\pi) = \chi(h_\ell)u(\rho, \phi),$$

where $h_\ell \in \Gamma$ is the unique (up to inversion) hyperbolic element associated to the funnel end X_f and $\ell \in (0, \infty)$ is the length of the central geodesic of X_f (see [DFP, Section 3.2.3] for details).

Let $\varepsilon > 0$ and let $\psi \in \mathcal{C}^\infty(\mathbb{R}_+)$ be real-valued with $\psi(t) = 0$ for $t \leq 1$ and $\psi(t) = 1$ for $t \geq 2$. Set $\psi_\varepsilon \in \mathcal{C}^\infty(\overline{X})$ with $\psi_\varepsilon(\rho, \theta) = \psi(\rho/\varepsilon)$ for $(\rho, \theta) \in X_f$ and $\psi_\varepsilon = 1$ on $X \setminus X_f$. Since $\operatorname{Re} s = 1/2$, we have that $s(1-s) \in \mathbb{R}$ and thus

$$\begin{aligned} 0 &= \int_X (\overline{s(1-s)}(\psi_\varepsilon u, u)_{E_\chi} - s(1-s)(u, \psi_\varepsilon u)_{E_\chi})\, d\mu_X \\ &= \int_X ([\Delta_{X,\chi}, \psi_\varepsilon \cdot \mathrm{id}_{E_\chi}]u, u)_{E_\chi}\, d\mu_X \\ &= \int_{X_f} ([\Delta_{X,\chi}, \psi_\varepsilon \cdot \mathrm{id}_{E_\chi}]u, u)_{E_\chi}\, d\mu_X, \end{aligned}$$

where $[\cdot, \cdot]$ denotes the commutator. The function u can be written as $u = \rho^s v$, where $v \in \mathcal{C}^\infty(\overline{X_f}, E_\chi)$. By assumption, we have that $\operatorname{Re} s = 1/2$, therefore $|u|^2 = \rho|v|^2$ and $(\rho\partial_\rho u, u)_{E_\chi} = s\rho|v|^2 + O(\rho^2)$. Writing $x = \rho/\varepsilon$, we obtain, using (9), that

$$\begin{aligned} ([\Delta_{X,\chi}, \psi_\varepsilon \cdot \mathrm{id}_{E_\chi}]u, u)_{E_\chi} &= -\left(\rho^2\partial_\rho^2\psi_\varepsilon\left(\frac{\rho}{\varepsilon}\right)u + 2\rho\partial_\rho\psi_\varepsilon\left(\frac{\rho}{\varepsilon}\right)\rho\partial_\rho u, u\right)_{E_\chi} + O(\varepsilon^2) \\ &= -\varepsilon x^2(x\psi''(x) + 2s\psi'(x))|v(0,\phi)|^2 + O(\varepsilon^2). \end{aligned}$$

as $\varepsilon \to 0$. It follows from (6) that the measure $d\mu_X$ restricted to X_f is given by

$$\begin{aligned} d\mu_X|_{X_f} &= \rho^{-2}\frac{\ell}{2\pi}\frac{d\rho\, d\phi}{\sqrt{1-\rho^2}} \\ &= \varepsilon^{-1}x^{-2}\frac{\ell}{2\pi}\frac{dx\, d\phi}{\sqrt{1-\varepsilon^2x^2}}. \end{aligned}$$

Therefore we have, as $\varepsilon \to 0$, that

$$\begin{aligned} &\int_{X_f} ([\Delta_{X,\chi}, \psi_\varepsilon \cdot \mathrm{id}_{E_\chi}]u, u)_{E_\chi}\, d\mu_X \\ &\qquad = -\frac{\ell}{2\pi}\int_1^2\int_0^{2\pi}(x\psi''(x) + 2s\psi'(x))|v(0,\phi)|^2\, dx\, d\phi + O(\varepsilon). \end{aligned}$$

We calculate that $\int_1^2 \psi'(x) = 1$ and $\int_1^2 x\psi''(x)\, dx = -1$ and therefore

$$\int_{X_f} ([\Delta_{X,\chi}, \psi_\varepsilon \cdot \mathrm{id}_{E_\chi}]u, u)_{E_\chi}\, d\mu_X = (1-2s)\frac{\ell}{2\pi}\int_0^{2\pi}|v(0,\phi)|^2\, d\phi + O(\varepsilon)$$

as $\varepsilon \to 0$. Combining this with the calculation of the commutator above yields $v|_{X_f} = 0$ and by Taylor expansion we obtain that $u|_{X_f} \in \rho_f^{s+1}\mathcal{C}^\infty(\overline{X_f}, E_\chi)$. Together with Proposition 4.4 this implies the claim. □

Proposition 4.5 implies that there are no resonances on the critical line with the exception of $s = 1/2$.

Corollary 4.6. *For* $\operatorname{Re} s = 1/2$ *and* $s \neq 1/2$*, the resolvent* $R_{X,\chi}$ *has no pole at* s*.*

Proof. By (42), we have that

$$R_{X,\chi}(s) = \sum_{j=1}^{p} \frac{A_j(s_0)}{(s(1-s) - s_0(1-s_0))^j} + H(s, s_0),$$

where $p \in \mathbb{N}$ is the order of the resonance, $A_j(s_0)$, $j = 1, \dots, p$ are finite rank operators and $H(s, s_0)$ is holomorphic in s near $s = s_0$. Let $\psi \in \mathcal{C}_c^\infty(X, E_\chi)$ and write $u = A_p(s_0)\psi$. By the definition of the resolvent, we have that

$$(\Delta_{X,\chi} - s_0(1-s_0))u = 0.$$

By Theorem 4.1, we have that $u \in \rho_f^{s_0} \rho_c^{s_0 - 1} \mathcal{C}^\infty(\overline{X}, E_\chi)$. For $\operatorname{Re} s_0 = 1/2$ and $s \neq 1/2$, Proposition 4.5 implies that $u = 0$ and consequently $A_p = 0$. This shows that $R_{X,\chi}(s)$ is holomorphic near s_0. □

Proposition 4.7. *The Laplacian* $\Delta_{X,\chi}$ *has no eigenvalues in the interval* $[1/4, \infty)$.

Proof. Let $\lambda \in [1/4, \infty)$ and set $s := 1/2 + i\sqrt{\lambda - 1/4}$. This implies that $\lambda = s(1-s)$. Assume that λ is an eigenvalue of $\Delta_{X,\chi}$, then there exists a function $u \in L^2(X, E_\chi)$ such that

$$(\Delta_{X,\chi} - s(1-s))u = 0.$$

Since X has infinite volume, there is at least one funnel end, which we will denote by X_f. We choose coordinates $(r, \phi) \in X_f$ as in Section 3.1. Choose $\psi \in \mathcal{C}_c^\infty(X_f, E_\chi|_{X_f})$ such that $\operatorname{supp} \psi \subset \{r \geq 2\}$. Then we have by (36) that

$$(\Delta_{X,\chi} - s(1-s))M_f(s)\psi = \psi - L_f(s)\psi.$$

Let $\varepsilon > 0$. Symmetry of $\Delta_{X,\chi}$ implies we have that

$$\begin{aligned}
\varepsilon(2s - 1 + \varepsilon) &\int_{X_f} \langle M_f(s+\varepsilon)\psi, u\rangle_{E_\chi} \, d\mu_X \\
&= \int_{X_f} \langle (\Delta_{X,\chi} - (s+\varepsilon)(1-s-\varepsilon))M_f(s+\varepsilon)\psi, u\rangle_{E_\chi} \, d\mu_X \\
&= \int_{X_f} \langle \psi - L_f(s+\varepsilon)\psi, u\rangle_{E_\chi} \, d\mu_X.
\end{aligned}$$

By the Cauchy–Schwarz inequality, we have that

$$\left| \int_{X_f} \langle M_f(s+\varepsilon)\psi, u\rangle_{E_\chi}\, d\mu_X \right| \leq \|M_f(s+\varepsilon)\psi\|\,\|u\|$$

and using (11) and (31), we obtain that

$$\rho^{-s-\varepsilon} M_f(s+\varepsilon)\psi \in \mathcal{C}^\infty(\overline{X_f}, E_\chi).$$

Therefore, we can estimate

$$\|M_f(s+\varepsilon)\psi\| \leq \sup_{X_f} |\rho^{-s-\varepsilon} M_f(s+\varepsilon)\psi|\,\|\rho^{s+\varepsilon}\|,$$

where the first factor in the right-hand side is bounded by a constant and the second factor is $O(\varepsilon^{-1/2})$ by a direct calculation. This implies that

$$\int_{X_f} \langle \psi - L_f(s+\varepsilon)\psi, u\rangle_{E_\chi}\, d\mu_X = O(\varepsilon^{1/2}) \quad \text{as } \varepsilon \to 0.$$

By the fundamental lemma of calculus of variations, this implies that

$$u(z) = (L_f(s)^T u)(z)$$

for $z \in X_f \cap \{r \geq 2\}$. By the definition of $L_f(s)$, (34), we have that $u|_{X_f} \in \rho_f^s \mathcal{C}^\infty(\overline{X_f}, E_\chi)$. Set $u_0(s) := \rho_f^{-s} u|_{\partial_\infty X_f}$. Since $u \in L^2(X, E_\chi)$ and $\operatorname{Re} s = 1/2$, it follows that $u_0(s) \equiv 0$ and therefore

$$u|_{X_f} \in \rho_f^{s+1} \mathcal{C}^\infty(\overline{X_f}, E_\chi).$$

Proposition 4.4 now finishes the proof. □

5 Scattering determinant

In this section, we prove Theorems A and B. We start with introducing the Poisson operator and studying its properties in Section 5.1. In Section 5.2, we define the scattering matrix and show the correspondence of resonances and poles of the scattering matrix for $\operatorname{Re} s < 1$ and $s \neq 1/2$. In Section 5.3, we study the behavior of $R_{X,\chi}(s)$ near $s = 1/2$ and prove Theorem B. In Section 5.4, we recall the basics of the Gohberg-Sigal theory and obtain a relation of scattering poles and resonances for $\operatorname{Re}(s) \leq 1$. In Section 5.5, we introduce the relative scattering matrix and the relative scattering determinant and, finally, prove Theorem A.

5.1 Poisson operator. Before we define the scattering matrix, we introduce the Poisson operator, which maps sections $\mathcal{C}^\infty(\partial_\infty X, E_\chi|_{\partial_\infty X})$ to solutions of the equation $(\Delta_{X,\chi} - s(1-s))u = 0$ with prescribed asymptotics at the boundary at infinity. We emphasize that these solutions do not have to be eigenfunctions of $\Delta_{X,\chi}$. The construction is similar to the one in the untwisted case [GZ97, (2.23)–(2.25)], but in our case the Poisson operator acts on sections of vector bundles and we have to be more careful due to the compactification in the cusp, which depends on the representation χ.

We recall that the ideal boundary at infinity $\partial_\infty X$ is a disjoint union of circles (representing funnel ends) and points (representing cusp ends) and that we have the decomposition

$$\partial_\infty X = \partial_f X \sqcup \partial_c X. \tag{48}$$

For $j \in \{1, \dots, n_f\}$ and $s \notin \mathcal{R}_{X,\chi}$ we define the map

$$E^{f,j}_{X,\chi}(s) \colon \mathcal{C}^\infty(\partial_\infty X_{f,j}, E_\chi|_{\partial_\infty X_{f,j}}) \to \mathcal{C}^\infty(X, E_\chi)$$

by its Schwartz kernel

$$E^{f,j}_{X,\chi}(s, z, \theta') := (\rho_f')^{-s} R_{X,\chi}(s; z, z')\Big|_{X\times\partial_\infty X_{f,j}}.$$

The restriction is well-defined by Theorem 4.1 and (11). In a similar way, for $j \in \{1, \dots, n_c\}$ and $s \notin \mathcal{R}_{X,\chi}$ we define

$$E^{c,j}_{X,\chi}(s) : \mathcal{C}^\infty(\partial_\infty X_{c,j}, E_\chi|_{\partial_\infty X_{c,j}}) \to \mathcal{C}^\infty(X, E_\chi),$$
$$E^{c,j}_{X,\chi}(s, z, \theta') := (\rho_c')^{1-s} R_{X,\chi}(s; z, z')\Big|_{X\times\partial_\infty X_{c,j}}.$$

The restriction is well-defined by Theorem 4.1 and (27). Further, by (40), (35), and (32), the map $E^{c,j}_{X,\chi}(s, z, \theta')$ is independent of θ' and defines an operator $\mathcal{C}^\infty(\partial_\infty X_{c,j}, E_\chi|_{\partial_\infty X_{c,j}}) \to \mathcal{C}^\infty(X, E_\chi)$. We denote this two-variable function by $E^{c,j}_{X,\chi}$ as well. We obtain the *Poisson operator* defined by its Schwartz kernel as follows:

$$E_{X,\chi}(s) : \mathcal{C}^\infty(\partial_\infty X, E_\chi|_{\partial_\infty X}) \to \mathcal{C}^\infty(X, E_\chi),$$
$$(E_{X,\chi}(s)\psi)(z) := \sum_{j=1}^{n_f} \frac{\ell_j}{2\pi} \int_0^{2\pi} E^{f,j}_{X,\chi}(s, z, \theta') f_j(\theta') d\theta' + \sum_{j=1}^{n_c} E^{c,j}_{X,\chi}(s, z) a_j,$$

where $\psi = (f_1, \dots, f_{n_f}, a_1, \dots, a_{n_c}) \in \mathcal{C}^\infty(\partial_\infty X, E_\chi|_{\partial_\infty X})$.

We define the transpose of $E_{X,\chi}(s)$ as follows. Let $u \in \mathcal{C}^\infty_c(X, E_\chi)$ and set

$$f_j(\theta) = \int_X E^{f,j}_{X,\chi}(s, z', \theta) u(z') d\mu_X(z'),$$
$$a_j = \int_X E^{c,j}_{X,\chi}(s, z') u(z') d\mu_X(z').$$

The operator

$$E_{X,\chi}(s)^T\colon \mathcal{C}_c^\infty(X, E_\chi) \to \mathcal{C}^\infty(\partial_\infty X, E_\chi|_{\partial_\infty X})$$

is defined by

$$E_{X,\chi}(s)^T u = (f_1, \dots, f_{n_f}, a_1, \dots, a_{n_c}).$$

The name is justified by the fact that for $u \in \mathcal{C}_c^\infty(X, E_\chi)$ and $\psi \in \mathcal{C}^\infty(\partial X, E_\chi|_{\partial X})$, we have that

$$\langle E_{X,\chi}(s)^T u, \psi\rangle_{L^2(\partial X, E_\chi|_{\partial X})} = \langle u, E_{X,\chi}(s)\psi\rangle_{L^2(X,E_\chi)},$$

where we recall that $\langle\cdot,\cdot\rangle$ is the bilinear inner product.

By the same arguments as in the proof of [DFP, Lemma 4.14], we can express the difference of resolvents in terms of the Poisson operator for general hyperbolic surfaces.

Proposition 5.1. *Let $X = \Gamma\backslash\mathbb{H}$ be a geometrically finite hyperbolic surface and*

$$\chi\colon \Gamma \to U(V)$$

a finite-dimensional unitary representation. For $s \notin \mathcal{R}_{X,\chi} \cup (1 - \mathcal{R}_{X,\chi})$, we have

$$R_{X,\chi}(s) - R_{X,\chi}(1-s) = (1-2s)E_{X,\chi}(s)E_{X,\chi}(1-s)^T.$$

Proof. We follow the proof of [DFP, Lemma 4.14], but we have to take care of the multiple ends.

We fix a fundamental domain $\mathcal{F} \subset \mathbb{H}$ of X. Then the bundle $E_\chi \boxtimes E'_\chi$ is trivial and can be identified with $\mathcal{F} \times \mathcal{F} \times \operatorname{End}(V)$. We fix $z, w \in \mathcal{F}$ and choose a basis $(e_k)_{k=1}^{\dim V}$ of V. We define the coefficients of $R_{X,\chi}(s; z, w)$ as

$$R_{jk}(s; z, w) \coloneqq \langle R_{X,\chi}(s; z, w)e_j, e_k\rangle_V.$$

We also set

$$R_{jk}^T(s; z, w) \coloneqq \langle R_{X,\chi}^T(s; z, w)e_j, e_k\rangle_V,$$

where $R_{X,\chi}^T(s; z, w)$ denotes the Schwartz kernel of the operator $R_{X,\chi}(s)^T$.

Let $X_\varepsilon \coloneqq \{z \in X\colon \rho(z) = \varepsilon\}$ and let $d\sigma_{X_\varepsilon}$ be the induced measure on X_ε. Denote by ∂_ν the outward pointing unit normal vector of X_ε. Formally, we have that

$$\begin{aligned}
R_{X,\chi}(s) - R_{X,\chi}(1-s) &= R_{X,\chi}(s)(\Delta_{X,\chi} - (1-s)s)R_{X,\chi}(1-s) \\
&\quad - (\Delta_{X,\chi} - s(1-s))R_{X,\chi}(s)R_{X,\chi}(1-s) \\
&= R_{X,\chi}(s)\Delta_{X,\chi}R_{X,\chi}(1-s) \\
&\quad - \Delta_{X,\chi}R_{X,\chi}(s)R_{X,\chi}(1-s).
\end{aligned}$$

We can make this calculation precise by using the Schwartz kernel. Together with the Green's formula, this implies that

$$\begin{aligned}
&R_{jk}(s;z,w) - R_{jk}(1-s;z,w)\\
&\qquad = \lim_{\varepsilon\to 0}\int_{\rho(z')>\varepsilon}\sum_{m=1}^{\dim V}(R_{jm}(s;z,z')\Delta_{X,\chi}R_{mk}(1-s;z',w)\\
&\qquad - \Delta_{X,\chi}R_{jm}(s;z,z')R_{mk}(1-s;z',w))d\mu_X(z')\\
&\qquad = \lim_{\varepsilon\to 0}\int_{\rho(z')>\varepsilon}\sum_{m=1}^{\dim V}(R_{jm}(s;z,z')\Delta_{X,\chi}R^T_{km}(1-s;w,z')\\
&\qquad - \Delta_{X,\chi}R_{jm}(s;z,z')R^T_{km}(1-s;w,z'))d\mu_X(z')\\
&\qquad = \lim_{\varepsilon\to 0}\int_{X_\varepsilon}\sum_{m=1}^{\dim V}(-R_{jm}(s;z,z')\partial_\nu R^T_{km}(1-s;w,z')\\
&\qquad + \partial_\nu R_{jm}(s;z,z')R^T_{km}(1-s;w,z'))d\sigma_{X_\varepsilon}(z'),
\end{aligned}$$

where ∂_ν and $\Delta_{X,\chi}$ act on the primed variables. If we pick $\varepsilon > 0$ sufficiently small, then the area of integration splits into a disjoint union of funnel and cusp ends. Without loss of generality, we suppose that $X_f = X_{f,j}$ and we set $X_{f,\varepsilon} := X_f \cap X_\varepsilon$. We take coordinates $(\rho,\theta) \in \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z} \cong X_f$ as in the proof of Proposition 4.5. From (6), we see that $\partial_\nu = -\rho\partial_\rho + O(\rho^2)$. For $z \in X$ with $\rho(z) > \varepsilon$ and $z' \in X_{f,\varepsilon}$, we have that

$$R_{X,\chi}(s;z,z') = (\rho')^s E_{X,\chi}(s;z,\phi') + O((\rho')^{s+1})$$

and

$$\begin{aligned}
\partial_\nu R_{X,\chi}(s;z,z') &= -\rho'\partial_{\rho'}R_{X,\chi}(s;z,z')\\
&= -s(\rho')^s E_{X,\chi}(s;z,\phi') + O((\rho')^{s+1}).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&-R_{jm}(s;z,z')\partial_\nu R^T_{km}(1-s;w,z')\\
&\qquad = (1-s)\varepsilon E_{jm}(s;z,\phi')E^T_{km}(1-s;w,\phi') + O(\varepsilon^2)
\end{aligned}$$

and

$$\begin{aligned}
&\partial_\nu R_{jm}(s;z,z')R^T_{km}(1-s;w,z')\\
&\qquad = -s\varepsilon E_{jm}(s;z,\phi')E^T_{km}(1-s;w,\phi') + O(\varepsilon^2).
\end{aligned}$$

Moreover, $d\sigma_{X_\varepsilon}|_{X_{f,\varepsilon}} = (2\pi\varepsilon)^{-1}\ell\, d\phi = \varepsilon^{-1}d\sigma_{\partial_\infty X_f}$, where $\ell \in (0,\infty)$ is the length of the central geodesic associated to X_f. Therefore, we obtain that

$$\begin{aligned}\int_{X_{f,\varepsilon}} &(- R_{jm}(s;z,z')\partial_\nu R^T_{km}(1-s;w,z') \\ &+ \partial_\nu R_{jm}(s;z,z')R^T_{km}(1-s;w,z'))d\sigma_{X_\varepsilon}(z') \\ &= (1-2s)\int_{\partial_\infty X_f}(E_{jm}(s;z,\phi')E^T_{km}(1-s;w,\phi') + O(\varepsilon))d\sigma_{\partial_\infty X_f}(\phi').\end{aligned}$$

Letting $\varepsilon \to 0$ proves the claim for the funnel ends.

For the cusp ends, we also suppose without loss of generality that $n_c = 1$ and $X_c = X_{c,1}$ is a single cusp end. We set $X_{c,\varepsilon} := X_c \cap X_\varepsilon$. We take coordinates $(\rho, \phi) \in \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z} \cong X_c$ as in Section 3.2 and calculate $g_{X_c}(\partial_\rho, \partial_\rho) = \rho^{-2}$ and therefore $\partial_\nu = -\rho'\partial_{\rho'}$. By the definition of the Poisson operator, we have for $z \in X$ with $\rho(z) > \varepsilon$ and $z' \in X_{c,\varepsilon}$ and as $\varepsilon \to 0$ (hence $\rho' \to 0$),

$$R_{X,\chi}(s;z,z') = (\rho')^{s-1}E_{X,\chi}(s;z,\phi') + O((\rho')^s)$$

and

$$\begin{aligned}\partial_\nu R_{X,\chi}(s;z,z') &= -\rho'\partial_{\rho'}R_{X,\chi}(s;z,z') \\ &= (1-s)(\rho')^{s-1}E_{X,\chi}(s;z,\phi') + O((\rho')^s).\end{aligned}$$

Therefore,

$$\begin{aligned}R_{jm}(s;z,z')&\partial_\nu R^T_{km}(1-s;w,z') \\ &= (\rho')^{s-1}E_{jm}(s;z,\phi)s(\rho')^{-s}E^T_{km}(1-s;w,\phi') \\ &\quad + O((\rho')^0)\end{aligned}$$

and

$$\begin{aligned}\partial_\nu R_{jm}&(s;z,z')R^T_{km}(1-s;w,z') \\ &= (1-s)(\rho')^{s-1}E_{jm}(s;z,\phi)(\rho')^{-s}E^T_{km}(1-s;w,\phi') + O((\rho')^0).\end{aligned}$$

By (26), we have that $d\sigma_{X_\varepsilon}|_{X_{c,\varepsilon}} = \varepsilon\frac{d\phi}{2\pi}$. Using that $E_{X,\chi}(s;z,\phi')$ is independent of ϕ', we arrive at

$$\begin{aligned}\int_{X_{c,\varepsilon}} &(- R_{jm}(s;z,z')\partial_\nu R^T_{km}(1-s;w,z') \\ &+ \partial_\nu R_{jm}(s;z,z')R^T_{km}(1-s;w,z'))d\sigma_{X_\varepsilon}(z') \\ &= (1-2s)\frac{1}{2\pi}\int_0^{2\pi} E_{jm}(s;z)E^T_{km}(1-s;w)\,d\phi' + O(\varepsilon) \\ &= (1-2s)E_{jm}(s;z)E^T_{km}(1-s;w) + O(\varepsilon).\end{aligned}$$

Taking $\varepsilon \to 0$ yields the result. □

The Poisson operator $E_{X,\chi}(s)$ provides generalized eigenfunctions in the following sense.

Proposition 5.2. *Let $s \notin \mathcal{R}_{X,\chi}$. For any $\psi \in \mathcal{C}^\infty(\partial_\infty X, E_\chi|_{\partial_\infty X})$, we have*

$$(\Delta_{X,\chi} - s(1-s))E_{X,\chi}(s)\psi = 0 \tag{49}$$

and

$$E_{X,\chi}(s)\psi \in \rho_f^{1-s}\rho_c^{-s}\mathcal{C}^\infty(\overline{X}, E_\chi) + \rho_f^s\rho_c^{s-1}\mathcal{C}^\infty(\overline{X}, E_\chi).$$

If $s \notin \mathbb{Z}/2$, then we have the asymptotics

$$(2s-1)E_{X,\chi}(s)\psi \sim \rho_f^{1-s}\rho_c^{-s}\psi + \rho_f^s\rho_c^{s-1}\phi_s, \tag{50}$$

where $\phi_s \in \mathcal{C}^\infty(\partial_\infty X, E_\chi|_{\partial_\infty X})$ depends meromorphically on $s \in \mathbb{C}$.

Proof. It is straightforward to see that $E_{X,\chi}(s)\psi$ solves the equation (49). To obtain (50), we use the result on the structure of the resolvent, Theorem 4.1. We have that

$$E_{X,\chi}^{f,j}(s; z, \theta') = \lim_{\rho'\to 0}(\rho')^{-s}(M_f(s; z, \rho', \theta') + Q(s; z, \rho', \theta'))$$

and by (31),

$$\lim_{\rho'\to 0}(\rho')^{-s}M_f(s; z, \rho', \theta') = (1-\eta_{f,0})E_{X_f,\chi}(s; z, \theta'),$$

where $E_{X_f,\chi}(s)$ is defined by (12). From the asymptotics of $Q(s)$, Theorem 4.1, we obtain

$$E_{X,\chi}^{f,j}f_j(z) - (1-\eta_{f,0})E_{X_f,\chi}(s)f_j \in \rho_f^s\rho_c^{s-1}\mathcal{C}^\infty(\overline{X}, E_\chi).$$

In the case of the model funnel, this result follows directly from the above equality and (18).

For the cusp ends we have to be more careful, because the compactification at the cusp of the bundle E_χ depends on the multiplicity of the eigenvalue 1 of $\chi(\gamma_j)$, where $\gamma_j \in \Gamma$ is a representative of the conjugacy class $[\gamma_j]$, associated to the cusp $X_{c,j}$. Let $a_j \in \mathcal{C}^\infty(\partial_c X_{c,j}, E_\chi|_{\partial_c X_{c,j}}) \cong \mathbb{C}^{n^\chi_{c,j}}$. We have that

$$E_{X,\chi}^{c,j}(s)a_j = \lim_{\rho'\to 0}(\rho')^{1-s}\frac{1}{2\pi}\int_0^{2\pi}(M_c(s; z, \rho', \theta') + Q(s; z, \rho', \theta'))a_j\, d\theta'.$$

By definition of M_c (see (32) and (28)), we have that

$$\lim_{\rho'\to 0}(\rho')^{1-s}\frac{1}{2\pi}\int_0^{2\pi} M_c(s; z, \rho', \theta')a_j\, d\theta' = (1-\eta_{c,0})\frac{\rho^{-s}}{2s-1}\, a_j.$$

For Q, we cannot directly use the smoothness of the Schwartz kernel, since due to the compactification this does not imply that

$$\lim_{\rho'\to 0}(\rho')^{1-s}\frac{1}{2\pi}\int_0^{2\pi} Q(s;z,\rho',\theta')a_j\,d\theta'$$

is smooth. Instead, we use that the limit does not change if we insert $1-\eta_{c,3}(\rho')$ and then we can apply (40) with $\varphi(\rho',\theta') = (1-\eta_{c,3}(\rho'))a_j$ to obtain that

$$(\rho')^{1-s}\frac{1}{2\pi}\int_0^{2\pi} Q(s;z,\rho',\theta')a_j = M(s)\eta_3(\mathrm{id}-L(s)\eta_3)^{-1}\psi,$$

where

$$\psi(\rho,\theta) = \lim_{\rho'\to 0}(\rho')^{1-s}\frac{1}{2\pi}\int_0^{2\pi} L_c(s;\rho,\theta,\rho',\theta')\varphi(\rho',\theta')\,d\theta'$$

is a compactly supported smooth function by (35). The function $\eta_3(\mathrm{id}-L(s)\eta_3)^{-1}\psi$ is smooth and compactly supported as well and we deduce from the integral kernel of $M(s)$ that

$$\lim_{\rho'\to 0}(\rho')^{1-s}\frac{1}{2\pi}\int_0^{2\pi} Q(s;z,\rho',\theta')a_j\,d\theta' \in \rho_f^s\rho_c^{s-1}\mathcal{C}^\infty(\overline{X},E_\chi),$$

which completes the proof. □

5.2 Scattering matrix. The scattering matrix interchanges the asymptotics of solutions of the equation $(\Delta_{X,\chi}-s(1-s))u=0$ as described in Proposition 5.2.

Definition 5.3. For $s\notin\mathcal{R}_{X,\chi}\cup\mathbb{Z}/2$, the **scattering matrix** is given by

$$S_{X,\chi}(s)\colon\ \mathcal{C}^\infty(\partial_\infty X, E_\chi|_{\partial_\infty X}) \to \mathcal{C}^\infty(\partial_\infty X, E_\chi|_{\partial_\infty X}),$$
$$S_{X,\chi}(s)\colon\ \psi\mapsto\phi_s,$$

where ϕ_s is defined by (50).

We observe that

$$S_{X,\chi}(s)^* = S_{X,\chi}(\overline{s}),\qquad S_{X,\chi}(s)^T = S_{X,\chi}(s),\tag{51}$$

where $S_{X,\chi}(s)^*$ is the adjoint of $S_{X,\chi}(s)$ with respect to the complex inner product on $L^2(\partial_\infty X, E_\chi|_{\partial_\infty X})$ and $S_{X,\chi}(s)^T$ is the transposed operator, that is the adjoint with respect to the bilinear inner product $\langle\cdot,\cdot\rangle_{L^2(\partial_\infty X, E_\chi|_{\partial_\infty X})}$.

Proposition 5.4. *For any* $s \in \mathbb{C}$, $s \notin \mathcal{R}_{X,\chi} \cup (1 - \mathcal{R}_{X,\chi})$ *and any element* $\psi \in \mathcal{C}^\infty(\partial_\infty X, E_\chi|_{\partial_\infty X})$, *we have*

$$E_{X,\chi}(1-s)S_{X,\chi}(s)\psi = -E_{X,\chi}(s)\psi,$$
$$S_{X,\chi}(1-s)S_{X,\chi}(s)\psi = \psi. \tag{52}$$

Proof. It is sufficient to prove the statement for $\operatorname{Re} s \leq 1/2$, $s \neq 1/2$ and $s \notin \mathcal{R}_{X,\chi} \cup (1 - \mathcal{R}_{X,\chi})$. By Proposition 5.1,

$$R_{X,\chi}(s) - R_{X,\chi}(1-s) = (1-2s)E_{X,\chi}(s)E_{X,\chi}(1-s)^T.$$

Multiplying this equation from the left with $\rho_f^{-s}\rho_c^{1-s}$ and restricting to the boundary yields

$$E_{X,\chi}(s)^T - 0 = -S_{X,\chi}(s)E_{X,\chi}(1-s)^T.$$

Using that $S_{X,\chi}(s)^T = S_{X,\chi}(s)$, we obtain the first claim. In order to obtain (52), we calculate

$$\begin{aligned} E_{X,\chi}(s)S_{X,\chi}(1-s)S_{X,\chi}(s)\psi &= -E_{X,\chi}(1-s)S_{X,\chi}(s)\psi \\ &= E_{X,\chi}(s)\psi. \end{aligned}$$

By (50), $E_{X,\chi}(s)$ is injective. This proves the claim. □

Proposition 5.4 together with Proposition 5.1 implies that

$$R_{X,\chi}(s) - R_{X,\chi}(1-s) = (1-2s)E_{X,\chi}(1-s)S_{X,\chi}(s)E_{X,\chi}(1-s)^T. \tag{53}$$

It is convenient to use the identification

$$\mathcal{C}^\infty(\partial_c X, E_\chi|_{\partial_c X}) \cong \mathbb{C}^{n_c^\chi},$$

where $n_c^\chi = \sum_{j=1}^{n_c} n_{c,j}^\chi$. Using the decomposition into funnel and cusp ends, we can write the scattering matrix as

$$S_{X,\chi}(s) = \begin{pmatrix} S_{X,\chi}^{ff}(s) & S_{X,\chi}^{fc} \\ S_{X,\chi}^{cf}(s) & S_{X,\chi}^{cc} \end{pmatrix},$$

where

$$\begin{aligned} S_{X,\chi}^{ff}(s)&\colon \mathcal{C}^\infty(\partial_f X, E_\chi|_{\partial_f X}) \to \mathcal{C}^\infty(\partial_f X, E_\chi|_{\partial_f X}), \\ S_{X,\chi}^{cf}(s)&\colon \mathcal{C}^\infty(\partial_f X, E_\chi|_{\partial_f X}) \to \mathbb{C}^{n_c^\chi}, \\ S_{X,\chi}^{fc}(s)&\colon \mathbb{C}^{n_c^\chi} \to \mathcal{C}^\infty(\partial_f X, E_\chi|_{\partial_f X}), \\ S_{X,\chi}^{cc}(s)&\colon \mathbb{C}^{n_c^\chi} \to \mathbb{C}^{n_c^\chi}. \end{aligned}$$

For $\operatorname{Re} s < 1/2$, we have that

$$S_{X,\chi}(s) = (2s-1)\,(\rho_f\rho_f')^{-s}(\rho_c\rho_c')^{1-s}R_{X,\chi}(s;z,z')\Big|_{\partial_\infty X\times\partial_\infty X}. \tag{54}$$

For $j = 1,\dots,n_f$ let $S_{X_{f,j},\chi}(s)$ be the scattering matrix for the funnel end $X_{f,j}$ as described in Section 3.1. The scattering matrix for funnel ends $S_{X_f,\chi}(s)$ is diagonal with respect to the decomposition of the boundary $\partial_\infty X$ and given by

$$S_{X_f,\chi}(s) : \mathcal{C}^\infty(\partial_f X, E_\chi|_{\partial_f X}) \to \mathcal{C}^\infty(\partial_f X, E_\chi|_{\partial_f X}),$$
$$S_{X_f,\chi}(s) \coloneqq S_{X_{f,1},\chi}(s)\oplus\ldots\oplus S_{X_{f,n_f},\chi}(s).$$

As it was already in the case for the resolvent, the scattering matrix $S_{X,\chi}(s)$ is closely related to the scattering matrix for the funnel ends, $S_{X_f,\chi}(s)$.

Lemma 5.5. *Let $Q^\#(s)\colon \mathcal{C}^\infty(\partial_\infty X, E_\chi|_{\partial_\infty X}) \to \mathcal{C}^\infty(\partial_\infty X, E_\chi|_{\partial_\infty X})$ be given by the matrix representation*

$$Q^\#(s) = \begin{pmatrix} Q^\#(s)^{ff} & Q^\#(s)^{fc}(s) \\ Q^\#(s)^{cf} & Q^\#(s)^{cc}(s)\end{pmatrix}, \tag{55}$$

where

$$Q^\#(s)^{ff} = E^T_{X_f,\chi}(s)(\eta_3-\eta_{f,1})(\operatorname{id}-L(s)\eta_3)^{-1}[\Delta_{X,\chi},\eta_{f,0}]E_{X_f,\chi}(s),$$
$$Q^\#(s)^{fc} = E^T_{X_f,\chi}(s)(\eta_3-\eta_{f,1})(\operatorname{id}-L(s)\eta_3)^{-1}[\Delta_{X,\chi},\eta_{c,0}]E_{X_c,\chi}(s),$$
$$Q^\#(s)^{cf} = E^T_{X_c,\chi}(s)(\eta_3-\eta_{c,1})(\operatorname{id}-L(s)\eta_3)^{-1}[\Delta_{X,\chi},\eta_{f,0}]E_{X_f,\chi}(s),$$
$$Q^\#(s)^{cc} = E^T_{X_c,\chi}(s)(\eta_3-\eta_{c,1})(\operatorname{id}-L(s)\eta_3)^{-1}[\Delta_{X,\chi},\eta_{c,0}]E_{X_c,\chi}(s).$$

Then the integral kernel of $Q^\#(s)$ is given by

$$Q^\#(s;\omega,\omega') = \lim_{\rho\to 0,\rho'\to 0}(\rho_f\rho_f')^{-s}(\rho_c\rho_c')^{1-s}Q(s;\rho,\omega,\rho',\omega')$$

for $\operatorname{Re} s < 1/2$.

Proof. By (40) we have that

$$Q(s)\varphi = M(s)\eta_3(\operatorname{id}-L(s)\eta_3)^{-1}L(s)\varphi$$

for $\varphi\in\mathcal{C}^\infty(\overline{X},E_\chi)$ with $\eta_3\varphi = 0$. If $\psi\in\mathcal{C}^\infty(\overline{X},E_\chi)$ with $\eta_1\psi = 0$, then we can write

$$(Q(s)\varphi,\psi)_{L^2} = \left(\begin{pmatrix} Q^{ff}(s) & Q^{fc}(s)\\ Q^{cf}(s) & Q^{cc}(s)\end{pmatrix}\begin{pmatrix}\varphi|_{X_f}\\ \varphi|_{X_c}\end{pmatrix}, \begin{pmatrix}\psi|_{X_f}\\ \psi|_{X_c}\end{pmatrix}\right)_{L^2(X,E_\chi)}.$$

From the definition of $M(s)$ and $L(s)$, we see that, for instance,

$$Q^{ff}(s) = R_{X_f,\chi}(s)(\eta_3 - \eta_{f,1})(\mathrm{id} - L(s)\eta_3)^{-1}[\Delta_{X,\chi}, \eta_{f,0}]R_{X_f,\chi}(s).$$

Using that the integral kernel of $E_{X_f,\chi}(s)^T$ is given by

$$E_{X_f,\chi}(s)^T(\phi, r', \phi') = \lim_{r\to\infty} \rho_f(r)^{-s} R_{X_f,\chi}(s; r, \phi, r', \phi')$$

and the integral kernel of $E_{X_f,\chi}(s)$ is given by (12), we obtain that

$$Q^{\#}(s; \omega, \omega')^{ff} = \lim_{\rho\to 0, \rho'\to 0} (\rho_f \rho_f')^{-s} Q^{ff}(s; \rho, \omega, \rho', \omega'). \qquad \square$$

Proposition 5.6. *The two scattering matrices, $S_{X,\chi}(s)$ and $S_{X_f,\chi}(s)$, are related by*

$$S_{X,\chi}(s) = S_{X_f,\chi}(s) \oplus 0 + (2s-1)Q^{\#}(s), \tag{56}$$

where $0\colon \mathbb{C}^{n_c^\chi} \to \mathbb{C}^{n_c^\chi}$ is the zero-map and $Q^{\#}(s)$ is given by Lemma 5.5. In particular,

$$S^{ff}_{X,\chi}(s) \in \Psi^{2\operatorname{Re} s - 1}(\partial_f X, E_\chi|_{\partial_f X}), \quad s \notin \mathcal{R}_{X,\chi} \cup (\mathbb{N}_0 + 1/2).$$

Remark 5.7. The appearance of the map $0\colon \mathbb{C}^{n_c^\chi} \to \mathbb{C}^{n_c^\chi}$ in (56) is due to the fact that for $\operatorname{Re} s > 1/2$, we have that

$$\lim_{y\to\infty} \rho_c(y)^{1-s} E_{C_\infty,\chi}(s; y) = 0.$$

Proof of Proposition 5.6. For $\operatorname{Re} s < 1/2$, this follows directly from the characterization of the scattering matrix as a limit of the resolvent, (54), Theorem 4.1. For $\operatorname{Re} s \geq 1/2$ we use meromorphic continuation. Note that $Q^{\#}(s)^{ff}$ is smoothing and hence a pseudodifferential operator of order $-\infty$. The second part then follows from $S^{ff}_{X,\chi}(s) = S_{X_f,\chi}(s) + Q^{\#}(s)^{ff}$ and (20). $\square$

As in the case of the resolvent, we want to investigate the structure of the scattering matrix near a resonance. For this we consider

$$\phi_\ell^{\#} \in \mathcal{C}^\infty(\partial_\infty X, E_\chi|_{\partial_\infty X})$$

defined by

$$\phi_\ell^{\#}(\omega) := \lim_{\rho\to 0} \rho_f^{-s_0} \rho_c^{1-s_0} \phi_\ell(\rho, \omega),$$

where ϕ_ℓ is as in (44). Let

$$\Phi^{\#}(v) := (\langle \phi_\ell^{\#}, v\rangle)_{\ell=1,\ldots,m_{X,\chi}(s_0)}, \quad v \in L^2(\partial_\infty X, E_\chi|_{\partial_\infty X}),$$

where $\langle\cdot,\cdot\rangle$ is the bilinear product on $L^2(\partial_\infty X, E_\chi|_{\partial_\infty X})$ defined by

$$\langle u, v\rangle = \int_{\partial_f X} \langle u, v\rangle_{E_\chi}\, d\sigma_{\partial_f X} + \sum_{j=1}^{n_c^\chi} u_j v_j.$$

Lemma 5.8. *Let $s_0 \in \mathbb{C}$ with $\operatorname{Re} s_0 < 1$ and $s_0 \neq 1/2$. The scattering matrix has a pole at s_0 if and only if $R_{X,\chi}(s)$ has a pole at s_0. In this case we have that*

$$\begin{aligned} S_{X,\chi}(s) = (\Phi^\#)^T E(s, s_0)\left(\sum_{j=1}^{n} (s(1-s) - s_0(1-s_0))^{-k_j} P_j\right) F(s, s_0)\Phi^\# \\ + H^\#(s, s_0), \end{aligned}$$

where for some $n, k_j > 0$ with

$$\sum_{j=1}^{n} k_j = m_{X,\chi}(s_0),$$

for each $j \in \{1, \ldots, n\}$ the matrices P_j are rank-1-projections from $\mathbb{C}^{m_{X,\chi}(s_0)}$ to mutually orthogonal subspaces, $E(\cdot, s_0)$ and $F(\cdot, s_0)$ are holomorphically invertible matrices of dimension $m_{X,\chi}(s_0)$, and

$$H^\#(\cdot, s_0)\colon L^2(\partial_\infty X, E_\chi|_{\partial_\infty X}) \to L^2(\partial_\infty X, E_\chi|_{\partial_\infty X})$$

is holomorphic near $s = s_0$.

Proof. Using (54) and (42), we have that

$$S_{X,\chi}(s) = \sum_{k=1}^{p} \frac{A_k^\#(s_0)}{(s(1-s) - s_0(1-s_0))^k} + H^\#(s, s_0)$$

for some (unique) $p \in \mathbb{N}_0$ such that $H^\#(\cdot, s_0)$ is holomorphic. For each $k \in \{1, \ldots, p\}$, the operator $A_k^\#(s_0)$ is determined by the integral kernel

$$A_k^\#(s_0, \omega, \omega') := (2s_0 - 1) \lim_{\rho\to 0} \lim_{\rho'\to 0} (\rho_f \rho_f')^{-s_0} (\rho_c \rho_c')^{1-s_0} A_k(s_0, \rho, \omega, \rho', \omega').$$

Recall from (45) that

$$A_k(s_0) = \sum_{\ell,m=1}^{m_{X,\chi}(s_0)} a_k^{\ell,m}(s_0)\phi_\ell\langle\phi_m, \cdot\rangle.$$

This implies

$$\begin{aligned} A_k^\#(s_0) &= \sum_{\ell,m=1}^{m_{X,\chi}(s_0)} a_k^{\ell,m}(s_0)\phi_\ell^\#\langle\phi_m^\#, \cdot\rangle \\ &= (\Phi^\#)^T a_k(s_0)\Phi^\#. \end{aligned}$$

Above, $a_k(s_0)$ is as in (45). Recall that $a_k(s_0) = a_1(s_0)d(s_0)^{k-1}$, where $d(s_0)$ is nilpotent. Hence, $S_{X,\chi}(s)$ can be written as

$$S_{X,\chi}(s) = (\Phi^{\#})^T a_1(s_0)\left(\sum_{k=0}^{p-1}(s(1-s)-s_0(1-s_0))^{-(k+1)}d(s_0)^k\right)\Phi^{\#} + H^{\#}(s,s_0)$$

in a sufficiently small neighborhood of s_0. Denote by N_k a Jordan block of dimension k with eigenvalue 0. The Jordan normal form of $d(s_0)$ is given by

$$J^{-1}d(s_0)J = \begin{pmatrix} N_{k_1} & 0 & \dots & 0 \\ 0 & N_{k_2} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & N_{k_n} \end{pmatrix},$$

where $\sum_{j=1}^n k_j = m_{X,\chi}(s_0)$ and J is invertible.

Since $d(s_0)^p = 0$, we have that $k_j \le p-1$ and therefore we can write

$$\sum_{m=0}^{p-1} x^{-(m+1)}N_{k_j}^m = \begin{pmatrix} x^{-1} & x^{-2} & \dots & x^{-k_j} \\ & x^{-1} & \ddots & \vdots \\ & & x^{-1} & x^{-2} \\ & & & x^{-1} \end{pmatrix}$$

which we may factorize as

$$\sum_{m=0}^{p-1} x^{-(m+1)}N_{k_j}^m = E_{k_j}(x)\begin{pmatrix} x^{-k_j} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}F_{k_j}(x),$$

where E_{k_j} and F_{k_j} are invertible matrices that depend polynomially on x (see also [GZ97, p. 622]).

Putting $x = s(1-s) - s_0(1-s_0)$ and applying the argumentation above to every Jordan block, we obtain invertible matrices $E(s,s_0)$, $F(s,s_0)$ depending polynomially on s and mutually orthogonal projections P_j of rank 1 such that

$$\sum_{k=0}^{p-1}(s(1-s)-s_0(1-s_0))^{-(k+1)}d(s_0)^k = E(s,s_0)\left(\sum_{j=1}^{n}(s(1-s)-s_0(1-s_0))^{-k_j}P_j\right)F(s,s_0) + \tilde{H}(s,s_0),$$

where $\tilde{H}(\cdot,s_0)$ is holomorphic. This proves the claim. □

5.3 The scattering matrix at $s = 1/2$.

Lemma 5.9. *The resolvent satisfies*

$$R_{X,\chi}(s) = \frac{1}{2s-1} \sum_{k=1}^{m_{X,\chi}(1/2)} \phi_k \langle \phi_k, \cdot \rangle + H(s), \tag{57}$$

where H is holomorphic near $1/2$, and, for each $k \in \{1, \ldots, m_{X,\chi}(1/2)\}$, the function

$$\phi_k \in \rho_f^{1/2} \rho_c^{1/2} \mathcal{C}^\infty(X, E_\chi) \tag{58}$$

satisfies

$$\Big(\Delta_{X,\chi} - \frac{1}{4}\Big)\phi_k = 0.$$

Proof. We note that $\mathrm{Im}(s^2 - s) = \mathrm{Im}((s-1/2)^2)$. Let $\psi \in \mathcal{C}_c^\infty(X, E_\chi)$. Using the self-adjointness of $\Delta_{X,\chi}$, we obtain the estimate

$$\begin{aligned} |((\Delta_{X,\chi} - s(1-s))u, u)_{L^2}| &\geq |\mathrm{Im}((\Delta_{X,\chi} - s(1-s))u, u)_{L^2}| \\ &= |\mathrm{Im}(s^2 - s)| \, \|u\|_{L^2}^2 \\ &= \left|\mathrm{Im}\Big(\Big(s - \frac{1}{2}\Big)^2\Big)\right| \|u\|_{L^2}^2. \end{aligned}$$

Therefore, we have

$$\|R_{X,\chi}(s)\|_{L^2 \to L^2} \leq \left|\mathrm{Im}\Big(\Big(s - \frac{1}{2}\Big)^2\Big)\right|^{-1}. \tag{59}$$

Hence, the order of the resonance at $s = 1/2$ is at most 2. This implies that

$$R_{X,\chi}(s) = \frac{A}{(2s-1)^2} + \frac{B}{2s-1} + h(s), \tag{60}$$

where h is holomorphic near $1/2$, and A and B are suitable operators, independent of s.

We note that

$$\Big(\Delta_{X,\chi} - \frac{1}{4}\Big) R_{X,\chi}(s) = \mathrm{id} - \frac{(2s-1)^2}{4} R_{X,\chi}(s).$$

Substituting (60) into this equation and taking the limit $s \to 1/2$ implies that every element u in the range of A and B satisfies $(\Delta_{X,\chi} - 1/4)u = 0$. We note that (59) implies that $A\colon L^2_{\mathrm{cpt}}(X, E_\chi) \to L^2(X, E_\chi)$. Hence, the range of A consists of eigenfunctions of $\Delta_{X,\chi}$ with eigenvalue $1/4$. By Proposition 4.7 there are no eigenfunctions if X has infinite volume, hence $A = 0$.

By the definition of the multiplicity, we have that $\operatorname{rank} B = m_{X,\chi}(1/2)$ and therefore we can write

$$B = \sum_{\ell,m=1}^{m_{X,\chi}(1/2)} a_1^{\ell,m} \tilde{\phi}_\ell \langle \tilde{\phi}_m, \cdot \rangle$$

for some symmetric invertible matrix

$$a_1 = (a_1^{\ell,m})_{\ell,m=1}^{m_{X,\chi}(1/2)}$$

and functions $\tilde{\phi}_k \in \rho_f^{1/2} \rho_c^{1/2} \mathcal{C}^\infty(X, E_\chi)$.

Since the resolvent at $1/2$ is self-adjoint and non-negative, a_1 is a positive matrix. Therefore we can find a matrix $(d_{k,\ell})_{k,\ell=1}^{m_{X,\chi}}$ such that

$$B = \sum_{k=1}^{m_{X,\chi}(1/2)} \phi_k \langle \phi_k, \cdot \rangle, \tag{61}$$

where $\phi_k = \sum_{\ell=1}^{m_{X,\chi}(1/2)} d_{k,\ell} \tilde{\phi}_\ell$ for $k = 1, \ldots, m_{X,\chi}(1/2)$. From the fact that the range of B is contained in the kernel of $(\Delta_{X,\chi} - \frac{1}{4})$ it follows that

$$(\Delta_{X,\chi} - \tfrac{1}{4})\phi_k = 0.$$

□

Proof of Theorem B. From Proposition 5.2 and Definition 5.3, we obtain that

$$(2s-1)E_{X,\chi}(s)u \sim \rho_f^{1-s} \rho_c^{-s} u + \rho_f^s \rho_c^{s-1} S_{X,\chi}(s)u.$$

At first glance, this does not make any sense for $s = 1/2$, but we will see that $E_{X,\chi}(s)$ has a simple pole at $s = 1/2$ and hence $(2s-1)E_{X,\chi}(s) \neq 0$ for $s = 1/2$.

By Theorem 4.1, we have the decomposition

$$R_{X,\chi}(s) = \tilde{M}_i(s) + M_f(s) + M_c(s) + Q(s)$$

and we recall that $\tilde{M}_i(s)$ and $M_f(s)$ are holomorphic near $s = 1/2$. We write the remainder term $Q(s)$ as

$$Q(s) = (2s-1)^{-1} \tilde{Q} + Q_{\mathrm{hol}}(s),$$

where Q_{hol} is holomorphic near $s = 1/2$. By (29) and (32), the term $M_c(s)$ is given by

$$(2s-1)M_c(s) = (1 - \eta_{c,0}) \rho_c^{s-1} (\rho_c')^{-s} \operatorname{id}_{\partial_c E_\chi} (1 - \eta_{c,1}) + (2s-1) M_{\mathrm{hol}}^c(s),$$

with $M^c_{\mathrm{hol}}(s)$ being holomorphic near $s = 1/2$. If we set

$$\tilde{M}_c = (1 - \eta_{c,0})(\rho_c\rho_c')^{-1/2}\,\mathrm{id}_{\partial_c E_\chi}(1 - \eta_{c,1}),$$

then we have that

$$R_{X,\chi}(s) = (2s-1)^{-1}(\tilde{Q} + \tilde{M}_c) + H(s)$$

and $H(s)$ is holomorphic near $s = 1/2$. Recall also that

$$S_{X,\chi}(s) = (S_{X_f,\chi}(s) \oplus 0) + (2s-1)Q^\#(s),$$

where

$$Q^\#(s;\omega,\omega') = (\rho_f\rho_f')^{-s}(\rho_c\rho_c')^{1-s}Q(s;\rho,\omega,\rho',\omega')\Big|_{\partial_\infty X\times\partial_\infty X}.$$

Pick $\tilde{Q}^\#$ such that

$$(2s-1)Q^\#(s) = \tilde{Q}^\# + Q^\#_{\mathrm{hol}}(s),$$

where $Q^\#_{\mathrm{hol}}(s)$ is holomorphic near $s = 1/2$. This implies that

$$\tilde{Q}^\# = (\rho_f\rho_f')^{-1/2}(\rho_c\rho_c')^{1/2}\tilde{Q}(\rho,\omega,\rho',\omega')\Big|_{\partial_\infty X\times\partial_\infty X}.$$

From the Fourier decomposition of $S_{X_f,\chi}(s)$, we see that $S_{X_f,\chi}(1/2) = -\,\mathrm{id}$. This implies that

$$\begin{aligned} P &:= \frac{1}{2}\left(S_{X,\chi}\left(\frac{1}{2}\right) + \mathrm{id}\right) \\ &= \frac{1}{2}((0 \oplus \mathrm{id}_{\partial_c E_\chi}) + \tilde{Q}^\#) \end{aligned}$$

is a compact operator. Using (51) and Proposition 5.4, we calculate $P^2 = P$ and $P^* = P$.

The residue of the resolvent at $s = 1/2$ is given by

$$B = \tilde{Q} + \tilde{M}_c.$$

This implies that

$$\begin{aligned} (\rho_f\rho_f')^{-1/2}(\rho_c\rho_c')^{1/2}B\Big|_{\partial_\infty X\times\partial_\infty X} &= \tilde{Q}^\# + (0 \oplus \mathrm{id}_{\partial_c E_\chi}) \\ &= 2P. \end{aligned}$$

With ϕ_k given by (57), we set

$$\phi_k^\# := \rho_f^{-1/2}\rho_c^{1/2}\phi_k\Big|_{\partial_\infty X}, \tag{62}$$

which defines a function $\phi_k^\# \in \mathcal{C}^\infty(\partial_\infty X, E_\chi)$ by (58). The functions $\phi_k^\#$ are linearly independent since, otherwise, a non-trivial linear combination would lead to an L^2-integrable solution of the eigenvalue equation in contradiction to $\Delta_{X,\chi}$ having no eigenvalues at $\lambda = 1/4$.

From (61) and (62) we obtain that

$$(\rho_f \rho_f')^{-1/2} (\rho_c \rho_c')^{1/2} B\Big|_{\partial_\infty X \times \partial_\infty X} = \sum_{k=1}^{m_{X,\chi}(1/2)} \phi_k^\# \langle \phi_k^\#, \cdot \rangle.$$

Thus the restriction of B to the boundary at infinity still has rank $m_{X,\chi}(1/2)$.

To see that P projects onto smooth functions that are invariant under $S_{X,\chi}(1/2)$, we note that since P is an orthogonal projection, the image of P is given by the sections u such that $Pu = u$. This is equivalent to $S_{X,\chi}(1/2)u = u$ by definition of P and

$$P = \frac{1}{2}((0 \oplus \mathrm{id}_{\partial_c E_\chi}) + \tilde{Q}^\#)$$

implies that if $Pu = u$, then $u \in \mathcal{C}^\infty(\partial_\infty X, E_\chi)$. This finishes the proof. □

Remark 5.10. The proof also shows that

$$S_{X,\chi}(\tfrac{1}{2}) = -\,\mathrm{id} + \sum_{k=1}^{m_{X,\chi}(1/2)} \phi_k^\# \langle \phi_k^\#, \cdot \rangle.$$

5.4 Scattering poles. Let $s_0 \in \mathbb{C}$ be a resonance, let $\varepsilon > 0$ and let $\gamma_{s_0,\varepsilon}$ be the path

(63) $$[0, 1] \ni t \mapsto s_0 + \varepsilon e^{2\pi i t}.$$

We suppose that ε is small enough such that there is no other resonance inside $\gamma_{s_0,\varepsilon}$ rather than s_0. Recall that the resonance multiplicity of s_0 is given as

$$m_{X,\chi}(s_0) := \mathrm{rank} \int_{\gamma_{s_0,\varepsilon}} R_{X,\chi}(t)\, dt, \quad s_0 \neq \frac{1}{2}.$$

The analogues of resonances for a scattering matrix are **scattering poles**. The definition of the multiplicity of a scattering pole is more involved.

We start with briefly recalling some definitions from the Gohberg–Sigal theory [GS71]: let $\mathcal{B}$ be a Banach space and let $\lambda_0 \in \mathbb{C}$. We further denote by $L(\mathcal{B})$ the algebra of all linear bounded operators from $\mathcal{B}$ to $\mathcal{B}$. We denote by $\mathcal{M}(\lambda_0)$ the germ of $L(\mathcal{B})$-valued functions that are holomorphic in some punctured neighborhood of λ_0 and have either a pole or a removable singularity at λ_0. In any concrete situation we will pick a suitable neighborhood.

An operator $B \in \mathcal{B}$ is called a **Φ-operator** if the set of values $\operatorname{Im} B$ is closed, $\ker B$ is finite-dimensional and $\dim(\mathcal{B}/\operatorname{Im} B) < \infty$. The integer

$$\operatorname{ind} B = \dim\ker B - \dim(\mathcal{B}/\operatorname{Im} B)$$

is called the **index** of the Φ-operator B.

Let $B \in \mathcal{M}(\lambda_0)$ be holomorphic at least in $\Omega_B \setminus \{\lambda_0\}$, where Ω_B is some open neighborhood of λ_0, and suppose that there exists a function $\psi\colon \Omega_B \to \mathcal{B}$ such that $\psi(\lambda_0) \not\equiv 0$, the functions ψ and $B\psi$ are holomorphic at λ_0; moreover, we suppose that $B\psi(\lambda_0) = 0$. We refer to $\psi(\lambda_0)$ as a **root vector** and to ψ as a **root function** of B at λ_0. The **rank** of a root vector $\psi(\lambda_0)$, further denoted as $\operatorname{rank}(\psi(\lambda_0))$, is the maximal order of vanishing of $B(\lambda)\phi(\lambda)$ at $\lambda = \lambda_0$ among all root functions ϕ with $\phi(\lambda_0) = \psi(\lambda_0)$. If these orders of vanishing are unbounded, we define $\operatorname{rank}(\psi(\lambda_0)) := \infty$. The set of all root vectors of B at λ_0 is a vector space. We refer to its closure in $\mathcal{B}$ as the **kernel** of $B(\lambda_0)$ and denote it by $\ker B(\lambda_0)$. In what follows, we suppose that $m := \dim\ker B(\lambda_0) < \infty$ and $\operatorname{rank}(v) < \infty$ for all $v \in \ker B(\lambda_0)$. We define a basis, $\{v^{(1)}, \ldots, v^{(m)}\}$, of $\ker B(\lambda_0)$ as follows: the rank of $v^{(1)}$ equals the maximal rank of all root vectors corresponding to λ_0 and the rank of $v^{(j)}$ for $j = 2, \ldots, m$ is the maximal rank of root vectors in some complementary subspace of $\{v^{(1)}, \ldots, v^{(j-1)}\}$ in $\ker B(\lambda_0)$. Let $r_j := \operatorname{rank} v^{(j)}$. We set

$$N_{\lambda_0}(B) := \sum_{j=1}^{m} r_j.$$

Example 5.11. As an example, let

$$A(\lambda) = Q + \sum_{j=1}^{n} (\lambda - \lambda_0)^{k_j} P_j,$$

where Q is a bounded operator and P_j are mutually orthogonal projections having rank 1 for $j > 0$, and k_j are non-zero integers. Then for each $j > 0$, the image of P_j is one-dimensional. For $k_j > 0$ each vector in its image is a root vector of rank k_j. Thus,

$$N_{\lambda_0}(A) = \sum_{k_j > 0} k_j.$$

If $A(\lambda)$ is invertible near $\lambda = 0$, then we additionally have that

$$N_{\lambda_0}(A^{-1}) = -\sum_{k_j<0} k_j.$$

We also recall (see, e.g., [Bor16, Definition 6.6]) that a set of bounded operators $A(\lambda)$ from $\mathcal{B}$ to $\mathcal{B}$, parametrized by $\lambda \in U \subset \mathbb{C}$, is a **finitely meromorphic family** if at each point $\lambda' \in U$, we have a Laurent series representation,

$$A(\lambda) = \sum_{k=-m}^{\infty} (\lambda - \lambda')^k A_k,$$

converging (in the operator topology) in some neighborhood of λ', where for $k < 0$, the coefficients A_k are finite rank operators. If $A(\lambda)$ is holomorphic at the point λ_0 and the operator $A(\lambda_0)$ is invertible, then λ_0 is called a **regular** point of $A(\lambda)$.

Additionally, $A(\lambda)$ is said to be **of Fredholm type** in U, if at each point $\lambda' \in U$, the operator A_0 in the expansion above is a Φ-operator.

The main result of Gohberg–Sigal [GS71, Theorem 2.1] is the following argument principle: Let $B \in \mathcal{M}(\lambda_0)$ be such that B is invertible in some neighborhood of λ_0. Suppose that B and B^{-1} are finitely meromorphic families of operators in this neighborhood of λ_0 and are of Fredholm type there. Suppose that all points inside a sufficiently small contour, γ, around λ_0 (except for, maybe, λ_0 itself) are regular for both B and B^{-1}. Then

$$N_{\lambda_0}(B) - N_{\lambda_0}(B^{-1}) = \frac{1}{2\pi i} \operatorname{Tr} \int_\gamma B(\lambda)^{-1} B'(\lambda)\, d\lambda. \tag{64}$$

If for such $B \in \mathcal{M}(\lambda_0)$ we define

$$M_{\lambda_0}(B) := \frac{1}{2\pi i} \operatorname{Tr} \int_\gamma B(\lambda)^{-1} B'(\lambda)\, d\lambda, \tag{65}$$

then for all $B_1, B_2 \in \mathcal{M}(\lambda_0)$ satisfying the conditions above we have

$$M_{\lambda_0}(B_1 B_2) = M_{\lambda_0}(B_1) + M_{\lambda_0}(B_2). \tag{66}$$

See [GS71, Theorem 5.2].

From (15) and Proposition 5.6, we obtain that $S_{X,\chi}(s)$ has poles of infinite rank at $s = 1/2 + \mathbb{N}_0$. As in the case of the model funnel (see Section 3.1), we want to normalize the scattering matrix so that it becomes a bounded operator with poles of **finite rank** at $\mathcal{R}_{X,\chi}$.

To cancel poles of infinite rank at $1/2 + \mathbb{N}_0$, we define the operator

$$\begin{aligned} &G(s)\colon \mathcal{C}^\infty(\partial_\infty X, E_\chi|_{\partial_\infty X}) \to \mathcal{C}^\infty(\partial_\infty X, E_\chi|_{\partial_\infty X}),\\ &G(s) := (\Gamma(s+\tfrac{1}{2})\, \mathrm{id}_{\mathcal{C}^\infty(\partial_f X, E_\chi|_{\partial_f X})}) \oplus \mathrm{id}_{\mathcal{C}^\infty(\partial_c X, E_\chi|_{\partial_c X})}\,. \end{aligned}$$

The second step is to normalize the scattering matrix so that it is a bounded operator for all $s \notin \mathcal{R}_{X,\chi} \cup (1/2 + \mathbb{N}_0)$. Denote by $\Lambda_{\partial_f X}$ the square-root of the Laplacian with respect to the bundle metric—or any other invertible elliptic operator $\Lambda_{\partial_f X} \in \Psi^1(\partial_f X, E_\chi|_{\partial_f X})$. Set

$$\Lambda(s)\colon \mathcal{C}^\infty(\partial_\infty X, E_\chi|_{\partial_\infty X}) \to \mathcal{C}^\infty(\partial_\infty X, E_\chi|_{\partial_\infty X}),$$
$$\Lambda(s) = \Lambda_{\partial_f X}^{-s+1/2} \oplus \mathrm{id}_{\mathcal{C}^\infty(\partial_c X, E_\chi|_{\partial_c X})}\,.$$

Note that $\Lambda(s)$ and $G(s)$ commute, and we have that $\Lambda(1-s)^{-1} = \Lambda(s)$. It follows from Proposition 5.6 that

$$\tilde{S}_{X,\chi}(s) \coloneqq G(s)\Lambda(s)S_{X,\chi}(s)\Lambda(1-s)^{-1}G(1-s)^{-1} \tag{67}$$

is a meromorphic family of pseudodifferential operators of order 0 with poles of finite rank. This definition is a generalization of (21). Note that both $G(s)$ and $G(1-s)^{-1}$ are invertible away from $s \in 1/2 \pm \mathbb{N}_0$. As for $S_{X,\chi}(s)$, we can write $\tilde{S}_{X,\chi}(s)$ as a 2×2 matrix,

$$\tilde{S}_{X,\chi}(s) = \begin{pmatrix} \tilde{S}^{ff}_{X,\chi}(s) & \tilde{S}^{fc}_{X,\chi} \\ \tilde{S}^{cf}_{X,\chi}(s) & \tilde{S}^{cc}_{X,\chi} \end{pmatrix}, \tag{68}$$

where

$$\tilde{S}^{ff}_{X,\chi}(s) \coloneqq \frac{\Gamma(s+\frac{1}{2})}{\Gamma(\frac{3}{2}-s)}\Lambda_{\partial_f X}^{-s+1/2} S^{ff}_{X,\chi}(s)\Lambda_{\partial_f X}^{-s+1/2},$$
$$\tilde{S}^{cf}_{X,\chi}(s) \coloneqq \Gamma(\tfrac{3}{2}-s)^{-1} S^{cf}_{X,\chi}(s)\Lambda_{\partial_f X}^{-s+1/2},$$
$$\tilde{S}^{fc}_{X,\chi}(s) \coloneqq \Gamma(s+\tfrac{1}{2})\Lambda_{\partial_f X}^{-s+1/2} S^{fc}_{X,\chi}(s),$$
$$\tilde{S}^{cc}_{X,\chi}(s) \coloneqq S^{cc}_{X,\chi}(s).$$

Moreover, we have that

$$\tilde{S}_{X,\chi}(1-s) = \tilde{S}_{X,\chi}(s)^{-1} \tag{69}$$

and $\tilde{S}_{X,\chi}(s)$ is a Fredholm operator. For the Fredholm property it suffices to consider $S^{ff}_{X,\chi}$, the other entries are finite rank and then it follows directly from Proposition 5.6 and the invertibility of $S_{X_f,\chi}(s)$, Proposition 3.1.

The multiplicity of a scattering pole $s_0 \in \mathbb{C}$ is defined as

$$\nu_{X,\chi}(s_0) \coloneqq -M_{s_0}(\tilde{S}_{X,\chi}) = -\frac{1}{2\pi i}\operatorname{Tr}\int_\gamma \tilde{S}_{X,\chi}(s)^{-1}\frac{d}{ds}\tilde{S}_{X,\chi}(s)\,ds. \tag{70}$$

By (66) it follows that $\nu_{X,\chi}(s)$ is independent of the specific choice of the operator $\Lambda_{\partial_f X}$.

Lemma 5.12. *For* $s_0 \in \mathcal{R}_{X,\chi}$ *with* $\operatorname{Re} s_0 < 1$, $s_0 \neq 1/2$, *we have that*

$$N_{1-s_0}(\tilde{S}_{X,\chi}) = N_{1-s_0}(\Lambda S_{X,\chi}\Lambda).$$

Moreover, for a resonance $s_0 \in \mathcal{R}_{X,\chi}$ *there exists* $n^\# > 0$ *and* $k_j^\# \in \mathbb{Z}$*, such that we have the decomposition near* $s_0 \in \mathcal{R}_{X,\chi}$*,*

$$\Lambda(s)S_{X,\chi}(s)\Lambda(s) = G_1(s)\left(\tilde{P}_0(s) + \sum_{j=1}^{n^\#}(s-s_0)^{-k_j^\#}P_j\right)G_2(s),$$

where G_1, G_2 *are holomorphically invertible near* $s_0 \in \mathcal{R}_{X,\chi}$*,*

$$\tilde{P}_0(s) = \begin{cases} (s-s_0)P_0, & s_0 \in \mathcal{R}_{X,\chi} \cap (\frac{1}{2} - \mathbb{N}), \\ P_0, & s_0 \in \mathcal{R}_{X,\chi} \setminus (\frac{1}{2} - \mathbb{N}), \end{cases}$$

and P_j *for* $j = 0, \dots, n^\#$ *are mutually orthogonal projections. For* $j = 1, \dots, n^\#$*, the projections* P_j *have rank 1.*

Proof. The first part of the statement for $s_0 \notin \frac{1}{2} - \mathbb{N}$ follows from (67) and the remark afterwards. Now let us consider $s_0 \in \frac{1}{2} - \mathbb{N}$ for which we follow [Bor16, Lemma 8.12]. We set

$$T(s) := \tilde{S}^{cc}(s) - \tilde{S}^{cf}(s)\tilde{S}^{ff}(s)^{-1}\tilde{S}^{fc}(s)$$

and note that it is well-defined near $1 - s_0$. We can then write

$$\tilde{S}_X(s) = \begin{pmatrix} \mathrm{id} & 0 \\ \tilde{S}^{cf}(s)\tilde{S}^{ff}(s)^{-1} & \mathrm{id} \end{pmatrix} \begin{pmatrix} \mathrm{id} & 0 \\ 0 & T(s) \end{pmatrix} \begin{pmatrix} \tilde{S}^{ff}(s) & \tilde{S}^{fc}(s) \\ 0 & \mathrm{id} \end{pmatrix}.$$

The first and last factors on the right-hand side of the previous equation are both invertible near $1 - s_0$. Together with [GS71, Section 1] this implies that

$$N_{1-s_0}(\tilde{S}_X) = N_{1-s_0}\left(\begin{pmatrix} \mathrm{id} & 0 \\ 0 & T \end{pmatrix}\right) = N_{1-s_0}(T).$$

Moreover,

$$\Lambda(s)S_{X,\chi}(s)\Lambda(s) = \begin{pmatrix} \mathrm{id} & 0 \\ \Gamma(s+\frac{1}{2})\tilde{S}^{cf}(s)\tilde{S}^{ff}(s)^{-1} & \mathrm{id} \end{pmatrix}$$
$$\times \begin{pmatrix} \frac{\Gamma(\frac{3}{2}-s)}{\Gamma(s+\frac{1}{2})}\mathrm{id} & 0 \\ 0 & T(s) \end{pmatrix} \begin{pmatrix} \tilde{S}^{ff}(s) & \frac{1}{\Gamma(\frac{3}{2}-s)}\tilde{S}^{fc}(s) \\ 0 & \mathrm{id} \end{pmatrix}.$$

We note that the first and third factors of the right hand side of the equality above are invertible near $s = 1 - s_0$. Hence,

$$N_{1-s_0}(\Lambda S_X \Lambda) = N_{1-s_0}\left(\begin{pmatrix} \frac{\Gamma(\frac{3}{2}-s)}{\Gamma(s+\frac{1}{2})}\,\mathrm{id} & 0 \\ 0 & T(s) \end{pmatrix}\right).$$

Since $1 + s_0 \in -\mathbb{N}_0$, the function $\Gamma(\frac{3}{2} - s)$ is singular at $s = 1 - s_0$ and hence $\Gamma(\frac{3}{2} - s)/\Gamma(s + \frac{1}{2})$ is singular as well and thus has no root vectors. Therefore

$$N_{1-s_0}\left(\begin{pmatrix} \frac{\Gamma(\frac{3}{2}-s)}{\Gamma(s+\frac{1}{2})}\,\mathrm{id} & 0 \\ 0 & T(s) \end{pmatrix}\right) = N_{1-s_0}(T)$$

which implies $N_{1-s_0}(\Lambda S_X \Lambda) = N_{1-s_0}(T)$ and proves the result.

The second part of the statement follows from the application of the Gohberg–Sigal Logarithmic Residue Theorem, (64), to $\Lambda S_{X,\chi}\Lambda$. □

Using (69) and comparing with Example 5.11, we have that

$$N_{1-s_0}(\Lambda S_{X,\chi}\Lambda) = \sum_{j:k_j^\# > 0} k_j^\#.$$

Lemma 5.8 implies that

$$\sum_{j:\,k_j^\# > 0} k_j^\# \le \sum_{j=1}^{n} k_j.$$

Proposition 5.13 (Relation between scattering poles and resonances)**.** *For* $s_0 \in \mathbb{C}$ *with* $\operatorname{Re} s_0 \le 1$ *we have*

$$\nu_{X,\chi}(s_0) = m_{X,\chi}(s_0) - m_{X,\chi}(1 - s_0).$$

Proof. First, we note that $S_{X,\chi}(1/2) = \tilde{S}_{X,\chi}(1/2)$ is unitary, and therefore $\nu_{X,\chi}(1/2) = 0$ by (70). Moreover, $m_{X,\chi}(1/2) - m_{X,\chi}(1 - 1/2) = 0$, which implies the claimed equality for $s_0 = 1/2$. Therefore it suffices to consider a resonance $s_0 \in \mathbb{C}$ with $\operatorname{Re} s_0 < 1$ and $s_0 \neq 1/2$. By (64), we have that

$$\nu_{X,\chi}(s_0) = -M_{s_0}(\tilde{S}_{X,\chi}) = N_{1-s_0}(\tilde{S}_{X,\chi}) - N_{s_0}(\tilde{S}_{X,\chi}).$$

It remains to show that $m_{X,\chi}(s_0) = N_{1-s_0}(\tilde{S}_{X,\chi})$. Note that the inequality

$$m_{X,\chi}(s_0) \ge N_{1-s_0}(\tilde{S}_{X,\chi})$$

follows from

$$N_{1-s_0}(\Lambda S_{X,\chi}\Lambda) = \sum_{j:\,k_j^\# > 0} k_j^\# \le \sum_{j=1}^{n} k_j = m_{X,\chi}(s_0).$$

Since the operator $\Phi^{\#}$ in Lemma 5.8 might not have full rank, we cannot directly deduce equality. To prove $m_{X,\chi}(s_0) \leq N_{1-s_0}(\tilde{S}_{X,\chi})$, we have to use (53). We will distinguish three different cases depending on whether $s_0(1-s_0)$ belongs to the discrete spectrum of $\Delta_{X,\chi}$ and whether $\operatorname{Re} s_0 < 1/2$ or $\operatorname{Re} s_0 > 1/2$.

Case 1. Assume that $s_0(1-s_0)$ does not belong to the discrete spectrum of $\Delta_{X,\chi}$. Then we have that $\operatorname{Re} s_0 < 1/2$ and $S_{X,\chi}(s)$ is holomorphic near $1-s_0$ by definition. Thus, $\tilde{S}_{X,\chi}(s)$ is holomorphic near $1-s_0$ and hence $N_{s_0}(\tilde{S}_{X,\chi}) = 0$, which follows by using that $\tilde{S}_{X,\chi}(s)\tilde{S}_{X,\chi}(1-s) = \mathrm{id}$. By Lemma 5.12 and (53), we have that

$$R_{X,\chi}(s) = R_{X,\chi}(1-s) + (2s-1)E_{X,\chi}(1-s)\Lambda(s)^{-1}G_1(s) \times \left(\tilde{P}_0(s) + \sum_{j=1}^{n^{\#}} (s-s_0)^{-k_j^{\#}} P_j\right) G_2(s)\Lambda(s)^{-1}E_{X,\chi}(1-s)^T.$$

Since all terms except for the factors $(s-s_0)^{-k_j^{\#}}$ are holomorphic and the P_j have rank 1, we have an upper bound for the rank of the residue A_1 of $R_{X,\chi}(s)$ in (42),

$$m_{X,\chi}(s_0) = \operatorname{rank} A_1 \leq \sum_{j\colon k_j^{\#}>0} k_j^{\#} = N_{1-s_0}(\Lambda S_{X,\chi}\Lambda).$$

Case 2. Assume that $s_0(1-s_0)$ belongs to the discrete spectrum of $\Delta_{X,\chi}$ and $\operatorname{Re} s_0 > 1/2$. The resolvent estimate implies that the order of the resonance at s_0 is 1. Straightforward argumentation shows that $A_1(s_0)$ in (42) is the projection onto the eigenspace. Let $(\phi_i)_{i=1}^{m_{X,\chi}(s_0)}$ be an orthonormal basis of the eigenspace and set

$$\phi_i^{\#} := \lim_{\rho\to 0} \rho_f^{-s_0}\rho_c^{1-s_0}\phi_i \in \mathcal{C}^\infty(\partial_\infty X, E_\chi|_{\partial_\infty X}).$$

The functions $\phi_i^{\#}$, $i \in \{1,\ldots,m\}$, are linearly independent, by a straightforward contradiction argument using Proposition 4.4. The Laurent expansion of $S_{X,\chi}(s)$ takes the form

$$S_{X,\chi}(s) = -(s-s_0)^{-1} \sum_{i=1}^{m_{X,\chi}(s_0)} \phi_i^{\#}\langle \phi_i^{\#}, \cdot\rangle + H_1(s),$$

where H_1 is holomorphic near $s = s_0$. Hence, $\tilde{S}_{X,\chi}^{-1}$ has $m_{X,\chi}(s_0)$ independent root vectors of rank 1 at $s = s_0$.

Case 3. Let $s_0(1-s_0) \in \sigma_d(\Delta_{X,\chi})$ with $\operatorname{Re} s_0 < 1/2$. In this case, we have to separate the contributions from the eigenfunctions with eigenvalue $s_0(1-s_0)$ and the remaining resonances. For $i \in \{1,\ldots,m\}$, let ϕ_i and $\phi_i^{\#}$ be as above. Denote the span of $\{\phi_i\}_{i=1}^m$ by W. Since $\operatorname{Re} s_0 < 1/2$, we have that $W \subset \rho_f^{1-s_0}\rho_c^{-s_0}\mathcal{C}^\infty(\overline{X}, E_\chi)$.

Using the Taylor expansion of $\rho_f^s \rho_c^{1-s}$ as a function of $s(1-s)$ near $s_0(1-s_0)$, we have that

$$\operatorname{ran} A_1(s_0) \subset \sum_{k=0}^{p-1} \rho_f^{s_0} \rho_c^{s_0-1} \log(\rho)^k \in \mathcal{C}^\infty(\overline{X}, E_\chi),$$

where we recall that $\rho = \rho_f \rho_c$. Using the unique continuation again, it follows that $\operatorname{ran} A_1(s_0)$ and W are disjoint. Therefore there exists a decomposition $\rho^{-1}L^2 = W \oplus W'$ with $\operatorname{ran} A_1(s_0) \subset W'$. Denote by Π the projection onto W' with $\ker \Pi = W$. We have that $\Pi\phi_i = 0$ and $\Pi A_1(s_0) = A_1(s_0)$. This means that after conjugating by Π, we can use a similar argument as in the first case to estimate $m_{X,\chi}(s_0)$ by $N_{1-s_0}(\tilde{S}_{X,\chi})$. To carry this out, we first use Case 2 to calculate the residue of $R_{X,\chi}(1-s)$.

The Laurent expansion of $R_{X,\chi}(1-s)$ near $s = s_0$ is given by

$$R_{X,\chi}(1-s) = (s_0 - s)^{-1} R_{-1} + R_{\mathrm{hol}}(1-s),$$

where $R_{\mathrm{hol}}(s)$ is holomorphic near s_0. To calculate the residue, we note that

$$s(1-s) - s_0(1-s_0) = -(s-s_0)(2s_0 - 1 + (s-s_0))$$

and hence

$$\begin{aligned} R_{-1} &= -\operatorname{res}_{s=s_0} R_{X,\chi}(1-s) \\ &= (2s_0-1)^{-1} \sum_{\ell,m}^{m_{X,\chi}(1-s_0)} a_1^{\ell,m}(1-s_0)\phi_k \langle \phi_k, \cdot \rangle, \end{aligned}$$

where $a_1^{\ell,m}(1-s_0)$ is defined as in (44).

We define the Laurent expansions

$$\begin{aligned} (2s-1)E_{X,\chi}(1-s)\Lambda(s)^{-1}G_1(s) &=: \sum_{l=-1}^{\infty} (s-s_0)^l E_l, \\ G_2(s)\Lambda(s)^{-1}E_{X,\chi}(1-s)^T &=: \sum_{m=-1}^{\infty} (s-s_0)^m F_m. \end{aligned}$$

The principal parts of these Laurent expansions are given by

$$\begin{aligned} E_{-1} &= \sum_i \phi_i \langle e_i, \cdot \rangle, \\ F_{-1} &= \sum_i f_i \langle \phi_i, \cdot \rangle, \end{aligned}$$

for some $e_i, f_i \in \mathcal{C}^\infty(\overline{X}, E_\chi)$. Consequently,

$$\Pi R_{-1} = 0, \quad \Pi E_{-1} = 0, \quad \text{and} \quad F_{-1}\Pi^T = 0.$$

Using (53) and Lemma 5.12, the residue of $R_{X,\chi}(s)$ at $s = s_0$ can be calculated as

$$A_1(s_0) = \operatorname{res}_{s_0} R_{X,\chi} = R_{-1} + \sum_{l+m-k_j^\#=-1} E_l P_j F_m.$$

Conjugating by Π yields

$$A_1(s_0) = \Pi A_1(s_0)\Pi^T = \sum_{j\colon k_j^\#>0} \sum_{l=0}^{k_j^\#-1} \Pi E_l P_j F_{k_j^\#-1-l}\Pi^T.$$

Since P_j is a projection of rank one, we arrive at

$$m_{X,\chi}(s_0) = \operatorname{rank} A_1(s_0) \leq \sum_{j\colon k_j^\#>0} k_j^\# = N_{1-s_0}(\tilde{S}_{X,\chi}).$$

□

5.5 Relative scattering matrix. Set

$$S_{X_{f,c},\chi}(s) := \begin{pmatrix} S_{X_f,\chi}(s) & \\ & -1 \end{pmatrix}.$$

The **relative scattering matrix** is defined by

(71) $$S^{\mathrm{rel}}_{X,\chi}(s) := S_{X_{f,c},\chi}(s)^{-1} S_{X,\chi}(s).$$

By Proposition 5.6, we have the decomposition

$$S_{X,\chi}(s) = \begin{pmatrix} S_{X_f,\chi}(s) & \\ & 0 \end{pmatrix} + (2s-1)\begin{pmatrix} Q^\#(s)^{ff} & Q^\#(s)^{fc} \\ Q^\#(s)^{cf} & Q^\#(s)^{cc} \end{pmatrix}.$$

Hence, the matrix coefficients of $S^{\mathrm{rel}}_{X,\chi}(s) = S_{X_{f,c},\chi}(s)^{-1} S_{X,\chi}(s)$ are given by

(72) $$\begin{aligned} S^{\mathrm{rel}}_{X,\chi}(s) &= \begin{pmatrix} S^{ff}_{\mathrm{rel}}(s) & S^{fc}_{\mathrm{rel}}(s) \\ S^{cf}_{\mathrm{rel}}(s) & S^{cc}_{\mathrm{rel}}(s) \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{id} & \\ & 0 \end{pmatrix} + (2s-1)\begin{pmatrix} S_{X_f,\chi}(s)^{-1} Q^\#(s)^{ff} & S_{X_f,\chi}(s)^{-1} Q^\#(s)^{fc} \\ -Q^\#(s)^{cf} & -Q^\#(s)^{cc} \end{pmatrix}. \end{aligned}$$

We can write $S^{\mathrm{rel}}_{X,\chi}(s) = \mathrm{id} + S_{X_{f,c},\chi}(s)^{-1} B(s)$, where

$$B(s) = (2s-1)Q^\#(s) + \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}.$$

Since $S_{X_{f,c},\chi}(s)^{-1}$ is a pseudodifferential operator and $B(s)$ is smoothing, it follows that $S^{\mathrm{rel}}_{X,\chi}(s) - \mathrm{id}$ is a smoothing operator on $\mathcal{C}^\infty(\partial_\infty X, E_\chi|_{\partial_\infty X})$. Therefore it makes sense to define the **relative scattering determinant**

$$\tau_{X,\chi}(s) \coloneqq \det S^{\mathrm{rel}}_{X,\chi}(s). \tag{73}$$

We have that

$$\begin{aligned}\tau_{X,\chi}(s) &= \det(\mathrm{id} + S_{X_{f,c},\chi}(s)^{-1}B(s))\\ &= \det(\mathrm{id} + B(s)S_{X_{f,c},\chi}(s)^{-1})\\ &= \det(S_{X,\chi}(s)S_{X_{f,c},\chi}(s)^{-1}).\end{aligned}$$

Together with the relations (16) and (52) this implies that

$$\tau_{X,\chi}(s)\tau_{X,\chi}(1-s) = 1 \tag{74}$$

and thus

$$|\tau_{X,\chi}(s)| = 1 \quad \text{for } \operatorname{Re} s = \frac{1}{2}. \tag{75}$$

Let

$$E_2(s) \coloneqq (1-s)\exp\Big(s + \frac{s^2}{2}\Big).$$

By [DFP, Theorem B] and [Boa54, Theorem 2.6.5], the Weierstrass product

$$\mathcal{P}_{X,\chi}(s) \coloneqq s^{m_{X,\chi}(0)} \prod_{\mu \in \mathcal{R}_{X,\chi}\setminus\{0\}} E_2\Big(\frac{s}{\mu}\Big) \tag{76}$$

is well-defined and holomorphic of order 2.

The Weierstrass product $\mathcal{P}_{X_f,\chi}(s)$ for X_f is defined analogously, only exchanging X for X_f in (76), i.e.,

$$\mathcal{P}_{X_f,\chi}(s) \coloneqq s^{m_{X_f,\chi}(0)} \prod_{\mu \in \mathcal{R}_{X_f,\chi}\setminus\{0\}} E_2\Big(\frac{s}{\mu}\Big). \tag{77}$$

We recall that $\mathcal{R}_{X_f,\chi}$ is given by (41) and for one funnel end, the resonances are given by (10). As in the untwisted case (see [GZ97, Proposition 2.14]) we prove the following result.

Proposition 5.14. *The relative scattering determinant admits a factorization*

$$\tau_{X,\chi}(s) = e^{q(s)} \frac{\mathcal{P}_{X,\chi}(1-s)}{\mathcal{P}_{X,\chi}(s)} \frac{\mathcal{P}_{X_f,\chi}(s)}{\mathcal{P}_{X_f,\chi}(1-s)}, \tag{78}$$

where $q\colon \mathbb{C} \to \mathbb{C}$ is an entire function.

Proof. We set

$$h(s) \coloneqq \frac{\mathcal{P}_{X,\chi}(1-s)}{\mathcal{P}_{X,\chi}(s)} \frac{\mathcal{P}_{X_f,\chi}(s)}{\mathcal{P}_{X_f,\chi}(1-s)} \tag{79}$$

for any $s \in \mathbb{C}$, for which the map on the right hand side is defined. Then h is meromorphic on all of $\mathbb{C}$, as is the map $\tau_{X,\chi}$. It suffices to show that the zeros and poles of the two maps h and $\tau_{X,\chi}$ coincide, including their multiplicities.

We first consider $s \in \mathbb{C}$ with $\operatorname{Re} s = 1/2$. From (75) it follows that $\tau_{X,\chi}$ has no zero or pole at s. For $s = 1/2$, we have that $h(1/2) = 1$ and, for $\operatorname{Re} s = 1/2$ and $s \neq 1/2$, both $\mathcal{P}_{X,\chi}$ and $\mathcal{P}_{X_f,\chi}$ have no zeros near s by Corollary 4.6 and (10), respectively. Hence, h has no zero or pole at s.

We consider now $s \in \mathbb{C}$ with $\operatorname{Re} s < 1/2$ and show that the multiplicities of s as a zero or pole of h and $\tau_{X,\chi}$ coincide. Since $\tau_{X,\chi}(1-s) = 1/\tau_{X,\chi}(s)$ by (74) as well as $h(1-s) = 1/h(s)$, this equality of multiplicities then extends immediately to the right half-plane $\{\operatorname{Re} s > 1/2\}$. We now pick $\varepsilon > 0$ such that the ball of radius ε around s contains no zeros of the Weierstrass products $\mathcal{P}_{X,\chi}$ and $\mathcal{P}_{X_f,\chi}$ except at s. Using the argument principle, it remains to show that

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma_{s,\varepsilon}} \frac{\tau'_{X,\chi}(t)}{\tau_{X,\chi}(t)} dt = m_{X,\chi}(1-s) - m_{X,\chi}(s) \\ + m_{X_f,\chi}(s) - m_{X_f,\chi}(1-s). \end{aligned} \tag{80}$$

Taking advantage of (73) and (65) and using Jacobi's formula [Yaf92, p. 43], we can write the left-hand side of (80) as

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma_{s,\varepsilon}} \frac{\tau'_{X,\chi}(t)}{\tau_{X,\chi}(t)} dt &= \frac{1}{2\pi i} \int_{\gamma_{s,\varepsilon}} \frac{(\det S^{\mathrm{rel}}_{X,\chi}(t))'}{\det S^{\mathrm{rel}}_{X,\chi}(t)} dt \\ &= \frac{1}{2\pi i} \operatorname{Tr} \int_{\gamma_{s,\varepsilon}} (S^{\mathrm{rel}}_{X,\chi}(t))^{-1} (S^{\mathrm{rel}}_{X,\chi}(t))' \, dt \\ &= M_s(S^{\mathrm{rel}}_{X,\chi}). \end{aligned}$$

Extending the definition for the model funnel, (21), we define the normalized model scattering matrix by

$$\tilde{S}_{X_f,\chi}(s) \coloneqq G(s)\Lambda(s)S_{X_f,\chi}(s)\Lambda(1-s)^{-1}G(1-s)^{-1}$$

and obtain, using (71), that

$$S^{\mathrm{rel}}_{X,\chi}(s) = G(1-s)^{-1}\Lambda(s)\tilde{S}_{X_f,\chi}(s)^{-1}\tilde{S}_{X,\chi}(s)\Lambda(s)^{-1}G(1-s).$$

We recall that $G(1-s)$ and $\Lambda(s)$ are holomorphic for $\operatorname{Re} s < 1/2$. By (66) we have that

$$M_s(S^{\mathrm{rel}}_{X,\chi}) = M_s(\tilde{S}_{X,\chi}) - M_s(\tilde{S}_{X_f,\chi}).$$

Proposition 5.13 implies

$$M_s(\tilde{S}_{X,\chi}) = \nu_{X,\chi}(s) = m_{X,\chi}(1-s) - m_{X,\chi}(s).$$

Now $m_{X_f,\chi}(1-s) = 0$ since $\operatorname{Re} s < 1/2$. We note that the equality

$$M_s(\tilde{S}_{X_f,\chi}) = -m_{X_f,\chi}(s)$$

follows directly from Proposition 3.1, that completes the proof. □

To prove Theorem A we have to show that q is a polynomial of degree at most 4. For this, we need a singular value estimate on the relative scattering matrix. This will give us an estimate on the scattering determinant. We define the set

$$\mathcal{L}_D^0 := \{s \in \mathbb{C} : D(s) = 0\},$$

where $D(s)$, as in [DFP, Lemma 6.1], is defined by

$$D(s) := \det(1 - (L(s)\eta_3)^3).$$

For $\delta > 0$ set

$$\mathcal{B}(\delta) := B_1\left(\frac{1}{2}\right) \cup \bigcup_{\zeta \in \mathcal{L}_D^0 \cup \mathcal{R}_{X_f,\chi} \cup (1-\mathcal{R}_{X_f,\chi})} B_{\langle\zeta\rangle^{-(2+\delta)}}(\zeta), \tag{81}$$

where $B_r(z)$ denotes the ball of radius r around z.

Lemma 5.15. *For $\delta > 0$ large enough there exists $C > 0$ and $c > 0$ such that for $s \notin \mathcal{B}(\delta)$ and $k \in \mathbb{N}$, we have*

$$\mu_k(S^{\mathrm{rel}}_{X,\chi}(s) - \mathrm{id})) \leq e^{C\langle s\rangle^{2+\varepsilon} - ck}.$$

Proof. By (17), we have that $S_{X_f,\chi}(s)^{-1}E_{X_f,\chi}(s)^T = -E_{X_f,\chi}(1-s)^T$ and together with (55) and (72), we obtain

$$\begin{aligned}
S^{ff}_{\mathrm{rel}}(s) &= \mathrm{id} - (2s-1)E_{X_f,\chi}(1-s)^T(\eta_3 - \eta_1) \\
&\qquad \times (\mathrm{id} - L(s)\eta_3)^{-1}[\Delta_{X,\chi}, \eta_0]E_{X_f,\chi}(s), \\
S^{fc}_{\mathrm{rel}}(s) &= -(2s-1)E_{X_f,\chi}(1-s)^T(\eta_3 - \eta_1) \\
&\qquad \times (\mathrm{id} - L(s)\eta_3)^{-1}[\Delta_{X,\chi}, \eta_0]E_{X_c,\chi}(s), \\
S^{cf}_{\mathrm{rel}}(s) &= -(2s-1)E_{X_c,\chi}(s)^T(\eta_3 - \eta_1)(\mathrm{id} - L(s)\eta_3)^{-1}[\Delta_{X,\chi}, \eta_0]E_{X_f,\chi}(s), \\
S^{cc}_{\mathrm{rel}}(s) &= -(2s-1)E_{X_c,\chi}(s)^T(\eta_3 - \eta_1)(\mathrm{id} - L(s)\eta_3)^{-1}[\Delta_{X,\chi}, \eta_0]E_{X_c,\chi}(s).
\end{aligned}$$

From (29) we obtain for every compactly supported $A \in \mathrm{Diff}^1(X_c, E_\chi|_{X_c})$ the bound

$$\|AE_{X_c,\chi}(s)\| \leq e^{C\langle s\rangle} \tag{82}$$

for $s \notin B_1(1/2)$.

Without loss of generality, we suppose that X_f is a single funnel, that is contained in the hyperbolic cylinder $C_\ell = \langle h_\ell \rangle \backslash \mathbb{H}$. If $\operatorname{Re} s > \varepsilon > 0$, we can directly use (14) to estimate the singular values of $AE_{X_f,\chi}(s)$, where $A \in \operatorname{Diff}^1(X_f, E_\chi|_{X_f})$ is compactly supported, which gives

$$\mu_k(AE_{X_f,\chi}(s)) \leq e^{C\langle s \rangle - ck}.$$

For $\operatorname{Re} s < 1/2 - \varepsilon$, we use (14) together with (17) to obtain

$$\begin{aligned}\mu_k(AE_{X_f,\chi}(s)) &= \mu_k(A(-E_{X_f,\chi}(1-s))S_{X_f,\chi}(s)) \\ &\leq \|AE_{X_f,\chi}(1-s)\| \, \mu_k(S_{X_f,\chi}(s)) \\ &\leq e^{C\langle s \rangle} \mu_k(S_{X_f,\chi}(s)).\end{aligned}$$

Using the estimate for the singular values of the scattering matrix (25), we obtain for all $s \in \mathbb{C}$ the estimate

$$\mu_k(AE_{X_f,\chi}(s)) \leq \begin{cases} d_k(s) e^{C\langle s \rangle \log \langle s \rangle}, & k \leq m_{\max}, \\ (\frac{\langle s \rangle}{k})^{2\langle s \rangle} e^{C\langle s \rangle - ck}, & k > m_{\max}, \end{cases} \tag{83}$$

where $m_{\max}$ was defined in (24), the function $d_k(s)$ was defined by (23), and $A \in \operatorname{Diff}^1(X_f, E_\chi)$ is compactly supported. We note that for every $\delta > 0$ and $s \notin \mathcal{B}(\delta)$, we have that

$$d_0(s) \lesssim \langle s \rangle^{2+\delta}$$

due to the fact that there are only finitely many resonances in a ball of radius 1 around s. Therefore, we can estimate the singular values by

$$\mu_k(AE_{X_f,\chi}(s)) \leq e^{C\langle s \rangle^{2+\varepsilon}}, \tag{84}$$

where we have also used that $\langle s \rangle^{2\langle s \rangle} \leq e^{2\langle s \rangle^2}$ for all $s \in \mathbb{C}$.

The estimate on the determinant $D(s)$ in [DFP, Section 6] implies—as in the untwisted case (see [GZ97, Lemma 3.6])—that for $\delta > 0$ large enough and any

$$s \notin \bigcup_{\zeta \in \mathcal{L}_D^0 \cup \mathcal{R}_{X_f,\chi}} B_{\langle \zeta \rangle^{-(2+\delta)}}(\zeta),$$

the following estimate holds for all $\varepsilon > 0$:

$$\|(\operatorname{id} - L(s)\eta_3)^{-1}\|_{L^2(X,E_\chi) \to L^2(X,E_\chi)} \leq e^{C\langle s \rangle^{2+\varepsilon}}. \tag{85}$$

Using (82), (84), and (85), we obtain

$$\mu_k(S^{\mathrm{rel}}_{X,\chi}(s)^\bullet) \leq e^{C\langle s \rangle^{2+\varepsilon}}, \quad \bullet \in \{fc, cf, cc\},$$

for $s \notin \mathcal{B}(\delta)$, where we have used that all matrix components involving cusp terms are finite rank operators. So in particular, $\mu_k(S^{\mathrm{rel}}_{X,\chi}(s)^{\bullet}) = 0$ for $k > N$ for some $N \in \mathbb{N}$.

For the funnel term, we estimate

$$\begin{aligned}
&\mu_k(S^{\mathrm{rel}}_{X,\chi}(s)^{ff} - \mathrm{id}) \\
&\quad \le \|(\eta_3 - \eta_1)E_{X_f,\chi}(1-s)\| \, \|(\mathrm{id} - L(s)\eta_3)^{-1}\| \, \mu_k([\Delta_{X,\chi}, \eta_0]E_{X_f,\chi}(s)) \\
&\quad \le d_0(1-s)e^{C\langle 1-s\rangle \log(1-s)} \, e^{C\langle s\rangle^{2+\varepsilon}} \, \mu_k([\Delta_{X,\chi}, \eta_0]E_{X_f,\chi}(s)).
\end{aligned}$$

From the remark above, we obtain that for $s \notin \mathcal{B}(\delta)$,

$$\mu_k(S^{\mathrm{rel}}_{X,\chi}(s)^{ff} - \mathrm{id}) \le e^{C\langle s\rangle^{2+\varepsilon}} \, \mu_k([\Delta_{X,\chi}, \eta_0]E_{X_f,\chi}(s)).$$

If $k \le m_{\max}$, then we can apply the same argument to obtain that

$$\mu_k(S^{\mathrm{rel}}_{X,\chi}(s)^{ff} - \mathrm{id}) \le e^{C\langle s\rangle^{2+\varepsilon}}.$$

For $k \ge m_{\max}$, we have that

$$\begin{aligned}
\mu_k(S^{\mathrm{rel}}_{X,\chi}(s)^{ff} - \mathrm{id}) &\le e^{C\langle s\rangle^{2+\varepsilon}} \left(\frac{\langle s\rangle}{k}\right)^{2\langle s\rangle} e^{C\langle s\rangle - ck} \\
&\le e^{C\langle s\rangle^{2+\varepsilon} - ck}.
\end{aligned}$$

□

With all these results at our disposal, the proof of Theorem A is analogous to the corresponding statement in the untwisted setting. For the convenience of the reader, we provide the details.

Proof of Theorem A. In Proposition 5.14 we established the factorization

$$\tau_{X,\chi}(s) \cdot \frac{\mathcal{P}_{X,\chi}(s)\mathcal{P}_{X_f,\chi}(1-s)}{\mathcal{P}_{X,\chi}(1-s)\mathcal{P}_{X_f,\chi}(s)} = e^{q(s)} \tag{86}$$

with q being an entire function. It remains to show that q is polynomial with degree bounded by 4, for which we will take advantage of the Hadamard factorization theorem [Tit58, 8.24]. To that end we let $\varphi\colon \mathbb{C} \to \mathbb{C}$,

$$\varphi(s) \coloneqq \tau_{X,\chi}(s) \cdot \frac{\mathcal{P}_{X,\chi}(s)\mathcal{P}_{X_f,\chi}(1-s)}{\mathcal{P}_{X,\chi}(1-s)\mathcal{P}_{X_f,\chi}(s)},$$

denote the function on the left-hand side of the equation in (86) and note that φ is entire and has no zeros (as q is entire). Therefore $e^{q(\cdot)}$ is the (full) Hadamard factorization of φ, and hence q is polynomial. In order to estimate the degree of q, we now provide a numerical bound on the order of φ.

Let $\delta > 0$ be as in Lemma 5.15 and set $\mathcal{B} := \mathcal{B}(\delta)$, where $\mathcal{B}(\delta)$ is defined in (81). We recall that $\mathcal{B}$ encloses all zeros of $\mathcal{P}_{X,\chi}$ and $\mathcal{P}_{X_f,\chi}$. By [Boa54, Theorem 2.6.5] and the upper bounds on the resonances, [DFP, Remark 4.13] and [DFP, Theorem B], we see that both Weierstrass products $\mathcal{P}_{X,\chi}$ and $\mathcal{P}_{X_f,\chi}$ are of order 2. In combination with the minimum modulus theorem [Tit58, 8.71] we obtain that for all $\varepsilon > 0$ we have

$$\log\left|\frac{\mathcal{P}_{X,\chi}(s)\mathcal{P}_{X_f,\chi}(1-s)}{\mathcal{P}_{X,\chi}(1-s)\mathcal{P}_{X_f,\chi}(s)}\right| \lesssim_\varepsilon \langle s\rangle^{2+\varepsilon} \quad \text{for all } s \notin \mathcal{B}.$$

We may estimate the scattering determinant $\tau_{X,\chi}$ using [GK69, IV.1.2] and Lemma 5.15 to obtain, for all $s \notin \mathcal{B}$,

$$\begin{aligned} |\tau_{X,\chi}(s)| &= |\det(\mathrm{id} + (S^{\mathrm{rel}}_{X,\chi}(s) - \mathrm{id})| \\ &\leq \prod_{k=1}^{\infty}(1 + \mu_k(S^{\mathrm{rel}}_{X,\chi}(s) - \mathrm{id})) \\ &\leq \prod_{k=1}^{\infty}(1 + e^{C\langle s\rangle^{2+\varepsilon} - ck}) \end{aligned}$$

for all $\varepsilon > 0$ and suitable $c, C > 0$ (possibly depending on ε). Choose $N(s) \in \mathbb{N}$ such that $cN(s) < C\langle s\rangle^{2+\varepsilon} < c(N(s)+1)$. We have that

$$\begin{aligned} \log|\tau_{X,\chi}(s)| &\leq \sum_{k=1}^{\infty} \log(1 + e^{C\langle s\rangle^{2+\varepsilon} - ck}) \\ &= \sum_{k=1}^{N(s)} \log(1 + e^{C\langle s\rangle^{2+\varepsilon} - ck}) + \sum_{k=N(s)+1}^{\infty} \log(1 + e^{C\langle s\rangle^{2+\varepsilon} - ck}) \\ &\lesssim_\varepsilon N(s)\langle s\rangle^{2+\varepsilon} + \sum_{j=0}^{\infty} \log(1 + e^{-cj} e^{C\langle s\rangle^{2+\varepsilon} - c(N(s)+1)}) \\ &\lesssim_\varepsilon \langle s\rangle^{4+2\varepsilon}. \end{aligned}$$

Therefore, for every $\varepsilon > 0$ and $s \notin \mathcal{B}$, we obtain $C > 0$ such that

$$\log|\varphi(s)| \lesssim_\varepsilon \langle s\rangle^{4+\varepsilon}. \tag{87}$$

By [DFP, Theorem B and Proposition 6.2], we have that

$$\#\{\zeta \in \mathcal{L}^0_D \cup \mathcal{R}_{X_f,\chi} \cup (1 - \mathcal{R}_{X_f,\chi}) \colon |\zeta| \in (r-1, r)\} \lesssim r^2$$

for any $r > 1$. Hence, we can estimate the area of $\mathcal{B}$ restricted to the annulus $\{r - 1 < |z| < r\}$ by

$$\begin{aligned} \mathrm{vol}(\mathcal{B} \cap \{z \in \mathbb{C} \colon |z| \in (r-1, r)\}) &\lesssim_\varepsilon r^2 \langle r-1\rangle^{-2(\delta+2)} \\ &= O(\langle r\rangle^{-2\delta-2}), \quad \text{as } r \to \infty. \end{aligned}$$

Hence, taking $R > 1$ large enough, for any $r > R$ and $s \in \mathbb{C}$ with $|s| \leq r$, we have the estimate

$$\log|\varphi(s)| \lesssim_{\varepsilon} \langle r \rangle^{4+\varepsilon}$$

by the maximum modulus principle (see for instance [Tit58, 5.1]). Thus, φ is of order 4 and hence q is a polynomial of degree at most 4. □

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