THE STEIN–TOMAS INEQUALITY UNDER THE EFFECT OF SYMMETRIES

By

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Abstract. We prove new Fourier restriction estimates to the unit sphere $\mathbb{S}^{d-1}$ on the class of $O(d-k)\times O(k)$-symmetric functions, for every $d\ge 4$ and $2\le k\le d-2$. As an application, we establish the existence of maximizers for the endpoint Stein–Tomas inequality within that class. Moreover, we construct examples showing that the range of Lebesgue exponents in our estimates is sharp.

1 Introduction

The Fourier restriction conjecture predicts the validity of the estimate

$$\left(\int_{\mathbb{S}^{d-1}} |\widehat{f}(\omega)|^q \,\mathrm{d}\sigma(\omega)\right)^{\frac{1}{q}} \le C(d,p,q)\|f\|_{L^p(\mathbb{R}^d)}, \tag{1.1}$$

as long as the dimension $d\ge 2$ of the ambient Euclidean space $\mathbb{R}^d$ and the Lebesgue exponents $p,q\in[1,\infty]$ satisfy the following conditions:

$$\frac{1}{p} > \frac{d+1}{2d} \quad\text{and}\quad \frac{d+1}{p}+\frac{d-1}{q}\ge d+1. \tag{1.2}$$

Integration on the left-hand side of (1.1) is with respect to the usual surface measure σ on the unit sphere $\mathbb{S}^{d-1} := \{\omega\in\mathbb{R}^d : |\omega| = 1\}$. The restriction conjecture has a rich history, and is remarkable in its numerous connections and applications. It exhibits deep links to Bochner–Riesz summation methods and to decoupling phenomena for the Fourier transform, and is known to imply the Kakeya conjecture. Despite the great deal of attention received by this circle of problems during the past four decades, the restriction conjecture has been established only when $d = 2$ (see [15] for the non-endpoint case $p' > 3q$, and [9, 49] for the endpoint $p' = 3q$) and remains an open question in dimensions $d\ge 3$. For further information on the restriction problem, we refer the interested reader to the survey [45] and the recent account [42].

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The special case $q = 2$ in (1.1) is well understood and of particular importance. If $d \geq 2$ and $1 \leq p \leq \frac{2(d+1)}{d+3}$, then the classical Stein–Tomas inequality [38, 46] states the existence of a constant $C(d, p) < \infty$, such that

$$\left(\int_{\mathbb{S}^{d-1}} |\widehat{f}(\omega)|^2 \,\mathrm{d}\sigma(\omega) \right)^{\frac{1}{2}} \leq C(d, p) \|f\|_{L^p(\mathbb{R}^d)}, \tag{1.3}$$

for every $f \in L^p(\mathbb{R}^d)$. A well-known construction of Knapp (see [44]) reveals that the range of Lebesgue exponents is sharp in this case. In particular, estimate (1.3) is of endpoint type when $p = \frac{2(d+1)}{d+3}$, in the sense that it becomes false if either of the exponents 2, $\frac{2(d+1)}{d+3}$ is increased. The Stein–Tomas inequality finds applications in harmonic analysis and PDEs. In particular, it underlies most of the early progress towards the Fourier restriction conjecture; see [45]. The argument is robust enough to be applied to other manifolds, e.g., to the paraboloid, the cone, and the hyperboloid, in which case it implies the foundational Strichartz estimates for the Schrödinger, Wave, and Klein–Gordon equations, respectively; see [44]. Moreover, inequality (1.3) has been generalized to a variety of other contexts, and found surprising applications ranging from fractal geometry [30] to number theory [19], among many others.

Set $p_d := \frac{2(d+1)}{d+3}$. The optimal Stein–Tomas constant,

$$\mathbf{T}_d := \mathbf{T}_d(p_d), \quad \text{where } \mathbf{T}_d(p) := \sup_{0 \neq f \in L^p(\mathbb{R}^d)} \frac{\left(\int_{\mathbb{S}^{d-1}} |\widehat{f}(\omega)|^2 \,\mathrm{d}\sigma(\omega)\right)^{\frac{1}{2}}}{\|f\|_{L^p(\mathbb{R}^d)}},$$

has attracted a great deal of attention in the recent literature. In the lowest dimensional cases $d \in \{2, 3\}$, the dual exponent $p_d' = \frac{2(d+1)}{d-1}$ is an even integer, and the adjoint Stein–Tomas inequality can be reformulated in terms of multilinear convolution operators. Following this path, Christ–Shao [12] and Shao [37] established the precompactness of maximizing sequences (modulo symmetries) for $\mathbf{T}_d$, and therefore the existence of maximizers, when $d = 3$ and $d = 2$, respectively. The exact form of the maximizers for $\mathbf{T}_3$ was subsequently determined by Foschi [16] via a remarkable geometric argument, but $d = 3$ remains the unique dimension for which such a characterization in known. In fact, even the mere existence of maximizers for $\mathbf{T}_d$ is an outstanding open problem for every $d \geq 4$. Partial progress was recently obtained by Frank–Lieb–Sabin [18], who proved that if a well-known conjecture about the optimal constant in the Strichartz inequality is true, then maximizers for $\mathbf{T}_d$ exist. More precisely, the main theorem in [18] states the following: If Gaussians maximize the Strichartz inequality for the Schrödinger equation in $\mathbb{R}^d$, then maximizing sequences for $\mathbf{T}_d$, normalized in $L^{p_d}(\mathbb{R}^d)$, are precompact in $L^{p_d}(\mathbb{R}^d)$ up to translations and, in particular, maximizers for $\mathbf{T}_d$ exist. It remains an open problem to turn this conditional result into an unconditional one.

1.1 Setting. Given a subgroup $G \subset O(d)$ of the orthogonal group, a function $f : \mathbb{R}^d \to \mathbb{C}$ is said to be ***G*-symmetric in** $\mathbb{R}^d$ if $f \circ A = f$ holds for every $A \in G$. An especially interesting situation arises when considering the subgroup $G_k := O(d-k) \times O(k)$ for some $k \in \{0, 1, \dots, d\}$. In this paper, we are interested in restriction estimates to the unit sphere,

$$\left(\int_{\mathbb{S}^{d-1}} |\widehat{f}(\omega)|^q \, \mathrm{d}\sigma(\omega) \right)^{\frac{1}{q}} \le C(k, d, p, q) \|f\|_{L^p(\mathbb{R}^d)}, \tag{1.4}$$

which hold in the class of G_k-symmetric functions. We are led to define the Banach space

$$L^p_{G_k}(\mathbb{R}^d) := \{f \in L^p(\mathbb{R}^d) : f \text{ is } G_k\text{-symmetric}\}.$$

The cases $k \in \{0, d\}$ correspond to radial functions on $\mathbb{R}^d$. If $f \in L^p(\mathbb{R}^d)$ is radial, then $\widehat{f}$ is continuous on $\mathbb{R}^d \setminus \{0\}$ whenever $1 \le p < \frac{2d}{d+1}$; see [39, Prop. 5.1]. In particular, inequality (1.4) holds for radial functions provided $1 \le p < \frac{2d}{d+1}$ and $1 \le q \le \infty$. This range of exponents is in fact optimal, since the radially symmetric counterexample from (6.1) reveals that the adjoint of the restriction operator fails to be bounded from $L^{q'}(\mathbb{S}^{d-1})$ to $L^{p'}(\mathbb{R}^d)$, for every $1 \le q \le \infty$ and $p \ge \frac{2d}{d+1}$. Thus the L^p–L^q mapping properties of the restriction operator in the radial cases $k \in \{0, d\}$ are completely understood. The cases $k \in \{1, d-1\}$ are likewise special. Since Knapp's construction in $\mathbb{R}^d$ is rotationally invariant with respect to $d-1$ variables, we do not believe that G_1-symmetry or G_{d-1}-symmetry allows for a larger range of Lebesgue exponents on which Fourier restriction estimates can hold. In fact, in Remark 6.1 below we adapt Knapp's construction to the G_k-symmetric setting, $k \in \{1, d-1\}$, thus revealing that no estimate beyond those predicted by the restriction conjecture is possible. For this reason, we will focus on the situation when $k \in \{2, 3, \dots, d-2\}$.

1.2 Results. Our first result addresses the Stein–Tomas regime $q = 2$.

Theorem 1.1. *Let $d \ge 4$, $k \in \{2, 3, \dots, d-2\}$, and $m := \min\{d-k, k\}$. Then the estimate*

$$\left(\int_{\mathbb{S}^{d-1}} |\widehat{f}(\omega)|^2 \, \mathrm{d}\sigma(\omega) \right)^{\frac{1}{2}} \le C(k, d, p) \|f\|_{L^p(\mathbb{R}^d)} \tag{1.5}$$

holds for every G_k-symmetric function $f : \mathbb{R}^d \to \mathbb{C}$ if $1 \le p \le \frac{2(d+m)}{d+m+2}$.

Given that $\frac{2(d+m)}{d+m+2}$ is strictly larger than the Stein–Tomas exponent $p_d = \frac{2(d+1)}{d+3}$, Theorem 1.1 improves upon (1.3). To the best of our knowledge, this result is new in every dimension $d \ge 4$.

As an immediate application of Theorem 1.1, one may deduce improved mapping properties of the Helmholtz resolvent acting on G_k-symmetric functions. More precisely, combining Theorem 1.1 with [48, Theorem 1.2] for the admissible extension triple $(G, q, Q) = (G_k, (\frac{2(d+m)}{d+m+2})', \mathbb{1})$ according to [48, Definition 1.1], we obtain boundedness of the Helmholtz resolvent

$$\mathcal{R} : L^p_{G_k}(\mathbb{R}^d) \to L^{p'}_{G_k}(\mathbb{R}^d) \text{ as long as } \frac{2d}{d+2} \leq p < \frac{2d(d+m)}{d^2+(m+2)d+m-1}.$$

If $k \in \{2, 3, \dots, d-2\}$, then $m = \min\{d-k, k\} \geq 2$ and this upper bound for p is strictly larger than $\frac{2(d+1)}{d+3}$, which in turn is best possible in the non-symmetric setting [22].

Next, we apply our G_k-symmetric Stein–Tomas inequality to establish the precompactness of maximizing sequences for the constrained optimization problem

$$\mathbf{T}_{d,k}(p) := \sup_{0 \neq f \in L^p_{G_k}(\mathbb{R}^d)} \frac{(\int_{\mathbb{S}^{d-1}} |\widehat{f}(\omega)|^2 \, \mathrm{d}\sigma(\omega))^{\frac{1}{2}}}{\|f\|_{L^p(\mathbb{R}^d)}},$$

and consequently the unconditional existence of maximizers for $\mathbf{T}_{d,k}(p)$. This is the content of our second result.

Theorem 1.2. *Let $d \geq 4$, $k \in \{2, 3, \dots, d-2\}$, $m := \min\{d-k, k\}$, and $1 \leq p < \frac{2(d+m)}{d+m+2}$. Maximizing sequences for $\mathbf{T}_{d,k}(p)$, normalized in $L^p(\mathbb{R}^d)$, are precompact in $L^p_{G_k}(\mathbb{R}^d)$. In particular, maximizers for $\mathbf{T}_{d,k}(p)$ exist.*

In particular, the set of all normalized maximizers for $\mathbf{T}_{d,k}(p)$ is itself compact as long as $1 \leq p < \frac{2(d+m)}{d+m+2}$. Note that Theorem 1.2 applies to $p = p_d$, and thus establishes the unconditional existence of maximizers for the classical Stein–Tomas inequality within the class of G_k-symmetric functions. We remark that, in contrast to the conclusion of [18, Theorem 1.1], precompactness of complex-valued maximizing sequences is not expected to hold modulo symmetries only, since G_k-symmetry eliminates the loss of compactness due to translations. There is still the danger that a maximizing sequence might conceivably converge weakly to zero. To show that this is not the case, the proof of Theorem 1.2 will make use of a decay property of the Fourier transform which is only available in the G_k-symmetric setting for $2 \leq k \leq d-2$; see Proposition 2.4 and Corollary 2.5 below. The endpoint case of Theorem 1.2 remains an interesting open question.

The range of exponents of Theorem 1.1 turns out to be optimal. This is a consequence of our third result, which exhibits the necessary conditions for the restriction estimate (1.4) to hold within the class of G_k-symmetric functions.

Theorem 1.3. *Let* $d \geq 4$*,* $k \in \{2, 3, \dots, d-2\}$*,* $m := \min\{d-k, k\}$*, and* $1 \leq p, q \leq \infty$*. If inequality* (1.4) *holds within the class of* G_k*-symmetric functions, then one of the following conditions is satisfied:*
(i) $\frac{d+1}{2d} < \frac{1}{p} < \frac{m+1}{2m}$ *and* $\frac{d+m}{p} + \frac{d-m}{q} \geq d+1$;
(ii) $\frac{1}{p} = \frac{m+1}{2m}$ *and* $\frac{1}{p} + \frac{1}{q} > 1$;
(iii) $\frac{1}{p} > \frac{m+1}{2m}$ *and* $\frac{1}{p} + \frac{1}{q} \geq 1$.

We shall see in Corollary 1.1 that the necessary conditions from (i) are sufficient when $\frac{d+m+2}{2(d+m)} \leq \frac{1}{p} < \frac{m+1}{2m}$. For larger values of p, the sufficiency of these conditions remains an open problem. On the other hand, conditions (ii) and (iii) turn out to be sufficient for estimate (1.4) to hold in the G_k-symmetric setting. The crux of the matter is a (mixed) Lorentz space estimate at the endpoint $p = q' = \frac{2m}{m+1}$, which is the content of our fourth result. If $m := \min\{d-k, k\} < \frac{d}{2}$ we set $X_p := L^{p,1}(\mathbb{R}^d)$, otherwise $m = \frac{d}{2}$ and let

$$X_p := L_x^{p,1}(\mathbb{R}^{d-k}; L_y^{p,1}(\mathbb{R}^k)) + L_y^{p,1}(\mathbb{R}^k; L_x^{p,1}(\mathbb{R}^{d-k})). \tag{1.6}$$

Theorem 1.4. *Let* $d \geq 4$, $k \in \{2, 3, \dots, d-2\}$*, and* $m := \min\{d-k, k\}$*. Then the estimate*

$$\|\widehat{f}\|_{L^{\frac{2m}{m-1},\infty}(\mathbb{S}^{d-1})} \leq C(k, d)\|f\|_{X_{\frac{2m}{m+1}}}$$

holds for every G_k*-symmetric function* $f : \mathbb{R}^d \to \mathbb{C}$*.*

Compactness of $\mathbb{S}^{d-1}$ and real interpolation of Lorentz [5, § 5.3] and mixed Lorentz spaces [27, Cor. 1] between Theorem 1.1, Theorem 1.4, and the trivial endpoint $(p, q) = (1, \infty)$ together imply a range of estimates which we now record; see Figure 1 for the corresponding Riesz diagram.

Corollary 1.1. *Let* $d \geq 4, k \in \{2, 3, \dots, d-2\}$*, and* $m := \min\{d-k, k\}$*. Then inequality* (1.4) *holds in the class of* G_k*-symmetric functions if one of the following conditions is satisfied*:
(i) $\frac{d+m+2}{2(d+m)} \leq \frac{1}{p} < \frac{m+1}{2m}$ *and* $\frac{d+m}{p} + \frac{d-m}{q} \geq d+1$;
(ii) $\frac{1}{p} = \frac{m+1}{2m}$ *and* $\frac{1}{p} + \frac{1}{q} > 1$;
(iii) $\frac{1}{p} > \frac{m+1}{2m}$ *and* $\frac{1}{p} + \frac{1}{q} \geq 1$.

As a concluding remark, note that Corollary 1.1 (i) implies the diagonal estimate

$$\|\widehat{f}\|_{L^p(\mathbb{S}^{d-1})} \lesssim \|f\|_{L^p(\mathbb{R}^d)} \tag{1.7}$$

at $p = \frac{2(d+m)}{d+m+2}$, for every G_k-symmetric $f : \mathbb{R}^d \to \mathbb{C}$. In this case, if $m = k = \lfloor \frac{d}{2} \rfloor$, then $p' = 2 + \frac{8}{3}d^{-1} + O(d^{-2})$. That (1.7) holds for general $f \in L^p(\mathbb{R}^d)$ in the range $p' > 2 + \frac{8}{3}d^{-1} + O(d^{-2})$ was recently proved by Guth [20], and later improved in [21]. This remains to date the state of the art in the high-dimensional restriction conjecture.

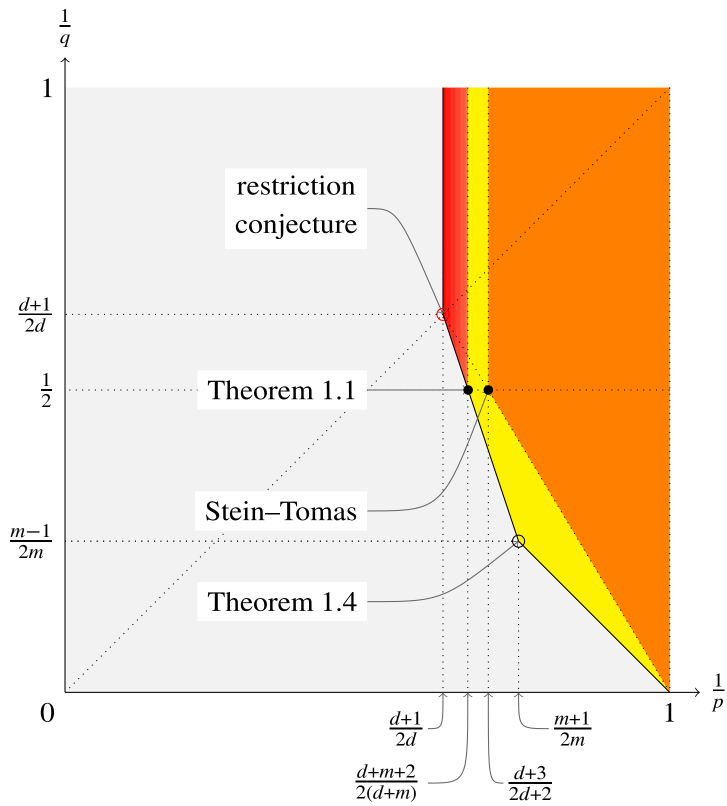

Figure 1. Riesz diagram for the G_k-symmetric restriction problem to $\mathbb{S}^{d-1}$. Estimates in the orange region follow from the Stein–Tomas inequality, estimates in the yellow region follow from Corollary 1.1 and, in light of Theorem 1.3, no estimates are possible within the grey region. The possibility of estimates in the red region remains an open problem.

1.3 Historical remarks. The expectation that further Fourier restriction estimates are available within certain classes of functions exhibiting additional symmetries has been extensively explored in the literature. For instance, the well-known fact that the restriction conjecture holds for radial functions has been generalized to the class of products of radial functions and spherical harmonics [13]. On the paraboloid and the cone, the restriction conjecture has been established for functions which are invariant under spatial rotations, and further estimates are known to hold in the cylindrically symmetric case, but only for dyadically supported functions; see [35, 36]. This was later generalized to the mixed norm setting;

see [28, 29]. Very recently, the G_k-symmetric setting has been proposed in [48], albeit in the context of adjoint restriction estimates on the unit sphere involving weight functions supported on sets of the form $\{(y, z) \in \mathbb{R}^{d-k} \times \mathbb{R}^k : |y| \leq C|z|^{-\alpha}\}$ for some $C, \alpha > 0$. With the present work, we also aim to initiate a systematic exploration of the more general $O(k_1) \times \cdots \times O(k_n)$-symmetric restriction problem to $\mathbb{S}^{d-1}$, with $\sum_{j=1}^n k_j = d$. This so-called **block-radial** symmetry has been extensively explored in the related context of Sobolev space embeddings, starting with the pioneering work of Lions [25].

In all of the above problems, it is very natural to ask about maximizers and optimal constants. Sharp restriction theory is a vibrant area of research which has flourished in the last decade. A natural first step towards sharp restriction inequalities is to establish the existence of maximizers. This provides a stepping stone towards a qualitative analysis, the discovery of explicit maximizers, and the corresponding full characterization (which up to now is only available in very special circumstances). Works addressing the existence of maximizers for inequalities of endpoint Fourier restriction type tend to be a tour de force in classical analysis, using a variety of sophisticated techniques. Besides the aforementioned precompactness results on the unit sphere [12, 18, 37], we highlight the general method developed in [14] together with the (unconditional) existence results on the paraboloid [4, 43], the cone [32, 34], the hyperboloid [10, 11, 33], and the moment curve [6]. For a more comprehensive discussion and further references, we refer the interested reader to the survey [17].

1.4 Structure of the paper. In §2, we discuss some analytic preliminaries about the interplay between G_k-symmetry and the Fourier transform. We also investigate a useful family of oscillatory integrals, and establish weighted versions of the classical inequalities of Hausdorff–Young and Hardy–Littlewood–Sobolev. In §3, we prove the weighted restriction estimates that will play a central role in the proof of Theorem 1.1, which is then the subject of §4. Theorems 1.2, 1.3, 1.4 are proved in §5, §6, §7, respectively.

1.5 Forthcoming notation. We reserve the letter d to denote the dimension of the ambient space $\mathbb{R}^d$. Given a Lebesgue exponent $p \in [1, \infty]$, its dual is $p' = p/(p-1)$. The usual Lebesgue and Lorentz spaces are denoted by $L^p(\mathbb{R}^d)$ and $L^{p,s}(\mathbb{R}^d)$, respectively, and the corresponding (quasi-)norms are indexed accordingly. The Schwartz space is denoted by $\mathcal{S}(\mathbb{R}^d)$. The Fourier transform on $\mathbb{R}^d$

is normalized in the following way:

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-i\xi\cdot x}\,\mathrm{d}x.$$

The indicator function of a set $E \subset \mathbb{R}^d$ is denoted by $\mathbb{1}_E$, and its Lebesgue measure by $|E|$. The usual surface measure on $\mathbb{S}^{d-1}$ is denoted by σ, and its surface area is given by $\sigma(\mathbb{S}^{d-1}) = \int_{\mathbb{S}^{d-1}} \mathrm{d}\sigma(\omega) = 2\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})^{-1}$. We shall write

$$\|F\|^p_{L^p(\mathbb{S}^{d-1})} = \int_{\mathbb{S}^{d-1}} |F(\omega)|^p\,\mathrm{d}\sigma(\omega).$$

Finally, we use the shorthand notation $X \lesssim Y$, $Y \gtrsim X$, $X = O(Y)$ to denote the estimate $|X| \le CY$ for some positive constant C which is only allowed to depend on the space dimension d, the symmetry index k, and possibly some other fixed parameters. We also write $X \simeq Y$ for $X \lesssim Y \lesssim X$.

2 Preliminaries

In this section, we discuss some analytic preliminaries related to duality, Bessel functions, G_k-symmetry, oscillatory integrals, and weighted variants of the classical inequalities of Hausdorff–Young and Hardy–Littlewood–Sobolev.

2.1 Duality. The adjoint of the restriction operator to the unit sphere, $\mathcal{R} : L^p(\mathbb{R}^d) \to L^q(\mathbb{S}^{d-1}), f \mapsto \widehat{f}\,|_{\mathbb{S}^{d-1}}$, is the **extension operator**,

$$\mathcal{R}^* : L^{q'}(\mathbb{S}^{d-1}) \to L^{p'}(\mathbb{R}^d),$$
$$F \mapsto \widehat{F\sigma}$$

defined at $x \in \mathbb{R}^d$ via the expression

$$\widehat{F\sigma}(x) := \int_{\mathbb{S}^{d-1}} F(\omega)e^{ix\cdot\omega}\,\mathrm{d}\sigma(\omega). \tag{2.1}$$

In particular, if $F \equiv 1$, then

$$\widehat{\sigma}(x) = (2\pi)^{\frac{d}{2}}|x|^{\frac{2-d}{2}}J_{\frac{d-2}{2}}(|x|), \tag{2.2}$$

where J_ν denotes the Bessel function of the first kind; this is a special case of the so-called Bochner–Hecke formula (see [38, p. 347]). From the classical asymptotic formulae for Bessel functions, see (2.4)–(2.5) below, or via a direct stationary phase argument, one has that

$$|\widehat{\sigma}(x)| = O((1+|x|)^{\frac{1-d}{2}}); \tag{2.3}$$

see [38, p. 348]. Estimate (2.3) is a manifestation of the well-known fact that curvature causes the Fourier transform to decay.

In this dual setting, a function $F : \mathbb{S}^{d-1} \to \mathbb{C}$ is said to be **G_k-symmetric on** $\mathbb{S}^{d-1}$ if $F \circ A = F$, for every $A \in G_k$. In particular, a set $S \subset \mathbb{S}^{d-1}$ will be called G_k-symmetric if its indicator function $\mathbb{1}_S$ is G_k-symmetric, and similarly for subsets $E \subset \mathbb{R}^d$.

2.2 Bessel functions. In view of identity (2.2), the Bessel function

$$J_\nu(r) := \sum_{j=0}^{\infty} \frac{(-1)^j(\frac{1}{2}r)^{2j+\nu}}{j!\Gamma(\nu+j+1)}$$

is expected to play a role in the analysis. Only $\nu, r \geq 0$ will be of interest. As is well-known, for any fixed $\nu \geq 0$, one has that

$$J_\nu(r) = \left(\frac{\pi r}{2}\right)^{-\frac{1}{2}} \cos\left(r - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O(r^{-\frac{3}{2}}), \quad \text{as } r \to \infty; \tag{2.4}$$

$$|J_\nu(r)| \leq \frac{r^\nu}{2^\nu\Gamma(\nu+1)}, \quad \text{for all } r \geq 0; \tag{2.5}$$

see [38, pp. 356–357] and [47, pp. 48–49]. From (2.4)–(2.5), it is natural to expect the following result.

Lemma 2.1. *Let $\nu \geq 0$. There exist a constant $A_\nu \in \mathbb{C} \setminus \{0\}$ and a function $R_\nu : (0, \infty) \to \mathbb{R}$, such that*

$$J_\nu(r) = (A_\nu e^{ir} + \overline{A_\nu} e^{-ir}) r^{-\frac{1}{2}} \mathbb{1}_{[1,\infty)}(r) + R_\nu(r), \tag{2.6}$$

where additionally $|R_\nu(r)| \lesssim r^\nu(1+r)^{-\nu-\frac{3}{2}}$, for every $r \geq 0$.

Proof. Let $A_\nu := (\frac{2}{\pi})^{\frac{1}{2}} e^{-i(\frac{\nu\pi}{2}+\frac{\pi}{4})}$, and define the function R_ν via identity (2.6) above. Then the desired estimate for R_ν follows from [47, p. 201]. □

2.3 G_k-symmetry. Given a G_k-symmetric function $f : \mathbb{R}^d \to \mathbb{C}$, we shall define $f_0 : (0, \infty)^2 \to \mathbb{C}$ via $f_0(|y|, |z|) := f(y, z)$, for $(y, z) \in \mathbb{R}^{d-k} \times \mathbb{R}^k$, and denote the corresponding Fourier variables by $(\eta, \zeta) \in \mathbb{R}^{d-k} \times \mathbb{R}^k$.

A function is radial if and only if its Fourier transform is radial. This well-known fact admits the following straightforward generalization: G_k-symmetry is preserved under the Fourier transform. Indeed, identity (2.2) implies the following result.

Lemma 2.2. *Let $f \in \mathcal{S}(\mathbb{R}^d)$ be G_k-symmetric, and set $f_0(|y|,|z|) := f(y,z)$. Then the following identity holds at every $(\eta,\zeta) \in \mathbb{R}^{d-k} \times \mathbb{R}^k$:*

$$\begin{aligned}&\widehat{f}(\eta,\zeta)\\&= (2\pi)^{\frac{d}{2}}|\eta|^{\frac{2-d+k}{2}}|\zeta|^{\frac{2-k}{2}}\int_0^\infty\int_0^\infty \rho_1^{\frac{d-k}{2}}\rho_2^{\frac{k}{2}} f_0(\rho_1,\rho_2) J_{\frac{d-k-2}{2}}(\rho_1|\eta|)J_{\frac{k-2}{2}}(\rho_2|\zeta|)\,\mathrm{d}\rho_1\,\mathrm{d}\rho_2.\end{aligned}$$

Proof. This follows from an explicit computation in polar coordinates and the G_k-invariance of f. Indeed, introducing coordinates (ρ_1,ω_1) in $\mathbb{R}^{d-k}$ and (ρ_2,ω_2) in $\mathbb{R}^k$, we find that

$$\begin{aligned}\widehat{f}(\eta,\zeta) = &\int_0^\infty\int_0^\infty \rho_1^{d-k-1}\rho_2^{k-1} f_0(\rho_1,\rho_2)\\&\times\left(\int_{\mathbb{S}^{d-k-1}} e^{-i\omega_1\cdot\rho_1\eta}\,\mathrm{d}\sigma(\omega_1)\right)\left(\int_{\mathbb{S}^{k-1}} e^{-i\omega_2\cdot\rho_2\zeta}\,\mathrm{d}\sigma(\omega_2)\right)\mathrm{d}\rho_2\,\mathrm{d}\rho_1.\end{aligned}$$

The antipodal change of variables $(\omega_1,\omega_2) \rightsquigarrow -(\omega_1,\omega_2)$ then reduces the claim to identity (2.2). The proof is complete. □

We will need to integrate G_k-symmetric functions in $\mathbb{R}^d$ over the unit sphere. The next result provides the corresponding formula.

Lemma 2.3. *Let $f : \mathbb{R}^d \to \mathbb{C}$ be G_k-symmetric and integrable on $\mathbb{S}^{d-1}$, and set $f_0(|\eta|,|\zeta|) := f(\eta,\zeta)$. Then the following identity holds:*

$$\int_{\mathbb{S}^{d-1}} f(\eta,\zeta)\,\mathrm{d}\sigma(\eta,\zeta) = \frac{\sigma(\mathbb{S}^{d-k-1})\sigma(\mathbb{S}^{k-1})}{\sigma(\mathbb{S}^{d-1})}\int_0^1 r^{d-k-1}(1-r^2)^{\frac{k-2}{2}} f_0(r,\sqrt{1-r^2})\,\mathrm{d}r.$$

Proof. This follows from slice integration [1, Theorem A.4] and the G_k-invariance of f. □

As far as pointwise bounds for the extension operator of a G_k-symmetric function on $\mathbb{S}^{d-1}$ are concerned, we have the following result.

Proposition 2.4. *There exists $C = C_{k,d} < \infty$ such that the pointwise bound*

$$|\widehat{F\sigma}(y,z)| \le C_{k,d}\|F\|_{L^2(\mathbb{S}^{d-1})}(1+|y|)^{\frac{k+1-d}{2}}(1+|z|)^{\frac{1-k}{2}} \tag{2.7}$$

holds for every $(y,z) \in \mathbb{R}^{d-k}\times\mathbb{R}^k$ and every G_k-symmetric $F \in L^2(\mathbb{S}^{d-1})$.

Proof. Let $F \in L^2(\mathbb{S}^{d-1})$ be G_k-symmetric and given. Start by noting that the function $\widehat{F\sigma}$ is real-analytic since it is the Fourier transform of a compactly supported distribution. Set $F_0(r) := F(r\omega,\sqrt{1-r^2}\nu)$ for $r \in [0,1]$, $\omega \in \mathbb{S}^{d-k-1}$, $\nu \in \mathbb{S}^{k-1}$; see [1, p. 241]. Since F is integrable on $\mathbb{S}^{d-1}$, we can appeal to the slice integration formula from [1, Theorem A.4] to conclude, via a passage to polar coordinates, that

$$\begin{aligned}&\widehat{F\sigma}(y,z)\\&\simeq \int_0^1 r^{d-k-1}(1-r^2)^{\frac{k-2}{2}}F_0(r)\Big(\int_{\mathbb{S}^{d-k-1}} e^{iy\cdot r\omega}\,\mathrm{d}\sigma(\omega)\Big)\Big(\int_{\mathbb{S}^{k-1}} e^{iz\cdot\sqrt{1-r^2}\nu}\,\mathrm{d}\sigma(\nu)\Big)\,\mathrm{d}r.\end{aligned}$$

Note that the implicit constant depends only on k, d. The Cauchy–Schwarz inequality, Lemma 2.3, and estimate (2.3) together imply

$$\begin{aligned}&|\widehat{F\sigma}(y,z)|^2\\&\quad\lesssim \|F\|^2_{L^2(\mathbb{S}^{d-1})}\int_0^1 r^{d-k-1}(1-r^2)^{\frac{k-2}{2}}(1+r|y|)^{k+1-d}(1+\sqrt{1-r^2}|z|)^{1-k}\,\mathrm{d}r.\end{aligned}$$

The pointwise bound (2.7) follows from this via another application of the Cauchy–Schwarz inequality together with elementary considerations in both regimes $|y|\le 1$, $|y|>1$ separately, and similarly for z. This concludes the proof of the proposition.□

For the purposes of the upcoming analysis in §5, we will be interested in the following straightforward consequence of Proposition 2.4 which, in the language of concentration compactness theory [26], will preclude vanishing (i.e., mass sent to infinity) of maximizing sequences.

Corollary 2.5. *Let $d \ge 4$ and $k \in \{2, 3, \dots, d-2\}$. Then, for every $\varepsilon > 0$, there exists $R = R(k, d, \varepsilon) < \infty$ for which $|\widehat{F\sigma}(x)| < \varepsilon$ if $|x| > R$, for every G_k-symmetric $F \in L^2(\mathbb{S}^{d-1})$ such that $\|F\|_{L^2(\mathbb{S}^{d-1})} = 1$.*

Remark 2.6. That no such property can hold for general $F \in L^2(\mathbb{S}^{d-1})$ follows at once from the fact that the extension operator intertwines modulation and translation:

$$\begin{aligned}\mathcal{R}^*(e^{iy\cdot}F)(x) &= \int_{\mathbb{S}^{d-1}} e^{iy\cdot\omega}F(\omega)e^{ix\cdot\omega}\,\mathrm{d}\sigma(\omega)\\&= \int_{\mathbb{S}^{d-1}} F(\omega)e^{i(x+y)\cdot\omega}\,\mathrm{d}\sigma(\omega) = \mathcal{R}^*(F)(x+y).\end{aligned}$$

Indeed, if a nonzero function F and its modulation $e^{iy\cdot}F$ are both G_k-symmetric on $\mathbb{S}^{d-1}$, then necessarily $y = 0$.

2.4 Oscillatory integrals. We will use the following simple bound on a certain class of oscillatory integrals.

Lemma 2.7. *Let* $0 < \gamma \neq 1$. *Then there exists a constant* $C = C_\gamma < \infty$ *such that, for every* $a \geq 1$ *and* $\lambda \in [-2, 2] \setminus \{0\}$, *the following inequality holds:*

$$\left| \int_a^\infty r^{-\gamma} e^{i\lambda r} \, \mathrm{d}r \right| \leq C \begin{cases} |\lambda|^{\gamma-1} & \text{if } 0 < \gamma < 1, \\ a^{1-\gamma} & \text{if } \gamma > 1. \end{cases}$$

Proof. No generality is lost in assuming that $\lambda \in (0, 2]$. Changing variables $\lambda r = \rho$,

$$\int_a^\infty r^{-\gamma} e^{i\lambda r} \, \mathrm{d}r = \lambda^{\gamma-1} \int_{\lambda a}^\infty \rho^{-\gamma} e^{i\rho} \, \mathrm{d}\rho,$$

we see that the desired conclusion follows from

$$\sup_{R>0} \left| \int_R^\infty \rho^{-\gamma} e^{i\rho} \, \mathrm{d}\rho \right| < \infty \quad \text{if } 0 < \gamma < 1,$$
$$\sup_{R>0} R^{\gamma-1} \left| \int_R^\infty \rho^{-\gamma} e^{i\rho} \, \mathrm{d}\rho \right| < \infty \quad \text{if } \gamma > 1.$$

The first estimate follows from integration by parts, and the second estimate follows even more simply from an application of the triangle inequality. □

2.5 Weighted Hausdorff–Young Inequality.

While Lemma 2.8 below is clearly related to Pitt's inequality (also known as Hardy's inequality; see [2, 3]), we choose to present a self-contained, short proof of the special one-dimensional case which will be directly relevant to our analysis. For convenience, set $\mathcal{F}(f) := \widehat{f}$.

Lemma 2.8. *Let* $1 < p \leq 2 \leq q < \infty$, *and* $\delta := 1 - \frac{1}{p} - \frac{1}{q}$. *If* $0 \leq \delta < 1$, *then the estimate*

$$\|\mathcal{F}(f|\cdot|^{-\delta})\|_{L^q(\mathbb{R})} \leq C(p, q)\|f\|_{L^p(\mathbb{R})}$$

holds for every $f \in L^p(\mathbb{R})$.

Proof. If $\delta = 0$, then the result amounts to the Hausdorff–Young inequality. If $\delta \in (0, 1)$, then we have that

$$\|\mathcal{F}(f|\cdot|^{-\delta})\|_{L^q(\mathbb{R})} = \|\widehat{f} * \mathcal{F}(|\cdot|^{-\delta})\|_{L^q(\mathbb{R})} \simeq \|\widehat{f} * (|\cdot|^{-(1-\delta)})\|_{L^q(\mathbb{R})}$$
$$\lesssim \|\widehat{f}\|_{L^{p'}(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}.$$

This chain of estimates results from consecutive applications of the Hardy–Littlewood–Sobolev [38, p. 354] and the Hausdorff–Young inequalities. □

2.6 Weighted Hardy–Littlewood–Sobolev Inequality. Our analysis will rely on the L^p–L^q mapping properties of the following family of integral operators, indexed by $a, b \in \mathbb{R}$ and acting on functions $f : \mathbb{R}_+ := [0, \infty) \to \mathbb{C}$ via

$$\mathcal{T}_{a,b}(f)(x) := x^{-a} \int_{y \leq x} y^{-b} f(y)\, \mathrm{d}y.$$

Lemma 2.9. *Let* $1 < p \leq q < \infty$ *and* $a, b \in \mathbb{R}$. *Then* $\mathcal{T}_{a,b} : L^p(\mathbb{R}_+) \to L^q(\mathbb{R}_+)$ *is bounded if* $bp' < 1$ *and* $\frac{1}{p'} + \frac{1}{q} = a + b$.

Proof. For any $x \geq 0$, from Hölder's inequality it follows that

$$|\mathcal{T}_{a,b}(f)(x)| \leq x^{-a} \left(\int_0^x y^{-bp'}\, \mathrm{d}y \right)^{\frac{1}{p'}} \|f\|_{L^p(\mathbb{R}_+)} \simeq x^{\frac{1}{p'} - a - b} \|f\|_{L^p(\mathbb{R}_+)},$$

where the implicit constant is finite as long as $bp' < 1$. This implies

$$\|\mathcal{T}_{a,b}(f)\|_{L^{q,\infty}(\mathbb{R}_+)} \lesssim \|(\cdot)^{\frac{1}{p'} - a - b}\|_{L^{q,\infty}(\mathbb{R}_+)} \|f\|_{L^p(\mathbb{R}_+)},$$

where the first term on the right-hand side is finite precisely when $\frac{1}{p'} + \frac{1}{q} = a + b$. To conclude, note that the claimed strong-type estimate for $1 < p \leq q < \infty$ follows from the Marcinkiewicz interpolation theorem [5, p. 9] applied to the bounds $L^p(\mathbb{R}_+) \to L^{q,\infty}(\mathbb{R}_+)$ with

$$\left(\frac{1}{p}, \frac{1}{q} \right) \in \{(1 - \max\{b, 0\} - \delta, a + \min\{b, 0\} - \delta), (1 - a - b + \delta, \delta)\},$$

for sufficiently small enough $\delta > 0$. Indeed, all exponents $1 < p \leq q < \infty$ satisfying $bp' < 1$ and $\frac{1}{p'} + \frac{1}{q} = a + b$ are covered by this interpolation procedure since $0 < a + b \leq 1$. This finishes the proof of the lemma. □

Given $\ell \in (0, \infty)$ and certain $a, b \in \mathbb{R}$, the need will arise to consider the following related family of integral operators, acting on functions $f : [0, \ell] \to \mathbb{C}$ via:

$$\mathcal{S}_{a,b}(f)(x) := x^{-a} \int_{y \leq x} (x - y)^{-b} f(y)\, \mathrm{d}y. \tag{2.8}$$

Useful[1] estimates for $\mathcal{S}_{a,b}$ follow from the Stein–Weiss inequality [40], which extends the Hardy–Littlewood–Sobolev inequality for fractional integrals.

Lemma 2.10. *Let* $\ell \in (0, \infty)$, $1 < p \leq q < \infty$, $a \geq 0$, *and* $0 < b < 1$. *Then* $\mathcal{S}_{a,b} : L^p([0, \ell]) \to L^q([0, \ell])$ *is bounded if* $\frac{1}{p'} + \frac{1}{q} \geq a + b$.

[1] Albeit non-optimal. We omit trivial improvements of Lemma 2.10 (obtainable, e.g., via Hölder's inequality) which are not directly relevant to the forthcoming analysis.

Remark 2.11. In the endpoint case $\frac{1}{p'} + \frac{1}{q} = a + b$, the assumption $p \le q$ is in fact necessary for the L^p–L^q boundedness of $\mathcal{S}_{a,b}$. Indeed, if $p > q$, then let $0 < \varepsilon < \frac{p}{q} - 1$ and

$$f_\varepsilon(x) := x^{-\frac{1}{p}} |\log(x)|^{-\frac{1+\varepsilon}{p}} \mathbb{1}_{[0,\frac{1}{2}]}(x).$$

It is easy to check that $f_\varepsilon \in L^p([0,1])$, but that $|S_{a,b}(f_\varepsilon)(x)| \gtrsim x^{\frac{1}{p'}-a-b} |\log(x)|^{-\frac{1+\varepsilon}{p}}$ for all sufficiently small $x > 0$. In particular, $S_{a,b}(f_\varepsilon) \notin L^q([0,1])$ in view of our choice of ε.

Proof of Lemma 2.10. For every $0 \le x \le \ell$, we have

$$|\mathcal{S}_{a,b}(f)(x)| \le |x|^{-\delta a} \int_{\mathbb{R}} |x-y|^{-b} |y|^{-(1-\delta)a} |(f\, \mathbb{1}_{[0,\ell]})(y)| \,\mathrm{d}y,$$

as long as $0 < \delta < 1$. The hypotheses make it possible to choose $\delta \in (0,1)$ and exponents $\tilde{p}, \tilde{q}$ satisfying $1 < \tilde{p} \le p \le q \le \tilde{q} < \infty$, in such a way that

$$\frac{1}{\tilde{p}'} > (1-\delta)a, \quad \frac{1}{\tilde{q}} > \delta a, \quad \frac{1}{\tilde{p}'} + \frac{1}{\tilde{q}} = \delta a + b + (1-\delta)a.$$

From the Stein–Weiss inequality [40, Theorem B_1^*], it then follows that

$$\|\mathcal{S}_{a,b}(f)\|_{L^{\tilde{q}}(\mathbb{R})} \lesssim \|f\, \mathbb{1}_{[0,\ell]}\|_{L^{\tilde{p}}(\mathbb{R})},$$

which implies the desired conclusion via the inclusion of Lebesgue spaces on bounded intervals since $\tilde{p} \le p$ and $q \le \tilde{q}$. This concludes the proof of the lemma. $\square$

In the course of the proof of Proposition 3.1 below, we will also invoke bounds for the adjoint of the operator $\mathcal{S}_{a,b}$, whose form we record here:

$$\mathcal{S}^*_{a,b}(g)(y) := \int_{x \ge y} (x-y)^{-b} x^{-a} g(x) \,\mathrm{d}x. \tag{2.9}$$

Naturally, $\mathcal{S}^*_{a,b} : L^{q'}([0,\ell]) \to L^{p'}([0,\ell])$ if and only if $\mathcal{S}_{a,b} : L^p([0,\ell]) \to L^q([0,\ell])$.

3 Weighted 2D Restriction-type estimates

In this section, we analyze the $L^p(\mathbb{R}^2)$–$L^q(\mathbb{S}^1)$ mapping properties of the operator $\mathcal{R}_{\alpha,\beta}$, defined as follows:

$$\begin{aligned}\mathcal{R}_{\alpha,\beta}(g)(\omega) \\ := \int_{\mathbb{R}^2} g(x) \mathbb{1}_{|\omega_1||x_1| \ge 1} \mathbb{1}_{|\omega_2||x_2| \ge 1} (1+|x_1|)^{-\alpha} (1+|x_2|)^{-\beta} e^{-ix\cdot(|\omega_1|,|\omega_2|)} \,\mathrm{d}x;\end{aligned} \tag{3.1}$$

here $\alpha, \beta > 0$, $\omega = (\omega_1, \omega_2) \in \mathbb{S}^1$, and $x = (x_1, x_2) \in \mathbb{R}^{1+1}$. We highlight the role of the indicator functions in the integrand of (3.1). Without them, the resulting operator would have similar, but not identical, mapping properties to that of $\mathcal{R}_{\alpha,\beta}$, which by themselves do not appear sufficient to prove Theorem 1.1. Considering the adjoint operator,

$$\mathcal{R}^*_{\alpha,\beta}(F)(x) = (1+|x_1|)^{-\alpha}(1+|x_2|)^{-\beta} \int_{\mathbb{S}^1} F(\omega) \mathbb{1}_{|\omega_1||x_1|\geq 1} \mathbb{1}_{|\omega_2||x_2|\geq 1} e^{ix\cdot(|\omega_1|,|\omega_2|)} \,\mathrm{d}\sigma(\omega),$$

we investigate the $L^{q'}(\mathbb{S}^1)$–$L^{p'}(\mathbb{R}^2)$ boundedness of $\mathcal{R}^*_{\alpha,\beta}$, and start with the important special case when $p' = p = 2$; see also [7].

Proposition 3.1. *Let $2 \leq q < \infty$ and $\alpha, \beta > 0$ be such that $\frac{1}{2} < \alpha + \beta < 1$. Then $\mathcal{R}^*_{\alpha,\beta} : L^{q'}(\mathbb{S}^1) \to L^2(\mathbb{R}^2)$ is bounded if $\alpha + \beta + \min\{\alpha, \beta\} \geq \frac{3}{2} - \frac{1}{q}$.*

The main tool is the oscillatory integral estimate from Lemma 2.7 which together with elementary considerations place us in the setting of the weighted Hardy–Littlewood–Sobolev inequality, Lemma 2.10.

Proof of Proposition 3.1. By symmetry, we may assume that $\alpha \geq \beta$, and by interpolation, that $\alpha, \beta \neq \frac{1}{2}$. We can further assume that $F \in L^{q'}(\mathbb{S}^1)$ is supported in the region $\{\omega \in \mathbb{S}^1 : \omega_1, \omega_2 \geq 0\}$, since the other contributions can be estimated in a similar way. For such F, set $F_\star(\phi) := F(\cos\phi, \sin\phi)$, and compute:

$$\begin{aligned}
&\|\mathcal{R}^*_{\alpha,\beta}(F)\|^2_{L^2(\mathbb{R}^2)} \\
&\quad= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} F_\star(\varphi)\overline{F_\star(\phi)} \left(\int_{\min\{\cos(\phi),\cos(\varphi)\}|x_1|\geq 1} (1+|x_1|)^{-2\alpha} e^{ix_1(\cos\phi - \cos\varphi)} \,\mathrm{d}x_1 \right) \\
&\qquad \times \left(\int_{\min\{\sin(\phi),\sin(\varphi)\}|x_2|\geq 1} (1+|x_2|)^{-2\beta} e^{ix_2(\sin\phi-\sin\varphi)} \,\mathrm{d}x_2 \right) \mathrm{d}\phi \,\mathrm{d}\varphi \\
&\quad= 2\sum_{j=1}^{3} \int_{(\varphi,\phi)\in I_j} |F_\star(\varphi)||F_\star(\phi)| \left| \int_{|x_1|\geq \cos(\varphi)^{-1}} (1+|x_1|)^{-2\alpha} e^{ix_1(\cos\phi-\cos\varphi)} \,\mathrm{d}x_1 \right| \\
&\qquad \times \left| \int_{|x_2|\geq \sin(\phi)^{-1}} (1+|x_2|)^{-2\beta} e^{ix_2(\sin\phi - \sin\varphi)} \,\mathrm{d}x_2 \right| \mathrm{d}\phi\,\mathrm{d}\varphi,
\end{aligned}$$

where the regions $\{I_j\}_{j=1}^3 \subset [0, \frac{\pi}{2}]^2$ are defined as follows:

$$\begin{aligned}
I_1 &:= \left\{0 \leq \varphi \leq \frac{\pi}{4}, 0 \leq \phi \leq \varphi\right\}, \\
I_2 &:= \left\{\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2}, 0 \leq \phi \leq \frac{\pi}{8}\right\}, \\
I_3 &:= \left\{\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2}, \frac{\pi}{8} \leq \phi \leq \varphi\right\}.
\end{aligned}$$

All the resulting oscillatory integrals will be estimated via Lemma 2.7. The ones over the region I_2 are easy to handle since $|\cos\phi - \cos\varphi|, |\sin\phi - \sin\varphi| \geq c$, for some $c > 0$ and every $(\varphi, \phi) \in I_2$. Therefore the contribution from I_2 is bounded by a constant multiple of $\|F\|^2_{L^1(\mathbb{S}^1)} = O(\|F\|^2_{L^{q'}(\mathbb{S}^1)})$.

To estimate the integrals over the regions I_1, I_3, we make use of the following lower bounds, valid for some $c > 0$ and every $0 \leq \phi \leq \varphi \leq \frac{\pi}{2}$:

$$
\begin{aligned}
|\cos\varphi - \cos\phi| &\geq \cos\Big(\frac{\varphi+\phi}{2}\Big) - \cos\varphi \geq \frac{\varphi-\phi}{2}\sin\Big(\frac{\varphi+\phi}{2}\Big) \\
&\geq c\frac{\varphi-\phi}{2}\frac{\varphi+\phi}{2} \geq \frac{c}{4}(\varphi-\phi)\varphi, \\
|\sin\varphi - \sin\phi| &\geq \frac{c}{4}(\varphi-\phi)\Big(\frac{\pi}{2}-\phi\Big).
\end{aligned} \tag{3.2}
$$

Similarly, for some $C < \infty$ and every $0 \leq \phi \leq \varphi \leq \frac{\pi}{2}$, we have that

$$
|\cos\varphi - \cos\phi| \leq C(\varphi-\phi)\varphi, \quad |\sin\varphi - \sin\phi| \leq C(\varphi-\phi)\Big(\frac{\pi}{2}-\phi\Big). \tag{3.3}
$$

Analysis on I_1: If $0 < \alpha, \beta < \frac{1}{2}$, then Lemma 2.7 and the bounds (3.2) together imply

$$
\begin{aligned}
&\int_{(\varphi,\phi)\in I_1} |F_\star(\varphi)||F_\star(\phi)| \left| \int_{|x_1|\geq\cos(\varphi)^{-1}} (1+|x_1|)^{-2\alpha} e^{ix_1(\cos\phi-\cos\varphi)}\,\mathrm{d}x_1 \right| \\
&\qquad \times \left| \int_{|x_2|\geq\sin(\phi)^{-1}} (1+|x_2|)^{-2\beta} e^{ix_2(\sin\phi-\sin\varphi)}\,\mathrm{d}x_2 \right| \mathrm{d}\phi\,\mathrm{d}\varphi \\
&\lesssim \int_{(\phi,\varphi)\in I_1} |F_\star(\varphi)||F_\star(\phi)||\cos\phi-\cos\varphi|^{2\alpha-1}|\sin\phi-\sin\varphi|^{2\beta-1}\,\mathrm{d}\phi\,\mathrm{d}\varphi \\
&\lesssim \int_{(\phi,\varphi)\in I_1} |F_\star(\varphi)||F_\star(\phi)||\varphi(\varphi-\phi)|^{2\alpha-1}\left|\Big(\frac{\pi}{2}-\phi\Big)(\varphi-\phi)\right|^{2\beta-1}\mathrm{d}\phi\,\mathrm{d}\varphi \\
&\lesssim \int_0^{\frac{\pi}{4}} |F_\star(\varphi)|\left(|\varphi|^{2\alpha-1}\int_0^{\varphi}|F_\star(\phi)||\varphi-\phi|^{2\alpha+2\beta-2}\,\mathrm{d}\phi\right)\mathrm{d}\varphi,
\end{aligned} \tag{3.4}
$$

where we have used that $(\frac{\pi}{2}-\phi)^{2\beta-1} \lesssim 1$ since $0 \leq \phi \leq \frac{\pi}{4}$. Setting

$$
(a, b) := (1-2\alpha, 2-2\alpha-2\beta),
$$

we have $a \geq 0, 0 < b < 1$ and $a + b \leq \frac{2}{q}$. Therefore definition (2.8), Hölder's inequality, and Lemma 2.10 (which can be applied since $q \geq 2$ and thus $q \geq q'$) together imply that (3.4) is bounded by

$$
\begin{aligned}
\int_0^{\frac{\pi}{4}} |F_\star(\varphi)|\mathcal{S}_{a,b}(|F_\star|)(\varphi)\,\mathrm{d}\varphi &\leq \|F_\star\|_{L^{q'}([0,\frac{\pi}{4}])}\|\mathcal{S}_{a,b}(|F_\star|)\|_{L^q([0,\frac{\pi}{4}])} \\
&\lesssim \|F_\star\|^2_{L^{q'}([0,\frac{\pi}{4}])} \leq \|F\|^2_{L^{q'}(\mathbb{S}^1)}.
\end{aligned}
$$

If $0 < \beta < \frac{1}{2} < \alpha$, then Lemma 2.7 and estimates (3.2)–(3.3) together imply

$$\begin{aligned}
&\int_{(\varphi,\phi)\in I_1} |F_\star(\varphi)||F_\star(\phi)| \left| \int_{|x_1|\geq \cos(\varphi)^{-1}} (1+|x_1|)^{-2\alpha} e^{ix_1(\cos\phi-\cos\varphi)}\,\mathrm{d}x_1 \right| \\
&\qquad \times \left| \int_{|x_2|\geq \sin(\phi)^{-1}} (1+|x_2|)^{-2\beta} e^{ix_2(\sin\phi-\sin\varphi)}\,\mathrm{d}x_2 \right| \mathrm{d}\phi\,\mathrm{d}\varphi \\
&\lesssim \int_{(\phi,\varphi)\in I_1} |F_\star(\varphi)||F_\star(\phi)||\cos\phi|^{2\alpha-1}|\sin\phi-\sin\varphi|^{2\beta-1}\,\mathrm{d}\phi\,\mathrm{d}\varphi \qquad (3.5)\\
&\lesssim \int_{(\phi,\varphi)\in I_1} |F_\star(\varphi)||F_\star(\phi)|\Big(\frac{\pi}{2}-\phi\Big)^{2\alpha-1}\Big|\Big(\frac{\pi}{2}-\phi\Big)(\varphi-\phi)\Big|^{2\beta-1}\,\mathrm{d}\phi\,\mathrm{d}\varphi \\
&\lesssim \int_0^{\frac{\pi}{4}} |F_\star(\varphi)|\Big(\int_0^{\phi} |F_\star(\varphi)||\varphi-\phi|^{2\beta-1}\,\mathrm{d}\phi\Big)\,\mathrm{d}\varphi.
\end{aligned}$$

Setting $(a,b) := (0, 1-2\beta)$, we have $a \geq 0$, $0 < b < 1$, and $a+b \leq \frac{2}{q}$; indeed, setting $\gamma := \alpha+\beta+\min\{\alpha,\beta\}$, it follows that

$$\frac{2}{q} - a - b \geq 3 - 2\gamma - (1-2\beta) = 2 - 2(\alpha+\beta) \geq 0.$$

Lemma 2.10 again implies that the integral (3.5) is $O(\|F\|^2_{L^{q'}(\mathbb{S}^1)})$.

Analysis on I_3: If $0 < \alpha, \beta < \frac{1}{2}$, then Lemma 2.7, the bounds (3.2), and the change of variables $(\frac{\pi}{2}-\varphi,\phi) \rightsquigarrow (\varphi, \frac{\pi}{2}-\phi)$ together yield

$$\begin{aligned}
&\int_{(\varphi,\phi)\in I_3} |F_\star(\varphi)||F_\star(\phi)| \left| \int_{|x_1|\geq \cos(\varphi)^{-1}} (1+|x_1|)^{-2\alpha} e^{ix_1(\cos\phi-\cos\varphi)}\,\mathrm{d}x_1 \right| \\
&\qquad \times \left| \int_{|x_2|\geq \sin(\phi)^{-1}} (1+|x_2|)^{-2\beta} e^{ix_2(\sin\phi-\sin\varphi)}\,\mathrm{d}x_2 \right| \mathrm{d}\phi\,\mathrm{d}\varphi \\
&\lesssim \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} |F_\star(\varphi)|\Big(\int_{\frac{\pi}{8}}^{\varphi} (\varphi-\phi)^{2\alpha+2\beta-2}\Big(\frac{\pi}{2}-\phi\Big)^{2\beta-1}|F_\star(\phi)|\,\mathrm{d}\phi\Big)\,\mathrm{d}\varphi \\
&= \int_0^{\frac{\pi}{4}} \Big|F_\star\Big(\frac{\pi}{2}-\varphi\Big)\Big|\Big(\int_{\varphi}^{\frac{\pi}{2}} (\varphi-\phi)^{2\alpha+2\beta-2}\phi^{2\beta-1}\Big|F_\star\Big(\frac{\pi}{2}-\phi\Big)\Big|\,\mathrm{d}\phi\Big)\,\mathrm{d}\varphi \\
&= \int_0^{\frac{\pi}{4}} |\tilde{F}_\star(\varphi)|\mathcal{S}^*_{a,b}(|\tilde{F}_\star|)(\varphi)\,\mathrm{d}\varphi \lesssim \|\tilde{F}_\star\|^2_{L^{q'}([0,\frac{\pi}{4}])} \leq \|F\|^2_{L^{q'}(\mathbb{S}^1)},
\end{aligned}$$

where $\tilde{F}_\star := F_\star(\frac{\pi}{2}-\cdot)$. Here we used the $L^{q'}$–L^q bound for the adjoint operator $\mathcal{S}^*_{a,b}$ (recall (2.9)), implied by Lemma 2.10 with $(a,b) := (1-2\beta, 2-2\alpha-2\beta)$. The analysis of the case $\beta < \frac{1}{2} < \alpha$ proceeds along similar lines, and is therefore omitted. This concludes the proof of the proposition. □

Next we extend the range of boundedness of $\mathcal{R}^*_{\alpha,\beta}$ given by Proposition 3.1 via interpolation with a trivial estimate for $\mathcal{R}^*_{0,0}$.

Proposition 3.2. *Let $1 < p \leq 2 \leq q < \infty$ and $\alpha, \beta > 0$ be such that $\frac{1}{p'} < \alpha + \beta < \frac{2}{p'}$. Then $\mathcal{R}^*_{\alpha,\beta} : L^{q'}(\mathbb{S}^1) \to L^{p'}(\mathbb{R}^2)$ is bounded if*

$$\alpha + \beta + \min\{\alpha, \beta\} \geq \frac{3}{p'} - \frac{1}{q}.$$

Proof. Set $\gamma := \alpha + \beta + \min\{\alpha, \beta\}$. We use complex interpolation for the analytic family of operators given by

$$\mathcal{E}_s := \mathcal{R}^*_{\frac{p'\alpha s}{2}, \frac{p'\beta s}{2}}, \tag{3.6}$$

where $s \in S := \{z \in \mathbb{C} : 0 \leq \Re(z) \leq 1\}$. Start by noting that, given simple functions $F \in L^1(\mathbb{S}^1)$ and $g \in L^1(\mathbb{R}^2)$, the map

$$s \mapsto \int_{\mathbb{R}^2} \mathcal{E}_s(F)(x) g(x) \, \mathrm{d}x \tag{3.7}$$

is analytic in the interior of the strip S, continuous on S, and moreover the function defined by (3.7) is uniformly bounded above in S.

Now, Proposition 3.1, with (α, β) replaced by $(p'\alpha/2, p'\beta/2)$, yields

$$\|\mathcal{E}_s(F)\|_{L^2(\mathbb{R}^2)} \lesssim \|F\|_{L^{q_1'}(\mathbb{S}^1)} \quad \text{for } \Re(s) = 1, \text{ if } \frac{1}{2} \leq \frac{1}{q_1'} \leq \frac{p'}{2}\gamma - \frac{1}{2}. \tag{3.8}$$

On the other hand, we have the following trivial estimate:

$$\|\mathcal{E}_s(F)\|_{L^\infty(\mathbb{R}^2)} \lesssim \|F\|_{L^{q_0'}(\mathbb{S}^1)} \quad \text{for } \Re(s) = 0, \text{ if } 0 \leq \frac{1}{q_0'} \leq 1. \tag{3.9}$$

Since $q \geq p$ and $\gamma \geq \frac{3}{p'} - \frac{1}{q}$, we may choose q_0, q_1 satisfying the above conditions, with the additional property that $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ for $\theta := \frac{2}{p'}$. Then Stein's interpolation theorem [41, p. 205] can be applied to the analytic family of operators $\{\mathcal{E}_s\}_{s \in S}$ given by (3.6), resulting from (3.8)–(3.9) that

$$\|\mathcal{E}_s(F)\|_{L^{p'}(\mathbb{R}^2)} \lesssim \|F\|_{L^{q'}(\mathbb{S}^1)}, \quad \text{for } \Re(s) = \theta = \frac{2}{p'}.$$

This amounts to the desired conclusion since $\theta = \frac{2}{p'}$ implies $\mathcal{E}_\theta = \mathcal{R}^*_{\alpha,\beta}$. □

Proposition 3.2 will later on be refined via interpolation with a non-trivial estimate for the operator $\mathcal{R}^*_{0,0}$. Since $\mathcal{R}^*_{0,0}$ is similar but not identical to the two-dimensional extension operator on the unit circle, we first need to prove the latter estimate. That is the content of our next result.

Proposition 3.3. *Let $1 \leq p < \frac{4}{3}$ and $1 \leq q \leq \frac{p'}{3}$. Then $\mathcal{R}^*_{0,0} : L^{q'}(\mathbb{S}^1) \to L^{p'}(\mathbb{R}^2)$ defines a bounded operator.*

Proof. We may assume $p > 1$, and will bound $\mathcal{R}_{0,0} : L^p(\mathbb{R}^2) \to L^q(\mathbb{S}^1)$ instead. Given $f : \mathbb{R}^2 \to \mathbb{C}$, set $g_{x_1}(x_2) := g(x_1, x_2) := f(x_1, x_2)\mathbb{1}_{x_1^{-2}+x_2^{-2}\le 1}$. Then

$$\begin{aligned}\mathcal{R}_{0,0}(f)(\omega) &= \int_{\frac{1}{|\omega_1|}}^{\infty}\int_{\frac{1}{|\omega_2|}}^{\infty} f(x_1,x_2)e^{-i(x_1|\omega_1|+x_2|\omega_2|)}\,\mathrm{d}x_1\,\mathrm{d}x_2\\ &= \int_{\frac{1}{|\omega_1|}}^{\infty}\int_{\mathbb{R}} g(x_1,x_2)e^{-i(x_1|\omega_1|+x_2|\omega_2|)}\,\mathrm{d}x_1\,\mathrm{d}x_2\\ &= \int_{\mathbb{R}^2} g(x)e^{-ix\cdot(|\omega_1|,|\omega_2|)}\,\mathrm{d}x - \int_1^{\frac{1}{|\omega_1|}} \widehat{g_{x_1}}(|\omega_2|)e^{-ix_1|\omega_1|}\,\mathrm{d}x_1.\end{aligned}$$

Within the desired range of exponents, bounds for the first term are well-known [9, 49], so we proceed to bound the second term in $L^q(\mathbb{S}^1)$:

$$\begin{aligned}&\left(\int_{\mathbb{S}^1}\left(\int_1^{\frac{1}{|\omega_1|}}|\widehat{g_{x_1}}(|\omega_2|)|\,\mathrm{d}x_1\right)^q\mathrm{d}\sigma(\omega)\right)^{\frac{1}{q}}\\ &\qquad= \left(\int_0^1(1-r^2)^{-\frac{1}{2}}\left(\int_1^{\frac{1}{\sqrt{1-r^2}}}|\widehat{g_{x_1}}(r)|\,\mathrm{d}x_1\right)^q\mathrm{d}r\right)^{\frac{1}{q}}\\ &\qquad= \sup_{\|h\|_{L^{q'}}=1}\int_0^1(1-r^2)^{-\frac{1}{2q}}\left(\int_1^{\frac{1}{\sqrt{1-r^2}}}|\widehat{g_{x_1}}(r)|\,\mathrm{d}x_1\right)h(r)\,\mathrm{d}r \qquad (3.10)\\ &\qquad= \sup_{\|h\|_{L^{q'}}=1}\int_1^{\infty}\left(\int_{\sqrt{1-x_1^{-2}}}^1(1-r^2)^{-\frac{1}{2q}}|\widehat{g_{x_1}}(r)|h(r)\,\mathrm{d}r\right)\mathrm{d}x_1\\ &\qquad\le \int_1^{\infty}\left(\int_{\sqrt{1-x_1^{-2}}}^1(1-r^2)^{-\frac{p'}{2(p'-q)}}\,\mathrm{d}r\right)^{\frac{1}{q}-\frac{1}{p'}}\|\widehat{g_{x_1}}\|_{L^{p'}}\,\mathrm{d}x_1.\end{aligned}$$

Here we changed variables, used duality, Fubini's Theorem, and Hölder's inequality. Another change of variables, the Hausdorff–Young inequality, and Hölder's inequality in Lorentz space [31, Theorem 3.4] together yield the following upper bound for (3.10):

$$\begin{aligned}&\int_1^{\infty}\left(\int_0^{1-\sqrt{1-x_1^{-2}}} s^{-\frac{p'}{2(p'-q)}}\,\mathrm{d}s\right)^{\frac{1}{q}-\frac{1}{p'}}\|g_{x_1}\|_{L^p}\,\mathrm{d}x_1\\ &\qquad\lesssim \int_1^{\infty}[(1-\sqrt{1-x_1^{-2}})^{1-\frac{p'}{2(p'-q)}}]^{\frac{1}{q}-\frac{1}{p'}}\|g_{x_1}\|_{L^p}\,\mathrm{d}x_1\\ &\qquad\lesssim \int_1^{\infty}(x_1^{-2})^{\frac{1}{q}-\frac{1}{p'}-\frac{1}{2q}}\|g_{x_1}\|_{L^p}\,\mathrm{d}x_1\\ &\qquad\le \|(\cdot)^{-\frac{1}{q}+\frac{2}{p'}}\|_{L^{p',\infty}([1,\infty))}\|g\|_{L^{p,1}(\mathbb{R}^2)}.\end{aligned}$$

The first term on the right-hand side is finite since $q \le \frac{p'}{3}$. Since $|g| \le |f|$, this establishes the $L^{p,1}(\mathbb{R}^2)$–$L^q(\mathbb{S}^1)$ boundedness of $\mathcal{R}_{0,0}$, provided $1 \le p < \frac{4}{3}$

and $1 \leq q \leq \frac{p'}{3}$. Real interpolation [5, Theorem 5.3.2] within this family of Lorentz space estimates and compactness of $\mathbb{S}^1$ together yield the claimed strong type estimates. This concludes the proof of the proposition. □

4 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. After some preliminary simplifications, we reduce the analysis to three main estimates, which we address separately.

Fix $d \geq 4$ and $k \in \{2, 3, \ldots, d-2\}$, and set $m := \min\{k, d-k\}$. By a routine density argument, it suffices to establish the estimate

$$\|\widehat{f}\|_{L^2(\mathbb{S}^{d-1})} \leq C(k, d, p)\|f\|_{L^p(\mathbb{R}^d)}, \quad 1 \leq p \leq \frac{2(d+m)}{d+m+2}, \tag{4.1}$$

for every G_k-symmetric Schwartz function $f : \mathbb{R}^d \to \mathbb{C}$. Our starting point is the following formula from Lemma 2.2:

$$\begin{aligned} \widehat{f}(\eta, \zeta) =& (2\pi)^{\frac{d}{2}} |\eta|^{\frac{2-d+k}{2}} |\zeta|^{\frac{2-k}{2}} \\ & \times \int_0^\infty \int_0^\infty \rho_1^{\frac{d-k}{2}} \rho_2^{\frac{k}{2}} f_0(\rho_1, \rho_2) J_{\frac{d-k-2}{2}}(\rho_1|\eta|) J_{\frac{k-2}{2}}(\rho_2|\zeta|) \, \mathrm{d}\rho_1 \, \mathrm{d}\rho_2, \end{aligned} \tag{4.2}$$

which holds at every $(\eta, \zeta) \in \mathbb{R}^{d-k} \times \mathbb{R}^k$, for any G_k-symmetric Schwartz function $f : \mathbb{R}^d \to \mathbb{C}$. In light of Lemma 2.1, there exist nonzero constants $A_1, A_2 \in \mathbb{C} \setminus \{0\}$ and functions $R_1, R_2 : (0, \infty) \to \mathbb{C}$, such that, for every $r \geq 0$,

$$J_{\frac{d-k-2}{2}}(r) = (A_1 e^{ir} + \overline{A_1} e^{-ir}) r^{-\frac{1}{2}} \mathbb{1}_{[1,\infty)}(r) + R_1(r), \tag{4.3}$$

$$J_{\frac{k-2}{2}}(r) = (A_2 e^{ir} + \overline{A_2} e^{-ir}) r^{-\frac{1}{2}} \mathbb{1}_{[1,\infty)}(r) + R_2(r). \tag{4.4}$$

Moreover, the following estimates hold, for every $r \geq 0$:

$$|R_1(r)| \lesssim r^{\frac{d-k-2}{2}} (1+r)^{\frac{k-d-1}{2}}, \tag{4.5}$$

$$|R_2(r)| \lesssim r^{\frac{k-2}{2}} (1+r)^{-\frac{k+1}{2}}. \tag{4.6}$$

The decomposition (4.3)–(4.4) induces a splitting in (4.2),

$$\widehat{f} = (2\pi)^{\frac{d}{2}} \left(\widehat{f_1} + \sum_{j=2}^{5} (\widehat{f_j} + \overline{\widehat{f_j}}) \right), \tag{4.7}$$

where each of the pieces is defined as follows:

$$\text{(4.8)}\quad \widehat{f_1}(\eta,\zeta) := |\eta|^{\frac{k-d+2}{2}}|\zeta|^{\frac{2-k}{2}}\int_0^\infty\int_0^\infty \rho_1^{\frac{d-k}{2}}\rho_2^{\frac{k}{2}} f_0(\rho_1,\rho_2)R_1(\rho_1|\eta|)R_2(\rho_2|\zeta|)\,\mathrm{d}\rho_2\,\mathrm{d}\rho_1;$$

$$\text{(4.9)}\quad \widehat{f_2}(\eta,\zeta) := A_2|\eta|^{\frac{k-d+2}{2}}|\zeta|^{\frac{1-k}{2}}\int_0^\infty\int_{\frac{1}{|\zeta|}}^\infty \rho_1^{\frac{d-k}{2}}\rho_2^{\frac{k-1}{2}} f_0(\rho_1,\rho_2)R_1(\rho_1|\eta|)e^{i\rho_2|\zeta|}\,\mathrm{d}\rho_2\,\mathrm{d}\rho_1;$$

$$\widehat{f_3}(\eta,\zeta) := A_1|\eta|^{\frac{k-d+1}{2}}|\zeta|^{\frac{2-k}{2}}\int_{\frac{1}{|\eta|}}^\infty\int_0^\infty \rho_1^{\frac{d-k-1}{2}}\rho_2^{\frac{k}{2}} f_0(\rho_1,\rho_2)e^{i\rho_1|\eta|}R_2(\rho_2|\zeta|)\,\mathrm{d}\rho_2\,\mathrm{d}\rho_1;$$

$$\text{(4.10)}\quad \widehat{f_4}(\eta,\zeta) := A_1A_2|\eta|^{\frac{k-d+1}{2}}|\zeta|^{\frac{1-k}{2}}\int_{\frac{1}{|\eta|}}^\infty\int_{\frac{1}{|\zeta|}}^\infty \rho_1^{\frac{d-k-1}{2}}\rho_2^{\frac{k-1}{2}} f_0(\rho_1,\rho_2)e^{i(\rho_1|\eta|+\rho_2|\zeta|)}\,\mathrm{d}\rho_2\,\mathrm{d}\rho_1;$$

$$\text{(4.11)}\quad \widehat{f_5}(\eta,\zeta) := A_1\overline{A_2}|\eta|^{\frac{k-d+1}{2}}|\zeta|^{\frac{1-k}{2}}\int_{\frac{1}{|\eta|}}^\infty\int_{\frac{1}{|\zeta|}}^\infty \rho_1^{\frac{d-k-1}{2}}\rho_2^{\frac{k-1}{2}} f_0(\rho_1,\rho_2)e^{i(\rho_1|\eta|-\rho_2|\zeta|)}\,\mathrm{d}\rho_2\,\mathrm{d}\rho_1.$$

Note that each $\widehat{f_j}$ is G_k-symmetric. We proceed to find suitable bounds for $\|\widehat{f_j}\|_{L^2(\mathbb{S}^{d-1})}$ for each $1 \le j \le 5$. By interchanging $d-k$ and k, the estimates for $\widehat{f_3}, \widehat{f_5}$ are analogous to those for $\widehat{f_2}, \widehat{f_4}$, respectively, and so the analysis actually reduces to three cases.

Recall that, for a given G_k-symmetric function $f : \mathbb{S}^{d-1} \to \mathbb{C}$, we set

$$f_0(|\eta|, |\zeta|) = f(\eta,\zeta),$$

and have, for some dimensional constant $c_d \in (0,\infty)$ whose exact value will be unimportant,

$$\text{(4.12)}\qquad \|f\|^p_{L^p(\mathbb{R}^d)} = c_d^p\int_0^\infty\int_0^\infty \rho_1^{d-k-1}\rho_2^{k-1}|f_0(\rho_1,\rho_2)|^p\,\mathrm{d}\rho_1\,\mathrm{d}\rho_2.$$

4.1 Estimating $\widehat{f_1}$.

This is by far the easiest case to handle.

Proposition 4.1. *For every $1 \le p < 2$, there exists $C = C(k,d,p) < \infty$ such that*

$$\|\widehat{f_1}\|_{L^2(\mathbb{S}^{d-1})} \le C\|f\|_{L^p(\mathbb{R}^d)},$$

for every G_k-symmetric Schwartz function $f : \mathbb{R}^d \to \mathbb{C}$.

Proof. Fix $p \in [1,2)$. From definition (4.8) of $\widehat{f_1}$ and estimates (4.5)–(4.6), it follows that

$$\begin{aligned}
&|\widehat{f_1}(\eta,\zeta)| \\
&\lesssim \int_0^\infty \int_0^\infty \rho_1^{d-k-1}\rho_2^{k-1}|f_0(\rho_1,\rho_2)|(1+\rho_1|\eta|)^{\frac{k-d-1}{2}}(1+\rho_2|\zeta|)^{-\frac{k+1}{2}}\,\mathrm{d}\rho_2\,\mathrm{d}\rho_1 \\
&\leq \left(\int_0^\infty \int_0^\infty \rho_1^{d-k-1}\rho_2^{k-1}(1+\rho_1|\eta|)^{\frac{p'(k-d-1)}{2}}(1+\rho_2|\zeta|)^{-\frac{p'(k+1)}{2}}\,\mathrm{d}\rho_2\,\mathrm{d}\rho_1\right)^{\frac{1}{p'}}\|f\|_{L^p(\mathbb{R}^d)},
\end{aligned}$$

where the last line follows from an application of Hölder's inequality and (4.12). The change of variables $(\rho_1|\eta|,\rho_2|\zeta|) \rightsquigarrow (\rho_1,\rho_2)$ then reveals that

$$|\widehat{f_1}(\eta,\zeta)| \lesssim |\eta|^{\frac{k-d}{p'}}|\zeta|^{-\frac{k}{p'}}\|f\|_{L^p(\mathbb{R}^d)}, \tag{4.13}$$

for every $(\eta,\zeta) \in \mathbb{R}^{d-k}\times\mathbb{R}^k$. The latter estimate can be integrated over the unit sphere $|\eta|^2+|\zeta|^2=1$ via Lemma 2.3, yielding

$$\begin{aligned}
&\int_{\mathbb{S}^{d-1}} |\widehat{f_1}(\eta,\zeta)|^2\,\mathrm{d}\sigma(\eta,\zeta) \\
&\quad\lesssim \int_0^1 r^{d-k-1}(1-r^2)^{\frac{k-2}{2}}\left(r^{\frac{k-d}{p'}}(1-r^2)^{-\frac{k}{2p'}}\|f\|_{L^p(\mathbb{R}^d)}\right)^2\mathrm{d}r \\
&\quad= \left(\int_0^1 r^{2(d-k)(\frac12-\frac1{p'})-1}(1-r^2)^{k(\frac12-\frac1{p'})-1}\,\mathrm{d}r\right)\|f\|^2_{L^p(\mathbb{R}^d)} \lesssim \|f\|^2_{L^p(\mathbb{R}^d)}
\end{aligned}$$

since $p<2$. This concludes the proof of Proposition 4.1. □

4.2 Estimating $\widehat{f_2}$ and $\widehat{f_3}$.

The estimates in this section will follow, in the spirit of Carleson–Sjölin [9], from a successive application of weighted versions of the Hausdorff–Young and the Hardy–Littlewood–Sobolev inequalities.

Proposition 4.2. *There exists $C = C(k,d,p) < \infty$ such that*

$$\begin{aligned}
\|\widehat{f_2}\|_{L^2(\mathbb{S}^{d-1})} &\leq C\|f\|_{L^p(\mathbb{R}^d)}, \quad \text{if } 1 \leq p \leq \frac{2(d+k)}{d+k+2}, \\
\|\widehat{f_3}\|_{L^2(\mathbb{S}^{d-1})} &\leq C\|f\|_{L^p(\mathbb{R}^d)}, \quad \text{if } 1 \leq p \leq \frac{2(2d-k)}{2d-k+2},
\end{aligned}$$

for every G_k-symmetric Schwartz function $f:\mathbb{R}^d\to\mathbb{C}$.

Proof. We focus on the estimate for f_2 because the analysis of f_3 is analogous up to interchanging the roles of $k, d-k$. By interpolation with the trivial estimate at $p=1$, it suffices to consider the endpoint case $p = p_\star := \frac{2(d+k)}{d+k+2}$. Set $\delta := (k-1)(\frac{1}{p_\star}-\frac12)$, and define

$$g(\rho_1,\rho_2) := \rho_1^{\frac{d-k-1}{p_\star}}\rho_2^{\frac{k-1}{p_\star}}f_0(\rho_1,\rho_2), \tag{4.14}$$

for each $\rho_1, \rho_2 > 0$, as well as the corresponding "slice" function $g_{\rho_1} : (0,\infty) \to \mathbb{C}$,

$$g_{\rho_1}(\rho_2) := \rho_2^{-\delta} g(\rho_1,\rho_2)\mathbb{1}_{[1,\infty)}(\rho_2). \tag{4.15}$$

From the definitions (4.9) of $\widehat{f_2}$ and (4.14)–(4.15) of g, g_{ρ_1}, respectively, it follows that

$$\begin{aligned}
&|\widehat{f_2}(\eta,\zeta)| \\
&\quad\lesssim |\eta|^{\frac{2-d+k}{2}}|\zeta|^{\frac{1-k}{2}}\left|\int_0^\infty \rho_1^{\frac{d-k}{2}-\frac{d-k-1}{p_\star}} R_1(\rho_1|\eta|)\left(\int_{\frac{1}{|\zeta|}}^\infty \rho_2^{-\delta} g(\rho_1,\rho_2)e^{i\rho_2|\zeta|}\,\mathrm{d}\rho_2\right)\mathrm{d}\rho_1\right| \\
&\quad\le |\eta|^{\frac{2-d+k}{2}}|\zeta|^{\frac{1-k}{2}}\int_0^\infty \rho_1^{\frac{d-k}{2}-\frac{d-k-1}{p_\star}}|R_1(\rho_1|\eta|)| \\
&\qquad\qquad\times\left(|\widehat{g}_{\rho_1}(|\zeta|)| + \int_1^{\frac{1}{|\zeta|}} \rho_2^{-\delta}|g(\rho_1,\rho_2)|\,\mathrm{d}\rho_2\right)\mathrm{d}\rho_1,
\end{aligned}$$

for every $(\eta,\zeta)\in\mathbb{R}^{d-k}\times\mathbb{R}^k$. Estimate (4.5) then implies

$$\begin{aligned}
&|\widehat{f_2}(\eta,\zeta)| \\
&\lesssim |\zeta|^{\frac{1-k}{2}}\int_0^\infty \rho_1^{\frac{d-k-1}{p_\star'}}(1+\rho_1|\eta|)^{\frac{k-d-1}{2}}\left(|\widehat{g}_{\rho_1}(|\zeta|)| + \int_1^{\frac{1}{|\zeta|}}\rho_2^{-\delta}|g(\rho_1,\rho_2)|\,\mathrm{d}\rho_2\right)\mathrm{d}\rho_1 \\
&\lesssim |\zeta|^{\frac{1-k}{2}}\int_0^\infty \rho_1^{\frac{d-k-1}{p_\star'}}(1+\rho_1|\eta|)^{\frac{k-d-1}{2}}(|\widehat{g}_{\rho_1}(|\zeta|)| + |\zeta|^{\delta-\frac{1}{p_\star'}}\|g(\rho_1,\cdot)\|_{L^{p_\star}_{\rho_2}(\mathbb{R}_+)})\,\mathrm{d}\rho_1,
\end{aligned}$$

where the last line follows from an application of Hölder's inequality. We break up the latter integral into two pieces, and analyze both summands I, II (defined in (4.17), (4.16) below) separately. The second one is easier to handle, and can be bounded using Hölder's inequality as follows:

$$\begin{aligned}
\mathrm{II}(\eta,\zeta) &:= |\zeta|^{\frac{1-k}{2}+\delta-\frac{1}{p_\star'}}\int_0^\infty \rho_1^{\frac{d-k-1}{p_\star'}}(1+\rho_1|\eta|)^{\frac{k-d-1}{2}}\|g(\rho_1,\cdot)\|_{L^{p_\star}_{\rho_2}(\mathbb{R}_+)}\,\mathrm{d}\rho_1 \\
&\lesssim |\eta|^{\frac{k-d}{p_\star'}}|\zeta|^{-\frac{k}{p_\star'}}\|g\|_{L^{p_\star}(\mathbb{R}^2_+)} \simeq |\eta|^{\frac{k-d}{p_\star'}}|\zeta|^{-\frac{k}{p_\star'}}\|f\|_{L^{p_\star}(\mathbb{R}^d)}.
\end{aligned}\tag{4.16}$$

Comparing with (4.13), we see that this piece can be handled exactly as in the proof of Proposition 4.1, resulting in the following bound:

$$\int_{\mathbb{S}^{d-1}} \mathrm{II}^2(\eta,\zeta)\,\mathrm{d}\sigma(\eta,\zeta) \lesssim \|f\|^2_{L^{p_\star}(\mathbb{R}^d)}.$$

Here we only used $p_\star < 2$. It remains to estimate the integral of (the square of) the first summand, given by

$$\mathrm{I}(\eta,\zeta) := |\zeta|^{\frac{1-k}{2}}\int_0^\infty \rho_1^{\frac{d-k-1}{p_\star'}}(1+\rho_1|\eta|)^{\frac{k-d-1}{2}}|\widehat{g}_{\rho_1}(|\zeta|)|\,\mathrm{d}\rho_1. \tag{4.17}$$

With that purpose in mind, specialize Lemma 2.8 to $p = p_\star$ and

$$\frac{1}{q} = \frac{1}{q_\star} := 1 - \frac{1}{p_\star} - \delta = \frac{k+1}{2} - \frac{k}{p_\star} = \frac{d-k}{2(d+k)},$$

to obtain the following estimate:

$$\|\widehat{g}_{\rho_1}\|_{L^{q_\star}(\mathbb{R}_+)} \lesssim \|g(\rho_1,\cdot)\mathbb{1}_{[1,\infty)}\|_{L^{p_\star}(\mathbb{R}_+)} \leq \|g(\rho_1,\cdot)\|_{L^{p_\star}(\mathbb{R}_+)}. \tag{4.18}$$

From Lemma 2.3, we have that

$$\begin{aligned} &\int_{\mathbb{S}^{d-1}} \mathrm{I}^2(\eta,\zeta)\,\mathrm{d}\sigma(\eta,\zeta) \\ &\quad \lesssim \int_0^1 r^{d-k-1}(1-r^2)^{-\frac{1}{2}} \\ &\qquad\quad \times \left(\int_0^\infty \rho_1^{\frac{d-k-1}{p_\star'}}(1+\rho_1 r)^{\frac{k-d-1}{2}}|\widehat{g}_{\rho_1}(\sqrt{1-r^2})|\,\mathrm{d}\rho_1\right)^2 \mathrm{d}r \\ &\quad = \int_0^1 (1-s^2)^{\frac{d-k-2}{2}}\left(\int_0^\infty \rho_1^{\frac{d-k-1}{p_\star'}}(1+\rho_1\sqrt{1-s^2})^{\frac{k-d-1}{2}}|\widehat{g}_{\rho_1}(s)|\,\mathrm{d}\rho_1\right)^2 \mathrm{d}s \\ &\quad = \int_0^\infty\int_0^\infty (\rho_1\tilde\rho_1)^{\frac{d-k-1}{p_\star'}}\left(\int_0^1 K_{\rho_1,\tilde\rho_1}(s)|\widehat{g}_{\rho_1}(s)||\widehat{g}_{\tilde\rho_1}(s)|\,\mathrm{d}s\right)\mathrm{d}\tilde\rho_1\,\mathrm{d}\rho_1 \\ &\quad = 2\int_0^\infty\int_0^{\rho_1} (\rho_1\tilde\rho_1)^{\frac{d-k-1}{p_\star'}}\left(\int_0^1 K_{\rho_1,\tilde\rho_1}(s)|\widehat{g}_{\rho_1}(s)||\widehat{g}_{\tilde\rho_1}(s)|\,\mathrm{d}s\right)\mathrm{d}\tilde\rho_1\,\mathrm{d}\rho_1, \end{aligned} \tag{4.19}$$

where, for each $(\rho_1, \tilde\rho_1) \in \mathbb{R}^2_+$, the function $K_{\rho_1,\tilde\rho_1} : [0,1] \to \mathbb{R}$ is defined via

$$K_{\rho_1,\tilde\rho_1}(s) := (1-s^2)^{\frac{d-k-2}{2}}(1+\rho_1\sqrt{1-s^2})^{\frac{k-d-1}{2}}(1+\tilde\rho_1\sqrt{1-s^2})^{\frac{k-d-1}{2}}.$$

Set $I := [0,1]$, and estimate the $L^{(\frac{q_\star}{2})'}(I)$-norm of $K_{\rho_1,\tilde\rho_1}$ as follows:

$$\begin{aligned} &\|K_{\rho_1,\tilde\rho_1}\|_{L^{(\frac{q_\star}{2})'}(I)}| \\ &\quad = \left(\int_0^1 r^{\frac{q_\star(d-k-2)}{q_\star-2}+1}(1-r^2)^{-\frac{1}{2}}(1+\rho_1 r)^{\frac{q_\star(k-d-1)}{2(q_\star-2)}}(1+\tilde\rho_1 r)^{\frac{q_\star(k-d-1)}{2(q_\star-2)}}\,\mathrm{d}r\right)^{\frac{q_\star-2}{q_\star}} \\ &\quad \lesssim \left(\int_0^{\frac{1}{2}} r^{\frac{q_\star(d-k-2)}{q_\star-2}+1}(1+\rho_1 r)^{\frac{q_\star(k-d-1)}{2(q_\star-2)}}(1+\tilde\rho_1 r)^{\frac{q_\star(k-d-1)}{2(q_\star-2)}}\,\mathrm{d}r\right)^{\frac{q_\star-2}{q_\star}} \\ &\qquad + (1+\rho_1)^{\frac{k-d-1}{2}}(1+\tilde\rho_1)^{\frac{k-d-1}{2}} \\ &\quad \leq \left(\tilde\rho_1^{\frac{q_\star(k-d+2)}{q_\star-2}-2}\int_0^\infty t^{\frac{q_\star(d-k-2)}{q_\star-2}+1}\left(1+\frac{\rho_1}{\tilde\rho_1}t\right)^{\frac{q_\star(k-d-1)}{2(q_\star-2)}}(1+t)^{\frac{q_\star(k-d-1)}{2(q_\star-2)}}\,\mathrm{d}t\right)^{\frac{q_\star-2}{q_\star}} \\ &\qquad + (1+\rho_1)^{\frac{k-d-1}{2}}(1+\tilde\rho_1)^{\frac{k-d-1}{2}} \\ &\quad \lesssim \rho_1^{\frac{k-d-1}{2}}\tilde\rho_1^{\frac{k-d+1}{2}+\frac{4}{q_\star}} + (1+\rho_1)^{\frac{k-d-1}{2}}(1+\tilde\rho_1)^{\frac{k-d-1}{2}}, \end{aligned} \tag{4.20}$$

where in the last line we used the trivial bound $1 + \frac{\rho_1}{\tilde\rho_1} t \geq \frac{\rho_1}{\tilde\rho_1} t$. The contribution from the second summand in (4.20) is straightforward to handle, and so we focus on the first one. Going back to (4.19), we then have that

$$\begin{aligned}
&\int_{\mathbb{S}^{d-1}} \mathrm{I}^2(\eta,\zeta)\,\mathrm{d}\sigma(\eta,\zeta)\\
&\quad\lesssim \int_0^\infty \int_0^{\rho_1} (\rho_1\tilde\rho_1)^{\frac{d-k-1}{p_\star'}} \|K_{\rho_1,\tilde\rho_1}\|_{L^{(\frac{q_\star}{2})'}(I)} \|\widehat{g}_{\rho_1}\|_{L^{q_\star}(I)} \|\widehat{g}_{\tilde\rho_1}\|_{L^{q_\star}(I)}\,\mathrm{d}\tilde\rho_1\,\mathrm{d}\rho_1\\
&\quad\lesssim \int_0^\infty \int_0^{\rho_1} (\rho_1\tilde\rho_1)^{\frac{d-k-1}{p_\star'}} \rho_1^{\frac{k-d-1}{2}} \tilde\rho_1^{\frac{k-d+1}{2}+\frac{4}{q_\star}} \|g(\rho_1,\cdot)\|_{L^{p_\star}(\mathbb{R}_+)} \|g(\tilde\rho_1,\cdot)\|_{L^{p_\star}(\mathbb{R}_+)}\,\mathrm{d}\tilde\rho_1\,\mathrm{d}\rho_1\\
&\quad= \int_0^\infty \|g(\rho_1,\cdot)\|_{L^{p_\star}(\mathbb{R}_+)} \Big(\rho_1^{-a}\int_0^{\rho_1} \tilde\rho_1^{-b} \|g(\tilde\rho_1,\cdot)\|_{L^{p_\star}(\mathbb{R}_+)}\,\mathrm{d}\tilde\rho_1\Big)\,\mathrm{d}\rho_1,
\end{aligned}$$

where the second estimate follows from (4.18) and (4.20), and we set[2]

$$a := \frac{d-k+1}{2} - \frac{d-k-1}{p_\star'}, \quad b := (d-k-1)\Big(\frac{1}{2} - \frac{1}{p_\star'}\Big) - \frac{4}{q_\star}.$$

It is straightforward to check that these exponents satisfy $a + b = \frac{2}{p_\star'}$ and $bp_\star' < 1$. Since we also have that $1 < p_\star \leq p_\star' < \infty$, Lemma 2.9 yields

$$\begin{aligned}
&\int_{\mathbb{S}^{d-1}} \mathrm{I}^2(\eta,\zeta)\,\mathrm{d}\sigma(\eta,\zeta)\\
&\quad\lesssim \big\|\|g(\rho_1,\cdot)\|_{L^{p_\star}(\mathbb{R}_+)}\big\|_{L^{p_\star}_{\rho_1}(\mathbb{R}_+)} \Big\|\rho_1^{-a}\int_0^{\rho_1} \tilde\rho_1^{-b}\|g(\tilde\rho_1,\cdot)\|_{L^{p_\star}(\mathbb{R}_+)}\,\mathrm{d}\tilde\rho_1\Big\|_{L^{p_\star'}_{\rho_1}(\mathbb{R}_+)}\\
&\quad\lesssim \big\|\|g(\rho_1,\cdot)\|_{L^{p_\star}(\mathbb{R}_+)}\big\|^2_{L^{p_\star}_{\rho_1}(\mathbb{R}_+)} = \|g\|^2_{L^{p_\star}(\mathbb{R}^2_+)} \simeq \|f\|^2_{L^{p_\star}(\mathbb{R}^d)}.
\end{aligned}$$

This concludes the proof of Proposition 4.2. ☐

4.3 Estimating $\widehat{f_4}$ and $\widehat{f_5}$. The estimates in this section are the most delicate ones, but most of the work has been done already. We heavily rely on the weighted estimates for the restriction-type operator $\mathcal{R}_{\alpha,\beta}$, recall (3.1), which we proved in §3, and record the following consequence.

Corollary 4.3. *Let $d \geq 4, k \in \{2, 3, \ldots, d-2\}, m := \min\{d-k, k\}$, and $1 \leq p \leq \frac{2(d+m)}{d+m+2}$. Set*

$$\alpha_p := (d-k-1)\Big(\frac{1}{p} - \frac{1}{2}\Big), \quad \beta_p := (k-1)\Big(\frac{1}{p} - \frac{1}{2}\Big). \tag{4.21}$$

Then $\mathcal{R}_{\alpha_p,\beta_p} : L^p(\mathbb{R}^2) \to L^2(\mathbb{S}^1)$ defines a bounded operator.

[2]Alternatively, $a = \frac{2d-1}{d+k}$ and $b = \frac{k-d-1}{d+k}$.

Proof. By interpolation and compactness of $\mathbb{S}^1$, it suffices to consider the endpoint case $p = \frac{2(d+m)}{d+m+2}$. Proposition 3.2 directly implies the desired conclusion in all situations of interest, except when $2k = 4 = d$, so we focus on that case. Proposition 3.1 implies

$$\|\mathcal{R}^*_{\frac{1}{3},\frac{1}{3}}(F)\|_{L^2(\mathbb{R}^2)} \lesssim \|F\|_{L^2(\mathbb{S}^1)}, \tag{4.22}$$

and from Proposition 3.3 we have that

$$\|\mathcal{R}^*_{0,0}(F)\|_{L^6(\mathbb{R}^2)} \lesssim \|F\|_{L^2(\mathbb{S}^1)}. \tag{4.23}$$

As in the proof of Proposition 3.2, Stein's interpolation theorem [41, p. 205] can be applied to the analytic family of operators $\{\mathcal{R}^*_{\frac{s}{3},\frac{s}{3}} : 0 \le \Re(s) \le 1\}$, yielding from (4.22)–(4.23)

$$\|\mathcal{R}^*_{\frac{1}{6},\frac{1}{6}}(F)\|_{L^3(\mathbb{R}^2)} \lesssim \|F\|_{L^2(\mathbb{S}^1)}.$$

This is equivalent, by duality, to the desired conclusion. □

The relevance of Corollary 4.3 becomes apparent once we note that from definitions (4.10), (4.11) it follows that

$$\begin{aligned} \widehat{f_4}(\eta,\zeta) &= A_1 A_2 |\eta|^{\frac{k-d+1}{2}} |\zeta|^{\frac{1-k}{2}} \mathcal{R}_{\alpha_p,\beta_p}(h)(-|\eta|, -|\zeta|), \\ \widehat{f_5}(\eta,\zeta) &= A_1 \overline{A_2} |\eta|^{\frac{k-d+1}{2}} |\zeta|^{\frac{1-k}{2}} \mathcal{R}_{\alpha_p,\beta_p}(h)(-|\eta|, |\zeta|). \end{aligned} \tag{4.24}$$

Here, α_p, β_p were defined in (4.21), and

$$h(\rho_1,\rho_2) := \rho_1^{\frac{d-k-1}{2}} \rho_2^{\frac{k-1}{2}} (1+\rho_1)^{\alpha_p} (1+\rho_2)^{\beta_p} f_0(\rho_1,\rho_2) \mathbb{1}_{[1,\infty)^2}(\rho_1,\rho_2). \tag{4.25}$$

Proposition 4.4. *For every* $1 \le p \le \frac{2(d+m)}{d+m+2}$, *there exists* $C = C(k,d,p) < \infty$ *such that*

$$\|\widehat{f_4}\|_{L^2(\mathbb{S}^{d-1})} + \|\widehat{f_5}\|_{L^2(\mathbb{S}^{d-1})} \le C\|f\|_{L^p(\mathbb{R}^d)},$$

for every G_k*-symmetric Schwartz function* $f : \mathbb{R}^d \to \mathbb{C}$.

Proof. By the usual considerations, it suffices to bound $\|\widehat{f_4}\|_{L^2(\mathbb{S}^{d-1})}$. Identity (4.24) and Lemma 2.3 together imply

$$\int_{\mathbb{S}^{d-1}} |\widehat{f_4}(\eta,\zeta)|^2 \, \mathrm{d}\sigma(\eta,\zeta) \lesssim \int_{\mathbb{S}^1} |\mathcal{R}_{\alpha_p,\beta_p}(h)(\omega)|^2 \, \mathrm{d}\sigma(\omega),$$

where the function h was defined in (4.25). For every $1 \le p \le \frac{2(d+m)}{d+m+2}$, Corollary 4.3 and (4.12) together yield

$$\begin{aligned} &\int_{\mathbb{S}^{d-1}} |\widehat{f_4}(\eta,\zeta)|^2 \, \mathrm{d}\sigma(\omega) \\ &\quad \lesssim \|h\|^2_{L^p(\mathbb{R}^2)} \lesssim \|\rho_1^{\frac{d-k-1}{p}} \rho_2^{\frac{k-1}{p}} f_0(\rho_1,\rho_2)\|^2_{L^p_{\rho_1,\rho_2}(\mathbb{R}^2_+)} \simeq \|f\|^2_{L^p(\mathbb{R}^d)}. \end{aligned}$$

In the second estimate, we used the fact that the support of h is contained in $[1,\infty)^2 \subset \mathbb{R}^2$, so that $1+\rho_j \sim \rho_j$, for each $j \in \{1,2\}$. This concludes the proof of Proposition 4.4. □

4.4 Conclusion of the proof. The aim is to verify estimate (4.1) for every G_k-symmetric Schwartz function $f : \mathbb{R}^d \to \mathbb{C}$. Splitting f as in (4.7), by the subsequent considerations it suffices to bound $\|\widehat{f_j}\|_{L^2(\mathbb{S}^{d-1})}$, $1 \leq j \leq 5$, appropriately in terms of $\|f\|_{L^p(\mathbb{R}^d)}$, whenever $1 \leq p \leq \frac{2(d+m)}{d+m+2}$. In turn, this is accomplished by Propositions 4.1, 4.2, 4.4. The proof of Theorem 1.1 is thus complete.

5 Proof of Theorem 1.2

In this section, we explain how Theorem 1.2 follows from Theorem 1.1. As in most optimization problems, the difficulty is to find a weak limit of a maximizing sequence which is non-zero. The key observation is that, in light of estimate (1.5), the Stein–Tomas exponent $p_d = \frac{2(d+1)}{d+3}$ is no longer an endpoint exponent within the class of G_k-symmetric functions, provided $2 \leq k \leq d-2$. Therefore Theorem 1.1 can be used in conjunction with the Fourier decay property from Corollary 2.5 to show that maximizing sequences do not converge weakly to zero.

Precompactness of maximizing sequences for non-endpoint, L^2-based adjoint restriction estimates is in general well-understood. We follow the approach of [10] which relies on a useful reformulation of the Brézis–Lieb lemma [8] given in [14], but with an important twist. Since translations are not symmetries of the G_k-symmetric problem, one may expect precompactness to hold, instead of precompactness modulo translations as in the general non-symmetric case [12, 18]. To facilitate the comparison with references [10, 14], and especially [18], we choose to formulate and prove Theorem 1.2 for the extension operator instead. Thus we are led to define the Hilbert space[3]

$$L^2_{G_k}(\mathbb{S}^{d-1}) := \{F \in L^2(\mathbb{S}^{d-1}) : F \text{ is } G_k\text{-symmetric}\},$$

and the quantity

$$\mathbf{T}^*_{d,k}(p) := \sup_{0 \neq F \in L^2_{G_k}(\mathbb{S}^{d-1})} \frac{\|\widehat{F\sigma}\|_{L^{p'}(\mathbb{R}^d)}}{\|F\|_{L^2(\mathbb{S}^{d-1})}}.$$

By duality, we naturally have that $\mathbf{T}^*_{d,k}(p) = \mathbf{T}_{d,k}(p)$, but we will use both designations in order to keep track of the extremal problem under consideration.

[3]That $L^2_{G_k}(\mathbb{S}^{d-1})$ is indeed a Hilbert space follows from the Riesz–Fischer Theorem, since G_k-symmetry is preserved under pointwise limits.

Theorem 5.1. *Let $d \geq 4$, $k \in \{2, 3, \ldots, d-2\}$, $m := \min\{d-k, k\}$, and $1 \leq p < \frac{2(d+m)}{d+m+2}$. Then maximizing sequences for $\mathbf{T}^*_{d,k}(p)$, normalized in $L^2(\mathbb{S}^{d-1})$, are precompact in $L^2_{G_k}(\mathbb{S}^{d-1})$. In particular, maximizers for $\mathbf{T}^*_{d,k}(p)$ exist.*

Theorem 5.1 is in fact equivalent to Theorem 1.2 via a well-known argument; see [43, § 6] for a more general statement along these lines. For the convenience of the reader, we present the details of the relevant implication in Appendix A. The proof of Theorem 5.1 relies on [14, Proposition 1.1], which is a useful reformulation of the Brézis–Lieb lemma [8, Theorem 1].

Proof of Theorem 5.1. Let $(F_n)_{n\in\mathbb{N}} \subset L^2_{G_k}(\mathbb{S}^{d-1})$ be an L^2-normalized maximizing sequence for $\mathbf{T}^*_{d,k}(p)$, i.e., F_n is G_k-symmetric, $\|F_n\|_{L^2(\mathbb{S}^{d-1})} = 1$ for all $n \in \mathbb{N}$, and

$$\|\widehat{F_n\sigma}\|_{L^{p'}(\mathbb{R}^d)} \to \mathbf{T}^*_{d,k}(p), \quad \text{as } n \to \infty. \tag{5.1}$$

From Theorem 1.1 in dual form and L^2-normalization of $(F_n)_{n\in\mathbb{N}}$, there exists $C_{k,d} < \infty$ such that

$$\sup_{n\in\mathbb{N}} \|\widehat{F_n\sigma}\|_{L^{p'_\star}(\mathbb{R}^d)} < C_{k,d}, \tag{5.2}$$

where $p_\star = \frac{2(d+m)}{d+m+2}$ and $m = \min\{d-k, k\}$. By convexity of L^p-norms, we further have

$$\|\widehat{F_n\sigma}\|_{L^{p'}(\mathbb{R}^d)} \leq \|\widehat{F_n\sigma}\|^{\theta}_{L^{p'_\star}(\mathbb{R}^d)} \|\widehat{F_n\sigma}\|^{1-\theta}_{L^\infty(\mathbb{R}^d)}, \tag{5.3}$$

where $\theta \in (0,1)$ is given by $\frac{1}{p'} = \frac{\theta}{p'_\star} + \frac{1-\theta}{\infty}$, i.e., $\theta = \frac{p'_\star}{p'}$. Estimates (5.1), (5.2), (5.3) together imply the existence of $\varepsilon_0 = \varepsilon_0(k,d) > 0$, depending only on k, d, for which $\|\widehat{F_n\sigma}\|_{L^\infty(\mathbb{R}^d)} \geq \varepsilon_0$ for every $n \in \mathbb{N}$ (possibly after extraction of a subsequence). Thus there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$, such that

$$|\widehat{F_n\sigma}(x_n)| \geq \varepsilon_0 > 0, \quad \text{for every } n \in \mathbb{N}. \tag{5.4}$$

Corollary 2.5 guarantees[4] the existence of a certain radius $R = R(k,d) < \infty$, depending only on k, d, for which $|x_n| \leq R$ for every $n \in \mathbb{N}$. On the other hand, by the Cauchy–Schwarz inequality together with the L^2-normalization of $(F_n)_{n\in\mathbb{N}}$, we have that

$$\|\widehat{F_n\sigma}\|_{L^\infty(\mathbb{R}^d)} \leq \|F_n\|_{L^1(\mathbb{S}^{d-1})} \leq \sigma(\mathbb{S}^{d-1})^{\frac{1}{2}} \|F_n\|_{L^2(\mathbb{S}^{d-1})} = \sigma(\mathbb{S}^{d-1})^{\frac{1}{2}};$$
$$\|\nabla_x(\widehat{F_n\sigma})\|_{L^\infty(\mathbb{R}^d)} \leq \|F_n| \cdot |\|_{L^1(\mathbb{S}^{d-1})} \leq \|F_n\|_{L^1(\mathbb{S}^{d-1})} \leq \sigma(\mathbb{S}^{d-1})^{\frac{1}{2}}.$$

[4]Note the tension between estimates (2.7) and (5.4) which, in particular, reveals that a maximizing sequence for $\mathbf{T}^*_{d,k}(p)$ cannot concentrate on a copy of $\mathbb{S}^{\min\{d-k,k\}-1}$ inside $\mathbb{S}^{d-1}$. If $k \in \{1, d-1\}$, this would amount to concentration at a pair of antipodal points on $\mathbb{S}^{d-1}$, which in [12, 18] was identified as the "most essential obstacle" to the precompactness of maximizing sequences modulo symmetries when $p = p_d$.

It follows that the sequence $(\widehat{F_n\sigma})_{n\in\mathbb{N}}$ is uniformly bounded and equicontinuous on the cube $Q_R := [-R,R]^d \subset \mathbb{R}^d$. The Arzelà–Ascoli Theorem on compact subsets of $\mathbb{R}^d$ then implies that the sequence $(\widehat{F_n\sigma})_{n\in\mathbb{N}}$ has a subsequence which converges uniformly to a limit in Q_R. That this limit is nonzero follows at once from (5.4) and the fact that $(x_n)_{n\in\mathbb{N}} \subset Q_R$.

Now, since the sequence $(F_n)_{n\in\mathbb{N}}$ is bounded on $L^2_{G_k}(\mathbb{S}^{d-1})$, it has a weakly convergent subsequence. In other words, there exists a function $F_\star \in L^2_{G_k}(\mathbb{S}^{d-1})$, such that $F_n \rightharpoonup F_\star$ weakly in $L^2_{G_k}(\mathbb{S}^{d-1})$, as $n\to\infty$. Since the extension operator is bounded from $L^2_{G_k}(\mathbb{S}^{d-1})$ to $L^{p'}(\mathbb{R}^d)$, it follows that $\widehat{F_n\sigma} \rightharpoonup \widehat{F_\star\sigma}$ weakly in $L^{p'}(\mathbb{R}^d)$, as $n\to\infty$. Since uniform convergence implies weak convergence, and weak limits are unique, from the previous paragraph it follows that $\widehat{F_\star\sigma}$ is nonzero, and so the function $F_\star$ is itself nonzero.

We are now in a position to apply [14, Proposition 1.1] to the extension operator on $\mathbb{S}^{d-1}$ with $\mathcal{H} = L^2_{G_k}(\mathbb{S}^{d-1})$, $p = p'_d \in (2,\infty)$, $(h_n)_{n\in\mathbb{N}} = (F_n)_{n\in\mathbb{N}}$, and $\overline{h} = F_\star$. Hypotheses (1) and (2) from [14, Proposition 1.1] hold by the assumptions on the sequence $(F_n)_{n\in\mathbb{N}}$, and hypothesis (3) follows from the previous paragraph. Finally, hypothesis (4) is an easy consequence of the compactness of $\mathbb{S}^{d-1}$. The conclusion is that, possibly after extraction of a subsequence, $F_n \to F_\star$ in $L^2_{G_k}(\mathbb{S}^{d-1})$, as $n\to\infty$. In particular, $F_\star$ is a maximizer for $\mathbf{T}^*_{d,k}(p)$. This finishes the proof of the theorem. □

6 Proof of Theorem 1.3

In this section, we construct appropriate examples to show the necessity of conditions (i)–(iii) in the statement of Theorem 1.3. We work with the extension operator (2.1) rather than with the restriction operator directly, and aim to show that the estimate

$$\|\widehat{F\sigma}\|_{L^{p'}(\mathbb{R}^d)} \le C(k,d,p,q)\|F\|_{L^{q'}(\mathbb{S}^{d-1})},$$

which is dual to (1.4), can only hold for every G_k-symmetric function $F : \mathbb{S}^{d-1} \to \mathbb{C}$ provided d, p, q, and $m = \min\{d-k, k\}$ are chosen in such a way that conditions (i)–(iii) in the statement of Theorem 1.3 hold. As in the general non-symmetric situation, the first condition $\frac{d+1}{2d} < \frac{1}{p}$ is dictated by the choice $F \equiv 1$, since

$$\widehat{\sigma} \in L^{p'}(\mathbb{R}^d) \text{ if and only if } p' > \frac{2d}{d-1}. \tag{6.1}$$

The latter equivalence follows from identity (2.2) together with the standard asymptotics of Bessel functions at zero and infinity; recall (2.4)–(2.5).

The remaining necessary conditions are obtained from analyzing a G_k-symmetric variant of Knapp's construction. Let $d \ge 4$, $k \in \{2,3,\dots,d-2\}$ be given, and assume without loss of generality that $m = k$, which we take as fixed from now onwards. Given $\delta \in (0, \frac{1}{2})$, consider the following union of two "spherical caps" of radius δ:

$$\mathcal{C}_\delta := \{(\eta, \zeta) \in \mathbb{R}^{d-k} \times \mathbb{R}^k : |\eta|^2 + |\zeta|^2 = 1, |\eta| < \delta\} \subset \mathbb{S}^{d-1}.$$

By construction, the set $\mathcal{C}_\delta$ is G_k-symmetric, and so are the indicator function $\mathbb{1}_\delta := \mathbb{1}_{\mathcal{C}_\delta}$ and its Fourier extension, $\widehat{\mathbb{1}_\delta \sigma}$. Using Lemma 2.3, we estimate

$$\|\mathbb{1}_\delta\|_{L^{q'}(\mathbb{S}^{d-1})} = \sigma(\mathcal{C}_\delta)^{\frac{1}{q'}} \simeq \left(\int_0^\delta r^{d-k-1}(1-r^2)^{\frac{k-2}{2}}\, \mathrm{d}r \right)^{\frac{1}{q'}} \simeq \delta^{\frac{d-k}{q'}}. \tag{6.2}$$

On the other hand, if $(y, z) \in \mathbb{R}^{d-k} \times \mathbb{R}^k$, then (2.2) together with a further application of Lemma 2.3 yield

$$\begin{aligned} \widehat{\mathbb{1}_\delta \sigma}(y, z) &= \int_{\mathcal{C}_\delta} e^{i(y,z)\cdot(\eta,\zeta)}\, \mathrm{d}\sigma(\eta, \zeta) \\ &\simeq \int_0^\delta r^{d-k-1}(1-r^2)^{\frac{k-2}{2}} (r|y|)^{\frac{2-d+k}{2}} \\ &\qquad \times J_{\frac{d-k-2}{2}}(r|y|)(\sqrt{1-r^2}|z|)^{\frac{2-k}{2}} J_{\frac{k-2}{2}}(\sqrt{1-r^2}|z|)\, \mathrm{d}r. \end{aligned} \tag{6.3}$$

Let $\{z_j\}_{j\geq 1}$ denote the increasing sequence of local maxima of the Bessel function $J_{(k-2)/2}$. By the asymptotic expansion (2.4), there exist constants $0 < c, C < \infty$, such that $|z_j - 2\pi j| \leq C$, as $j \to \infty$, and moreover $J_{\frac{k-2}{2}}(z_j) \geq c j^{-\frac{1}{2}}$, for every $j \geq 0$. Recalling (2.5) and shrinking c if necessary, we obtain

$$z^{\frac{2-d+k}{2}} J_{\frac{d-k-2}{2}}(z) \geq c, \quad \text{for every } z \in (0, c), \tag{6.4}$$

$$z^{\frac{2-k}{2}} J_{\frac{k-2}{2}}(z) \geq c j^{\frac{1-k}{2}}, \quad \text{for every } z \in [z_j - c, z_j + c] \text{ and } j \geq 0. \tag{6.5}$$

Consider the disjoint union $E := \bigcup_{j=1}^{\lfloor c\delta^{-2} \rfloor} E_j$, where each set E_j is defined as follows:

$$E_j := \left\{ (y, z) \in \mathbb{R}^{d-k} \times \mathbb{R}^k : 0 \leq |y| \leq c\delta^{-1}, \frac{z_j - c}{\sqrt{1-\delta^2}} \leq |z| \leq z_j + c \right\}.$$

Each set E_j is G_k-symmetric, and so is E. If $(y, z) \in E$, then estimates (6.4)–(6.5) applied to (6.3) imply the following lower bound:

$$|\widehat{\mathbb{1}_\delta \sigma}(y, z)| \gtrsim \delta^{d-k} j^{\frac{1-k}{2}},$$

provided $\delta \in (0, c)$ is chosen sufficiently small. On the other hand, there exists an index j_0 with the following property: for each $j \in \{j_0, j_0 + 1, \ldots, \lfloor c\delta^{-2} \rfloor\}$,

$$(z_j + c)^k - \left(\frac{z_j - c}{\sqrt{1-\delta^2}} \right)^k \gtrsim j^{k-1},$$

provided $\delta > 0$ is chosen sufficiently small. This follows directly from Taylor expansion, and readily implies the size estimate $|E_j| \gtrsim \delta^{k-d} j^{k-1}$, for each $j_0 \le j \le \lfloor c\delta^{-2} \rfloor$. As a consequence,

$$\begin{aligned}\|\widehat{\mathbb{1}_\delta \sigma}\|^{p'}_{L^{p'}(\mathbb{R}^d)} \ge \|\widehat{\mathbb{1}_\delta \sigma}\|^{p'}_{L^{p'}(E)} &\gtrsim \sum_{j=j_0}^{\lfloor c\delta^{-2}\rfloor} (\delta^{d-k} j^{\frac{1-k}{2}})^{p'} |E_j| \\ &\gtrsim \sum_{j=j_0}^{\lfloor c\delta^{-2}\rfloor} (\delta^{d-k} j^{\frac{1-k}{2}})^{p'} (\delta^{k-d} j^{k-1}) \\ &= \delta^{(d-k)(p'-1)} \sum_{j=j_0}^{\lfloor c\delta^{-2}\rfloor} j^{-\frac{(k-1)(p'-2)}{2}}.\end{aligned}$$

According to $\frac{(k-1)(p'-2)}{2}$ being smaller than, equal to, or larger than 1, we thus obtain

$$\|\widehat{\mathbb{1}_\delta \sigma}\|_{L^{p'}(\mathbb{R}^d)} \gtrsim \begin{cases} \delta^{\frac{d+k}{p}-k-1}, & \text{if } \frac{1}{p} < \frac{k+1}{2k}, \\ \delta^{\frac{d+k}{p}-k-1} |\log(\delta)|^{\frac{1}{p'}}, & \text{if } \frac{1}{p} = \frac{k+1}{2k}, \\ \delta^{\frac{d-k}{p}}, & \text{if } \frac{1}{p} > \frac{k+1}{2k}. \end{cases}$$

The latter estimate is valid also when $p' = \infty$. Together with (6.2), we finally conclude:

$$\frac{\|\widehat{\mathbb{1}_\delta \sigma}\|_{L^{p'}(\mathbb{R}^d)}}{\|\mathbb{1}_\delta\|_{L^{q'}(\mathbb{S}^{d-1})}} \gtrsim \begin{cases} \delta^{\frac{d+k}{p}+\frac{d-k}{q}-d-1}, & \text{if } \frac{1}{p} < \frac{k+1}{2k}, \\ \delta^{\frac{d+k}{p}+\frac{d-k}{q}-d-1} |\log(\delta)|^{\frac{1}{p'}}, & \text{if } \frac{1}{p} = \frac{k+1}{2k}, \\ \delta^{\frac{d-k}{p}+\frac{d-k}{q}-d+k}, & \text{if } \frac{1}{p} > \frac{k+1}{2k}. \end{cases}$$

The proof of Theorem 1.3 is completed by letting $\delta \to 0^+$.

Remark 6.1. If $m = \min\{k, d-k\} = 1$, then the following G_1-symmetric version of Knapp's construction reveals that estimates beyond those predicted by the restriction conjecture, recall (1.1)–(1.2), are not possible within the G_1-symmetric setting. Define

$$\mathcal{C}_\delta := \{(\eta, \zeta) \in \mathbb{R}^{d-1} \times \mathbb{R} : |\eta|^2 + \zeta^2 = 1, |\eta| < \delta\} \subset \mathbb{S}^{d-1}$$

and $E := \bigcup_{j=1}^{\lfloor (2\delta)^{-2} \rfloor} E_j$, where

$$E_j := \Big\{(y, z) \in \mathbb{R}^{d-1} \times \mathbb{R} : 0 \le |y| \le \frac{\pi}{4\delta}, \frac{2\pi j - \frac{\pi}{4}}{\sqrt{1-\delta^2}} \le |z| \le 2\pi j + \frac{\pi}{4}\Big\}.$$

Here, $\delta > 0$ is a sufficiently small parameter, and the values $\{2\pi j\}_{j\ge 1}$ are the counterparts of the Bessel maxima $\{z_j\}_{j\ge 1}$ considered above. Naturally, the sets $\mathcal{C}_\delta$

and E are both G_1-symmetric. Repeating the steps from the proof of Theorem 1.3, one finds that $|E_j| \gtrsim \delta^{1-d}$ and thus $\|\widehat{\mathbb{1}_\delta \sigma}\|_{L^{p'}(\mathbb{R}^d)}^{p'} \gtrsim \delta^{(d-1)p'-(d+1)}$. In turn, this implies the following lower bound:

$$\frac{\|\widehat{\mathbb{1}_\delta \sigma}\|_{L^{p'}(\mathbb{R}^d)}}{\|\mathbb{1}_\delta\|_{L^{q'}(\mathbb{S}^{d-1})}} \gtrsim \delta^{\frac{d+1}{p}+\frac{d-1}{q}-d-1}.$$

The latter quotient remains bounded as $\delta \to 0^+$ if and only if $\frac{d+1}{p} + \frac{d-1}{q} \geq d+1$, which matches the second condition in (1.2). All in all, we recover the same necessary conditions as in the general, non-symmetric case.

7 Proof of Theorem 1.4

In this section, we provide a short proof of Theorem 1.4. No generality is lost in assuming $m = k$, $f \in \mathcal{S}(\mathbb{R}^d)$, and throughout the proof we set $p := \frac{2k}{k+1}$. The representation formula from Lemma 2.2 and the bounds (2.4)–(2.5) for Bessel functions together imply

$$\begin{aligned}
&|\widehat{f}(\eta,\zeta)| \\
&\simeq |\eta|^{\frac{2-d+k}{2}}|\zeta|^{\frac{2-k}{2}}\left|\int_0^\infty\int_0^\infty \rho_1^{\frac{d-k}{2}}\rho_2^{\frac{k}{2}} f_0(\rho_1,\rho_2) J_{\frac{d-k-2}{2}}(\rho_1|\eta|) J_{\frac{k-2}{2}}(\rho_2|\zeta|)\,\mathrm{d}\rho_1\,\mathrm{d}\rho_2\right| \\
&\lesssim \int_0^\infty\int_0^\infty \rho_1^{d-k-1}\rho_2^{k-1}|f_0(\rho_1,\rho_2)|(1+\rho_1|\eta|)^{\frac{k-d+1}{2}}(1+\rho_2|\zeta|)^{\frac{1-k}{2}}\,\mathrm{d}\rho_1\,\mathrm{d}\rho_2.
\end{aligned}$$

If $k < \frac{d}{2}$, then Hölder's inequality in Lorentz spaces [31, Theorem 3.4] then implies the following pointwise bound for $|\widehat{f}(\eta,\zeta)|$:

$$\begin{aligned}
&\|\rho_1^{\frac{d-k-1}{p}}\rho_2^{\frac{k-1}{p}} f_0(\rho_1,\rho_2)\|_{L^{p,1}_{\rho_1,\rho_2}(\mathbb{R}^2_+)}\|\rho_1^{\frac{d-k-1}{p'}}\rho_2^{\frac{k-1}{p'}}(1+\rho_1|\eta|)^{\frac{k-d+1}{2}}(1+\rho_2|\zeta|)^{\frac{1-k}{2}}\|_{L^{p',\infty}_{\rho_1,\rho_2}(\mathbb{R}^2_+)} \\
&\qquad = \|f\|_{L^{p,1}(\mathbb{R}^d)}\|(1+|y||\eta|)^{\frac{k-d+1}{2}}(1+|z||\zeta|)^{\frac{1-k}{2}}\|_{L^{p',\infty}(\mathbb{R}^d)} \\
&\qquad \simeq \|f\|_{L^{p,1}(\mathbb{R}^d)}|\eta|^{\frac{k-d}{p'}}|\zeta|^{-\frac{k}{p'}}, \qquad (7.1)
\end{aligned}$$

where the last line follows from changing variables $y \rightsquigarrow |\eta|y \in \mathbb{R}^{d-k}$ and $z \rightsquigarrow |\zeta|z \in \mathbb{R}^k$, and using the facts that $2k < d$ and $p' = \frac{2k}{k-1}$ in order to control the corresponding weak quasi-norm. If $k = \frac{d}{2}$, then H ölder's inequality in mixed Lorentz spaces [23, Prop. 6.1] implies[5] the following pointwise bound for $|\hat{f}(\eta,\zeta)|$:

$$\begin{aligned}
&\|\rho_1^{\frac{d-k-1}{p}}\rho_2^{\frac{k-1}{p}} f_0(\rho_1,\rho_2)\|_{L^{p,1}_{\rho_1}(\mathbb{R}_+;L^{p,1}_{\rho_2}(\mathbb{R}_+))}\\
&\quad\times \|\rho_1^{\frac{d-k-1}{p'}}\rho_2^{\frac{k-1}{p'}}(1+\rho_1|\eta|)^{\frac{k-d+1}{2}}(1+\rho_2|\zeta|)^{\frac{1-k}{2}}\|_{L^{p',\infty}_{\rho_1}(\mathbb{R}_+;L^{p',\infty}_{\rho_2}(\mathbb{R}_+))}\\
&\quad\lesssim \|f\|_{X_p}\|(1+|y||\eta|)^{\frac{k-d+1}{2}}(1+|z||\zeta|)^{\frac{1-k}{2}}\|_{L^{p',\infty}(\mathbb{R}^{d-k};L^{p',\infty}(\mathbb{R}^k))}\\
&\quad\simeq \|f\|_{X_p}|\eta|^{\frac{k-d}{p'}}|\zeta|^{-\frac{k}{p'}}. \qquad (7.2)
\end{aligned}$$

Estimates (7.1)–(7.2) estimate can be integrated over the unit sphere, resulting in

$$\|\widehat{f}\|_{L^{p',\infty}(\mathbb{S}^{d-1})}\lesssim \|f\|_{X_p}\||\eta|^{\frac{k-d}{p'}}|\zeta|^{-\frac{k}{p'}}\|_{L^{p',\infty}(\mathbb{S}^{d-1})}\lesssim \|f\|_{X_p}.$$

This concludes the proof of Theorem 1.4.

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Appendix A Theorem 5.1 implies Theorem 1.2

Let $(f_n)_{n\in\mathbb{N}}\subset L^p_{G_k}(\mathbb{R}^d)$ be a maximizing sequence for $\mathbf{T}_{d,k}(p)$, normalized in $L^p(\mathbb{R}^d)$. In other words, each f_n is G_k-symmetric in $\mathbb{R}^d$, $\|f_n\|_{L^p(\mathbb{R}^d)}=1$ for every $n\in\mathbb{N}$, and

$$\|\widehat{f_n}\|_{L^2(\mathbb{S}^{d-1})}\to \mathbf{T}_{d,k}(p),\quad \text{as } n\to\infty. \qquad (A.1)$$

Let $F_n:=\|\widehat{f_n}\|^{-1}_{L^2(\mathbb{S}^{d-1})}\widehat{f_n}\,|_{\mathbb{S}^{d-1}}$. Then $(F_n)_{n\in\mathbb{N}}\subset L^2_{G_k}(\mathbb{S}^{d-1})$ is an L^2-normalized maximizing sequence for $\mathbf{T}^*_{d,k}(p)$, i.e., each F_n is G_k-symmetric on $\mathbb{S}^{d-1}$, $\|F_n\|_{L^2(\mathbb{S}^{d-1})}=1$ for every $n\in\mathbb{N}$, and $\|\widehat{F_n\sigma}\|_{L^{p'}(\mathbb{R}^d)}\to\mathbf{T}^*_{d,k}(p)$, as $n\to\infty$. By Theorem 5.1, there exists $F_\star\in L^2_{G_k}$ such that, possibly after extraction of a subsequence, $F_n\to F_\star$ in $L^2(\mathbb{S}^{d-1})$, as $n\to\infty$. In particular, observe that $\|F_\star\|_{L^2(\mathbb{S}^{d-1})}=1$ and

$$\|\widehat{F_\star\sigma}\|_{L^{p'}(\mathbb{R}^d)}=\mathbf{T}^*_{d,k}(p). \qquad (A.2)$$

Since the sequence $(f_n)_{n\in\mathbb{N}}$ is bounded on $L^p_{G_k}(\mathbb{R}^d)$, it has a weakly convergent subsequence. In other words, there exists a function $f_\star\in L^p_{G_k}(\mathbb{R}^d)$, such that $f_n\rightharpoonup f_\star$

[5] Recall the definition (1.6) of the space X_p.

weakly in $L^p_{G_k}(\mathbb{R}^d)$, as $n \to \infty$. We claim that $f_\star$ is a maximizer for $\mathbf{T}_{d,k}(p)$, and that in fact $f_n \to f_\star$ strongly in $L^p_{G_k}(\mathbb{R}^d)$, as $n \to \infty$. To see this, note that

$$\begin{aligned}\mathbf{T}_{d,k}(p)^2 &= \lim_{n\to\infty} \|\widehat{f_n}\|^2_{L^2(\mathbb{S}^{d-1})} = \mathbf{T}_{d,k}(p) \lim_{n\to\infty} |\langle f_n, \widehat{F_n\sigma}\rangle| \\ &= \mathbf{T}_{d,k}(p)|\langle f_\star, \widehat{F_\star\sigma}\rangle| \leq \mathbf{T}_{d,k}(p)\|f_\star\|_{L^p(\mathbb{R}^d)}\|\widehat{F_\star\sigma}\|_{L^{p'}(\mathbb{R}^d)} \\ &= \|f_\star\|_{L^p(\mathbb{R}^d)}\mathbf{T}^2_{d,k}(p). \end{aligned} \tag{A.3}$$

Here we used (A.1), duality, weak convergence of $(f_n)_{n\in\mathbb{N}}$ and continuity of the extension operator, Hölder's inequality, and (A.2) together with $\mathbf{T}^*_{d,k}(p) = \mathbf{T}_{d,k}(p)$. From the chain of inequalities (A.3), we read off that

$$\|f_\star\|_{L^p(\mathbb{R}^d)} \geq 1 = \lim_{n\to\infty} \|f_n\|_{L^p(\mathbb{R}^d)}. \tag{A.4}$$

Since the reverse inequality holds since $f_n \rightharpoonup f_\star$ weakly in $L^{p'}_{G_k}(\mathbb{R}^d)$, as $n \to \infty$, we actually have equality in (A.4). But weak convergence together with convergence of norms implies strong convergence; see [24, Theorem 2.11]. Therefore $f_n \to f_\star$ in $L^p_{G_k}(\mathbb{R}^d)$, as $n \to \infty$. By continuity of the restriction operator, it follows that $f_\star$ is a maximizer for $\mathbf{T}_{d,k}(p)$, as desired. This concludes the proof that Theorem 5.1 implies Theorem 1.2.

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