UNIQUENESS THEOREMS FOR WEIGHTED HARMONIC FUNCTIONS IN THE UPPER HALF-PLANE

By

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Abstract. We consider a class of weighted harmonic functions in the open upper half-plane known as α-harmonic functions. Of particular interest is the uniqueness problem for such functions subject to a vanishing Dirichlet boundary value on the real line and an appropriate vanishing condition at infinity. We find that the non-classical case ($\alpha \neq 0$) allows for a considerably more relaxed vanishing condition at infinity compared to the classical case ($\alpha = 0$) of usual harmonic functions in the upper half-plane. The reason behind this dichotomy is different geometry of zero sets of certain polynomials naturally derived from the classical binomial series. These findings shed new light on the theory of harmonic functions, for which we provide sharp uniqueness results under vanishing conditions at infinity along geodesics or along rays emanating from the origin.

1 Introduction

Let $\mathbb{H}$ be the open upper half-plane in the complex plane $\mathbb{C}$ and consider the weighted Laplace differential operator

$$(1.1) \qquad \Delta_{\mathbb{H};\alpha,z} = \partial_z (\operatorname{Im} z)^{-\alpha} \bar{\partial}_z, \quad z \in \mathbb{H},$$

where ∂ and $\bar{\partial}$ are the usual complex partial derivatives and $\alpha > -1$. Here $\operatorname{Im} z$ is the imaginary part of the complex number $z \in \mathbb{C}$ and the function $w_{\mathbb{H};\alpha}(z) = (\operatorname{Im} z)^{\alpha}$ for $z \in \mathbb{H}$ has an interpretation of a standard weight function for $\mathbb{H}$. The study of differential operators of the form (1.1) is suggested by a classical paper of Paul Garabedian [11].

We refer to the differential operator $\Delta_{\mathbb{H};\alpha}$ in (1.1) as the α-Laplacian for $\mathbb{H}$. An α-harmonic function u in $\mathbb{H}$ is a twice continuously differentiable function u in $\mathbb{H}$ (in symbols: $u \in C^2(\mathbb{H})$) such that

$$\Delta_{\mathbb{H};\alpha} u = 0 \quad \text{in } \mathbb{H}.$$

Notice that $\Delta_{\mathbb{H};0} = \partial\bar{\partial}$ is the usual Laplacian and that a 0-harmonic function in $\mathbb{H}$ is a harmonic function in $\mathbb{H}$ in the usual sense. The restriction $\alpha > -1$ on the

weight parameter ensures good supply of α-harmonic functions with well-behaved boundary values.

In this paper we address the uniqueness problem for α-harmonic functions in $\mathbb{H}$, that is, we wish to characterize the identically zero function $u \equiv 0$ within the class of α-harmonic functions in $\mathbb{H}$. A natural condition is that of a vanishing boundary value

$$(1.2) \qquad \lim_{\mathbb{H} \ni z \to x} u(z) = 0, \quad x \in \mathbb{R},$$

on the real line $\mathbb{R}$. A condition of this type is often referred to as a vanishing Dirichlet boundary value. There are plenty of non-trivial α-harmonic functions in $\mathbb{H}$ satisfying (1.2). A simple such example is the function

$$u(z) = (\operatorname{Im} z)^{\alpha+1}, \quad z \in \mathbb{H},$$

which is α-harmonic in $\mathbb{H}$ and satisfies (1.2). In order to obtain satisfactory results on the uniqueness problem above it is thus natural to complement (1.2) with some condition(s) taking into account the behavior of the α-harmonic function u at infinity. Here we think of the point at infinity ∞ as a boundary point of $\mathbb{H}$ in the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. In the case of usual harmonic functions in $\mathbb{H}$ ($\alpha = 0$) this problem setup is classical. We mention here a recent contribution by Carlsson and Wittsten [9] concerned with a uniqueness result in this flavor for α-harmonic functions in $\mathbb{H}$ with $\alpha > -1$. This result of Carlsson and Wittsten has served as a guidance for the present investigations.

We say that a function u in $\mathbb{H}$ is of temperate growth at infinity if it satisfies an estimate of the form

$$|u(z)| \leq C(|z|^2 / \operatorname{Im}(z))^N$$

for $z \in \mathbb{H}$ with $|z| > R$, where C, R and N are positive constants. We refer to the parameter N as an order of growth at infinity for the function u.

A first main result concerns α-harmonic functions u in $\mathbb{H}$ that are of temperate growth at infinity. We prove that such a function u satisfies (1.2) if and only if it has the form

$$(1.3) \qquad u(z) = \sum_{k=0}^{n} c_k (\operatorname{Im} z)^{\alpha+1} p_{k,\alpha}(z), \quad z \in \mathbb{H},$$

for some $n \in \mathbb{N} = \{0, 1, 2, \dots\}$ and $c_0, \dots, c_n \in \mathbb{C}$, where

$$p_{k,\alpha}(z) = \sum_{j=0}^{k} \frac{(\alpha+1)_j}{j!} z^{k-j} \bar{z}^j$$

for $k = 0, 1, \ldots$ (see Corollary 5.4). We point out that the polynomials $p_{k,\alpha}$ are naturally derived from the classical binomial series. We also establish (1.3) under a weaker distributional version of (1.2) (see Theorem 5.7).

We denote by $\mathcal{V}_\alpha$ the set of all functions u of the form (1.3) for some $n \in \mathbb{N}$ and $c_0, \ldots, c_n \in \mathbb{C}$. The set $\mathcal{V}_\alpha$ is naturally filtered in the sense that

$$\mathcal{V}_\alpha = \bigcup_{n=0}^{\infty} \mathcal{V}_{\alpha,n},$$

where $\mathcal{V}_{\alpha,n}$ is the set of all functions of the form (1.3) for some $c_0, \ldots, c_n \in \mathbb{C}$. The set $\mathcal{V}_{\alpha,n}$ has a natural structure of a complex vector space of finite dimension $n + 1$. The space $\mathcal{V}_{\alpha,n}$ admits a natural description within the class $\mathcal{V}_\alpha$ using order of growth at infinity (see Lemma 5.6 and Proposition 5.9). The finer study of behaviors at infinity of functions in the class $\mathcal{V}_\alpha$ depends on the parameter $\alpha > -1$ and divides naturally into cases whether $\alpha \neq 0$ or $\alpha = 0$. The reason behind this dichotomy is a different geometry of zero sets for the polynomials $p_{k,\alpha}$.

We consider next the problem of characterizing the null function in the class $\mathcal{V}_\alpha$ using a vanishing condition at infinity. For parameters $\alpha > -1$ with $\alpha \neq 0$, we prove that if $u \in \mathcal{V}_\alpha$ is such that

$$(1.4) \qquad \lim_{j \to \infty} \frac{u(z_j)}{(\mathrm{Im}(z_j))^{\alpha+1}} = 0$$

for some sequence $\{z_j\}$ in $\mathbb{H}$ with $z_j \to \infty$ in $\mathbb{C}_\infty$ as $j \to \infty$, then $u(z) = 0$ for all $z \in \mathbb{H}$ (see Theorem 6.4). We emphasize the big freedom allowed in the choice of sequence $\{z_j\}$ in (1.4) above. This analysis leads to the following highly flexible uniqueness result for α-harmonic functions in the case $\alpha \neq 0$ as well as a distributional version thereof (see Theorem 6.5).

Theorem 1.1. *Let $\alpha > -1$ and $\alpha \neq 0$. Let u be an α-harmonic function in $\mathbb{H}$ which is of temperate growth at infinity.*
  (1) *Assume that (1.2) holds.*
  (2) *Assume that there exists a sequence $\{z_j\}$ in $\mathbb{H}$ with $z_j \to \infty$ in $\mathbb{C}_\infty$ as $j \to \infty$ such that (1.4) holds.*
Then $u(z) = 0$ for all $z \in \mathbb{H}$.

We now turn our attention to the case $\alpha = 0$ of usual harmonic functions in $\mathbb{H}$. A function u belongs to the class $\mathcal{V}_0$ if and only if it has the form

$$u(z) = \sum_{k=1}^{n} c_k \, \mathrm{Im}(z^k), \quad z \in \mathbb{H},$$

for some $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ and $c_1, \ldots, c_n \in \mathbb{C}$ (see Proposition 5.2). (Thus, $\mathbb{Z}^+$ are the positive integers, while $\mathbb{N}$, introduced above, are the non-negative integers.) Observe that the harmonic polynomial

$$u_k(z) = \text{Im}(z^k), \quad z \in \mathbb{C},$$

vanishes on a union of k lines passing through the origin. Notice also that the zero set of u_k intersects the unit circle at the $2k$-th roots of unity. These examples make evident that the flexible uniqueness results for α-harmonic functions in $\mathbb{H}$ with $\alpha \neq 0$ are no longer true when $\alpha = 0$ (compare with Theorem 1.1 above).

In order to obtain satisfactory uniqueness results for usual harmonic functions in $\mathbb{H}$ we shall restrict condition (1.4) to suitable classes of curves. We consider two such classes of curves, namely, geodesics in $\mathbb{H}$ and rays in $\mathbb{H}$ emanating from the origin.

By a geodesic in $\mathbb{H}$ we understand a ray in $\mathbb{H}$ which is parallel to the imaginary axis. We prove that if $u \in \mathcal{V}_0$ is such that

$$\lim_{y \to +\infty} u(x + iy)/y = 0$$

for $x = x_j \in \mathbb{R}$ ($j = 1, 2$) with $x_1 \neq x_2$, then $u(z) = 0$ for all $z \in \mathbb{H}$ (see Theorem 7.1). This analysis leads to corresponding uniqueness results for usual harmonic functions in $\mathbb{H}$ (see Theorems 7.2 and 7.3). We emphasize that those geodesic uniqueness results require vanishing on two distinct geodesics which is best possible. Together with the results for $\alpha \neq 0$, this marks a significant advancement from an earlier uniqueness result of Carlsson and Wittsten [9, Corollary 1.9] for α-harmonic functions which required vanishing on an interval of geodesics.

By a ray in $\mathbb{H}$ emanating from a point $a \in \mathbb{R}$ we understand a set of the form $\{a + te^{i\theta} : t > 0\}$, where $0 < \theta < \pi$. We shall restrict our attention to rays emanating from the origin ($a = 0$). In view of translation invariance of the class of harmonic functions this restriction is minor. In order to discuss vanishing of functions along rays we introduce a notion of admissible function of angles which is a function element (E, η) with $E \subset (0, \pi)$ and $\eta : E \to \mathbb{Z}^+$ having the property that for every $k \in \mathbb{Z}^+$ there exists $\theta \in E$ such that $\sin(k\theta) \neq 0$ and $k \geq \eta(\theta)$ (see Definition 8.1). We prove that if $u \in \mathcal{V}_0$ and there exists an admissible function of angles (E, η) such that

$$\lim_{t \to +\infty} u(te^{i\theta})/t^{\eta(\theta)} = 0$$

for every $\theta \in E$, then $u(z) = 0$ for all $z \in \mathbb{H}$ (see Theorem 8.2). This analysis leads to corresponding uniqueness results for usual harmonic functions in $\mathbb{H}$ (see Theorems 8.3 and 8.4).

The notion of admissible function of angles involves an arithmetic element. In order to illuminate this fact we mention the following result.

Theorem 1.2. *Let u be a harmonic function in the open upper half-plane $\mathbb{H}$ which is of temperate growth at infinity.*
(1) *Assume that* (1.2) *holds.*
(2) *Assume that*

$$\lim_{t \to +\infty} u(te^{i\theta})/t = 0 \tag{1.5}$$

for some $0 < \theta < \pi$ which is not a rational multiple of π.
Then $u(z) = 0$ for all $z \in \mathbb{H}$.

The examples u_k above show that the arithmetic condition on θ in (1.5) is to the point. Behind Theorem 1.2 lies a construction of a one point admissible function of angles (E, η), where $E = \{\theta\}$ and $\eta(\theta) = 1$. The arithmetic condition on θ in (1.5) ensures that this latter function element (E, η) is an admissible function of angles. More elaborate constructions of admissible functions of angles lead to corresponding uniqueness results for usual harmonic functions in $\mathbb{H}$ (see Corollaries 8.5, 8.6, 8.7 and 8.8).

In the final section we provide constructions of admissible functions of angles (see Theorems 9.1 and 9.2). The set $\mathcal{A}$ of admissible functions of angles is structured by a natural partial order. We show that every element in $\mathcal{A}$ has a lower bound which is minimal (see Theorem 9.3 and Lemma 9.4). Our constructions of admissible functions of angles yield precisely the minimal elements in $\mathcal{A}$ (see Theorem 9.5).

There is a substantial literature on boundary uniqueness problems for (sub-)harmonic functions with contributions of Wolf [26], Shapiro [22], Dahlberg [10], Berman and Cohn [5] and Borichev with collaborators [6, 8] to name a few. As far as usual harmonic functions in $\mathbb{H}$ are concerned our uniqueness results supersede an earlier uniqueness result of Siegel and Talvila [23, Corollary 3.1].

The α-Laplacian (or rather its symmetric part) is also related to the Laplace-Beltrami equation in the Riemannian space defined by the metric

$$ds^2 = x_n^{-\alpha/(n-2)} \sum_1^n dx_i^2, \quad n > 2,$$

as studied by Weinstein [24] and Huber [15], among others. For historic reasons, solutions of said Laplace–Beltrami equation are referred to as generalized axially symmetric potentials. For a discussion on this connection and more recent applications in this direction we refer to Wittsten [25]. Another related area of

interest is the recent study of higher order Laplacians initiated by Borichev and Hedenmalm [7].

The present paper is rooted in previous investigations. In an earlier paper [20] we initiated a theory of α-harmonic functions in the open unit disc $\mathbb{D}$ in the complex plane. A function $u \in C^2(\mathbb{D})$ is called α-harmonic in $\mathbb{D}$ if $\Delta_{\mathbb{D};\alpha} u = 0$ in $\mathbb{D}$, where

$$\Delta_{\mathbb{D};\alpha,z} = \partial_z (1 - |z|^2)^{-\alpha} \bar{\partial}_z, \quad z \in \mathbb{D},$$

is the α-Laplacian for $\mathbb{D}$ and $\alpha \in \mathbb{R}$. A main concern in this theory is the representation of an α-harmonic function in $\mathbb{D}$ as a Poisson integral $u = P_\alpha[f]$ in $\mathbb{D}$ with respect to the kernel

$$(1.6) \qquad P_\alpha(z) = \frac{(1 - |z|^2)^{\alpha+1}}{(1 - z)(1 - \bar{z})^{\alpha+1}}, \quad z \in \mathbb{D}.$$

We refer to the function P_α as the α-harmonic Poisson kernel for $\mathbb{D}$.

We prove that a Poisson integral representation $u = P_\alpha[f]$ with f a distribution on the unit circle $\mathbb{T} = \partial\mathbb{D}$ exists if and only if u is α-harmonic in $\mathbb{D}$, u has temperate growth in $\mathbb{D}$ and a certain spectral condition is satisfied (see Theorem 3.3). This latter spectral condition is automatically satisfied when α is not a negative integer (see Corollary 3.4). For $\alpha > -1$, the distribution $f \in \mathscr{D}'(\mathbb{T})$ has a natural interpretation as a (distributional) boundary limit of the function u. On the other hand, for $\alpha \leq -1$, existence of a distributional boundary limit of an α-harmonic function u in $\mathbb{D}$, forces u to be analytic in $\mathbb{D}$ (see Theorem 3.5). We thus establish Poisson integral representations $u = P_\alpha[f]$ with $f \in \mathscr{D}'(\mathbb{T})$ in situations where a distributional boundary value of u is non-existent. To overcome those difficulties we resort to a study of related hypergeometric functions, found in Section 2.

A link between the settings of the upper half-plane $\mathbb{H}$ and the unit disc $\mathbb{D}$ is provided by a certain conformal invariance property of α-harmonic functions which was recently studied by the first author [18]. Let $\varphi : \mathbb{D} \to \mathbb{H}$ be a biholomorphic map. For a function u in $\mathbb{H}$ we consider the weighted pull-back

$$(1.7) \qquad v(z) = u_{\varphi,\alpha}(z) = \varphi'(z)^{-\alpha/2} u(\varphi(z)), \quad z \in \mathbb{D},$$

of u by φ with respect to the parameter α. Here the power in (1.7) is defined in the usual way using a logarithm of φ' in $\mathbb{D}$. We shall use the fact that the function v is α-harmonic in $\mathbb{D}$ if and only if the function u is α-harmonic in $\mathbb{H}$. This latter fact follows easily from [18, Theorem 1.1]. We refer to Geller [13] or Ahern et al. [1, 2] for earlier results.

Let us return to an α-harmonic function u in $\mathbb{H}$ satisfying some appropriate conditions, notably (1.2) and temperate growth at infinity. We consider a weighted

pull-back v of u of the form (1.7). The function v is α-harmonic in $\mathbb{D}$ and we propose to study this function by means of its Poisson integral representation $v = P_\alpha[f]$ in $\mathbb{D}$. An ambiguity lies in the choice of biholomorphic map $\varphi : \mathbb{D} \to \mathbb{H}$ which we chose as the Möbius transformation

$$\varphi(z) = i\frac{1+z}{1-z}.$$

Notice that $\varphi(0) = i$ and $\varphi(1) = \infty$.

From (1.2) we have that the boundary value f for v vanishes on the set $\mathbb{T} \setminus \{1\}$. Standard distribution theory then dictates that the distribution $f \in \mathscr{D}'(\mathbb{T})$ is a finite linear combination of derivatives of a Dirac mass δ_1 located at the point 1 on $\mathbb{T}$: $f = \sum_{k=0}^n c_k \delta_1^{(k)}$ in $\mathscr{D}'(\mathbb{T})$ (see Hörmander [14, Theorem 2.3.4]). The Poisson integral $P_\alpha[\delta_1']$ is naturally interpreted as an angular derivative iAP_α of the Poisson kernel P_α, and by iteration we find that $v = \sum_{k=0}^n c_k (iA)^k P_\alpha$ (see Corollary 3.7). In Section 4, those angular derivatives $(iA)^k P_\alpha$ of P_α are carefully investigated using the Möbius transformation φ above, and in Section 5 this part of the analysis culminates in the proof of the representation formula (1.3) (see Theorem 5.1).

2 Series expansion of α-harmonic functions in $\mathbb{D}$

In this section we revisit the series expansion of α-harmonic functions in $\mathbb{D}$. Of particular concern is a characterization of temperate growth of an α-harmonic function in $\mathbb{D}$ in terms of polynomial growth of coefficients (see Theorem 2.3). The proof of this latter result depends on properties of related hypergeometric functions.

For $a \in \mathbb{C}$, we set $(a)_0 = 1$ and

$$(a)_k = \prod_{j=0}^{k-1}(a+j)$$

for $k = 1, 2, \ldots$. The numbers $(a)_k$ are known as Pochhammer symbols. Notice that $(a)_k = \Gamma(a+k)/\Gamma(a)$ for $k \in \mathbb{N}$, where Γ is the standard Gamma function.

The hypergeometric function is the function defined by the power series expansion

$$(2.1) \qquad F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad z \in \mathbb{D},$$

for parameters $a, b, c \in \mathbb{C}$ with $c \neq 0, -1, -2, \ldots$. Convergence in (2.1) follows by the standard ratio test.

Recall also the classical binomial series

$$(2.2) \qquad (1 - z)^{-a} = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k, \quad z \in \mathbb{D},$$

for $a \in \mathbb{C}$. Notice that

$$F(a, b; b; z) = (1 - z)^{-a},$$

which follows from (2.1) and (2.2).

We take as our starting point a certain series expansion of α-harmonic functions in $\mathbb{D}$. It is known that a function u is α-harmonic in $\mathbb{D}$ if and only if it has the form

$$(2.3) \qquad u(z) = \sum_{k=0}^{\infty} c_k z^k + \sum_{k=1}^{\infty} c_{-k} F(-\alpha, k; k+1; |z|^2) \bar{z}^k, \quad z \in \mathbb{D},$$

for some sequence $\{c_k\}_{k=-\infty}^{\infty}$ of complex numbers such that

$$(2.4) \qquad \limsup_{|k| \to \infty} |c_k|^{1/|k|} \leq 1,$$

where F is the hypergeometric function (2.1) (see [20, Theorem 1.2]). Condition (2.4) ensures that the series expansion (2.3) is absolutely convergent in the space $C^{\infty}(\mathbb{D})$ of indefinitely differentiable functions in $\mathbb{D}$. As a consequence we have that $u \in C^{\infty}(\mathbb{D})$. We refer to Klintborg and Olofsson [16] for an updated account on these matters.

The property $F(a, b; c; 0) = 1$ of the hypergeometric function (2.1) yields a normalization of the expansion (2.3). As a consequence we have the coefficient formulas

$$c_k = \partial^k u(0)/k! \quad \text{and} \quad c_{-k} = \bar{\partial}^k u(0)/k!$$

for $k \in \mathbb{N}$, where u is as in (2.3) (see [16, Theorem 5.3]). In particular, an α-harmonic function in $\mathbb{D}$ is uniquely determined by its germ at the origin.

As an example of a series expansion of the form (2.3) we mention that of the α-harmonic Poisson kernel

$$(2.5) \qquad P_{\alpha}(z) = \sum_{k=0}^{\infty} z^k + \sum_{k=1}^{\infty} \frac{(\alpha + 1)_k}{k!} F(-\alpha, k; k+1; |z|^2) \bar{z}^k$$

for $z \in \mathbb{D}$ (see [16, Theorem 6.3]).

We shall make use of a classical result known as Euler's integral formula for the hypergeometric function. This result says that

$$(2.6) \qquad F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} \, dt$$

for $z \in \mathbb{D}$ when Re $c >$ Re $b > 0$, where Γ is the standard Gamma function (see [3, Theorem 2.2.1]).

In particular, from (2.6) we have that

$$(2.7) \qquad F(-\alpha, k; k+1; x) = k \int_0^1 t^{k-1}(1-xt)^\alpha \, dt, \quad 0 \le x < 1,$$

for $k \in \mathbb{Z}^+$ and $\alpha \in \mathbb{R}$. From this latter formula we see that the function $F(-\alpha, k; k+1; \cdot)$ is non-negative on the interval $[0, 1)$. Furthermore, the function $F(-\alpha, k; k+1; \cdot)$ is decreasing on the interval $[0, 1)$ if $\alpha \ge 0$ and, similarly, the function $F(-\alpha, k; k+1; \cdot)$ is increasing on the interval $[0, 1)$ if $\alpha \le 0$.

A passage to the limit in (2.7) shows that

$$(2.8) \qquad \lim_{x \to 1} F(-\alpha, k; k+1; x) = \frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)}$$

for $\alpha > -1$, where we have used a standard formula for the Beta function. When $\alpha \le -1$, the quantity $F(-\alpha, k; k+1; x)$ diverges to $+\infty$ as $x \to 1$.

We shall need some more detailed estimates of the hypergeometric functions appearing in (2.3).

Lemma 2.1. *Let* $k \in \mathbb{Z}^+$*. Then*

$$F(1, k; k+1; x) \le \frac{k}{x} \log \left(\frac{1}{1-x} \right)$$

for $0 < x < 1$*.*

Proof. From (2.7) we have that

$$F(1, k; k+1; x) = k \int_0^1 \frac{t^{k-1}}{1-xt} \, dt$$

for $0 \le x < 1$. An integration by parts shows that

$$F(1, k; k+1; x) = \frac{k}{x} \log \left(\frac{1}{1-x} \right) + \frac{k(k-1)}{x} \int_0^1 t^{k-2} \log(1-xt) \, dt$$

for $0 < x < 1$. Observe that the logarithm in the rightmost integral is negative. This yields the conclusion of the lemma. $\qquad \square$

We shall use also another result of Euler which says that

$$(2.9) \qquad F(a, b; c; z) = (1-z)^{c-a-b} F(c-a, c-b; c; z)$$

for $z \in \mathbb{D}$ (see [3, Theorem 2.2.5]).

Lemma 2.2. *Let $\alpha < -1$ and $k \in \mathbb{Z}^+$. Then*

$$F(-\alpha, k; k+1; x) \leq \max\left(1, -\frac{k}{\alpha+1}\right)(1-x)^{\alpha+1}$$

for $0 \leq x < 1$.

Proof. We first apply (2.9) to see that

(2.10) $$F(-\alpha, k; k+1; x) = (1-x)^{\alpha+1} F(k+\alpha+1, 1; k+1; x)$$

for $0 \leq x < 1$. We shall divide into cases depending on whether $k + \alpha + 1 > 0$ or $k + \alpha + 1 \leq 0$.

Assume first that $k + \alpha + 1 \leq 0$. By (2.6) we have that

$$F(k+\alpha+1, 1; k+1; x) = k \int_0^1 (1-t)^{k-1}(1-xt)^{-(k+\alpha+1)} \, dt$$

for $0 \leq x < 1$. Since $k+\alpha+1 \leq 0$, we have that this latter hypergeometric function $F(k+\alpha+1, 1; k+1; \cdot)$ is decreasing on $[0, 1)$. Therefore $F(k+\alpha+1, 1; k+1; x) \leq 1$ for $0 \leq x < 1$. In view of (2.10) this yields the conclusion of the lemma for $k+\alpha+1 \leq 0$.

We next assume that $k + \alpha + 1 > 0$. By symmetry and (2.6) we have that

$$F(k+\alpha+1, 1; k+1; x) = F(1, k+\alpha+1; k+1; x)$$
$$= \frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)\Gamma(-\alpha)} \int_0^1 t^{k+\alpha}(1-t)^{-(\alpha+1)}\frac{1}{1-xt} \, dt$$

for $0 \leq x < 1$. By monotonicity we have that

$$F(k+\alpha+1, 1; k+1; x) \leq \frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)\Gamma(-\alpha)} \int_0^1 t^{k+\alpha}(1-t)^{-(\alpha+1)-1} \, dt$$
$$= \frac{\Gamma(k+1)\Gamma(-(\alpha+1))}{\Gamma(-\alpha)\Gamma(k)} = -\frac{k}{\alpha+1}$$

for $0 \leq x < 1$, where the last two equalities follow by standard formulas for the Beta and Gamma functions. In view of (2.10) this yields the conclusion of the lemma for $k + \alpha + 1 > 0$. $\square$

We say that a function u in $\mathbb{D}$ is of temperate growth in $\mathbb{D}$ if it satisfies an estimate of the form

(2.11) $$|u(z)| \leq C(1 - |z|^2)^{-N}, \quad z \in \mathbb{D},$$

for some positive constants C and N.

Theorem 2.3. *Let $\alpha \in \mathbb{R}$. Let u be an α-harmonic function in $\mathbb{D}$ and consider the expansion (2.3). Then the function u is of temperate growth in $\mathbb{D}$ if and only if the sequence of coefficients $\{c_k\}_{k=-\infty}^{\infty}$ in (2.3) has at most polynomial growth.*

Proof. Assume that u has temperate growth in $\mathbb{D}$. We first show that the coefficient c_k has at most polynomial growth as $k \to +\infty$. Let $k \in \mathbb{N}$ and $0 < r < 1$. From (2.3) we have that

$$c_k r^k = \frac{1}{2\pi} \int_{\mathbb{T}} u(re^{i\theta}) e^{-ik\theta} \, d\theta.$$

From the triangle inequality and (2.11) we have that

$$|c_k| r^k \leq C/(1 - r^2)^N \leq C/(1 - r)^N$$

for $k \in \mathbb{N}$ and $0 < r < 1$, where C is as in (2.11). Choosing $r = 1 - 1/k$ with k big in this latter inequality, we see that $|c_k| \leq C' k^N$ as $k \to +\infty$.

We next show that the coefficient c_k has at most polynomial growth as $k \to -\infty$. Let $k \in \mathbb{Z}^- = \mathbb{Z} \setminus \mathbb{N}$ and $0 < r < 1$. From (2.3) we have that

$$(2.12) \qquad c_k F(-\alpha, |k|; |k| + 1; r^2) r^{|k|} = \frac{1}{2\pi} \int_{\mathbb{T}} u(re^{i\theta}) e^{-ik\theta} \, d\theta.$$

Let us first consider the case $\alpha \leq 0$. Since the function $F(-\alpha, |k|; |k| + 1; \cdot)$ is increasing on $[0, 1)$, we have from (2.12) and the triangle inequality that

$$|c_k| r^{|k|} \leq C/(1 - r)^N,$$

where C is as in (2.11). Choosing $r = 1 - 1/|k|$ in this latter inequality we see that $|c_k| \leq C' |k|^N$ as $k \to -\infty$.

Let us now consider the case $\alpha \geq 0$. Since the function $F(-\alpha, |k|; |k| + 1; \cdot)$ is decreasing on $[0, 1)$, we have from (2.12) and the triangle inequality that

$$(2.13) \qquad |c_k| \frac{\Gamma(|k| + 1)\Gamma(\alpha + 1)}{\Gamma(|k| + \alpha + 1)} r^{|k|} \leq C/(1 - r)^N,$$

where we have used (2.8) and C is as in (2.11). Stirling's formula ensures that a quotient of Gamma functions $\Gamma(x)/\Gamma(x + \alpha)$ behaves asymptotically as $1/x^\alpha$ when $x \to +\infty$ ([3, Section 1.4]). Choosing $r = 1 - 1/|k|$ in (2.13) we see that $|c_k| \leq C' |k|^{N+\alpha}$ as $k \to -\infty$.

Assume now that the sequence of coefficients $\{c_k\}_{k=-\infty}^{\infty}$ in (2.3) has at most polynomial growth, that is, $|c_k| \leq C(1 + |k|)^N$ for $k \in \mathbb{Z}$, where $N \geq 0$. We shall show that the function u has temperate growth in $\mathbb{D}$. We consider first the leftmost sum

$$f(z) = \sum_{k=0}^{\infty} c_k z^k, \quad z \in \mathbb{D},$$

appearing in (2.3). Observe that

$$(1 + k)^N \leq \frac{(k + N)!}{k!} = N! \frac{1}{k!} \prod_{j=1}^{k} (N + j) = N! \frac{(N + 1)_k}{k!}$$

for $k \in \mathbb{N}$. By the triangle inequality we have that

$$|f(z)| \leq C \sum_{k=0}^{\infty} (1+k)^N |z|^k \leq CN! \sum_{k=0}^{\infty} \frac{(N+1)_k}{k!} |z|^k = CN!(1-|z|)^{-(N+1)}$$

for $z \in \mathbb{D}$, where in the last equality we have used (2.2). This proves that the function f above has temperate growth in $\mathbb{D}$.

We consider next the function

$$g(z) = \sum_{k=1}^{\infty} c_{-k} F(-\alpha, k; k+1; |z|^2) \bar{z}^k, \quad z \in \mathbb{D},$$

appearing in (2.3). Assume first that $\alpha \geq 0$. Then the function $F(-\alpha, k; k+1; \cdot)$ is decreasing on $[0, 1)$. From the triangle inequality we have that

$$|g(z)| \leq \sum_{k=1}^{\infty} |c_{-k}||z|^k \leq C \sum_{k=1}^{\infty} (1+k)^N |z|^k \leq CN!(1-|z|)^{-(N+1)}$$

for $z \in \mathbb{D}$, where the last two inequalities follow as in the previous paragraph. This proves that the function g above has temperate growth in $\mathbb{D}$ if $\alpha \geq 0$.

Assume next that $-1 < \alpha \leq 0$. In this case the function $F(-\alpha, k; k+1; \cdot)$ is increasing on $[0, 1)$. By the triangle inequality and (2.8) we have that

$$|g(z)| \leq \sum_{k=1}^{\infty} |c_{-k}| \frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)} |z|^k$$

for $z \in \mathbb{D}$. Since $\Gamma(x)/\Gamma(x+\alpha)$ behaves asymptotically as $1/x^\alpha$ when $x \to +\infty$ we deduce as above that the function g has temperate growth in $\mathbb{D}$ if $-1 < \alpha \leq 0$.

Assume next that $\alpha = -1$. By the triangle inequality and Lemma 2.1 we have that

$$|g(z)| \leq \sum_{k=1}^{\infty} |c_{-k}| \frac{k}{|z|^2} \log\left(\frac{1}{1-|z|^2}\right) |z|^k \leq \frac{C}{|z|^2} \log\left(\frac{1}{1-|z|^2}\right) \sum_{k=1}^{\infty} (1+k)^{N+1} |z|^k$$

for $z \in \mathbb{D}$. We can now deduce as above that the function g has temperate growth in $\mathbb{D}$ if $\alpha = -1$.

Assume finally that $\alpha < -1$. By the triangle inequality and Lemma 2.2 we have that

$$|g(z)| \leq \sum_{k=1}^{\infty} |c_{-k}| kC(\alpha)(1-|z|^2)^{\alpha+1} |z|^k \leq CC(\alpha)(1-|z|^2)^{\alpha+1} \sum_{k=1}^{\infty} (1+k)^{N+1} |z|^k$$

for $z \in \mathbb{D}$, where $C(\alpha) = \max(1, -1/(\alpha+1))$. We can then once again deduce as above that the function g has temperate growth in $\mathbb{D}$ if $\alpha < -1$. $\qquad\square$

The case division in the proof of Theorem 2.3 depends on different behaviors of the hypergeometric functions appearing in (2.3). One can notice that the quantity $F(-\alpha, k; k + 1; x)$ is decreasing in $\alpha \in \mathbb{R}$. This observation leads to a shorter proof of Theorem 2.3 with less precision. An earlier result can be found in Olofsson [17, Proposition 3.2].

3 Poisson integrals of distributions

The purpose of this section is to further develop a theory of Poisson integral representations of α-harmonic functions in $\mathbb{D}$. Of particular interest is a characterization of Poisson integrals of distributions on $\mathbb{T}$ (see Theorem 3.3 and Corollary 3.4). Along the way we introduce some notation needed later.

We denote by $\mathscr{D}'(\mathbb{T})$ the space of distributions on $\mathbb{T}$. An integrable function $f \in L^1(\mathbb{T})$ on $\mathbb{T}$ is identified with the distribution

$$\langle f, \varphi \rangle = \frac{1}{2\pi} \int_{\mathbb{T}} f(e^{i\theta}) \varphi(e^{i\theta}) \, d\theta, \quad \varphi \in C^\infty(\mathbb{T}),$$

where $C^\infty(\mathbb{T})$ is the space of indefinitely differentiable test functions on $\mathbb{T}$. The space $\mathscr{D}'(\mathbb{T})$ is topologized in the usual way using the semi-norms

$$\mathscr{D}'(\mathbb{T}) \ni f \mapsto |\langle f, \varphi \rangle|$$

for $\varphi \in C^\infty(\mathbb{T})$. Notice that $f_k \to f$ in $\mathscr{D}'(\mathbb{T})$ as $k \to \infty$ means that the limit $\lim_{k\to\infty} \langle f_k, \varphi \rangle = \langle f, \varphi \rangle$ holds for every $\varphi \in C^\infty(\mathbb{T})$.

Let

$$\phi_k(e^{i\theta}) = e^{ik\theta}, \quad e^{i\theta} \in \mathbb{T},$$

for $k \in \mathbb{Z}$ be the exponential monomials on $\mathbb{T}$. The Fourier coefficients of a distribution $f \in \mathscr{D}'(\mathbb{T})$ are defined by

$$\hat{f}(k) = \langle f, \phi_{-k} \rangle, \quad k \in \mathbb{Z}.$$

A distribution on $\mathbb{T}$ is uniquely determined by its sequence of Fourier coefficients. It is well-known that a sequence of complex numbers $\{c_k\}_{k=-\infty}^{\infty}$ is of at most polynomial growth if and only if it is the sequence of Fourier coefficients for some distribution on $\mathbb{T}$, that is, there exists $f \in \mathscr{D}'(\mathbb{T})$ such that $\hat{f}(k) = c_k$ for $k \in \mathbb{Z}$.

The derivative of a distribution $f \in \mathscr{D}'(\mathbb{T})$ is the distribution f' in $\mathscr{D}'(\mathbb{T})$ determined by $(f')\hat{\,}(k) = ik\hat{f}(k)$ for $k \in \mathbb{Z}$.

The convolution $h = f * g$ of the distributions $f \in \mathscr{D}'(\mathbb{T})$ and $g \in \mathscr{D}'(\mathbb{T})$ is the distribution $h \in \mathscr{D}'(\mathbb{T})$ determined by $\hat{h}(k) = \hat{f}(k)\hat{g}(k)$ for $k \in \mathbb{Z}$. Notice that $f * g \in C^\infty(\mathbb{T})$ if $f \in \mathscr{D}'(\mathbb{T})$ and $g \in C^\infty(\mathbb{T})$.

For a suitably smooth function u in $\mathbb{D}$ we set

$$(3.1) \qquad u_r(e^{i\theta}) = u(re^{i\theta}), \quad e^{i\theta} \in \mathbb{T},$$

for $0 \leq r < 1$. Notice that the function u_r is essentially the restriction of u to the circle $\{z \in \mathbb{C} : |z| = r\}$.

Recall formula (1.6). The α-harmonic Poisson integral of a distribution $f \in \mathscr{D}'(\mathbb{T})$ is defined by

$$P_\alpha[f](z) = (P_{\alpha,r} * f)(e^{i\theta}), \quad z = re^{i\theta} \in \mathbb{D},$$

where $*$ denotes convolution, $P_{\alpha,r}$ is defined in accordance with (3.1), $r \geq 0$ and $e^{i\theta} \in \mathbb{T}$. Observe that

$$P_\alpha[f](z) = \frac{1}{2\pi} \int_{\mathbb{T}} P_\alpha(ze^{-i\tau})f(e^{i\tau})\,d\tau, \quad z \in \mathbb{D},$$

if $f \in L^1(\mathbb{T})$.

We next calculate the series expansion of the Poisson integral.

Proposition 3.1. *Let $\alpha \in \mathbb{R}$ and $f \in \mathscr{D}'(\mathbb{T})$. Then*

$$P_\alpha[f](z) = \sum_{k=0}^{\infty} \hat{f}(k)z^k + \sum_{k=1}^{\infty} \hat{f}(-k)\frac{(\alpha+1)_k}{k!}F(-\alpha, k; k+1; |z|^2)\bar{z}^k$$

for $z \in \mathbb{D}$, where $\hat{f}(k)$ is the k-th Fourier coefficient of f.

Proof. Let $0 \leq r < 1$. From (2.5) we have that

$$P_{\alpha,r}(e^{i\theta}) = \sum_{k=0}^{\infty} r^k e^{ik\theta} + \sum_{k=1}^{\infty} \frac{(\alpha+1)_k}{k!}F(-\alpha, k; k+1; r^2)r^k e^{-ik\theta}$$

for $e^{i\theta} \in \mathbb{T}$. Passing to the convolution we have that

$$P_\alpha[f](z) = (P_{\alpha,r} * f)(e^{i\theta}) = \sum_{k=0}^{\infty} r^k \hat{f}(k)e^{ik\theta}$$

$$+ \sum_{k=1}^{\infty} \frac{(\alpha+1)_k}{k!}F(-\alpha, k; k+1; r^2)r^k \hat{f}(-k)e^{-ik\theta}$$

for $z = re^{i\theta} \in \mathbb{D}$ with $r \geq 0$ and $e^{i\theta} \in \mathbb{T}$. This yields the conclusion of the proposition. $\square$

Notice that the expansion in Proposition 3.1 is a series expansion of the form (2.3). As a consequence, we have that the function $P_\alpha[f]$ is α-harmonic in $\mathbb{D}$. We record also that

$$(3.2) \qquad \frac{(\alpha+1)_k}{k!} = \frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+1)\Gamma(k+1)} \sim \frac{1}{\Gamma(\alpha+1)}(1+k)^\alpha$$

as $k \to +\infty$ which follows by Stirling's formula.

Recall the notion of Fourier spectrum of a distribution on $\mathbb{T}$ defined by

$$\mathrm{Spec}(f) = \{ k \in \mathbb{Z} : \hat{f}(k) \neq 0 \}$$

for $f \in \mathscr{D}'(\mathbb{T})$. We shall need a similar notion of spectrum of an α-harmonic function in $\mathbb{D}$. Recall that an α-harmonic function in $\mathbb{D}$ is uniquely determined by its sequence of coefficients in (2.3). For such a function u we set

$$\mathrm{Spec}(u) = \{ k \in \mathbb{Z} : c_k \neq 0 \},$$

where the c_k's are as in (2.3). Observe that $\mathrm{Spec}(u) = \emptyset$ if and only if $u = 0$.

Let us denote by $\mathbb{Z}^-$ the set of negative integers. In view of the series expansion (2.5) of the function P_α we have that $\mathrm{Spec}(P_\alpha) = \mathbb{Z}$ if $\alpha \in \mathbb{R} \setminus \mathbb{Z}^-$ whereas

$$\mathrm{Spec}(P_\alpha) = \{ \alpha + 1, \ \alpha + 2, \dots \}$$

if $\alpha \in \mathbb{Z}^-$.

Corollary 3.2. *Let $\alpha \in \mathbb{R}$ and $f \in \mathscr{D}'(\mathbb{T})$. Then*

$$\mathrm{Spec}(P_\alpha[f]) = \mathrm{Spec}(P_\alpha) \cap \mathrm{Spec}(f).$$

In particular, $P_\alpha[f] = 0$ if $\alpha \in \mathbb{Z}^-$ and $\mathrm{Spec}(f) \subset \{ k \in \mathbb{Z} : k \leq \alpha \}$.

Proof. The result is evident from Proposition 3.1. When $\alpha \in \mathbb{Z}^-$ the inclusion $\mathrm{Spec}(f) \subset \{ k \in \mathbb{Z} : k \leq \alpha \}$ ensures that $\mathrm{Spec}(P_\alpha) \cap \mathrm{Spec}(f) = \emptyset$.   □

We next turn our attention to a characterization of Poisson integrals of distributions.

Theorem 3.3. *Let $\alpha \in \mathbb{R}$. Then a function u in $\mathbb{D}$ has the form of a Poisson integral $u = P_\alpha[f]$ in $\mathbb{D}$ for some $f \in \mathscr{D}'(\mathbb{T})$ if and only if u is α-harmonic in $\mathbb{D}$, u has temperate growth in $\mathbb{D}$, and $\mathrm{Spec}(u) \subset \mathrm{Spec}(P_\alpha)$.*

Proof. Assume first that $u = P_\alpha[f]$ in $\mathbb{D}$ for some $f \in \mathscr{D}'(\mathbb{T})$. From a well-known characterization of Fourier coefficients of distributions on $\mathbb{T}$ we know that the sequence of Fourier coefficients $\{\hat{f}(k)\}_{k=-\infty}^\infty$ has at most polynomial growth. Proposition 3.1 supplies us with the series expansion of u. Clearly u is α-harmonic in $\mathbb{D}$. From Corollary 3.2 we have that $\mathrm{Spec}(u) \subset \mathrm{Spec}(P_\alpha)$. By Theorem 2.3 we conclude that u has at most temperate growth in $\mathbb{D}$.

Assume next that u is α-harmonic in $\mathbb{D}$, u has temperate growth in $\mathbb{D}$, and $\mathrm{Spec}(u) \subset \mathrm{Spec}(P_\alpha)$. Consider the series expansion (2.3). By Theorem 2.3 we

conclude that the sequence of coefficients $\{c_k\}_{k=-\infty}^{\infty}$ in (2.3) has at most polynomial growth. We set $a_k = c_k$ for $k \in \mathbb{N}$,

$$a_k = \frac{(-k)!}{(\alpha + 1)_{-k}} c_k$$

for $k \in \mathbb{Z}^-$ and $k \in \mathrm{Spec}(P_\alpha)$, and $a_k = 0$ for $k \in \mathbb{Z}^-$ and $k \notin \mathrm{Spec}(P_\alpha)$. From (3.2) we have that the sequence $\{a_k\}_{k=-\infty}^{\infty}$ is of polynomial growth. From a well-known characterization of Fourier coefficients of distributions on $\mathbb{T}$ there exists $f \in \mathscr{D}'(\mathbb{T})$ such that $\hat{f}(k) = a_k$ for $k \in \mathbb{Z}$. Since $\mathrm{Spec}(u) \subset \mathrm{Spec}(P_\alpha)$, we have by Proposition 3.1 that $u = P_\alpha[f]$ in $\mathbb{D}$. $\qquad \square$

The spectral condition in Theorem 3.3 is redundant when α is not a negative integer.

Corollary 3.4. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Z}^-$. Then a function u in $\mathbb{D}$ has the form of a Poisson integral $u = P_\alpha[f]$ in $\mathbb{D}$ for some $f \in \mathscr{D}'(\mathbb{T})$ if and only if it is α-harmonic in $\mathbb{D}$ and of temperate growth there.*

Proof. Recall that $\mathrm{Spec}(P_\alpha) = \mathbb{Z}$ in the present situation. The result follows from Theorem 3.3. $\qquad \square$

The α-harmonic Poisson kernel P_α has bounded L^1-means when $\alpha > -1$ (see Olofsson [17, Theorem 3.1]). As a consequence, for $\alpha > -1$, one has that $u_r \to f$ in $\mathscr{D}'(\mathbb{T})$ as $r \to 1$ if $f \in \mathscr{D}'(\mathbb{T})$ and $u = P_\alpha[f]$, where u_r is as in (3.1).

The boundary behavior of α-harmonic functions in $\mathbb{D}$ is conceptually different when $\alpha \leq -1$. The next result exemplifies this fact.

Theorem 3.5. *Let $\alpha \leq -1$ and let u be an α-harmonic function in $\mathbb{D}$. Assume that the limit $f = \lim_{r \to 1} u_r$ in $\mathscr{D}'(\mathbb{T})$ exists, where u_r is as in (3.1). Then u is analytic in $\mathbb{D}$.*

Proof. Consider the series expansion (2.3). We shall prove that $c_k = 0$ for $k < 0$. Let $k \in \mathbb{Z}^+$. From (2.3) we have that

$$(3.3) \qquad c_{-k} F(-\alpha, k; k+1; r^2) r^k = \frac{1}{2\pi} \int_{\mathbb{T}} u(re^{i\theta}) e^{ik\theta} \, d\theta$$

for $0 < r < 1$. By assumption the right-hand side in (3.3) has a limit as $r \to 1$. Recall from the discussion following formulas (2.7)–(2.8) that the quantity $F(-\alpha, k; k+1; x)$ increases to $+\infty$ as $x \to 1$. Passing to the limit in (3.3) as $r \to 1$ we conclude that $c_{-k} = 0$. This yields the conclusion of the theorem. $\qquad \square$

We refer to Olofsson [17, Theorem 2.3] for a result analogous to Theorem 3.5 phrased in another setting of generalized harmonic functions in $\mathbb{D}$. Under an assumption of pointwise boundary limit, a similar result also exists for the class of generalized axially symmetric potentials mentioned in the introduction; see Huber [15, Theorem 2].

A traditional approach to the Poisson integral representation

$$(3.4) \qquad u(z) = P_\alpha[f](z), \quad z \in \mathbb{D},$$

is to exhibit the distribution $f \in \mathscr{D}'(\mathbb{T})$ as a limit point of the u_r's as $r \to 1$ and then deduce the Poisson integral representation (3.4) from a uniqueness argument. See for instance Olofsson and Wittsten [20, Theorem 5.5] for an elaboration on this theme. An interesting point of Theorem 3.3 is that this result establishes Poisson integral representations (3.4) in cases where boundary limits are non-existent.

Structure in (2.3) suggests use of the differential operator $A = z\partial - \bar{z}\bar{\partial}$. Observe that

$$(3.5) \qquad Au(z) = \sum_{k=1}^{\infty} kc_k z^k - \sum_{k=1}^{\infty} kc_{-k} F(-\alpha, k; k+1; |z|^2)\bar{z}^k, \quad z \in \mathbb{D},$$

if u has the form (2.3). In particular, the function Au is α-harmonic in $\mathbb{D}$ if u is.

Proposition 3.6. *Let $\alpha \in \mathbb{R}$ and $f \in \mathscr{D}'(\mathbb{T})$. Then*

$$iAP_\alpha[f](z) = P_\alpha[f'](z)$$

for $z \in \mathbb{D}$, where f' is the distributional derivative of f.

Proof. Recall that $(f')\hat{}(k) = ik\hat{f}(k)$ for $k \in \mathbb{Z}$. By Proposition 3.1 we have that

$$P_\alpha[f'](z) = \sum_{k=0}^{\infty} ik\hat{f}(k)z^k + \sum_{k=1}^{\infty} (-ik)\hat{f}(-k)\frac{(\alpha+1)_k}{k!} F(-\alpha, k; k+1; |z|^2)\bar{z}^k$$

$$= iAP_\alpha[f](z)$$

for $z \in \mathbb{D}$, where in the last equality we have used (3.5). $\qquad\square$

We refer to the differential operator $iA = i(z\partial - \bar{z}\bar{\partial})$ as the angular derivative. Our interest in this operator arose in connection to the paper Olofsson [19].

Let $\delta_1 \in \mathscr{D}'(\mathbb{T})$ be the unit Dirac mass located at the point $1 \in \mathbb{T}$, that is, $\langle \delta_1, \varphi \rangle = \varphi(1)$ for $\varphi \in C^\infty(\mathbb{T})$.

Corollary 3.7. *Let $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$. Then*

$$P_\alpha[\delta_1^{(k)}](z) = (iA)^k P_\alpha(z)$$

for $z \in \mathbb{D}$, where $\delta_1^{(k)}$ denotes the k-th distributional derivative of δ_1.

Proof. It is straightforward to check that $(\delta_1)\hat{}(k) = 1$ for $k \in \mathbb{Z}$. By Proposition 3.1 and formula (2.5) we have that $P_\alpha[\delta_1](z) = P_\alpha$. The result now follows by Proposition 3.6. $\qquad\square$

4   Angular derivatives of Poisson kernels

This section is devoted to a careful analysis of angular derivatives of Poisson kernels. We shall make good use of the Möbius transformation

$$(4.1) \qquad\qquad \varphi(z) = i\frac{1+z}{1-z}$$

in our calculations. Notice that φ maps $\mathbb{D}$ one-to-one onto $\mathbb{H}$, $\varphi(0) = i$ and $\varphi(1) = \infty$. From standard theory we have that the Möbius transformation φ is uniquely determined by these three properties. For the sake of easy reference we record also the formulas

$$(4.2) \qquad \operatorname{Im}\varphi(z) = \frac{1-|z|^2}{|1-z|^2} \quad\text{and}\quad \varphi'(z) = \frac{2i}{(1-z)^2},$$

which are straightforward to check.

A most natural α-harmonic function in $\mathbb{H}$ is the $(\alpha+1)$-th power of the imaginary part:

$$u(z) = (\operatorname{Im} z)^{\alpha+1}, \quad z \in \mathbb{H}.$$

We first calculate the weighted pull-back $u_{\varphi,\alpha}$ from (1.7) of this function by φ.

Theorem 4.1. *Let $\alpha \in \mathbb{R}$. Then*

$$P_\alpha(z) = c\varphi'(z)^{-\alpha/2}(\operatorname{Im}\varphi(z))^{\alpha+1}, \quad z \in \mathbb{D},$$

where φ is as in (4.1) and $c\varphi'(0)^{-\alpha/2} = 1$.

Proof. Recall formula (1.6). The two formulas in (4.2) make evident that

$$P_\alpha(z) = c\varphi'(z)^{-\alpha/2}(\operatorname{Im}\varphi(z))^{\alpha+1}, \quad z \in \mathbb{D},$$

where $c \in \mathbb{C}$. From $P_\alpha(0) = 1$ we see that $c\varphi'(0)^{-\alpha/2} = 1$. $\qquad\square$

Notice that Theorem 4.1 with $\alpha = 0$ yields the well-known formula

$$P_0(z) = \operatorname{Re}\left(\frac{1+z}{1-z}\right), \quad z \in \mathbb{D},$$

for the usual Poisson kernel for $\mathbb{D}$.

Our next task is to calculate angular derivatives of Poisson kernels. We begin with a preparatory lemma.

Lemma 4.2. *Let φ be as in* (4.1). *Then $iA\varphi = \frac{1}{2}(\varphi^2 + 1)$.*

Proof. From formula (4.2) we have that

$$iA\varphi(z) = iz\varphi'(z) = -2z/(1 - z)^2.$$

Using some elementary algebra we now calculate that

$$iA\varphi(z) = \frac{1}{2}\frac{(1 - z)^2 - (1 + z)^2}{(1 - z)^2} = \frac{1}{2}\left(1 - \left(\frac{1 + z}{1 - z}\right)^2\right) = \frac{1}{2}(1 + \varphi(z)^2),$$

where the last equality follows by (4.1). $\qquad\square$

Let us record some properties of the angular derivative. The differential operator iA satisfies the product rule for differentiation: $iA(fg) = giA(f) + fiA(g)$ for suitable f and g. Denote by $\bar{f}$ the pointwise complex conjugate of a complex-valued function f. We shall use the chain rule in the form

(4.3) $$\qquad iA(h \circ f) = ((\partial h) \circ f)iAf + ((\bar{\partial} h) \circ f)\overline{iAf}$$

for suitable f and h. Formula (4.3) is straightforward to check. As a consequence of (4.3) we have that the differential operator iA commutes with the action of complex conjugation of functions: $iA(\bar{f}) = \overline{iAf}$ for suitable f.

We now turn to differentiation of powers.

Lemma 4.3. *Let φ be as in* (4.1) *and $\alpha \in \mathbb{R}$. Then*

$$iA((\operatorname{Im}\varphi)^{\alpha+1}) = (\alpha + 1)(\operatorname{Re}\varphi)(\operatorname{Im}\varphi)^{\alpha+1} \quad in \ \mathbb{D}.$$

Proof. Let $u = (\operatorname{Im}\varphi)^{\alpha+1}$ in $\mathbb{D}$. By a standard rule for differentiation we have that

$$iAu = (\alpha + 1)(\operatorname{Im}\varphi)^{\alpha}iA\left(\frac{1}{2i}(\varphi - \bar{\varphi})\right)$$

in $\mathbb{D}$, compare with (4.3). Since the differential operator iA commutes with the action of complex conjugation of functions we have that

$$iAu = (\alpha + 1)(\operatorname{Im}\varphi)^{\alpha}\frac{1}{2i}(iA\varphi - \overline{iA\varphi})$$

in $\mathbb{D}$. We now use Lemma 4.2 and calculate that

$$iAu = (\alpha + 1)(\operatorname{Im}\varphi)^{\alpha}\frac{1}{2i}\left(\frac{1}{2}(\varphi^2 + 1) - \frac{1}{2}(\bar{\varphi}^2 + 1)\right) = (\alpha + 1)(\operatorname{Im}\varphi)^{\alpha}\frac{1}{2i}\frac{1}{2}(\varphi^2 - \bar{\varphi}^2)$$

in $\mathbb{D}$, where the last equality follows by cancellation. By some elementary algebra we now have that

$$iAu = (\alpha + 1)(\operatorname{Im}\varphi)^{\alpha}\frac{1}{2i}\frac{1}{2}(\varphi - \bar{\varphi})(\varphi + \bar{\varphi}) = (\alpha + 1)(\operatorname{Im}\varphi)^{\alpha}(\operatorname{Im}\varphi)(\operatorname{Re}\varphi)$$

in $\mathbb{D}$. This yields the conclusion of the lemma. $\qquad\square$

Lemma 4.4. *Let φ be as in* (4.1) *and* $\alpha \in \mathbb{R}$. *Then* $iA((\varphi')^{-\alpha/2}) = \frac{\alpha}{2}(i-\varphi)(\varphi')^{-\alpha/2}$ *in* $\mathbb{D}$.

Proof. We first show that $iA\varphi' = (\varphi - i)\varphi'$. From (4.2) we have that

$$iA\varphi'(z) = iz\varphi''(z) = (2i)^2 \frac{z}{(1-z)^3} = i\frac{2z}{1-z}\varphi'(z),$$

where the last equality again follows by (4.2). By some elementary algebra we now have that

$$iA\varphi'(z) = i\frac{(1+z)-(1-z)}{1-z}\varphi'(z) = i\Big(\frac{1+z}{1-z} - 1\Big)\varphi'(z) = (\varphi(z)-i)\varphi'(z),$$

where the last equality follows by (4.1).

We now consider the general case. By a well-known differentiation formula we have that

$$iA((\varphi')^{-\alpha/2}) = -\frac{\alpha}{2}(\varphi')^{-\alpha/2-1}iA\varphi'$$

in $\mathbb{D}$, compare with (4.3). We now use the result of the previous paragraph to conclude that

$$iA((\varphi')^{-\alpha/2}) = -\frac{\alpha}{2}(\varphi')^{-\alpha/2-1}(\varphi-i)\varphi' = \frac{\alpha}{2}(i-\varphi)(\varphi')^{-\alpha/2}$$

in $\mathbb{D}$. □

We next calculate the angular derivative of the Poisson kernel.

Theorem 4.5. *Let* $\alpha \in \mathbb{R}$ *and let* φ *be as in* (4.1). *Then*

$$iAP_\alpha(z) = \frac{1}{2}(\varphi(z) + (\alpha+1)\overline{\varphi(z)} + i\alpha)P_\alpha(z)$$

for $z \in \mathbb{D}$.

Proof. Let $u = (\varphi')^{-\alpha/2}(\operatorname{Im}\varphi)^{\alpha+1}$ in $\mathbb{D}$. In view of Theorem 4.1 it suffices to prove the conclusion of the theorem with P_α replaced by the function u. By the product rule for differentiation we have that

$$iAu = (\operatorname{Im}\varphi)^{\alpha+1}iA((\varphi')^{-\alpha/2}) + (\varphi')^{-\alpha/2}iA((\operatorname{Im}\varphi)^{\alpha+1})$$

in $\mathbb{D}$. We now use Lemmas 4.3 and 4.4 to conclude that

$$iAu = (\operatorname{Im}\varphi)^{\alpha+1}\frac{\alpha}{2}(i-\varphi)(\varphi')^{-\alpha/2} + (\varphi')^{-\alpha/2}(\alpha+1)(\operatorname{Re}\varphi)(\operatorname{Im}\varphi)^{\alpha+1}$$

$$= \Big(\frac{\alpha}{2}(i-\varphi) + (\alpha+1)\frac{1}{2}(\varphi+\bar{\varphi})\Big)u$$

in $\mathbb{D}$, where the last equality is straightforward to check. This yields the conclusion of the theorem. □

An earlier version of Theorem 4.5 appears in Olofsson [19, Theorem 1.11]; see also Klintborg and Olofsson [16, Corollary 1.3] for a generalization along similar lines. Here our focus is on applications to the setting of the upper half-plane and we express matters using the Möbius transformation φ.

We denote by $\mathbb{C}[z, \bar{z}]$ the algebra of polynomials in z and $\bar{z}$ with complex coefficients. Define polynomials $\{h_{k,\alpha}\}_{k=0}^{\infty}$ in $\mathbb{C}[z, \bar{z}]$ by $h_{0,\alpha} = 1$ and

$$(4.4) \qquad h_{k+1,\alpha}(z) = \frac{1}{2}(z^2 + 1)\partial h_{k,\alpha}(z) + \frac{1}{2}(\bar{z}^2 + 1)\bar{\partial} h_{k,\alpha}(z)$$
$$+ \frac{1}{2}(z + (\alpha + 1)\bar{z} + i\alpha)h_{k,\alpha}(z)$$

for $k \geq 0$. It is straightforward to check that $h_{k,\alpha}$ has degree at most k for $k = 0, 1, \ldots$.

Theorem 4.6. *Let $\alpha \in \mathbb{R}$ and let φ be as in (4.1). Then*

$$(iA)^k P_\alpha(z) = h_{k,\alpha}(\varphi(z))P_\alpha(z), \quad z \in \mathbb{D},$$

for $k = 0, 1, 2, \ldots$, where the $h_{k,\alpha}$'s are as in (4.4).

Proof. For $k = 0$ the result is evident. For $k = 1$ the result follows by Theorem 4.5. We proceed by induction and assume that $(iA)^k P_\alpha = (h_{k,\alpha} \circ \varphi)P_\alpha$ for some $k \geq 0$. Applying the operator iA we have that

$$(iA)^{k+1}P_\alpha = P_\alpha iA(h_{k,\alpha} \circ \varphi) + (h_{k,\alpha} \circ \varphi)iAP_\alpha,$$

where we have used the product rule. From the chain rule (4.3) and Lemma 4.2 we have that

$$iA(h_{k,\alpha} \circ \varphi) = ((\partial h_{k,\alpha}) \circ \varphi)\frac{1}{2}(\varphi^2 + 1) + ((\bar{\partial} h_{k,\alpha}) \circ \varphi)\frac{1}{2}(\bar{\varphi}^2 + 1).$$

We now return to the function $(iA)^{k+1}P_\alpha$ and use Theorem 4.5 to conclude that

$$(iA)^{k+1}P_\alpha = P_\alpha\Big(((\partial h_{k,\alpha}) \circ \varphi)\frac{1}{2}(\varphi^2 + 1) + ((\bar{\partial} h_{k,\alpha}) \circ \varphi)\frac{1}{2}(\bar{\varphi}^2 + 1)\Big)$$
$$+ (h_{k,\alpha} \circ \varphi)\frac{1}{2}(\varphi + (\alpha + 1)\bar{\varphi} + i\alpha)P_\alpha = (h_{k+1,\alpha} \circ \varphi)P_\alpha,$$

where in the last equality we have used (4.4). The result now follows by invoking the induction principle. □

We shall study the polynomials $h_{k,\alpha}$ in some more detail.

Lemma 4.7. *Let $h_{k,\alpha} \in \mathbb{C}[z, \bar{z}]$ be as in (4.4) for some $\alpha \in \mathbb{R}$ and $k \geq 0$. Then the function $\mathbb{H} \ni z \mapsto (\operatorname{Im} z)^{\alpha+1}h_{k,\alpha}(z)$ is α-harmonic in $\mathbb{H}$.*

Proof. Recall that the function iAu is α-harmonic in $\mathbb{D}$ if u is, see (3.5). From Theorem 4.6 we thus have that the function $(h_{k,\alpha} \circ \varphi)P_\alpha$ is α-harmonic in $\mathbb{D}$. Theorem 4.1 displays the Poisson kernel P_α as a weighed pull-back by the function φ. From Olofsson [18, Theorem 1.1] we now conclude that the function $z \mapsto (\operatorname{Im} z)^{\alpha+1} h_{k,\alpha}(z)$ is α-harmonic in $\mathbb{H}$; see discussion following formula (1.7). $\square$

Let $\alpha \in \mathbb{R}$ and let us denote by y the imaginary part. A calculation shows that

$$\Delta_{\alpha;\mathbb{H}} y^{\alpha+1} = \frac{1}{2i} D_\alpha$$

in the sense of differential operators, where

$$(4.5) \qquad D_\alpha = D_{\alpha,z} = (z - \bar{z})\partial_z \bar{\partial}_z + \bar{\partial}_z - (\alpha + 1)\partial_z.$$

As a consequence, we have that a product $y^{\alpha+1} h$ is α-harmonic in $\mathbb{H}$ if and only if $D_\alpha h = 0$ in $\mathbb{H}$, where D_α is as in (4.5).

Notice that if $p \in \mathbb{C}[z, \bar{z}]$ is homogeneous of degree m, then the polynomial $D_\alpha p$ is homogeneous of degree $m - 1$.

Consider the partial sums

$$(4.6) \qquad s_{k,\alpha}(z) = \sum_{j=0}^{k} \frac{(\alpha + 1)_j}{j!} z^j$$

for $k = 0, 1, \ldots$ of a binomial series (2.2) with $a = \alpha + 1$. We shall need the associated homogeneous polynomials

$$(4.7) \qquad p_{k,\alpha}(z) = z^k s_{k,\alpha}(\bar{z}/z) = \sum_{j=0}^{k} \frac{(\alpha + 1)_j}{j!} z^{k-j} \bar{z}^j$$

for $k = 0, 1, \ldots$, where $\alpha \in \mathbb{R}$. Notice that $p_{k,\alpha} \in \mathbb{C}[z, \bar{z}]$ is homogeneous of degree k.

We now return to the differential operator D_α in (4.5).

Theorem 4.8. *Let $\alpha \in \mathbb{R}$. Let p in $\mathbb{C}[z, \bar{z}]$ be homogeneous of degree k. Then $D_\alpha p = 0$ if and only if p is a constant multiple of $p_{k,\alpha}$, where $p_{k,\alpha}$ is as in (4.7).*

Proof. We shall evaluate the differential operator D_α on a polynomial p of the form

$$(4.8) \qquad p(z) = \sum_{j=0}^{k} a_j z^{k-j} \bar{z}^j$$

for some $a_0, \ldots, a_k \in \mathbb{C}$. Calculations show that

$$\bar{\partial}p(z) = \sum_{j=0}^{k-1}(j+1)a_{j+1}z^{k-1-j}\bar{z}^j, \quad \partial p(z) = \sum_{j=0}^{k-1}(k-j)a_j z^{k-1-j}\bar{z}^j,$$

and

$$(z-\bar{z})\partial\bar{\partial}p(z) = \sum_{j=0}^{k-2}(j+1)(k-1-j)a_{j+1}z^{k-1-j}\bar{z}^j - \sum_{j=1}^{k-1}j(k-j)a_j z^{k-1-j}\bar{z}^j.$$

From these formulas we have that

$$D_\alpha p(z) = \sum_{j=0}^{k-1}(k-j)((j+1)a_{j+1} - (j+\alpha+1)a_j)z^{k-1-j}\bar{z}^j$$

for p of the form (4.8).

From the result of the previous paragraph we have that $D_\alpha p = 0$ if and only if

$$(j+1)a_{j+1} - (j+\alpha+1)a_j = 0$$

for $j = 0, \ldots, k-1$. We conclude that $D_\alpha p = 0$ if and only if $p = a_0 p_{k,\alpha}$. $\qquad\square$

Corollary 4.9. *Let $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$. Let $p_{k,\alpha}$ be as in (4.7). Then the function $\mathbb{H} \ni z \mapsto (\operatorname{Im} z)^{\alpha+1}p_{k,\alpha}(z)$ is α-harmonic in $\mathbb{H}$.*

Proof. By Theorem 4.8 we have that $D_\alpha p_{k,\alpha} = 0$. Therefore $\Delta_\alpha y^{\alpha+1}p_{k,\alpha} = 0$ in $\mathbb{H}$ by the factorization formula preceding (4.5). $\qquad\square$

We now return to the $h_{k,\alpha}$'s.

Theorem 4.10. *Let $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$. Let $h_{k,\alpha}$ be as in (4.4). Then $h_{k,\alpha}$ is a linear combination of $p_{0,\alpha}, \ldots, p_{k,\alpha}$, where the $p_{j,\alpha}$'s are as in (4.7).*

Proof. From Lemma 4.7 we have that the product $y^{\alpha+1}h_{k,\alpha}$ is α-harmonic in $\mathbb{H}$. Therefore $D_\alpha h_{k,\alpha} = 0$ in $\mathbb{C}[z, \bar{z}]$, where D_α is as in (4.5). We next write the polynomial $h_{k,\alpha}$ as a sum of homogeneous polynomials: $h_{k,\alpha} = \sum_{j=0}^{k}p_j$, where p_j is homogeneous of degree j. Clearly p_0 is a constant multiple of $p_{0,\alpha}$. Applying the differential operator D_α we see that

$$0 = D_\alpha h_{k,\alpha} = \sum_{j=1}^{k}D_\alpha p_j$$

in $\mathbb{C}[z, \bar{z}]$. Since $D_\alpha p_j$ is homogeneous of degree $j-1$, we have that $D_\alpha p_j = 0$ for $j = 1, \ldots k$. From Theorem 4.8 we have that p_j is a constant multiple of $p_{j,\alpha}$ for $j = 1, \ldots k$. We conclude that $h_{k,\alpha}$ is a linear combination of $p_{0,\alpha}, \ldots, p_{k,\alpha}$. $\qquad\square$

5 A general representation theorem

In this section we consider α-harmonic functions in $\mathbb{H}$ which are of temperate growth at infinity and vanish on the real line. A main task is to establish a representation of such functions. The analysis uses results from Section 4.

Theorem 5.1. *Let $\alpha > -1$. Let u be an α-harmonic function in $\mathbb{H}$ which is of temperate growth at infinity. Assume that $u(z) \to 0$ as $\mathbb{H} \ni z \to x$ for every $x \in \mathbb{R}$. Then*

$$(5.1) \qquad u(z) = \sum_{k=0}^{n} c_k (\operatorname{Im} z)^{\alpha+1} p_{k,\alpha}(z), \quad z \in \mathbb{H},$$

for some $n \in \mathbb{N}$ and $c_0, \ldots, c_n \in \mathbb{C}$, where the $p_{k,\alpha}$'s are as in (4.7).

Proof. We consider the weighted pull-back

$$v(z) = \varphi'(z)^{-\alpha/2} u(\varphi(z)), \quad z \in \mathbb{D},$$

where φ is as in (4.1). From [18, Theorem 1.1] it follows that v is α-harmonic in $\mathbb{D}$. From temperate growth of u at infinity and vanishing of u on $\mathbb{R}$ we have by a compactness argument that v is of temperate growth in $\mathbb{D}$. By Theorem 3.3 we conclude that $v = P_\alpha[g]$ for some distribution $g \in \mathscr{D}'(\mathbb{T})$. Since $\alpha > -1$, it follows that $v_r \to g$ in $\mathscr{D}'(\mathbb{T})$ as $r \to 1$, where the v_r's are defined as in (3.1) (see for instance [20, Theorem 5.4]). Since $u(z) \to 0$ as $\mathbb{H} \ni z \to x$ for $x \in \mathbb{R}$, we see that g vanishes on $\mathbb{T} \setminus \{1\}$ in the sense of distributions. Now since $\operatorname{supp}(g) \subset \{1\}$, standard distribution theory dictates that $g = \sum_{k=0}^{n} a_k \delta_1^{(k)}$ in $\mathscr{D}'(\mathbb{T})$ for some $n \in \mathbb{N}$ and complex numbers $a_0, \ldots, a_n \in \mathbb{C}$ (see Hörmander [14, Theorem 2.3.4]). By Corollary 3.7 we have that $P_\alpha[\delta_1^{(k)}] = (iA)^k P_\alpha$ for $k \geq 0$. Passing to the Poisson integral we have that

$$(5.2) \qquad v(z) = \sum_{k=0}^{n} a_k (iA)^k P_\alpha(z) = \left(\sum_{k=0}^{n} a_k h_{k,\alpha}(\varphi(z)) \right) P_\alpha(z)$$

for $z \in \mathbb{D}$, where the last equality follows by Theorem 4.6. By Theorem 4.10 the function $h_{k,\alpha}$ is a linear combination of $p_{0,\alpha}, \ldots, p_{k,\alpha}$. From (5.2) we conclude that

$$v(z) = \left(\sum_{k=0}^{n} b_k p_{k,\alpha}(\varphi(z)) \right) P_\alpha(z), \quad z \in \mathbb{D},$$

for some $b_0, \ldots, b_n \in \mathbb{C}$. Invoking Theorem 4.1 we see that

$$v(z) = \varphi'(z)^{-\alpha/2} \left(\sum_{k=0}^{n} c b_k (\operatorname{Im} \varphi(z))^{\alpha+1} p_{k,\alpha}(\varphi(z)) \right)$$

for $z \in \mathbb{D}$, where $c\varphi'(0)^{-\alpha/2} = 1$. A passage back to the function u yields (5.1) with $c_k = cb_k$ for $k = 0, \ldots, n$. This completes the proof of the theorem. $\square$

For $\alpha > -1$ and $n \in \mathbb{N}$, we denote by $\mathcal{V}_{\alpha,n}$ the set of all functions u of the form (5.1) for some $c_0, \ldots, c_n \in \mathbb{C}$, where the $p_{k,\alpha}$'s are as in (4.7). We also set

$$\mathcal{V}_{\alpha} = \bigcup_{n=0}^{\infty} \mathcal{V}_{\alpha,n}.$$

The space $\mathcal{V}_{\alpha,n}$ is a complex vector space of finite dimension $n + 1$. Notice that the set $\mathcal{V}_{\alpha}$ is a complex vector space which is naturally filtered by the sets $\mathcal{V}_{\alpha,n}$ for $n = 0, 1, \ldots$.

Observe that the conclusion of Theorem 5.1 says that $u \in \mathcal{V}_{\alpha}$. We next describe the class $\mathcal{V}_0$ in some more detail.

Proposition 5.2. *Let $n \in \mathbb{N}$. Then a function u belongs to the space $\mathcal{V}_{0,n}$ if and only if it has the form*

$$(5.3) \qquad u(z) = \sum_{k=1}^{n+1} c_k \operatorname{Im}(z^k), \quad z \in \mathbb{H},$$

for some $c_1, \ldots, c_{n+1} \in \mathbb{C}$.

Proof. A calculation using the formula for a finite geometric sum shows that

$$p_{k,0}(z) = \sum_{j=0}^{k} z^{k-j}\bar{z}^{j} = \frac{z^{k+1} - \bar{z}^{k+1}}{z - \bar{z}}.$$

Therefore

$$(\operatorname{Im} z)p_{k,0}(z) = \frac{z - \bar{z}}{2i}\frac{z^{k+1} - \bar{z}^{k+1}}{z - \bar{z}} = \operatorname{Im}(z^{k+1}).$$

The result is now evident from definition of the space $\mathcal{V}_{0,n}$. $\square$

We shall next investigate the order of growth of a function of the form

$$(5.4) \qquad u_{k,\alpha}(z) = (\operatorname{Im} z)^{\alpha+1}p_{k,\alpha}(z), \quad z \in \mathbb{H},$$

for some $k \in \mathbb{N}$ and $\alpha > -1$, where $p_{k,\alpha}$ is as in (4.7). From the proof of Proposition 5.2 we have that

$$u_{k,0}(z) = \operatorname{Im}(z^{k+1}), \quad z \in \mathbb{C},$$

for $k \in \mathbb{N}$.

Proposition 5.3. *Let $u_{k,\alpha}$ be as in (5.4) for some $k \in \mathbb{N}$ and $\alpha > -1$. Then*

$$|u_{k,\alpha}(z)| \leq \max_{0 < \theta < \pi} \sin^{k+2\alpha+2}(\theta)|p_{k,\alpha}(e^{i\theta})|(|z|^2/\operatorname{Im}(z))^{k+\alpha+1}$$

for $z \in \mathbb{H}$. In particular, if $u \in \mathcal{V}_{\alpha,n}$ for some $n \in \mathbb{N}$, then u has order of growth at most $n + \alpha + 1$ at infinity.

Proof. Let $z \in \mathbb{H}$ and write $z = te^{i\theta}$ with $t > 0$ and $0 < \theta < \pi$. By homogeneity we have that

$$u_{k,\alpha}(z) = \sin^{\alpha+1}(\theta)p_{k,\alpha}(e^{i\theta})t^{k+\alpha+1}.$$

Notice also that $|z|^2/\operatorname{Im}(z) = t/\sin(\theta)$. From these two facts we have that

$$u_{k,\alpha}(z) = \sin^{k+2\alpha+2}(\theta)p_{k,\alpha}(e^{i\theta})(|z|^2/\operatorname{Im}(z))^{k+\alpha+1}$$

for $z \in \mathbb{H}$, where $\theta = \arg z$. This yields the conclusion of the proposition. □

The next result points out the significance of the class $\mathcal{V}_\alpha$.

Corollary 5.4. *Let $\alpha > -1$. Then a function u belongs to the class $\mathcal{V}_\alpha$ if and only if it is α-harmonic in $\mathbb{H}$, has temperate growth at infinity and vanishes on the real line in the sense that $u(z) \to 0$ as $\mathbb{H} \ni z \to x$ for every $x \in \mathbb{R}$.*

Proof. The if part is a restatement of Theorem 5.1. From Corollary 4.9 we have that every function $u \in \mathcal{V}_\alpha$ is α-harmonic in $\mathbb{H}$. From Proposition 5.3 we have that every function $u \in \mathcal{V}_\alpha$ is of temperate growth at infinity. It is evident that every function $u \in \mathcal{V}_\alpha$ vanishes on the real line. □

Corollary 5.4 explains how the space $\mathcal{V}_\alpha$ can be thought of as the class of obstructions for the uniqueness problem for α-harmonic functions in $\mathbb{H}$ with respect to a vanishing Dirichlet boundary value on the real line and temperate growth at infinity.

Lemma 5.5. *Let $u \in \mathcal{V}_{\alpha,n}$ be of the form (5.1) for some $\alpha > -1$ and $n \in \mathbb{N}$. Then*

$$\lim_{t \to +\infty} u(te^{i\theta})/t^{n+\alpha+1} = c_n \sin^{\alpha+1}(\theta)p_{n,\alpha}(e^{i\theta})$$

for $0 < \theta < \pi$.

Proof. Let $t > 0$ and $0 < \theta < \pi$. By homogeneity we have that

$$u(te^{i\theta}) = \sum_{k=0}^{n} c_k(\sin\theta)^{\alpha+1}p_{k,\alpha}(e^{i\theta})t^{k+\alpha+1}.$$

The result now follows by a passage to the limit. □

We shall dissect the space $\mathcal{V}_\alpha$ using orders of growth at infinity. We say that a function u in $\mathbb{H}$ has order of growth $n + \alpha + 1$ at infinity if it satisfies an estimate of the form

$$(5.5) \qquad |u(z)| \leq C(|z|^2 / \operatorname{Im}(z))^{n+\alpha+1}$$

for $z \in \mathbb{H}$ with $|z| > R$, where $C \geq 0$ and $R > 1$ are finite constants.

Lemma 5.6. *Let $u \in \mathcal{V}_\alpha$ for some $\alpha > -1$. Let $n \in \mathbb{N}$. Then $u \in \mathcal{V}_{\alpha,n}$ if and only if (5.5) holds. Moreover, there exists a constant $C' = C'_{\alpha,n,R}$ such that $|c_k| \leq C'C$ for $k = 0, 1, \ldots, n$ whenever $u \in \mathcal{V}_{\alpha,n}$ has the form (5.1) and C is as in (5.5).*

Proof. From Proposition 5.3 we know that (5.5) holds if $u \in \mathcal{V}_{\alpha,n}$. Assume next that $u \in \mathcal{V}_\alpha$ satisfies (5.5). We shall prove that $u \in \mathcal{V}_{\alpha,n}$. If $u \in \mathcal{V}_{\alpha,0}$ there is nothing to prove. Assume therefore that $u \in \mathcal{V}_{\alpha,m}$ and $u \notin \mathcal{V}_{\alpha,m-1}$ for some $m \in \mathbb{Z}^+$. An application of Lemma 5.5 with $0 < \theta < \pi$ chosen such that $p_{m,\alpha}(e^{i\theta}) \neq 0$ shows that u has order of growth at least $m + \alpha + 1$. Therefore $m \leq n$, so that $u \in \mathcal{V}_{\alpha,n}$.

For fixed $R > 1$ we equip the space $\mathcal{V}_{\alpha,n}$ with the semi-norm given by the best constant C in (5.5). Another application of Lemma 5.5 shows that this latter semi-norm is in fact a norm. The space $\mathcal{V}_{\alpha,n}$ is thus a normed complex vector space of finite dimension $n + 1$. Since any two norms on such a space are equivalent, we conclude that $|c_k| \leq C'C$ for $k = 0, 1, \ldots, n$ whenever $u \in \mathcal{V}_{\alpha,n}$. $\qquad\square$

We denote by $\mathscr{D}'(\mathbb{R})$ the space of distributions on the real line $\mathbb{R}$. An integrable function $f \in L^1(\mathbb{R})$ is identified with the distribution

$$\langle f, \varphi \rangle = \int_{-\infty}^{\infty} f(x)\varphi(x)\,dx, \quad \varphi \in C_0^\infty(\mathbb{R}),$$

where $C_0^\infty(\mathbb{R})$ is the space of indefinitely differentiable test functions on $\mathbb{R}$ with compact support. By $u_j \to u$ in $\mathscr{D}'(\mathbb{R})$ as $j \to \infty$ we understand that

$$\lim_{j \to \infty} \langle u_j, \varphi \rangle = \langle u, \varphi \rangle$$

for every $\varphi \in C_0^\infty(\mathbb{R})$. A standard reference for distribution theory is Hörmander [14].

We shall next extend the validity of Theorem 5.1 to allow for a distributional boundary value.

Theorem 5.7. *Let $\alpha > -1$. Let u be an α-harmonic function in $\mathbb{H}$ which is of temperate growth at infinity. Assume that $\lim_{y \to 0+} u(\cdot + iy) = 0$ in $\mathscr{D}'(\mathbb{R})$. Then $u \in \mathcal{V}_\alpha$.*

Proof. The theorem is proved by a regularization argument. Let $\psi \in C_0^\infty(\mathbb{R})$ be a non-negative compactly supported test function with $\int_{-\infty}^\infty \psi(x)\,dx = 1$ and set $\psi_\varepsilon(x) = \psi(x/\varepsilon)/\varepsilon$ for $x \in \mathbb{R}$ and $\varepsilon > 0$. We consider the regularizations

$$u_\varepsilon(z) = \int_{-\infty}^\infty u(z - t)\psi_\varepsilon(t)\,dt, \quad z \in \mathbb{H},$$

for $\varepsilon > 0$. A differentiation under the integral shows that the function u_ε is α-harmonic in $\mathbb{H}$. It is straightforward to check that $u_\varepsilon(z) \to 0$ as $\mathbb{H} \ni z \to x$ for every $x \in \mathbb{R}$. Using that u is of temperate growth at infinity it is straightforward to check that

$$(5.6) \qquad\qquad |u_\varepsilon(z)| \le C(|z|^2/\operatorname{Im}(z))^{n+\alpha+1}$$

for $z \in \mathbb{H}$ with $|z| > R$ and $0 < \varepsilon < 1$, where $C \ge 0$ and $R > 1$ are finite constants and $n \in \mathbb{N}$. Notice also that $u = \lim_{\varepsilon \to 0} u_\varepsilon$ in $\mathbb{H}$ in the sense of normal convergence.

We now proceed to details. By Theorem 5.1 we have that $u_\varepsilon \in \mathcal{V}_\alpha$. From (5.6) and Lemma 5.6 we have that $u_\varepsilon \in \mathcal{V}_{\alpha,n}$. Therefore

$$(5.7) \qquad\qquad u_\varepsilon(z) = \sum_{k=0}^n c_k(\varepsilon)(\operatorname{Im} z)^{\alpha+1} p_{k,\alpha}(z), \quad z \in \mathbb{H},$$

for some $c_0(\varepsilon), \ldots, c_n(\varepsilon) \in \mathbb{C}$. Another application of Lemma 5.6 gives that $|c_k(\varepsilon)| \le C'C$ for $k = 0, 1, \ldots, n$, where C is as in (5.6) and $C' = C'_{\alpha,n,R}$ is a positive constant. By a compactness argument we can extract a subsequence $\varepsilon = \varepsilon_j \to 0$ of positive real numbers such that $c_k(\varepsilon_j) \to c_k$ as $j \to \infty$ for $k = 0, 1, \ldots, n$. The conclusion of the theorem now follows by setting $\varepsilon = \varepsilon_j$ in (5.7) and letting $j \to \infty$. $\square$

We point out that the assumption in Theorem 5.7 is that

$$\lim_{y \to 0} \int_{-\infty}^\infty u(x + iy)\varphi(x)\,dx = 0$$

for every $\varphi \in C_0^\infty(\mathbb{R})$.

The case of usual harmonic functions deserves special mention.

Corollary 5.8. *Let u be a harmonic function in $\mathbb{H}$ which is of temperate growth at infinity. Assume that $\lim_{y \to 0+} u(\cdot + iy) = 0$ in $\mathscr{D}'(\mathbb{R})$. Then $u \in \mathcal{V}_0$.*

We record also the following slight improvement of the if part of Lemma 5.6.

Proposition 5.9. *Let $u \in \mathcal{V}_\alpha$ for some $\alpha > -1$. Let $n \in \mathbb{N}$. Assume that*

$$u(z) = o((|z|^2/\operatorname{Im}(z))^{n+\alpha+2})$$

as $\mathbb{H} \ni z \to \infty$ in the extended complex plane. Then $u \in \mathcal{V}_{\alpha,n}$.

Proof. If $u \in \mathcal{V}_{\alpha,0}$ there is nothing to prove. Assume therefore that $u \in \mathcal{V}_{\alpha,m}$ and $u \notin \mathcal{V}_{\alpha,m-1}$ for some $m \in \mathbb{Z}^+$. An application of Lemma 5.5 with $0 < \theta < \pi$ chosen such that $p_{m,\alpha}(e^{i\theta}) \neq 0$ shows that u has order of growth at least $m + \alpha + 1$ at infinity. Therefore $m < n + 1$, so that $u \in \mathcal{V}_{\alpha,n}$. $\square$

Proposition 5.9 is included for the sake of convenience. We shall prove more refined versions of Proposition 5.9 in later sections.

6   A uniqueness result for the case $\alpha \neq 0$

We now turn our attention to uniqueness results for α-harmonic functions in $\mathbb{H}$. In this section we study the case of parameters $\alpha > -1$ such that $\alpha \neq 0$. The special case of usual harmonic functions is investigated in later sections. We begin our analysis with a study of the zeros of the polynomials $p_{k,\alpha}$.

A classical result of Eneström–Kakeya says that if

(6.1) $$p(z) = \sum_{k=0}^{n} a_k z^k$$

is an analytic polynomial of degree $n \geq 0$ such that $0 \leq a_k \leq a_{k+1}$ for $0 \leq k < n$, then the zeros of p are all located in the closed unit disc $\mathbb{D}$. An easy proof of this result can be found in Gardner and Govil [12, Theorem 1.3].

Corollary 6.1. *Let p be an analytic polynomial of the form* (6.1) *with positive coefficients: $a_k > 0$ for $0 \leq k \leq n$. Set*

$$r = \min_{0 \leq k < n} a_k/a_{k+1} \quad and \quad R = \max_{0 \leq k < n} a_k/a_{k+1}.$$

Then the zeroes of p are all located in the annulus $\{z \in \mathbb{C} : r \leq |z| \leq R\}$.

Proof. We first show that the zeros of p are all located in the disc $\{z \in \mathbb{C} : |z| \leq R\}$. Consider the polynomial $g(z) = p(Rz)$. By assumption the polynomial g satisfies the assumptions of the Eneström–Kakeya theorem quoted above. We conclude that the zeros of g are all located in $\overline{\mathbb{D}}$. This yields that the zeros of p are all located in the disc $\{z \in \mathbb{C} : |z| \leq R\}$.

We next show that the zeros of p are all located in the exterior disc $\{z \in \mathbb{C} : |z| \geq r\}$. Consider the polynomial $h(z) = z^n p(1/z)$. Notice that

$$h(z) = \sum_{k=0}^{n} a_{n-k} z^k.$$

By the result of the previous paragraph, the zeros of h are all located in the disc $\{z \in \mathbb{C} : |z| \leq 1/r\}$. Therefore the zeros of p are all located in the exterior disc $\{z \in \mathbb{C} : |z| \geq r\}$. $\square$

Our interest in zeros of polynomials stems from the following result.

Theorem 6.2. *Let $\alpha > -1$ and $\alpha \neq 0$. Let $p_{k,\alpha}$ be as in (4.7) for some $k \in \mathbb{N}$. Then $p_{k,\alpha}(e^{i\theta}) \neq 0$ for $e^{i\theta} \in \mathbb{T}$.*

Proof. By (4.7) it suffices to show that $s_{k,\alpha}(e^{i\theta}) \neq 0$ for $e^{i\theta} \in \mathbb{T}$, where $s_{k,\alpha}$ is as in (4.6). Set $a_j(\alpha) = \frac{1}{j!}(\alpha + 1)_j$. Observe that

$$(6.2) \qquad \frac{a_j(\alpha)}{a_{j+1}(\alpha)} = \frac{j+1}{\alpha+j+1} = 1 - \frac{\alpha}{\alpha+j+1}$$

for $0 \leq j < k$.

Assume next that $\alpha > 0$. In this case the quotients in (6.2) increase in j and we have that $a_j(\alpha)/a_{j+1}(\alpha) \leq k/(\alpha+k)$ for $0 \leq j < k$. By Corollary 6.1 the zeros of $s_{k,\alpha}$ are all located in the disc $\{z \in \mathbb{C} : |z| \leq k/(\alpha + k)\}$ which is compactly contained in $\mathbb{D}$.

Assume next that $-1 < \alpha < 0$. In this case the quotients in (6.2) decrease in j and we have that $a_j(\alpha)/a_{j+1}(\alpha) \geq 1/(\alpha + 1)$ for $0 \leq j < k$. By Corollary 6.1 the zeros of $s_{k,\alpha}$ are all located in the exterior disc $\{z \in \mathbb{C} : |z| \geq 1/(\alpha + 1)\}$ which is disjoint from $\overline{\mathbb{D}}$. $\qquad\square$

Remark 6.3. We point out that the zero set of $p_{k,0}$ intersects the unit circle if $k \geq 1$. In fact, from (4.6) we have that

$$s_{k,0}(z) = \sum_{j=0}^{k} z^j = (1 - z^{k+1})/(1 - z)$$

for $k \in \mathbb{N}$.

We now consider the class $\mathcal{V}_\alpha$.

Theorem 6.4. *Let $u \in \mathcal{V}_\alpha$ for some $\alpha > -1$ with $\alpha \neq 0$. Assume that there is a sequence $\{z_j\}$ in $\mathbb{H}$ with $|z_j| \to \infty$ as $j \to \infty$ such that*

$$(6.3) \qquad \lim_{j \to \infty} \frac{u(z_j)}{(\mathrm{Im}(z_j))^{\alpha+1}} = 0.$$

Then $u(z) = 0$ for all $z \in \mathbb{H}$.

Proof. We express $u \in \mathcal{V}_\alpha$ in the form (5.1) for some $n \in \mathbb{N}$ and constants $c_0, \ldots, c_n \in \mathbb{C}$, where the $p_{k,\alpha}$'s are as in (4.7). Recall that the polynomial $p_{k,\alpha}$ is homogeneous of degree $k \in \mathbb{N}$. By Theorem 6.2 the polynomial $p_{k,\alpha}$ has no zeros on the unit circle. From these two properties of $p_{k,\alpha}$ we have that the

quotient $|p_{k,\alpha}(z)|/|z|^k$ is bounded from above and below by finite positive constants uniformly in the punctured plane. From (6.3) we now have that

$$0 = \lim_{j \to \infty} \frac{u(z_j)}{p_{n,\alpha}(z_j)(\mathrm{Im}(z_j))^{\alpha+1}} = c_n.$$

Repeating the argument we conclude that $c_k = 0$ for $0 \le k \le n$. Therefore $u(z) = 0$ for all $z \in \mathbb{H}$. ▫

We emphasize that condition (6.3) can be checked along any sequence $\{z_j\}$ in $\mathbb{H}$ such that $z_j \to \infty$ in the extended complex plane $\mathbb{C}_\infty$ as $j \to \infty$.

We now turn to uniqueness theorems for α-harmonic functions with $\alpha \ne 0$. We first prove Theorem 1.1 stated in the introduction.

Proof of Theorem 1.1. By assumption (1) (formula (1.2)) we can apply Theorem 5.1 to conclude that $u \in \mathcal{V}_\alpha$. We next apply Theorem 6.4 to conclude that $u(z) = 0$ for $z \in \mathbb{H}$. ▫

We next extend the validity of Theorem 1.1 to allow for a distributional boundary value instead of (1.2).

Theorem 6.5. *Let $\alpha > -1$ and $\alpha \ne 0$. Let u be an α-harmonic function in $\mathbb{H}$ which is of temperate growth at infinity.*
(1)′ *Assume that $\lim_{y \to 0+} u(\cdot + iy) = 0$ in $\mathscr{D}'(\mathbb{R})$.*
(2) *Assume that there is a sequence $\{z_j\}$ in $\mathbb{H}$ with $z_j \to \infty$ in $\mathbb{C}_\infty$ as $j \to \infty$ such that (6.3) holds.*
Then $u(z) = 0$ for all $z \in \mathbb{H}$.

Proof. In view of assumption (1)′ we can apply Theorem 5.7 to conclude that $u \in \mathcal{V}_\alpha$. We next apply Theorem 6.4 to conclude that $u(z) = 0$ for $z \in \mathbb{H}$. ▫

We close this section with a relaxed version of Theorem 6.4.

Theorem 6.6. *Let $u \in \mathcal{V}_\alpha$ for some $\alpha > -1$ with $\alpha \ne 0$. Let $n \in \mathbb{N}$. Assume that there is a sequence $\{z_j\}$ in $\mathbb{H}$ with $z_j \to \infty$ in $\mathbb{C}_\infty$ as $j \to \infty$ such that*

$$\lim_{j \to \infty} \frac{u(z_j)}{|z_j|^{n+1}(\mathrm{Im}(z_j))^{\alpha+1}} = 0.$$

Then $u \in \mathcal{V}_{\alpha,n}$.

The proof of Theorem 6.6 follows along the same lines as the proof of Theorem 6.4. We omit the details.

7 Harmonic functions vanishing along geodesics

We shall now turn to uniqueness results for classical harmonic functions in $\mathbb{H}$ ($\alpha = 0$). Recall the harmonic polynomials

$$u_{k,0}(z) = \text{Im}(z^{k+1}), \quad z \in \mathbb{C},$$

for $k \in \mathbb{N}$ which all vanish on the real line $\mathbb{R}$. Observe that the zero set of $u_{k,0}$ is a union of $k + 1$ lines passing through the origin. A treatment of the uniqueness problem for classical harmonic functions along the lines of what we did in Section 6 thus calls for a more demanding vanishing condition at infinity. In this section we shall consider vanishing conditions along geodesics in $\mathbb{H}$, that is, along rays in $\mathbb{H}$ that are parallel to the positive imaginary axis.

Let us first consider the class $\mathcal{V}_0$.

Theorem 7.1. *Let $u \in \mathcal{V}_0$. Assume that*

$$(7.1) \qquad \lim_{y \to +\infty} u(x + iy)/y = 0$$

for $x = x_j \in \mathbb{R}$ ($j = 1, 2$) with $x_1 \neq x_2$. Then $u(z) = 0$ for all $z \in \mathbb{H}$.

Proof. We put $u \in \mathcal{V}_0$ on the form (5.3) for some $c_1, \ldots, c_{n+1} \in \mathbb{C}$ and $n \in \mathbb{N}$. By adding an extra term in (5.3) if necessary, we can arrange that $n \geq 1$ is odd. We shall prove that $c_n = c_{n+1} = 0$. This will complete the proof of the theorem.

We now proceed to details. Let $x \in \mathbb{R}$. From the binomial theorem we have that

$$(x + iy)^{n+1} = (iy)^{n+1} + (n + 1)x(iy)^n + O(y^{n-1})$$
$$= (-1)^{(n+1)/2}y^{n+1} + i(n + 1)(-1)^{(n-1)/2}xy^n + O(y^{n-1})$$

as $y \to +\infty$. Passing to the imaginary part we have that

$$(7.2) \qquad \text{Im}((x + iy)^{n+1}) = (n + 1)(-1)^{(n-1)/2}xy^n + O(y^{n-1})$$

as $y \to +\infty$. A similar consideration shows that

$$(7.3) \qquad \text{Im}((x + iy)^n) = (-1)^{(n-1)/2}y^n + O(y^{n-1})$$

as $y \to +\infty$.

We now return to the function u. From (7.2) and (7.3) we have that

$$\lim_{y \to +\infty} u(x + iy)/y^n = c_{n+1}(n + 1)(-1)^{(n-1)/2}x + c_n(-1)^{(n-1)/2}$$

for $x \in \mathbb{R}$. From (7.1) we have that

$$c_{n+1}(n+1)(-1)^{(n-1)/2}x + c_n(-1)^{(n-1)/2} = 0$$

for $x = x_j$ $(j = 1, 2)$. Since $x_1 \neq x_2$, we conclude that $c_{n+1} = c_n = 0$. $\qquad\square$

We next turn to uniqueness theorems for harmonic functions.

Theorem 7.2. *Let u be a harmonic function in the open upper half-plane $\mathbb{H}$ which is of temperate growth at infinity.*
 (1) *Assume that $u(z) \to 0$ as $\mathbb{H} \ni z \to x$ for every $x \in \mathbb{R}$.*
 (2) *Assume that*

$$\lim_{y \to +\infty} u(x + iy)/y = 0$$

 for $x = x_j \in \mathbb{R}$ $(j = 1, 2)$ with $x_1 \neq x_2$.
Then $u(z) = 0$ for all $z \in \mathbb{H}$.

Proof. From Theorem 5.1 we have that $u \in \mathcal{V}_0$. An application of Theorem 7.1 yields that $u(z) = 0$ for $z \in \mathbb{H}$. $\qquad\square$

Recall the definition of $u_{k,\alpha}$ in (5.4) and let $a \in \mathbb{R}$. We point out that the function

$$u(z) = u_{1,0}(z - a) = \operatorname{Im}((z - a)^2) = 2(x - a)y, \quad z = x + iy \in \mathbb{C},$$

is of quadratic growth, harmonic, and vanishes on the lines $x = a$ and $y = 0$. Thus, the vanishing assumption (2) in Theorem 7.2 can not be relaxed to vanishing along one geodesic $x = a$ only.

We next extend the validity of Theorem 7.2 to allow for a distributional boundary value in (1).

Theorem 7.3. *Let u be a harmonic function in the open upper half-plane $\mathbb{H}$ which is of temperate growth at infinity.*
 (1)′ *Assume that $\lim_{y \to 0+} u(\cdot + iy) = 0$ in $\mathscr{D}'(\mathbb{R})$.*
 (2) *Assume that*

$$\lim_{y \to +\infty} u(x + iy)/y = 0$$

 for $x = x_j \in \mathbb{R}$ $(j = 1, 2)$ with $x_1 \neq x_2$.
Then $u(z) = 0$ for all $z \in \mathbb{H}$.

Proof. From Theorem 5.7 we have that $u \in \mathcal{V}_0$. An application of Theorem 7.1 yields that $u(z) = 0$ for $z \in \mathbb{H}$. $\qquad\square$

The results of this section together with corresponding results from Section 6 yield considerable advancement from a recent result by Carlsson and Wittsten [9, Corollary 1.9]. In fact, instead of (6.3) or (7.1) those authors considered functions such that

$$\lim_{y \to \infty} u(x + iy)/y^{\alpha+1} = 0$$

for all $|x| < \delta$, where $\delta > 0$.

The importance of condition (7.1) for two distinct geodesics brings to mind an analogous situation in the study non-tangential limits. If u is a bounded harmonic function in $\mathbb{H}$ such that

$$\lim_{r \to 0} u(re^{i\theta_1}) = L = \lim_{r \to 0} u(re^{i\theta_2})$$

for some $0 < \theta_1 < \theta_2 < \pi$, then u has non-tangential limit L at the origin (see Axler et al. [4, Theorem 2.10]). This latter result can be thought of as a harmonic function version of a classical result of Lindelöf (see Rudin [21, Theorem 12.10]).

8   Harmonic functions vanishing along rays

In this section we continue our study of uniqueness results for classical harmonic functions in $\mathbb{H}$ ($\alpha = 0$). We now consider functions vanishing along rays in $\mathbb{H}$ emanating from the origin, that is, along rays of the form $\{te^{i\theta} : t > 0\}$, where $0 < \theta < \pi$.

We shall use a concept of admissible function of angles that we proceed to define.

Definition 8.1. Let $E \subset (0, \pi)$ be a set and $\eta : E \to \mathbb{Z}^+$ a positive integer valued function on E. We say that the function element (E, η) is an **admissible function of angles** if it has the property that for every $k \in \mathbb{Z}^+$ there exists $\theta \in E$ such that $\sin(k\theta) \neq 0$ and $k \geq \eta(\theta)$.

Our interest in admissible functions of angles stems from the following theorem.

Theorem 8.2. *Let $u \in \mathcal{V}_0$. Assume that there is an admissible function of angles (E, η) such that*

$$\lim_{t \to +\infty} u(te^{i\theta})/t^{\eta(\theta)} = 0$$

for every $\theta \in E$. Then $u(z) = 0$ for all $z \in \mathbb{H}$.

Proof.   We put $u \in \mathcal{V}_0$ on the form

$$u(z) = \sum_{k=1}^{n} c_k \operatorname{Im}(z^k), \quad z \in \mathbb{H},$$

for some $c_1, \ldots, c_n \in \mathbb{C}$ and $n \in \mathbb{Z}^+$. It suffices to show that $c_n = 0$. For such functions,

$$\lim_{t \to +\infty} u(e^{i\theta}t)/t^n = \lim_{t \to +\infty} \sum_{k=1}^n c_k \operatorname{Im}(e^{ik\theta}t^k)/t^n = c_n \sin(n\theta).$$

Since (E, η) is an admissible function of angles we may choose $\theta \in E$ such that $\sin(n\theta) \neq 0$ and $\eta(\theta) \le n$. It follows that

$$c_n \sin(n\theta) = \lim_{t \to +\infty} \frac{u(te^{i\theta})}{t^{\eta(\theta)}} \frac{t^{\eta(\theta)}}{t^n} = 0.$$

Since $\sin(n\theta) \neq 0$ we conclude that $c_n = 0$. □

We next turn to uniqueness results.

Theorem 8.3. *Let u be a harmonic function in the open upper half-plane $\mathbb{H}$ which is of temperate growth at infinity.*
  (1) *Assume that $u(z) \to 0$ as $\mathbb{H} \ni z \to x$ for every $x \in \mathbb{R}$.*
  (2) *Assume that there is an admissible function of angles (E, η) such that*

$$\lim_{t \to +\infty} u(te^{i\theta})/t^{\eta(\theta)} = 0$$

  for every $\theta \in E$.
Then $u(z) = 0$ for all $z \in \mathbb{H}$.

Proof. From Theorem 5.1 we have that $u \in \mathcal{V}_0$. An application of Theorem 8.2 yields that $u(z) = 0$ for $z \in \mathbb{H}$. □

We next extend the validity of Theorem 8.3 to allow for a distributional boundary value in (1).

Theorem 8.4. *Let u be a harmonic function in the open upper half-plane $\mathbb{H}$ which is of temperate growth at infinity.*
  (1)′ *Assume that $\lim_{y \to 0+} u(\cdot + iy) = 0$ in $\mathscr{D}'(\mathbb{R})$.*
  (2) *Assume that there is an admissible function of angles (E, η) such that*

$$\lim_{t \to +\infty} u(te^{i\theta})/t^{\eta(\theta)} = 0$$

  for every $\theta \in E$.
Then $u(z) = 0$ for all $z \in \mathbb{H}$.

Proof. From Theorem 5.7 we have that $u \in \mathcal{V}_0$. An application of Theorem 8.2 yields that $u(z) = 0$ for $z \in \mathbb{H}$. □

Let us comment on the generic situation.

Corollary 8.5. *Let u be a harmonic function in the open upper half-plane* $\mathbb{H}$ *which is of temperate growth at infinity.*
(1)′ *Assume that* $\lim_{y \to 0+} u(\cdot + iy) = 0$ *in* $\mathscr{D}'(\mathbb{R})$.
(2) *Assume that u satisfies the vanishing condition that* $\lim_{t \to +\infty} u(te^{i\theta})/t = 0$ *for some* $0 < \theta < \pi$ *which is not a rational multiple of* π.
Then $u(z) = 0$ *for all* $z \in \mathbb{H}$.

Proof. Let $E = \{\theta\}$ and $\eta(\theta) = 1$. Then (E, η) is an admissible function of angles for which condition (2) of Theorem 8.4 holds. Thus, an application of the mentioned theorem gives the result. □

Theorem 1.2 in the introduction is proved in the same way as Corollary 8.5 by using Theorem 8.3 instead of Theorem 8.4.

The examples

$$u_k(z) = \text{Im}(z^k),$$

for $k = 1, 2, \ldots$, show that the arithmetic condition on $\theta \in (0, \pi)$ in Corollary 8.5 is to the point. Clearly, the polynomial u_k vanishes on $\mathbb{R}$. Moreover, if $\theta = \frac{m}{n}\pi$ for some integers m and n with $n \geq 1$, then the function u_n vanishes on the line $e^{i\theta}\mathbb{R}$.

Corollary 8.6. *Let u be a harmonic function in the open upper half-plane* $\mathbb{H}$ *which is of temperate growth at infinity. Let* $\tau = \frac{m}{n}\pi$ *with* $m, n \in \mathbb{Z}^+$ *relatively prime and* $m < n$.
(1)′ *Assume that* $\lim_{y \to 0+} u(\cdot + iy) = 0$ *in* $\mathscr{D}'(\mathbb{R})$.
(2) *Assume that* $\lim_{t \to +\infty} u(te^{i\tau})/t = 0$.
(3) *Assume that* $\lim_{t \to +\infty} u(te^{i\theta})/t^n = 0$ *for some* $0 < \theta < \pi$ *which is not a rational multiple of* π.
Then $u(z) = 0$ *for all* $z \in \mathbb{H}$.

Proof. Let $E = \{\tau, \theta\}$ and set $\eta(\tau) = 1$ and $\eta(\theta) = n$. Then (E, η) is an admissible function of angles such that condition (2) of Theorem 8.4 is satisfied. Applying the mentioned theorem thus gives the result. □

It might be worthwhile to state the special case of Corollary 8.6 obtained when $\tau = \pi/2$:

Corollary 8.7. *Let u be a harmonic function in the open upper half-plane* $\mathbb{H}$ *which is of temperate growth at infinity.*
(1)′ *Assume that* $\lim_{y \to 0+} u(\cdot + iy) = 0$ *in* $\mathscr{D}'(\mathbb{R})$.
(2) *Assume that u satisfies the vanishing condition that* $\lim_{y \to +\infty} u(iy)/y = 0$.

(3) *Assume that u satisfies the vanishing condition that* $\lim_{t\to+\infty} u(te^{i\theta})/t^2 = 0$
  for some $0 < \theta < \pi$ *which is not a rational multiple of* π.
Then $u(z) = 0$ *for all* $z \in \mathbb{H}$.

Let us elaborate some more on the theme of Corollary 8.7.

Corollary 8.8. *Let u be a harmonic function in the open upper half-plane* $\mathbb{H}$
which is of temperate growth at infinity.
 (1)′ *Assume that* $\lim_{y\to 0+} u(\cdot + iy) = 0$ *in* $\mathscr{D}'(\mathbb{R})$.
 (2) *Assume that*
$$\lim_{t\to+\infty} u(te^{i2^{-k}\pi})/t^{2^{k-1}} = 0$$
  for $k = 1, 2, \ldots$.
Then $u(z) = 0$ *for all* $z \in \mathbb{H}$.

Proof. Let $E = \{2^{-k}\pi : k = 1, 2, \ldots\}$ and set $\eta(2^{-k}\pi) = 2^{k-1}$ for
$k = 1, 2, \ldots$. From Theorem 9.1 in the next section we have that (E, η) is an
admissible function of angles. Clearly condition (2) of Theorem 8.4 is satisfied.
Applying the mentioned theorem thus gives the result. $\qquad\square$

9 Admissible functions of angles

In this section we shall provide some constructions of admissible functions of
angles (see Theorems 9.1 and 9.2). Examples of such function elements are
needed for successful applications of Theorems 8.3 or 8.4. The set of admissible
functions of angles is equipped with a natural partial order. We show that the
admissible functions of angles constructed are in fact the minimal elements in the
partial order (see Theorem 9.5). We also show that every admissible function of
angles has a lower bound which is minimal (see Theorem 9.3). The construction
uses some notions from ideal theory for the ring of integers $\mathbb{Z}$.

Let $\theta \in \mathbb{R}$ be a real number and consider the set of integers

$$\mathfrak{I}(\theta) = \{m \in \mathbb{Z} : \sin(m\theta) = 0\}.$$

Using standard properties of the sine function, it is straightforward to check that
the set $\mathfrak{I}(\theta)$ is an ideal in $\mathbb{Z}$. Since the ring $\mathbb{Z}$ is a principal ideal domain, we have
that

$$\mathfrak{I}(\theta) = d(\theta)\mathbb{Z}$$

for some integer $d(\theta)$. Since the units in $\mathbb{Z}$ are ± 1, this integer $d(\theta)$ is uniquely
determined by the condition that $d(\theta) \geq 0$. This defines a function $d : \theta \mapsto d(\theta)$
from $\mathbb{R}$ to $\mathbb{N}$.

We observe that $d(\theta) = 0$ if and only if $\theta \in \mathbb{R}$ is not a rational multiple of π. Furthermore, if $\theta = \frac{r}{s}\pi$ with $r \in \mathbb{Z}$ and $s \in \mathbb{Z}^+$ relatively prime, then $d(\theta) = s$. These assertions are straightforward to check.

We denote by $\mathrm{lcm}(a_1, \ldots, a_n)$ the least common multiple of integers $a_1, \ldots, a_n$. Recall that $\mathrm{lcm}(a_1, \ldots, a_n)$ is the non-negative generator of the ideal $\bigcap_{k=1}^{n} a_k\mathbb{Z}$:

$$\mathrm{lcm}(a_1, \ldots, a_n)\mathbb{Z} = \bigcap_{k=1}^{n} a_k\mathbb{Z}.$$

The symbol $a|b$ means that the integer b is divisible by the integer a in the usual sense that $b = ac$ for some integer c.

We are now ready for the construction of admissible functions of angles.

Theorem 9.1. *Let $\{\theta_k\}_{k=1}^{\infty}$ be an infinite sequence of rational multiples of π from the interval $(0, \pi)$ such that*

$$d(\theta_k) \nmid \mathrm{lcm}(d(\theta_1), \ldots, d(\theta_{k-1}))$$

for $k = 2, 3, \ldots$. Set $E = \{\theta_k : k = 1, 2, \ldots\}$ and define $\eta : E \to \mathbb{Z}^+$ by $\eta(\theta_1) = 1$ and

$$\eta(\theta_k) = \mathrm{lcm}(d(\theta_1), \ldots, d(\theta_{k-1}))$$

for $k = 2, 3, \ldots$. Then (E, η) is an admissible function of angles.

Proof. Observe first that the requirements on the sequence $\{\theta_k\}_{k=1}^{\infty}$ guarantee that
$$2 \leq \mathrm{lcm}(d(\theta_1), \ldots, d(\theta_j)) < \mathrm{lcm}(d(\theta_1), \ldots, d(\theta_k))$$
for $1 \leq j < k$. Thus $\theta_j \neq \theta_k$ if $j \neq k$ and $\bigcap_{k=1}^{\infty} d(\theta_k)\mathbb{Z} = \{0\}$.

The function $\eta : E \to \mathbb{Z}^+$ is well-defined since the θ_k's are distinct. We proceed to prove that (E, η) is an admissible function of angles. Let $m \in \mathbb{Z}^+$ be a positive integer. Since $\bigcap_{k=1}^{\infty} d(\theta_k)\mathbb{Z} = \{0\}$, there exists $k \in \mathbb{Z}^+$ such that $m \notin d(\theta_k)\mathbb{Z}$. Furthermore, by the well-ordering of positive integers we can also arrange that $m \in d(\theta_j)\mathbb{Z}$ for $1 \leq j < k$. Since $m \notin d(\theta_k)\mathbb{Z}$, we have that $\sin(m\theta_k) \neq 0$. If $k = 1$, we clearly have that $m \geq 1 = \eta(\theta_1)$. Assume next that $k \geq 2$. Since $m \in d(\theta_j)\mathbb{Z}$ for $1 \leq j < k$, we have

$$m \in \bigcap_{j=1}^{k-1} d(\theta_j)\mathbb{Z} = \mathrm{lcm}(d(\theta_1), \ldots, d(\theta_{k-1}))\mathbb{Z},$$

which allows us to conclude that $m \geq \mathrm{lcm}(d(\theta_1), \ldots, d(\theta_{k-1})) = \eta(\theta_k)$. This completes the proof of the theorem. $\qquad\square$

Theorem 9.1 provides a multitude of examples of admissible functions of angles. For example, let $\theta_k = \frac{n_k}{2^k}\pi$ with $1 \le n_k < 2^k$ odd for $k = 1, 2, \ldots$. Now $d(\theta_k) = 2^k$ for $k = 1, 2, \ldots$, which makes evident that the assumption of Theorem 9.1 is satisfied. In this case $\eta(\theta_k) = 2^{k-1}$ for $k = 1, 2, \ldots$.

We next turn to the construction of admissible functions of angles (E, η) with E finite.

Theorem 9.2. *Let $\{\theta_k\}_{k=1}^n$ be a finite sequence of rational multiples of π from the interval $(0, \pi)$ such that*

$$d(\theta_k) \nmid \operatorname{lcm}(d(\theta_1), \ldots, d(\theta_{k-1}))$$

for $2 \le k \le n$, where $n \in \mathbb{N}$. Let $\theta_{n+1} \in (0, \pi)$ be an irrational multiple of π. Set $E = \{\theta_k : 1 \le k \le n+1\}$ and define $\eta : E \to \mathbb{Z}^+$ by $\eta(\theta_1) = 1$ and

$$\eta(\theta_k) = \operatorname{lcm}(d(\theta_1), \ldots, d(\theta_{k-1}))$$

for $2 \le k \le n+1$. Then (E, η) is an admissible function of angles.

Proof. The function $\eta : E \to \mathbb{Z}^+$ is well-defined since the θ_k's are distinct. We proceed to prove that (E, η) is an admissible function of angles. Let $m \in \mathbb{Z}^+$ be a positive integer. If $m \ge \operatorname{lcm}(d(\theta_1), \ldots, d(\theta_n))$, then it is straightforward to check that $\sin(m\theta_{n+1}) \ne 0$ and $m \ge \eta(\theta_{n+1})$.

It remains to consider the case $1 \le m < \operatorname{lcm}(d(\theta_1), \ldots, d(\theta_n))$. Since

$$\bigcap_{k=1}^n d(\theta_k)\mathbb{Z} = \operatorname{lcm}(d(\theta_1), \ldots, d(\theta_n))\mathbb{Z},$$

there exists $k \in \mathbb{Z}^+$ with $1 \le k \le n$ such that $m \notin d(\theta_k)\mathbb{Z}$ and $m \in d(\theta_j)\mathbb{Z}$ for $1 \le j < k$. Since $m \notin d(\theta_k)\mathbb{Z}$, we have that $\sin(m\theta_k) \ne 0$. If $k = 1$, we clearly have that $m \ge 1 = \eta(\theta_1)$. Assume next that $k \ge 2$. Since $m \in d(\theta_j)\mathbb{Z}$ for $1 \le j < k$, we have

$$m \in \bigcap_{j=1}^{k-1} d(\theta_j)\mathbb{Z} = \operatorname{lcm}(d(\theta_1), \ldots, d(\theta_{k-1}))\mathbb{Z},$$

which allows us to conclude that $m \ge \operatorname{lcm}(d(\theta_1), \ldots, d(\theta_{k-1})) = \eta(\theta_k)$. This yields the conclusion of the theorem. $\qquad\square$

We point out that the case of an empty sequence $\{\theta_k\}_{k=1}^n$ ($n = 0$) in Theorem 9.2 yields the admissible function of angles used in the proof of Corollary 8.5.

Let $E_j \subset (0, \pi)$ and $\eta_j : E_j \to \mathbb{Z}^+$ ($j = 1, 2$) be such that $E_1 \subset E_2$ and $\eta_1(\theta) \ge \eta_2(\theta)$ for $\theta \in E_1$. If (E_1, η_1) is an admissible function of angles, then so is (E_2, η_2). For applications of Theorems 8.3 or 8.4 it is of interest to have an admissible function of angles (E, η) which is as economical as possible in this respect.

Let us denote by $\mathcal{A}$ the set of all admissible functions of angles. We equip the set $\mathcal{A}$ with a relation $(\leq)$ defined by $(E_1, \eta_1) \leq (E_2, \eta_2)$ if $E_1 \subset E_2$ and $\eta_1(\theta) \geq \eta_2(\theta)$ for $\theta \in E_1$. It is straightforward to check that this relation gives the set $\mathcal{A}$ a structure of a partial order.

We denote by $\mathcal{A}_0$ the subset of $\mathcal{A}$ consisting of those admissible functions of angles (E, η) that are constructed as in Theorem 9.1 or 9.2.

We next prove that every element in $\mathcal{A}$ has a lower bound from the set $\mathcal{A}_0$.

Theorem 9.3. *Let $(E_1, \eta_1) \in \mathcal{A}$ be arbitrary. Then there exists $(E, \eta) \in \mathcal{A}_0$ such that $(E, \eta) \leq (E_1, \eta_1)$ in $\mathcal{A}$.*

Proof. The proof relies on a construction of a suitable sequence $\{\theta_k\}$ from the set $E_1 \subset (0, \pi)$. Since (E_1, η_1) is an admissible function of angles there exists $\theta \in E_1$ such that $\eta_1(\theta) = 1$. We set $\theta_1 = \theta$. If θ_1 is an irrational multiple of π, then $(E, \eta) \leq (E_1, \eta_1)$ in $\mathcal{A}$, where $E = \{\theta_1\}$ and $\eta(\theta_1) = 1$ are as in Theorem 9.2 with $n = 0$ there. If θ_1 is a rational multiple of π, then we proceed to construct an element θ_2 as described in the following paragraph.

Assume that $\theta_1, \ldots, \theta_n \in E_1$ are rational multiples of π such that

$$d(\theta_k) \nmid \operatorname{lcm}(d(\theta_1), \ldots, d(\theta_{k-1}))$$

and

$$\eta_1(\theta_k) \leq \operatorname{lcm}(d(\theta_1), \ldots, d(\theta_{k-1}))$$

for $2 \leq k \leq n$, where $n \geq 1$. Let $m = \operatorname{lcm}(d(\theta_1), \ldots, d(\theta_n))$. Clearly $m \in \mathbb{Z}^+$. Since (E_1, η_1) is an admissible function of angles there exists $\theta \in E_1$ such that $\sin(m\theta) \neq 0$ and $m \geq \eta_1(\theta)$. We set $\theta_{n+1} = \theta$. If θ_{n+1} is an irrational multiple of π, then the construction from Theorem 9.2 provides us with a function of angles $(E, \eta) \in \mathcal{A}_0$ such that $(E, \eta) \leq (E_1, \eta_1)$ in $\mathcal{A}$.

Assume next that θ_{n+1} is a rational multiple of π. Since $\sin(m\theta_{n+1}) \neq 0$, we have

$$m \notin \mathfrak{I}(\theta_{n+1}) = d(\theta_{n+1})\mathbb{Z},$$

which gives that $d(\theta_{n+1}) \nmid m = \operatorname{lcm}(d(\theta_1), \ldots, d(\theta_n))$. Notice also that the inequality $\eta_1(\theta_{n+1}) \leq \operatorname{lcm}(d(\theta_1), \ldots, d(\theta_n))$ is evident from construction. We are now in position to repeat the procedure from the previous paragraph.

Unless the above procedure terminates in a finite number of steps, the principle of induction provides us with an infinite sequence $\{\theta_k\}_{k=1}^{\infty}$ of rational multiples of π from the set E_1 satisfying the assumptions of Theorem 9.1. The construction from Theorem 9.1 now provides us with an admissible function of angles $(E, \eta) \in \mathcal{A}_0$ such that $(E, \eta) \leq (E_1, \eta_1)$ in $\mathcal{A}$. $\qquad\square$

Recall that an element m in a partial order $\mathcal{M}$ is called minimal (in $\mathcal{M}$) if it has the property that $x \in \mathcal{M}$ and $x \leq m$ implies that $x = m$.

We next prove that the admissible functions of angles (E, η) from the set $\mathcal{A}_0$ are minimal elements in $\mathcal{A}$.

Lemma 9.4. *Let $(E, \eta) \in \mathcal{A}_0$. Let $(E_1, \eta_1) \in \mathcal{A}$ be such that $(E_1, \eta_1) \leq (E, \eta)$ in $\mathcal{A}$. Then $(E_1, \eta_1) = (E, \eta)$.*

Proof. We consider the case when (E, η) is as in Theorem 9.1. The remaining case when (E, η) is as in Theorem 9.2 is similar and therefore omitted. By construction we have that $\eta(\theta_1) = 1$, $\eta(\theta_j) < \eta(\theta_k)$ for $1 \leq j < k$ and $\sin(\eta(\theta_k)\theta_j) = 0$ for $1 \leq j < k$. We need to prove that $\theta_k \in E_1$ and $\eta_1(\theta_k) = \eta(\theta_k)$ for $k = 1, 2, \ldots$. We proceed by induction.

Since (E_1, η_1) is an admissible function of angles there exists $\theta \in E_1$ such that $\eta_1(\theta) = 1$. Since $(E_1, \eta_1) \leq (E, \eta)$ we have that $\theta = \theta_j$ for some $j \in \mathbb{Z}^+$ and $\eta(\theta_j) = 1$. The condition $\eta(\theta_j) = 1$ forces $j = 1$. We have shown that $\theta_1 \in E_1$ and $\eta_1(\theta_1) = \eta(\theta_1) = 1$.

Assume next that $\theta_j \in E_1$ and $\eta_1(\theta_j) = \eta(\theta_j)$ for $1 \leq j < k$, where $k \geq 2$. We shall prove that $\theta_k \in E_1$ and $\eta_1(\theta_k) = \eta(\theta_k)$. Let $m = \eta(\theta_k) \in \mathbb{Z}^+$. Since (E_1, η_1) is an admissible function of angles, there exists $\theta \in E_1$ such that $\sin(m\theta) \neq 0$ and $m \geq \eta_1(\theta)$. Since $(E_1, \eta_1) \leq (E, \eta)$ we have that $\theta = \theta_j$ for some $j \in \mathbb{Z}^+$ with $\eta_1(\theta_j) \geq \eta(\theta_j)$. The inequality $\eta(\theta_j) \leq \eta(\theta_k)$ gives that $1 \leq j \leq k$. The condition $\sin(m\theta_j) \neq 0$ now forces $j = k$. We have shown that $\theta_k \in E_1$ and $\eta_1(\theta_k) = \eta(\theta_k)$. The conclusion of the lemma now follows by induction. $\square$

We are now in position to calculate the minimal elements in $\mathcal{A}$.

Theorem 9.5. *The minimal elements in $\mathcal{A}$ are exactly those that belong to the set $\mathcal{A}_0$.*

Proof. By Lemma 9.4 we know that the admissible functions of angles (E, η) from the set $\mathcal{A}_0$ are minimal elements in $\mathcal{A}$. We proceed to show that every minimal element in $\mathcal{A}$ has such form.

Let $(E_1, \eta_1) \in \mathcal{A}$ be minimal in $\mathcal{A}$. By Theorem 9.3 there exists $(E, \eta) \in \mathcal{A}_0$ such that $(E, \eta) \leq (E_1, \eta_1)$ in $\mathcal{A}$. Since (E_1, η_1) is minimal in $\mathcal{A}$, we conclude that $(E, \eta) = (E_1, \eta_1)$. $\square$

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