

DIMENSION-FREE L^p -ESTIMATES FOR VECTORS OF RIESZ TRANSFORMS IN THE RATIONAL DUNKL SETTING

By

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Abstract. In this article, we prove a dimension-free upper bound for the L^p -norms of the vector of Riesz transforms in the rational Dunkl setting. Our main technique is the Bellman function method adapted to the Dunkl setting.

1 Introduction

In the seminal article [22], Charles F. Dunkl defined new commuting differential-difference operators

$$T_{\xi}f(\mathbf{x}) = \partial_{\xi}f(\mathbf{x}) + \sum_{\alpha \in R} \frac{k(\alpha)}{2} \langle \alpha, \xi \rangle \frac{f(\mathbf{x}) - f(\sigma_{\alpha}(\mathbf{x}))}{\langle \alpha, \mathbf{x} \rangle}$$

associated with a finite reflection group G which is related to a root system R on a Euclidean space \mathbb{R}^N . Here $\xi \in \mathbb{R}^N$, σ_{α} denotes the reflection with respect to the hyperplane orthogonal to the root $\alpha \in R$, and $k: R \rightarrow \mathbb{C}$ is a G -invariant function (see Section 2 for details). The Dunkl operators are generalizations of the directional derivatives (in fact, they are ordinary partial derivatives for $k \equiv 0$), however, in general, they are non-local operators. They turn out to be a key tool in the study of special functions with reflection symmetries and allow to build up the framework for the theory of special functions and integral transforms in several variables related with reflection groups in [21]–[23]. Afterwards, the theory was studied and developed by many mathematicians from many different points of view. Beside the special functions and mathematical analysis, the Dunkl theory has deep connections with the other branches of mathematics, for instance probability theory, mathematical physics, and algebra.

The aim of the article is to study the Riesz transforms in the rational Dunkl setting defined as follows.

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Definition 1.1. Let $f \in \mathcal{S}(\mathbb{R}^N)$ and $j \in \{1, \dots, N\}$. The **Riesz transforms** R_j in the Dunkl setting are defined by

$$(1.1) \quad \mathcal{F}(R_j f)(\zeta) = -i \frac{\check{\zeta}_j}{\|\zeta\|} (\mathcal{F}f)(\zeta),$$

where \mathcal{F} is the Dunkl transform (see (2.9)). The **vector of the Riesz transforms** in the Dunkl setting is defined by

$$(1.2) \quad \mathcal{R}f(\mathbf{x}) = \left(\sum_{j=1}^N |R_j f(\mathbf{x})|^2 \right)^{1/2}.$$

Here and subsequently, $\mathcal{S}(\mathbb{R}^N)$ denotes the Schwartz class functions.

The Riesz transforms in the Dunkl setting were introduced in [56, Theorem 5.3]. The following theorem was proved in [1, Theorem 3.3].

Theorem 1.2 ([1, Theorem 3.3]). *Let $1 < p < \infty$. The Riesz transforms, defined initially on $\mathcal{S}(\mathbb{R}^N)$, extend to bounded operators $L^p(dw) \rightarrow L^p(dw)$, where dw is the measure associated with the root system R and the multiplicity function k (see (2.2) and Section 2 for details).*

Moreover, it can be checked by using the Dunkl transform (see Lemma 2.5) that

$$(1.3) \quad R_j f = -T_j(-\Delta_k)^{-1/2} f$$

for $f \in L^2(dw)$, where $\Delta_k = \sum_{j=1}^N T_{e_j}^2$ is the Dunkl Laplacian. Here and subsequently, $\{e_j\}_{1 \leq j \leq N}$ denote the canonical orthonormal basis in \mathbb{R}^N .

A well-known result concerning the classical Riesz transforms, proved by E. M. Stein in [52], stated that in the case $k \equiv 0$, there are upper bounds for the L^p -norm of the vector of the Riesz transforms independent of the dimension N . Then it was proved that, in fact, the L^p -norm of the vector of the Riesz transforms is controlled by $C \max(p, \frac{p}{p-1})$, where $C > 0$ is independent of p and the dimension N ; see [4, 24]. At this point, it is worth mentioning that in the case $k \equiv 0$, the norms of the vector of the Riesz transforms are still not known (see [5, 18, 33] for the some results concerning the subject).

The aim of the current paper is to prove the bounds for $L^p(dw)$ -norms of the vector of Riesz transforms in that spirit in the rational Dunkl setting, i.e., the case of $k \not\equiv 0$ and for any root system R .

The main goal of this paper is to prove the following theorem. Recall that the measurable function is **G -invariant**, if for almost all $\mathbf{x} \in \mathbb{R}^N$ and $\sigma \in G$ we have

$$f(\sigma(\mathbf{x})) = f(\mathbf{x})$$

(see Section 2 for the definition of the Weyl group G).

Theorem 1.3. *Let $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Set $p^* = \max(p, q)$. Then for all $f \in L^p(dw)$ we have*

$$(1.4) \quad \|\mathfrak{R}f\|_{L^p(dw)} \leq 144(p^* - 1) \left(\sum_{\alpha \in R} k(\alpha) + 2^7 \right) \|f\|_{L^p(dw)}.$$

Moreover, for all $f \in L^p(dw)$, which are G -invariant, we have

$$(1.5) \quad \|\mathfrak{R}f\|_{L^p(dw)} \leq 144(p^* - 1) \|f\|_{L^p(dw)}.$$

Our second main goal in this paper will be to prove a different version of Theorem 1.3 in the one-dimensional case. If $N = 1$, then there is just one Riesz transform (Dunkl Hilbert transform), which will be denoted by \mathcal{H} , i.e.,

$$(1.6) \quad \mathcal{F}(\mathcal{H}f)(\zeta) = -i \frac{\zeta}{|\zeta|} (\mathcal{F}f)(\zeta), \quad \zeta \in \mathbb{R}^N.$$

Theorem 1.4. *Assume that $N = 1$. Let $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Set $p^* = \max(p, q)$. Then for all $f \in L^p(dw)$ we have*

$$(1.7) \quad \|\mathcal{H}f\|_{L^p(dw)} \leq 1440(p^* - 1) \|f\|_{L^p(dw)}.$$

The dimension free estimate for a vector of Riesz transforms has been studied by many authors. For instance, the estimates of that spirit for the Riesz transforms and the vector of the Riesz transforms are considered in the following contexts:

- Ornstein–Uhlenbeck operator (see [3, 10, 15, 18, 28, 29, 30, 39, 47]);
- Laguerre operator (see [26, 31, 37, 42, 54, 59]);
- Jacobi operator (see [43, 44, 45, 53]);
- Harmonic oscillator (see [14, 20, 32, 35, 36]);
- Bessel operator (see [6]);
- Grushin operator (see [51]);
- General context of orthogonal expansions (see [25, 60]);
- Weighted Riesz transforms (see [17]);
- Noncommutative Riesz transforms (see [34]).

The main tool that is used in the current paper is the Bellman function method (see Section 4). The founder of the Bellman function technique is Donald L. Burkholder (see [7, 8, 9]) who developed this technique in probability theory and found a lot of very important Bellman functions (which were usually called “auxiliary functions” there). Then this method was used and developed by many mathematicians in probability theory. The term “Bellman function” was introduced by Nazarov, Treil and Volberg in the mid-90s. Their paper [40] was important from the point of view of implementation of the methods in analysis and

explanation of its links with the control theory. For more information concerning the history, development, and many various contexts in which the Bellman function technique was explored we refer the reader to the books [46] and [57]. From the point of view of the analytic results discussed in this article, it is worth mentioning the series of papers by Dragičević and Volberg [18, 19, 20], and Carbonaro and Dragičević [10, 11, 12, 13], where the approach based on carefully studying the properties of the concrete Bellmann function was developed.

Let us discuss some difficulties in Dunkl analysis, which distinguish it from the classical setting $k \equiv 0$. As it was pointed out in [55], one of the most serious problems in the Dunkl analysis lay in the lack of knowledge about generalized translations $\tau_{\mathbf{x}}, \mathbf{x} \in \mathbb{R}^N$, which generalize the ordinary translation of the function $f \mapsto f(\cdot - \mathbf{x})$. It was proved that for some root systems R the operators $\tau_{\mathbf{x}}$ do not preserve positive functions and the boundedness of $\tau_{\mathbf{x}}$ on $L^p(dw)$ -spaces ($p \neq 2$) becomes an open problem in the Dunkl analysis. In the context of this paper, we overcome this difficulty using recently proved upper bounds for the Dunkl Poisson kernel (see [2]).

Looking from the point of view of the current paper, let us discuss another difficulty regarding the Dunkl operators. The Dunkl operators T_{ξ} do not satisfy the Leibniz rule in the usual sense, i.e., the formula

$$T_{\xi}(fg)(\mathbf{x}) = f(\mathbf{x})T_{\xi}g(\mathbf{x}) + T_{\xi}f(\mathbf{x})g(\mathbf{x})$$

holds just in specific cases, e.g., if f or g is radial. In general case, the formula for $T_{\xi}(fg)$ contains summands of local and non-local character. The analysis becomes more complicated when we compose two or more Dunkl operators, which is the case when we are trying to adapt the Bellman function method.

At this point, it is also worth mentioning that in the Dunkl setting the explicit formulas for $\Delta_k u^p$ for $p \in [1, \infty)$ and $u \in \mathcal{S}(\mathbb{R}^N)$ seem to be of a quite different nature than in the case $k \equiv 0$. In order to elaborate the case of $p = 2$, let us consider the Dunkl version of the carré du champ operator:

$$\Gamma_k(f, g) = \frac{1}{2}(\Delta_k(fg) - f \Delta_k g - g \Delta_k f).$$

As noticed in [58], we have

$$\int_{\mathbb{R}^N} \Gamma_k(f, g) dw = \int_{\mathbb{R}^N} \sum_{j=1}^N T_{j^*} f T_j g dw,$$

but the identity $\sum_{j=1}^N T_{j^*} f T_j g \equiv \Gamma_k(f, g)$ is not true if $k \neq 0$, which can be checked

by the explicit calculation:

$$\Gamma_k(f, g)(\mathbf{x}) = \langle \nabla f(\mathbf{x}), \nabla g(\mathbf{x}) \rangle + \sum_{\alpha \in R} \frac{k(\alpha)}{2} \frac{(f(\mathbf{x}) - f(\sigma_\alpha(\mathbf{x}))(g(\mathbf{x}) - g(\sigma_\alpha(\mathbf{x})))}{\langle \alpha, \mathbf{x} \rangle^2}$$

(see also [27] for a more general calculation). In the current paper, following the approach presented in [20], we obtain an explicit formula for Δ_k applied to the Bellman function, which turns out to be closely related to the known formulas for $\Delta_k u^p$. Therefore, the Bellman approach has to be adapted to this specific setting.

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2 Basic definitions of the Dunkl theory

In this section, for the convenience of the reader, we present basic facts concerning the theory of the Dunkl operators. For details we refer the reader to [22], [49], and [50]. The reader who is familiar with the Dunkl theory can omit this section and proceed to Subsection 3.2.

We consider the Euclidean space \mathbb{R}^N with the scalar product $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=1}^N x_j y_j$, where $\mathbf{x} = (x_1, \dots, x_N)$, $\mathbf{y} = (y_1, \dots, y_N)$, and the norm $\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle$. The number N will be fixed throughout this paper. For a nonzero vector $\alpha \in \mathbb{R}^N$, the reflection σ_α with respect to the hyperplane α^\perp orthogonal to α is given by

$$(2.1) \quad \sigma_\alpha(\mathbf{x}) = \mathbf{x} - 2 \frac{\langle \mathbf{x}, \alpha \rangle}{\|\alpha\|^2} \alpha.$$

In this paper we fix a normalized root system in \mathbb{R}^N , that is, a finite set $R \subset \mathbb{R}^N \setminus \{0\}$ such that $R \cap \alpha \mathbb{R} = \{\pm \alpha\}$, $\sigma_\alpha(R) = R$, and $\|\alpha\| = \sqrt{2}$ for all $\alpha \in R$. The finite group G generated by the reflections σ_α , $\alpha \in R$ is called the **Weyl group (reflection group)** of the root system. A **multiplicity function** is a G -invariant function $k: R \rightarrow \mathbb{C}$ which will be ≥ 0 throughout this paper. Let

$$(2.2) \quad dw(\mathbf{x}) = \prod_{\alpha \in R} |\langle \mathbf{x}, \alpha \rangle|^{k(\alpha)} d\mathbf{x}$$

be the associated measure in \mathbb{R}^N , where, here and subsequently, $d\mathbf{x}$ stands for the Lebesgue measure in \mathbb{R}^N . For a Lebesgue measurable set A we denote

$$w(A) = \int_A dw(\mathbf{x}).$$

There is a constant $C > 0$ such that

$$(2.3) \quad C^{-1} w(B(\mathbf{x}, r)) \leq r^N \prod_{\alpha \in R} (|\langle \mathbf{x}, \alpha \rangle| + r)^{k(\alpha)} \leq C w(B(\mathbf{x}, r)),$$

so $dw(\mathbf{x})$ is doubling, that is, there is a constant $C > 0$ such that

$$(2.4) \quad w(B(\mathbf{x}, 2r)) \leq C w(B(\mathbf{x}, r)) \quad \text{for all } \mathbf{x} \in \mathbb{R}^N, r > 0.$$

Moreover, since the function w is G -invariant, for all $\sigma \in G$ we have

$$(2.5) \quad \int_{\mathbb{R}^N} f(\sigma(\mathbf{x})) dw(\mathbf{x}) = \int_{\mathbb{R}^N} f(\mathbf{x}) dw(\mathbf{x}).$$

For $\zeta \in \mathbb{R}^N$, the **Dunkl operators** T_ζ are the following k -deformations of the directional derivatives ∂_ζ by a difference operator:

$$(2.6) \quad T_\zeta f(\mathbf{x}) = \partial_\zeta f(\mathbf{x}) + \sum_{\alpha \in R} \frac{k(\alpha)}{2} \langle \alpha, \zeta \rangle \frac{f(\mathbf{x}) - f(\sigma_\alpha(\mathbf{x}))}{\langle \alpha, \mathbf{x} \rangle}.$$

The Dunkl operators T_ζ , which were introduced in [22], commute and are skew-symmetric with respect to the G -invariant measure dw . Let $\{e_j\}_{1 \leq j \leq N}$ denote the canonical orthonormal basis in \mathbb{R}^N and let $T_j = T_{e_j}$. As usual, for every multi-index

$$\beta = (\beta_1, \beta_2, \dots, \beta_N) \in \mathbb{N}_0^N = (\mathbb{N} \cup \{0\})^N,$$

we set $|\beta| = \sum_{j=1}^N \beta_j$ and

$$\partial^\beta = \partial_{e_1}^{\beta_1} \circ \partial_{e_2}^{\beta_2} \circ \dots \circ \partial_{e_N}^{\beta_N},$$

where $\{e_1, e_2, \dots, e_N\}$ is the canonical basis of \mathbb{R}^N . The additional subscript \mathbf{x} in $\partial_{\mathbf{x}}^\alpha$ means that the partial derivative ∂^α is taken with respect to the variable $\mathbf{x} \in \mathbb{R}^N$. By $\nabla_{\mathbf{x}} f$ we denote the gradient of the function f with respect to the variable \mathbf{x} .

The following fundamental theorem was proved by Ch. Dunkl.

Theorem 2.1 ([23]). *The Dunkl operators are skew-symmetric with respect to the measure dw . More precisely, for any $\zeta \in \mathbb{R}^N$, $f \in \mathcal{S}(\mathbb{R}^N)$, and $g \in C_b^1(\mathbb{R}^N)$ (here and subsequently, $C_b^1(\mathbb{R}^N)$ denotes the set of bounded functions with bounded and continuous partial derivatives), we have the following integration-by-parts formula:*

$$(2.7) \quad \int_{\mathbb{R}^N} T_\zeta f(\mathbf{x}) g(\mathbf{x}) dw(\mathbf{x}) = - \int_{\mathbb{R}^N} f(\mathbf{x}) T_\zeta g(\mathbf{x}) dw(\mathbf{x}).$$

Remark 2.2. Note that (2.7) holds also if $f \in C_c^\infty(\mathbb{R}^N)$ and $g \in C^1(\mathbb{R}^N)$. In order to justify this fact, it is enough to take $\varphi \in C_c^\infty(\mathbb{R}^N)$ such that $\varphi \equiv 1$ on

$$A = \bigcup_{\sigma \in G} \sigma(\text{supp} f).$$

It follows from (2.6) that $T_\xi f \equiv 0$ and $f \equiv 0$ on $\mathbb{R}^N \setminus A$. Hence, by (2.7) we have

$$\begin{aligned} \int_{\mathbb{R}^N} T_\xi f(\mathbf{x})g(\mathbf{x}) \, d\omega(\mathbf{x}) &= \int_{\mathbb{R}^N} T_\xi f(\mathbf{x})(\varphi g)(\mathbf{x}) \, d\omega(\mathbf{x}) \\ &= - \int_{\mathbb{R}^N} f(\mathbf{x})T_\xi(\varphi g)(\mathbf{x}) \, d\omega(\mathbf{x}) = - \int_{\mathbb{R}^N} f(\mathbf{x})T_\xi g(\mathbf{x}) \, d\omega(\mathbf{x}). \end{aligned}$$

We will also need the following technical lemma, which is well-known. We provide the sketch of its proof for the sake of completeness.

Lemma 2.3. For any $\beta \in \mathbb{N}_0^N$ there is a constant $C_\beta > 0$ such that for all $f \in C^\infty(\mathbb{R}^N)$ we have

$$\|T^\beta f\|_{L^\infty} \leq C_\beta \sum_{\beta' \in \mathbb{N}_0^N, |\beta'|=|\beta|} \|\partial^{\beta'} f\|_{L^\infty}.$$

Proof. By the definition of T_j and by the fundamental theorem of calculus, for all $f \in C^1(\mathbb{R}^N)$, we have

$$\begin{aligned} T_j f(\mathbf{x}) &= \partial_j f(\mathbf{x}) - \sum_{\alpha \in R} \frac{k(\alpha)}{2} \alpha_j \langle \mathbf{x}, \alpha \rangle^{-1} \int_0^1 \frac{d}{dt} (\phi(\mathbf{x} - 2t\alpha \|\alpha\|^{-2} \langle \mathbf{x}, \alpha \rangle)) \, dt \\ &= \partial_j f(\mathbf{x}) + \sum_{\alpha \in R} \frac{k(\alpha)}{2} \alpha_j \int_0^1 \langle (\nabla_{\mathbf{x}} f)(\mathbf{x} - 2t\alpha \|\alpha\|^{-2} \langle \mathbf{x}, \alpha \rangle), \alpha \rangle \, dt \end{aligned}$$

(cf. [50, page 9]). Consequently, for any $\beta \in \mathbb{N}_0^N$ there is a constant $C > 0$ such that for all $f \in C^{|\beta|+1}(\mathbb{R}^N)$ and $j \in \{1, \dots, N\}$ we have

$$(2.8) \quad \sup_{\mathbf{x} \in \mathbb{R}^N} |\partial^\beta T_j f(\mathbf{x})| \leq C \sup_{\mathbf{x} \in \mathbb{R}^N} \|\nabla_{\mathbf{x}} \partial^\beta f(\mathbf{x})\|.$$

The claim follows from (2.8) by the induction on $|\beta|$. □

For fixed $\mathbf{y} \in \mathbb{R}^N$ the **Dunkl kernel** $E(\mathbf{x}, \mathbf{y})$ is a unique analytic solution to the system

$$T_\xi f = \langle \xi, \mathbf{y} \rangle f, \quad f(0) = 1.$$

The function $E(\mathbf{x}, \mathbf{y})$, which generalizes the exponential function $e^{\langle \mathbf{x}, \mathbf{y} \rangle}$, has the unique extension to a holomorphic function on $\mathbb{C}^N \times \mathbb{C}^N$.

The **Dunkl transform** is defined by

$$(2.9) \quad \mathcal{F}f(\zeta) = c_k^{-1} \int_{\mathbb{R}^N} E(-i\zeta, \mathbf{x})f(\mathbf{x}) \, dw(\mathbf{x}),$$

where

$$c_k = \int_{\mathbb{R}^N} e^{-\frac{\|\mathbf{x}\|^2}{2}} \, dw(\mathbf{x}) > 0,$$

for $f \in L^1(dw)$. It was introduced in [23] for $k \geq 0$ and further studied in [16] in the more general context. It was proved in [23, Corollary 2.7] (see also [16, Theorem 4.26]) that it is an isometry on $L^2(dw)$, i.e.,

$$(2.10) \quad \|f\|_{L^2(dw)} = \|\mathcal{F}f\|_{L^2(dw)} \quad \text{for all } f \in L^2(dw).$$

We have also the following inversion theorem.

Theorem 2.4 (Inversion theorem, see [16, Theorem 4.20]). *For all $f \in L^1(dw)$ such that $\mathcal{F}f \in L^1(dw)$ we have*

$$(2.11) \quad f(\mathbf{x}) = (\mathcal{F})^2f(-\mathbf{x}) \quad \text{for almost all } \mathbf{x} \in \mathbb{R}^N.$$

The inverse \mathcal{F}^{-1} of \mathcal{F} has the form

$$(2.12) \quad \mathcal{F}^{-1}f(\mathbf{x}) = c_k^{-1} \int_{\mathbb{R}^N} f(\zeta)E(i\zeta, \mathbf{x}) \, dw(\zeta) = \mathcal{F}f(-\mathbf{x}).$$

Below we list some properties of \mathcal{F} .

Lemma 2.5. *Suppose that $f \in \mathcal{S}(\mathbb{R}^N)$ and $j \in \{1, \dots, N\}$. Then we have*

- (A) $\mathcal{F}f \in \mathcal{S}(\mathbb{R}^N)$;
- (B) $T_j(\mathcal{F}f)(\zeta) = \mathcal{F}g(\zeta)$, where $g(\zeta) = -i\zeta_j f(\zeta)$;
- (C) $\mathcal{F}(T_j f)(\zeta) = i\zeta_j \mathcal{F}f(\zeta)$.

2.1 Dunkl Laplacian.

Definition 2.6. The **Dunkl Laplacian** associated with G and k is the differential-difference operator

$$(2.13) \quad \Delta_k = \sum_{j=1}^N T_j^2.$$

It was introduced in [22], where it was also proved that Δ_k acts on $C^2(\mathbb{R}^N)$ functions by

$$(2.14) \quad \begin{aligned} \Delta_k f(\mathbf{x}) &= \Delta f(\mathbf{x}) + \sum_{\alpha \in R} k(\alpha) \delta_\alpha f(\mathbf{x}), \\ \delta_\alpha f(\mathbf{x}) &= \frac{\partial_\alpha f(\mathbf{x})}{\langle \alpha, \mathbf{x} \rangle} - \frac{f(\mathbf{x}) - f(\sigma_\alpha(\mathbf{x}))}{\langle \alpha, \mathbf{x} \rangle^2}. \end{aligned}$$

Here and subsequently, $\Delta = \sum_{j=1}^N \partial_j^2$.

We have the following theorem, which allows us to define $\sqrt{-\Delta_k}$ by spectral theorem.

Theorem 2.7 ([48, Theorem 4.8]). *The operator $(-\Delta_k, \mathcal{S}(\mathbb{R}^N))$ in $L^2(dw)$ is densely defined and closable. Its closure will be denoted by the same symbol $-\Delta_k$, is self-adjoint and its domain is*

$$\mathcal{D}(-\Delta_k) = \{f \in L^2(dw) : \|\xi\|^2(\mathcal{F}f)(\xi) \in L^2(dw(\xi))\}.$$

It is a unique positive self-adjoint extension of $(-\Delta_k, \mathcal{S}(\mathbb{R}^N))$.

Note that, thanks to Lemma 2.5 (C), for all $\xi \in \mathbb{R}^N$ and $f \in \mathcal{S}(\mathbb{R}^N)$ we have

$$(2.15) \quad \mathcal{F}(\Delta_k f)(\xi) = -\|\xi\|^2 \mathcal{F}f(\xi),$$

therefore

$$(2.16) \quad \mathcal{F}((-\Delta_k)^{1/2}f)(\xi) = -\|\xi\| \mathcal{F}f(\xi).$$

3 The Dunkl Poisson semigroup and $L^p(dw)$ -norm of the Riesz transform in terms of an integral involving the Dunkl Poisson semigroup

3.1 The k -Cauchy kernel and Dunkl Poisson semigroup.

Definition 3.1. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ and $t > 0$. We define the k -Cauchy kernel $p_t(\mathbf{x}, \mathbf{y})$ to be the integral kernel of the operator $P_t = e^{-t\sqrt{-\Delta_k}}$ on $L^2(dw)$ (see Theorem 2.7), that is

$$P_t f(\mathbf{x}) = \int_{\mathbb{R}^N} p_t(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) dw(\mathbf{y}).$$

The kernel $p_t(\mathbf{x}, \mathbf{y})$ was introduced and studied in [50].

Theorem 3.2 ([50, Theorem 5.6]). *Let f be a bounded continuous function on \mathbb{R}^N . Then the function given by $v(\mathbf{x}, t) = P_t f(\mathbf{x})$ is continuous and bounded. Moreover, it solves the Cauchy problem*

$$\begin{cases} \partial_t^2 v(\mathbf{x}, t) + \Delta_{k,\mathbf{x}} v(\mathbf{x}, t) = 0 & \text{on } \mathbb{R}^N \times (0, \infty), \\ v(\mathbf{x}, 0) = f(\mathbf{x}) & \text{for all } \mathbf{x} \in \mathbb{R}^N. \end{cases}$$

The k -Cauchy kernel is also called the **generalized Poisson kernel** (or **Dunkl Poisson kernel**) by the analogy with the classical Poisson semigroup. We have the following lemma.

Lemma 3.3. *Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ and $t > 0$. The generalized Poisson kernel $p_t(\mathbf{x}, \mathbf{y})$ has the following properties;*

- (A) $p_t(\mathbf{x}, \mathbf{y}) = p_t(\mathbf{y}, \mathbf{x})$;
- (B) $\int_{\mathbb{R}^N} p_t(\mathbf{x}, \mathbf{z}) \, d\omega(\mathbf{z}) = 1$;
- (C) $p_t(\mathbf{x}, \mathbf{y}) > 0$;
- (D) $p_t(\sigma(\mathbf{x}), \sigma(\mathbf{y})) = p_t(\mathbf{x}, \mathbf{y})$ for all $\sigma \in G$.

It follows by Theorem 3.2, (2.16), and the inversion theorem for the Dunkl transform (see Theorem 2.4) that for all $f \in \mathcal{S}(\mathbb{R}^N)$, $\mathbf{x} \in \mathbb{R}^N$, and $t > 0$ we have

$$(3.1) \quad P_t f(\mathbf{x}) = c_k^{-1} \int_{\mathbb{R}^N} e^{-t\|\zeta\|} E(i\zeta, \mathbf{x}) \mathcal{F}f(\zeta) \, d\omega(\zeta).$$

We also have the following upper and lower bound for the generalized Poisson kernel.

Proposition 3.4 ([2, Proposition 5.1]). *For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ and $t, r > 0$ we denote*

$$(3.2) \quad \begin{aligned} V(\mathbf{x}, \mathbf{y}, r) &= \max\{w(B(\mathbf{x}, r)), w(B(\mathbf{y}, r))\}, \\ d(\mathbf{x}, \mathbf{y}) &= \min_{\sigma \in G} \|\sigma(\mathbf{x}) - \mathbf{y}\|. \end{aligned}$$

(a) *Upper and lower bounds: there is a constant $C \geq 1$ such that*

$$(3.3) \quad \begin{aligned} \frac{C^{-1}}{V(\mathbf{x}, \mathbf{y}, t + \|\mathbf{x} - \mathbf{y}\|)} \frac{t}{t + \|\mathbf{x} - \mathbf{y}\|} &\leq p_t(\mathbf{x}, \mathbf{y}) \\ &\leq \frac{C}{V(\mathbf{x}, \mathbf{y}, t + d(\mathbf{x}, \mathbf{y}))} \frac{t}{t + d(\mathbf{x}, \mathbf{y})} \end{aligned}$$

for all $t > 0$ and for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$.

(b) *Dunkl gradient: for every $\zeta \in \mathbb{R}^N$, there is a constant $C > 0$ such that*

$$(3.4) \quad |T_{\zeta, \mathbf{y}} p_t(\mathbf{x}, \mathbf{y})| \leq \frac{C}{V(\mathbf{x}, \mathbf{y}, t + d(\mathbf{x}, \mathbf{y}))} \frac{1}{t + d(\mathbf{x}, \mathbf{y})}$$

for all $t > 0$ and for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$.

(c) *Mixed derivatives: for any nonnegative integer m and for any multi-index $\beta \in \mathbb{N}_0^N$, there is a constant $C \geq 0$ such that, for all $t > 0$ and for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$,*

$$(3.5) \quad |\partial_t^m \partial_{\mathbf{y}}^\beta p_t(\mathbf{x}, \mathbf{y})| \leq C p_t(\mathbf{x}, \mathbf{y}) (t + d(\mathbf{x}, \mathbf{y}))^{-m-|\beta|} \times \begin{cases} 1 & \text{if } m = 0, \\ 1 + \frac{d(\mathbf{x}, \mathbf{y})}{t} & \text{if } m > 0. \end{cases}$$

Moreover, for any nonnegative integer m and for any multi-indices $\beta, \beta' \in \mathbb{N}_0^N$, there is a constant $C \geq 0$ such that, for all $t > 0$ and for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$,

$$(3.6) \quad |\partial_t^m \partial_{\mathbf{x}}^\beta \partial_{\mathbf{y}}^{\beta'} p_t(\mathbf{x}, \mathbf{y})| \leq C t^{-m-|\beta|-|\beta'|} p_t(\mathbf{x}, \mathbf{y}).$$

Note that the estimates in Proposition 3.4 are given in the spirit of spaces of homogeneous type, except that the metric $\|\mathbf{x} - \mathbf{y}\|$ is replaced by the distance of the orbits $d(\mathbf{x}, \mathbf{y})$ (see (3.2)). One of the reason why the estimates of Proposition 3.4 are suitable in many contexts is explained in the next lemma. We omit its standard proof.

Lemma 3.5. *Let $1 \leq p \leq \infty$. If $f \in L^p(dw)$, then $(\mathbf{x}, t) \mapsto P_t f(\mathbf{x})$ belongs to $C^\infty(\mathbb{R}^N \times (0, \infty))$ and for all $(m, \beta) \in \mathbb{N}_0 \times \mathbb{N}_0^N$ we have*

$$\partial_t^m \partial_{\mathbf{x}}^\beta P_t f(\mathbf{x}) = \int_{\mathbb{R}^N} \partial_t^m \partial_{\mathbf{x}}^\beta p_t(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) dw(\mathbf{y}).$$

Moreover, for any $m \in \mathbb{N}_0$ there is a constant $C = C_{p,m} > 0$ such that for all $t > 0$ and $f \in L^p(dw)$ we have

$$(3.7) \quad \|\partial_t^m P_t f\|_{L^p(dw)} \leq C t^{-m} \|f\|_{L^p(dw)}.$$

3.2 $L^p(dw)$ -norm of Riesz transforms in terms of an integral involving the Dunkl Poisson semigroup. The next proposition is well-known (see [52], [18, Lemma 2.1]). We provide its version in the Dunkl setting for the sake of completeness.

Proposition 3.6. *For all $j \in \{1, \dots, N\}$ and $f, g \in \mathcal{S}(\mathbb{R}^N)$ we have*

$$(3.8) \quad \left| \int_{\mathbb{R}^N} R_j f(\mathbf{x}) g(\mathbf{x}) dw(\mathbf{x}) \right| = 4 \left| \int_{\mathbb{R}^N} \int_0^\infty t \partial_t P_t g(\mathbf{x}) T_j P_t f(\mathbf{x}) dt dw(\mathbf{x}) \right|.$$

Proof. For $1 \leq j \leq N$, $\mathbf{x} \in \mathbb{R}^N$, and $t > 0$ we define

$$\varphi(\mathbf{x}, t) := P_t R_j f(\mathbf{x}) P_t g(\mathbf{x}).$$

It follows by Proposition 3.4 that for fixed $\mathbf{x} \in \mathbb{R}^N$ there is a constant $C > 0$ independent of \mathbf{x} such that for all $\mathbf{y} \in \mathbb{R}^N$ and $t > 0$ we have

$$p_t(\mathbf{x}, \mathbf{y}) \leq \frac{C}{w(B(\mathbf{x}, t))}.$$

Hence, for all $F \in \mathcal{S}(\mathbb{R}^N)$ we have

$$|P_t F(\mathbf{x})| \leq \int_{\mathbb{R}^N} p_t(\mathbf{x}, \mathbf{y}) |F(\mathbf{y})| dw(\mathbf{y}) \leq \frac{C}{w(B(\mathbf{x}, t))} \|F\|_{L^1(dw)}.$$

Moreover, by (2.3), for all $\mathbf{x} \in \mathbb{R}^N$ we have

$$\lim_{t \rightarrow \infty} \frac{1}{w(B(\mathbf{x}, t))} = 0.$$

Consequently, by (3.6) we get that for fixed $\mathbf{x} \in \mathbb{R}^N$ we have $\varphi(\mathbf{x}, \cdot) \in C^2((0, \infty))$ and

$$\lim_{t \rightarrow \infty} \varphi(\mathbf{x}, t) = \lim_{t \rightarrow \infty} t \partial_t \varphi(\mathbf{x}, t) = 0.$$

Therefore, by the fundamental theorem of calculus and Theorem 3.2, for all $\mathbf{x} \in \mathbb{R}^N$ we have

$$(3.9) \quad R_j f(\mathbf{x}) g(\mathbf{x}) = \varphi(\mathbf{x}, 0) = \int_0^\infty t \partial_t^2 \varphi(\mathbf{x}, t) dt.$$

Since, by the definition of $\{P_t\}_{t \geq 0}$, $\partial_t P_t = \sqrt{-\Delta_k} P_t$, and the operator $\sqrt{-\Delta_k}$ is self-adjoint on $L^2(dw)$, by (3.9) we have

$$(3.10) \quad \begin{aligned} & \int_{\mathbb{R}^N} R_j f(\mathbf{x}) g(\mathbf{x}) dw(\mathbf{x}) \\ &= \int_{\mathbb{R}^N} \int_0^\infty t \partial_t^2 \varphi(\mathbf{x}, t) dt dw(\mathbf{x}) \\ &= \int_{\mathbb{R}^N} \int_0^\infty t ((\partial_t^2 P_t) R_j f(\mathbf{x}) P_t g(\mathbf{x}) + 2 \partial_t P_t R_j f(\mathbf{x}) \partial_t P_t g(\mathbf{x}) \\ &\quad + P_t R_j f(\mathbf{x}) (\partial_t^2 P_t) g(\mathbf{x})) dt dw(\mathbf{x}) \\ &= 4 \int_{\mathbb{R}^N} \int_0^\infty t \sqrt{-\Delta_k} P_t R_j f(\mathbf{x}) \partial_t P_t g(\mathbf{x}) dt dw(\mathbf{x}). \end{aligned}$$

Finally, note that by the definition of the Riesz transform (see (1.1)), (2.16), (3.1), and Lemma 2.5 (C), for all $1 \leq j \leq N$ we have

$$\begin{aligned} \sqrt{-\Delta_k} (P_t R_j) f(\mathbf{x}) &= c_k^{-1} \int_{\mathbb{R}^N} (-\|\xi\|) e^{-t\|\xi\|} E(i\xi, \mathbf{x}) \mathcal{F}(R_j f)(\xi) dw(\xi) \\ &= c_k^{-1} \int_{\mathbb{R}^N} (-\|\xi\|) e^{-t\|\xi\|} E(i\xi, \mathbf{x}) \frac{-i\xi_j}{\|\xi\|} \mathcal{F}f(\xi) dw(\xi) \\ &= T_j P_t f(\mathbf{x}), \end{aligned}$$

so the claim follows by (3.10). □

As the consequence of Proposition 3.6, we obtain the following corollary.

Corollary 3.7. *Let $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for all $f \in \mathcal{S}(\mathbb{R}^N)$ we have*

$$(3.11) \quad \begin{aligned} & \| \mathcal{R}f \|_{L^p(dw)} \\ &= 4 \sup_{g_j \in \mathcal{S}(\mathbb{R}^N), \| \mathbf{g}(\mathbf{y}) \|_{L^q(dw(\mathbf{y}))} \leq 1} \left| \sum_{j=1}^N \int_{\mathbb{R}^N} \int_0^\infty t \partial_t P_t g_j(\mathbf{x}) T_j P_t f(\mathbf{x}) dt dw(\mathbf{x}) \right|. \end{aligned}$$

Here and subsequently, for $g_j \in \mathcal{S}(\mathbb{R}^N)$, $1 \leq j \leq N$, and $\mathbf{x} \in \mathbb{R}^N$ we denote

$$\mathbf{g}(\mathbf{x}) = (g_1(\mathbf{x}), \dots, g_N(\mathbf{x})),$$

$$\|\mathbf{g}(\mathbf{x})\| = \left(\sum_{j=1}^N |g_j(\mathbf{x})|^2 \right)^{1/2}.$$

4 Bellman function

In this section, we introduce the Bellman function, which will be the main ingredient in the proof of Theorem 1.3.

Definition 4.1. Let $p \geq 2$ and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $N_1, N_2 \in \mathbb{N}$. We define the **Bellman function** $\beta: [0, \infty)^2 \rightarrow [0, \infty)$ by the formula

$$(4.1) \quad \beta(s, t) = s^p + t^q + \gamma \begin{cases} s^2 t^{2-q} & \text{if } s^p < t^q \\ \frac{2}{p} s^p + (\frac{2}{q} - 1) t^q & \text{if } s^p \geq t^q, \end{cases} \quad \gamma := \frac{q(q-1)}{8}.$$

The number γ will be fixed throughout the paper. Next, we define the **Nazarov–Treil Bellman function** $B: \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \rightarrow [0, \infty)$ by the formula

$$(4.2) \quad B(\eta, \zeta) = \frac{1}{2} \beta(\|\eta\|, \|\zeta\|).$$

The function $B(\eta, \zeta)$ was introduced by Nazarov and Treil in [41], then used and simplified in [10, 11, 18, 19, 20].

Note that the function B is differentiable but not smooth. We will need the smooth version of B . For $N_1, N_2 \in \mathbb{N}$ let $\phi: \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \rightarrow [0, \infty)$ be a smooth radial function supported in $B(0, 1) \subset \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ defined by the formula

$$\phi(\mathbf{x}_1, \mathbf{x}_2) = c_{N_1, N_2} \chi_{B(0,1)}(\mathbf{x}_1, \mathbf{x}_2) \exp(-(1 - \|\mathbf{x}_1\|^2 - \|\mathbf{x}_2\|^2)^{-1}),$$

where $c_{N_1, N_2} > 0$ is a constant such that

$$\int_{\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}} \phi(\mathbf{x}_1, \mathbf{x}_2) d\mathbf{x}_1 d\mathbf{x}_2 = 1.$$

For $\kappa > 0$ and $(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ we set

$$(4.3) \quad \phi_\kappa(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{\kappa^{N_1+N_2}} \phi(\mathbf{x}_1/\kappa, \mathbf{x}_2/\kappa).$$

Definition 4.2. Let $p \geq 2$ and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $N_1, N_2 \in \mathbb{N}$ and $\kappa > 0$. We define $B_\kappa: \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \rightarrow [0, \infty)$ by the formula

$$(4.4) \quad \begin{aligned} B_\kappa(\eta, \zeta) &= B \star \phi_\kappa(\eta, \zeta) := \frac{1}{2} \beta_\kappa(\|\eta\|, \|\zeta\|) \\ &= \int_{\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}} \phi_\kappa(\eta - \eta_1, \zeta - \zeta_1) B(\eta_1, \zeta_1) d\eta_1 d\zeta_1. \end{aligned}$$

Remark 4.3. In order to avoid misunderstanding, we would like to emphasize that the convolution “ \star ” in (4.4) is the ordinary one (not the Dunkl generalized convolution). Let us also point out that in the proof of Theorem 1.3 we will set $N_1 = 1$ and $N_2 = N$.

The following properties of the functions β_κ and B_κ were proved in [20, Theorems 3 and 4] and [37].

Proposition 4.4. *Let $p \geq 2$ and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. There is a constant $C_p > 0$ such that for all $\kappa \in (0, 1]$ and $s, t > 0$ we have*

$$\begin{aligned} 0 \leq \partial_s \beta_\kappa(s, t) &\leq C_p \max((s + \kappa)^p, (t + \kappa)), \\ 0 \leq \partial_t \beta_\kappa(s, t) &\leq C_p (t + \kappa)^{q-1}. \end{aligned}$$

Theorem 4.5. *Let $p \geq 2$ and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $\kappa \in (0, 1]$. Then $B_\kappa \in C^\infty(\mathbb{R}^{N_1} \times \mathbb{R}^{N_2})$. Moreover, there is a function $\tau: \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \rightarrow [0, \infty)$ such that for all $\eta \in \mathbb{R}^{N_1}$, $\zeta \in \mathbb{R}^{N_2}$, and $\omega = (\omega_1, \omega_2) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ we have*

$$(4.5) \quad 0 \leq B_\kappa(\eta, \zeta) \leq \frac{1 + \gamma}{2} ((\|\eta\| + \kappa)^p + (\|\zeta\| + \kappa)^q),$$

$$(4.6) \quad \langle \text{Hess}(B_\kappa)(\eta, \zeta)\omega, \omega \rangle \geq \frac{\gamma}{2} \left((\tau \star \phi_\kappa)(\eta, \zeta) \|\omega_1\|^2 + \left(\frac{1}{\tau} \star \phi_\kappa \right)(\eta, \zeta) \|\omega_2\|^2 \right).$$

It follows from the proof of [20, Theorem 3] that one can take

$$(4.7) \quad \tau(\eta, \zeta) = \|\zeta\|^{2-q}.$$

Remark 4.6. In our further considerations, we will need the explicit form of τ (see (4.7)). This form of τ follows directly from the proofs presented in [20, Theorem 3] and [37, Proposition 6.3], although it is not given explicitly there. Therefore, for the convenience of the reader, we repeat the proof from [37] in Appendix A with τ given by (4.7).

In our further considerations, we will need the following elementary lemma, which concerns the properties of τ in (4.7).

Lemma 4.7. *Let $1 < q \leq 2$ and $N_3 \in \mathbb{N}$. Then for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{N_3}$ we have*

$$(4.8) \quad \int_0^1 s \|\mathbf{sa} + (1 - s)\mathbf{b}\|^{2-q} ds \geq 2^{-6} \max(\|\mathbf{a}\|, \|\mathbf{b}\|)^{2-q},$$

$$(4.9) \quad \int_0^1 s \|\mathbf{sa} + (1 - s)\mathbf{b}\|^{q-2} ds \geq 2^{-1} \max(\|\mathbf{a}\|, \|\mathbf{b}\|)^{q-2}.$$

Proof. The proof is standard, but we provide it for the sake of completeness. We will prove (4.8) first. Let us consider two cases.

Case 1. $\|\mathbf{a}\| \geq \|\mathbf{b}\|$. Then we have

$$\begin{aligned} \int_0^1 s\|\mathbf{sa} + (1-s)\mathbf{b}\|^{2-q} ds &\geq \int_{3/4}^1 s\|\mathbf{sa} + (1-s)\mathbf{b}\|^{2-q} ds \\ &\geq \int_{3/4}^1 s(s\|\mathbf{a}\| - (1-s)\|\mathbf{b}\|)^{2-q} ds \\ &\geq \int_{3/4}^1 s(3\|\mathbf{a}\|/4 - \|\mathbf{b}\|/4)^{2-q} ds \geq 2^{-6}\|\mathbf{a}\|^{2-q}. \end{aligned}$$

Case 2. $\|\mathbf{b}\| > \|\mathbf{a}\|$. By the change of variables we have

$$\int_0^1 s\|\mathbf{sa} + (1-s)\mathbf{b}\|^{2-q} ds = \int_0^1 s\|\mathbf{sb} + (1-s)\mathbf{a}\|^{2-q} ds,$$

so we are reduced to Case 1.

In order to prove (4.9), we write

$$\begin{aligned} \int_0^1 s\|\mathbf{sa} + (1-s)\mathbf{b}\|^{q-2} ds &\geq \int_0^1 s(s\|\mathbf{a}\| + (1-s)\|\mathbf{b}\|)^{q-2} ds \\ &\geq \max(\|\mathbf{a}\|, \|\mathbf{b}\|)^{q-2} \int_0^1 s ds \\ &= 2^{-1} \max(\|\mathbf{a}\|, \|\mathbf{b}\|)^{q-2}. \end{aligned} \quad \square$$

4.1 The Dunkl Laplacian on the Bellman function.

Definition 4.8. Let $p \geq 2$ and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$, $\kappa \in (0, 1]$. For $f \in L^p(dw)$ and $g_j \in L^q(dw)$, $1 \leq j \leq N$, $\mathbf{x} \in \mathbb{R}^N$, and $t > 0$ we define

$$(4.10) \quad u(\mathbf{x}, t) := (P_t f(\mathbf{x}), P_t g_1(\mathbf{x}), \dots, P_t g_N(\mathbf{x})),$$

$$(4.11) \quad \tilde{u}(\mathbf{x}, t) := (P_t f(\mathbf{x}), P_t \mathbf{g}(\mathbf{x})) = (P_t f(\mathbf{x}), (P_t g_1(\mathbf{x}), \dots, P_t g_N(\mathbf{x}))),$$

$$(4.12) \quad b_\kappa(\mathbf{x}, t) := B_\kappa(P_t f(\mathbf{x}), P_t \mathbf{g}(\mathbf{x})) = B_\kappa(P_t f(\mathbf{x}), (P_t g_1(\mathbf{x}), \dots, P_t g_N(\mathbf{x}))),$$

where $\{P_t\}_{t \geq 0}$ is the Dunkl Poisson semigroup (see Definition 3.1).

Lemma 4.9. Assume that $f, g_j \in \mathcal{S}(\mathbb{R}^N)$, $1 \leq j \leq N$, and $\kappa \in (0, 1]$. Then

(A) $b_\kappa \in C^\infty(\mathbb{R}^N \times (0, \infty))$;

(B) there is a constant $C_{f,\mathbf{g}} > 0$, which depends on f and \mathbf{g} and is independent of κ , such that for all $\mathbf{x} \in \mathbb{R}^N$ and $t > 0$ we have

$$(4.13) \quad |\partial_t b_\kappa(\mathbf{x}, t)| \leq \frac{1}{t} C_{f,\mathbf{g}}.$$

Proof. By Lemma 3.5, for $f, g_j \in \mathcal{S}(\mathbb{R}^N)$, $1 \leq j \leq N$, the functions $P_t f, P_t g_j$ belong to $C^\infty(\mathbb{R}^N \times (0, \infty))$. Therefore, by Theorem 4.5 and (4.12), b_κ is a composition of smooth functions, so (A) follows. In order to prove (B), note that by the chain rule we have

$$\partial_t b_\kappa(\mathbf{x}, t) = \langle \nabla B_\kappa(\tilde{u}(\mathbf{x}, t)), \partial_t u(\mathbf{x}, t) \rangle.$$

Consequently, by Proposition 4.4 and the Cauchy–Schwarz inequality we get that there is a constant $C_p > 0$, which depends just on p , such that

$$(4.14) \quad |\partial_t b_\kappa(\mathbf{x}, t)| \leq C_p \|\partial_t u(\mathbf{x}, t)\| (|P_t f(\mathbf{x})|^{p-1} + \|P_t \mathbf{g}(\mathbf{x})\|^{q-1} + \|P_t \mathbf{g}(\mathbf{x})\| + \kappa^{q-1}).$$

Note that by Lemma 3.5 and the fact that $f, g_j \in \mathcal{S}(\mathbb{R}^N)$ there is a constant $C = C_{f,g} > 0$ such for all $\mathbf{x} \in \mathbb{R}^N$, $t > 0$, and $1 \leq j \leq N$ we have

$$|P_t f(\mathbf{x})| \leq C, \quad |P_t g_j(\mathbf{x})| \leq C, \\ \|\partial_t u(\mathbf{x}, t)\| \leq \frac{C}{t},$$

so, by (4.14), the proof of (B) is finished. □

In the next proposition we obtain an explicit formula for $\Delta_k b_\kappa$ (cf. [27, Section 4]).

Proposition 4.10. *Assume that $f, g_j \in \mathcal{S}(\mathbb{R}^N)$, $1 \leq j \leq N$, and $\kappa \in (0, 1]$. Let u, \tilde{u} , and b_κ be as in Definition 4.8. Then for all $\mathbf{x} \in \mathbb{R}^N$ and $t > 0$ we have*

$$(4.15) \quad \begin{aligned} &(\partial_t^2 + \Delta_{k,\mathbf{x}})b_\kappa(\mathbf{x}, t) \\ &= \langle \text{Hess}(B_\kappa)(\tilde{u}(\mathbf{x}, t))\partial_t u(\mathbf{x}, t), \partial_t u(\mathbf{x}, t) \rangle \\ &\quad + \sum_{j=1}^N \langle \text{Hess}(B_\kappa)(\tilde{u}(\mathbf{x}, t))\partial_{j,\mathbf{x}} u(\mathbf{x}, t), \partial_{j,\mathbf{x}} u(\mathbf{x}, t) \rangle \\ &\quad + \sum_{\alpha \in R} k(\alpha) \int_0^1 s \langle \text{Hess}(B_\kappa)(s\tilde{u}(\mathbf{x}, t) \\ &\quad \quad \quad + (1-s)\tilde{u}(\sigma_\alpha(\mathbf{x}), t))\rho_\alpha u(\mathbf{x}, t), \rho_\alpha u(\mathbf{x}, t) \rangle ds, \end{aligned}$$

where

$$(4.16) \quad \rho_\alpha u(\mathbf{x}, t) := \frac{u(\mathbf{x}, t) - u(\sigma_\alpha(\mathbf{x}), t)}{\langle \mathbf{x}, \alpha \rangle}.$$

Proof. It follows by the chain rule (see, e.g., [18, Lemma 1.4]) that

$$(4.17) \quad \begin{aligned} \partial_t^2 b_\kappa(\mathbf{x}, t) &= \langle \text{Hess}(B_\kappa)(\tilde{u}(\mathbf{x}, t))\partial_t u(\mathbf{x}, t), \partial_t u(\mathbf{x}, t) \rangle \\ &\quad + \langle \nabla B_\kappa(\tilde{u}(\mathbf{x}, t)), \partial_t^2 u(\mathbf{x}, t) \rangle, \end{aligned}$$

and

$$(4.18) \quad \begin{aligned} \Delta_{\mathbf{x}} b_{\kappa}(\mathbf{x}, t) &= \sum_{j=1}^N \langle \text{Hess}(B_{\kappa})(\tilde{u}(\mathbf{x}, t)) \partial_{j,\mathbf{x}} u(\mathbf{x}, t), \partial_{j,\mathbf{x}} u(\mathbf{x}, t) \rangle \\ &\quad + \langle \nabla B_{\kappa}(\tilde{u}(\mathbf{x}, t)), \Delta_{\mathbf{x}} u(\mathbf{x}, t) \rangle. \end{aligned}$$

Moreover, for $\alpha \in R$ we have

$$(4.19) \quad \frac{\partial_{\alpha,\mathbf{x}} b_{\kappa}(\mathbf{x}, t)}{\langle \mathbf{x}, \alpha \rangle} = \left\langle \nabla B_{\kappa}(\tilde{u}(\mathbf{x}, t)), \frac{\partial_{\alpha,\mathbf{x}} u(\mathbf{x}, t)}{\langle \mathbf{x}, \alpha \rangle} \right\rangle.$$

Recall that, by the definition of the Dunkl Poisson semigroup (see Definition 3.1), $(\partial_t^2 + \Delta_{k,\mathbf{x}})u(\mathbf{x}, t) = 0$. Therefore, by (2.14), (4.17), (4.18), and (4.19) we get

$$\begin{aligned} (\partial_t^2 + \Delta_{k,\mathbf{x}})b_{\kappa}(\mathbf{x}, t) &= \langle \text{Hess}(B_{\kappa})(\tilde{u}(\mathbf{x}, t)) \partial_t u(\mathbf{x}, t), \partial_t u(\mathbf{x}, t) \rangle \\ &\quad + \sum_{j=1}^N \langle \text{Hess}(B_{\kappa})(\tilde{u}(\mathbf{x}, t)) \partial_{j,\mathbf{x}} u(\mathbf{x}, t), \partial_{j,\mathbf{x}} u(\mathbf{x}, t) \rangle \\ &\quad + \sum_{\alpha \in R} k(\alpha) \left\langle \nabla B_{\kappa}(\tilde{u}(\mathbf{x}, t)), \frac{u(\mathbf{x}, t) - u(\sigma_{\alpha}(\mathbf{x}), t)}{\langle \mathbf{x}, \alpha \rangle^2} \right\rangle \\ &\quad - \sum_{\alpha \in R} k(\alpha) \frac{b_{\kappa}(\mathbf{x}, t) - b_{\kappa}(\sigma_{\alpha}(\mathbf{x}), t)}{\langle \mathbf{x}, \alpha \rangle^2}. \end{aligned}$$

Finally, note that by the Taylor’s expansion of the function $b_{\kappa}(\mathbf{x}, t)$, for all $\alpha \in R$ we have

$$\begin{aligned} &\left\langle \nabla B_{\kappa}(\tilde{u}(\mathbf{x}, t)), \frac{u(\mathbf{x}, t) - u(\sigma_{\alpha}(\mathbf{x}), t)}{\langle \mathbf{x}, \alpha \rangle^2} \right\rangle - \frac{b_{\kappa}(\mathbf{x}, t) - b_{\kappa}(\sigma_{\alpha}(\mathbf{x}), t)}{\langle \mathbf{x}, \alpha \rangle^2} \\ &= \int_0^1 s \langle \text{Hess}(B_{\kappa})(s\tilde{u}(\mathbf{x}, t) + (1-s)\tilde{u}(\sigma_{\alpha}(\mathbf{x}), t)) \rho_{\alpha} u(\mathbf{x}, t), \rho_{\alpha} u(\mathbf{x}, t) \rangle ds, \end{aligned}$$

so the proof is finished. □

Corollary 4.11. *Assume that $f, g_j \in \mathcal{S}(\mathbb{R}^N)$, $1 \leq j \leq N$. There is a constant $C = C_{f,g} > 0$ such that for all $\mathbf{x} \in \mathbb{R}^N$, $\kappa \in (0, 1]$, and $t > 0$ we have*

$$|\Delta_{k,\mathbf{x}}(b_{\kappa})(\mathbf{x}, t)| \leq \frac{C_{f,g}}{t^2}.$$

Proof. Since $f, g_j \in \mathcal{S}(\mathbb{R}^N)$, $1 \leq j \leq N$, by Lemma 3.5 there is a constant $C > 0$, which depends on f, g_j , such that for all $\mathbf{x} \in \mathbb{R}^N$ and $t > 0$ we have

$$|P_t f(\mathbf{x})| \leq C, \quad \|P_t \mathbf{g}(\mathbf{x})\| \leq C.$$

Consequently, by the fact that ∇B_κ and $\text{Hess}(B_\kappa)$ are smooth and by (4.12), we obtain that there is a constant $C' = C'_{f,g} > 0$ such that for all $\mathbf{x} \in \mathbb{R}^N$, $t > 0$, $\alpha \in R$, and $s \in [0, 1]$ we have

$$\|\nabla B_\kappa(\tilde{u}(\mathbf{x}, t))\| \leq C', \quad \|\text{Hess}(B_\kappa)(s\tilde{u}(\mathbf{x}, t) + (1 - s)\tilde{u}(\sigma_\alpha(\mathbf{x}), t))\|_{\text{HS}} \leq C',$$

where $\|\cdot\|_{\text{HS}}$ is the Hilbert–Schmidt norm. Moreover, by Lemma 3.5, there is a constant $C'' = C''_{f,g} > 0$ such that for all $\mathbf{x} \in \mathbb{R}^N$ and $t > 0$ we have

$$|\partial_t u(\mathbf{x}, t)| \leq \frac{C''}{t}, \quad |\partial_t^2 u(\mathbf{x}, t)| \leq \frac{C''}{t^2}.$$

Recall that for all $\mathbf{x} \in \mathbb{R}^N$ and $\alpha \in R$ we have $\sqrt{2}|\langle \mathbf{x}, \alpha \rangle| = \|\mathbf{x} - \sigma_\alpha(\mathbf{x})\|$ (see (2.1)). Hence, by (3.6) and the mean value theorem we have that there is a constant $C''' = C'''_{f,g} > 0$ such that for all $\mathbf{x} \in \mathbb{R}^N$ and $t > 0$ we have

$$|\rho_\alpha u(\mathbf{x}, t)| \leq \frac{C'''}{t}.$$

Finally, the claim is a consequence of (4.15), (4.17), and the Cauchy–Schwarz inequality. □

5 Proof of Theorem 1.3

In this section, we prove Theorem 1.3. We closely follow the reasoning from [10] and [20].

5.1 Definition of $I(n, \varepsilon, \kappa)$. The proof of Theorem 1.3 is based on the upper and lower estimates of the quantities $I(n, \varepsilon, \kappa)$, which approximate the integral

$$\int_{\mathbb{R}^N} \int_0^\infty t(\partial_t^2 + \Delta_{k,\mathbf{x}})b_\kappa(\mathbf{x}, t) dt d\omega(\mathbf{x}).$$

Definition 5.1. Let $\Phi \in C_c^\infty(\mathbb{R}^N)$ be a radial radially decreasing function such that $\text{supp } \Phi \subseteq B(0, 2)$, $0 \leq \Phi \leq 1$, and $\Phi(\mathbf{x}) = 1$ for all $\mathbf{x} \in B(0, 1)$. The function Φ will be fixed throughout the paper.

For $a > 0$ we define the function $v_a: (0, \infty) \rightarrow (0, \infty)$ by the formula

$$(5.1) \quad v_a(t) := t \exp(-a(t + t^{-1})).$$

Let $n \in \mathbb{N}$ and $\varepsilon > 0$. For a function $\kappa: \mathbb{N} \rightarrow (0, 1]$ we set

$$(5.2) \quad I(n, \varepsilon, \kappa) := \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty v_\varepsilon(t)(\partial_t^2 + \Delta_{k,\mathbf{x}})(b_{\kappa(n)})(\mathbf{x}, t) dt d\omega(\mathbf{x}).$$

5.2 Lower estimate of $I(n, \varepsilon, \kappa)$.

Lemma 5.2. *Assume that $f, g_j \in \mathcal{S}(\mathbb{R}^N)$, $1 \leq j \leq N$. Then for any $\kappa: \mathbb{N} \rightarrow (0, 1]$ we have*

$$(5.3) \quad \sum_{j=1}^N \int_{\mathbb{R}^N} \int_0^\infty t |\partial_t P_t g_j(\mathbf{x}) T_j P_t f(\mathbf{x})| dt d\omega(\mathbf{x}) \leq \frac{2}{\gamma} \left(\sum_{\alpha \in R} k(\alpha) + 2^7 \right) \liminf_{\varepsilon \rightarrow 0^+} \liminf_{n \rightarrow \infty} I(n, \varepsilon, \kappa).$$

Moreover, if the function f is G -invariant, then

$$(5.4) \quad \sum_{j=1}^N \int_{\mathbb{R}^N} \int_0^\infty t |\partial_t P_t g_j(\mathbf{x}) T_j P_t f(\mathbf{x})| dt d\omega(\mathbf{x}) \leq \frac{2}{\gamma} \liminf_{\varepsilon \rightarrow 0^+} \liminf_{n \rightarrow \infty} I(n, \varepsilon, \kappa).$$

Proof. Let us prove (5.3) first. Fix $\kappa: \mathbb{N} \rightarrow (0, 1]$. By the monotone convergence theorem we have

$$\begin{aligned} & \sum_{j=1}^N \int_{\mathbb{R}^N} \int_0^\infty t |\partial_t P_t g_j(\mathbf{x}) T_j P_t f(\mathbf{x})| dt d\omega(\mathbf{x}) \\ &= \lim_{\varepsilon \rightarrow 0^+} \lim_{n \rightarrow \infty} \sum_{j=1}^N \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty v_\varepsilon(t) |\partial_t P_t g_j(\mathbf{x}) T_j P_t f(\mathbf{x})| dt d\omega(\mathbf{x}). \end{aligned}$$

Fix $n \in \mathbb{N}$ and $\varepsilon > 0$. By the definition of T_j (see (2.6)) we get

$$(5.5) \quad \begin{aligned} & \sum_{j=1}^N \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty v_\varepsilon(t) |\partial_t P_t g_j(\mathbf{x}) T_j P_t f(\mathbf{x})| dt d\omega(\mathbf{x}) \\ & \leq \sum_{j=1}^N \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty v_\varepsilon(t) |\partial_t P_t g_j(\mathbf{x}) \partial_{j,\mathbf{x}} P_t f(\mathbf{x})| dt d\omega(\mathbf{x}) \\ & \quad + \sum_{j=1}^N \sum_{\alpha \in R} \frac{k(\alpha)}{2} |\alpha_j| \\ & \quad \times \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty v_\varepsilon(t) \left| \partial_t P_t g_j(\mathbf{x}) \frac{P_t f(\mathbf{x}) - P_t f(\sigma_\alpha(\mathbf{x}))}{\langle \mathbf{x}, \alpha \rangle} \right| dt d\omega(\mathbf{x}) \\ & =: I_1 + I_2. \end{aligned}$$

We will estimate I_1 and I_2 separately.

Estimate of I_1 . In order to estimate I_1 , for $\mathbf{y} = (y_1, \mathbf{y}_2) \in \mathbb{R} \times \mathbb{R}^N$ we set

$$(5.6) \quad \tau_1(\mathbf{y}) = \tau_1(y_1, \mathbf{y}_2) = \|\mathbf{y}_2\|^{2-q}.$$

Recall that $\phi_{\kappa(n)}$ for $n \in \mathbb{N}$ is defined in (4.3). By the fact that $\int_{\mathbb{R}^{N+1}} \phi_{\kappa(n)}(\mathbf{y}) \, d\mathbf{y} = 1$, the inequality between the arithmetic and geometric mean, and (4.6) with

$$\begin{cases} N_1 = 1, N_2 = N, \\ \eta = P_t f(\mathbf{x}), \\ \zeta = P_t \mathbf{g}(\mathbf{x}), \\ \omega = \partial_t u(\mathbf{x}, t) \text{ or } \omega = \partial_{j,\mathbf{x}} u(\mathbf{x}, t) \end{cases}$$

(see (4.10)) we get

$$\begin{aligned} & \sum_{j=1}^N \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty v_\varepsilon(t) |\partial_t P_t g_j(\mathbf{x}) \partial_{j,\mathbf{x}} P_t f(\mathbf{x})| \, dt \, d\omega(\mathbf{x}) \\ &= \sum_{j=1}^N \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty v_\varepsilon(t) \\ & \quad \times \left(\int_{\mathbb{R}^{N+1}} \phi_{\kappa(n)}(u(\mathbf{x}, t) - \mathbf{y}) \, d\mathbf{y} \right) |\partial_t P_t g_j(\mathbf{x}) \partial_{j,\mathbf{x}} P_t f(\mathbf{x})| \, dt \, d\omega(\mathbf{x}) \\ &\leq \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty v_\varepsilon(t) \\ & \quad \times \left(\int_{\mathbb{R}^{N+1}} \phi_{\kappa(n)}(u(\mathbf{x}, t) - \mathbf{y}) \left(\tau_1(\mathbf{y})^{-1} \left(\sum_{j=1}^N |\partial_t P_t g_j(\mathbf{x})|^2 \right) \right) \, d\mathbf{y} \right) dt \, d\omega(\mathbf{x}) \\ (5.7) \quad & + \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty v_\varepsilon(t) \\ & \quad \times \left(\int_{\mathbb{R}^{N+1}} \phi_{\kappa(n)}(u(\mathbf{x}, t) - \mathbf{y}) \left(\tau_1(\mathbf{y}) \left(\sum_{j=1}^N |\partial_{j,\mathbf{x}} P_t f(\mathbf{x})|^2 \right) \right) \, d\mathbf{y} \right) dt \, d\omega(\mathbf{x}) \\ &\leq \frac{2}{\gamma} \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty v_\varepsilon(t) \langle \text{Hess } B_{\kappa(n)}(\tilde{u}(\mathbf{x}, t)) \partial_t u_t(\mathbf{x}), \partial_t u_t(\mathbf{x}) \rangle \, dt \, d\omega(\mathbf{x}) \\ & \quad + \sum_{j=1}^N \frac{2}{\gamma} \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \\ & \quad \times \int_0^\infty v_\varepsilon(t) \langle \text{Hess } B_{\kappa(n)}(\tilde{u}(\mathbf{x}, t)) \partial_{j,\mathbf{x}} u_t(\mathbf{x}), \partial_{j,\mathbf{x}} u_t(\mathbf{x}) \rangle \, dt \, d\omega(\mathbf{x}). \end{aligned}$$

Estimate of I_2 . In order to estimate I_2 , by the fact that $\|\alpha\| = \sqrt{2}$,

$$\int_{\mathbb{R}^{N+1}} \phi_{\kappa(n)}(\mathbf{y}) \, d\mathbf{y} = 1,$$

and the inequality between the arithmetic and geometric mean, we have

$$\begin{aligned} & \sum_{j=1}^N \sum_{\alpha \in R} \frac{k(\alpha)}{2} |\alpha_j| \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty v_\varepsilon(t) \left| \partial_t P_t g_j(\mathbf{x}) \frac{P_t f(\mathbf{x}) - P_t f(\sigma_\alpha(\mathbf{x}))}{\langle \mathbf{x}, \alpha \rangle} \right| dt \, d\mathbf{w}(\mathbf{x}) \\ & \leq \sum_{\alpha \in R} k(\alpha) \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty v_\varepsilon(t) \\ & \quad \times \left(\sum_{j=1}^N |\partial_t P_t g_j(\mathbf{x})|^2 \right)^{1/2} \left| \frac{P_t f(\mathbf{x}) - P_t f(\sigma_\alpha(\mathbf{x}))}{\langle \mathbf{x}, \alpha \rangle} \right| dt \, d\mathbf{w}(\mathbf{x}) \\ (5.8) \quad & \leq \sum_{\alpha \in R} k(\alpha) \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty v_\varepsilon(t) \\ & \quad \times \left(\int_{\mathbb{R}^{N+1}} \phi_{\kappa(n)}(u(\mathbf{x}, t) - \mathbf{y}) \tau_1(\mathbf{y})^{-1} \, d\mathbf{y} \right) \left(\sum_{j=1}^N |\partial_t P_t g_j(\mathbf{x})|^2 \right) dt \, d\mathbf{w}(\mathbf{x}) \\ & \quad + \sum_{\alpha \in R} k(\alpha) \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty v_\varepsilon(t) \left(\int_{\mathbb{R}^{N+1}} \phi_{\kappa(n)}(u(\mathbf{x}, t) - \mathbf{y}) \tau_1(\mathbf{y}) \, d\mathbf{y} \right) \\ & \quad \times \left| \frac{P_t f(\mathbf{x}) - P_t f(\sigma_\alpha(\mathbf{x}))}{\langle \mathbf{x}, \alpha \rangle} \right|^2 dt \, d\mathbf{w}(\mathbf{x}) \\ & =: I_{2,1} + I_{2,2}. \end{aligned}$$

The summand $I_{2,1}$ is the same as the first summand in (5.7), but it is multiplied by $\sum_{\alpha \in R} k(\alpha)$. Recall that $\phi_{\kappa(n)} \geq 0$. In order to estimate $I_{2,2}$ we write

$$\begin{aligned} & \int_{\mathbb{R}^{N+1}} \phi_{\kappa(n)}(u(\mathbf{x}, t) - \mathbf{y}) \tau_1(\mathbf{y}) \, d\mathbf{y} \\ & \leq \int_{\mathbb{R}^{N+1}} (\phi_{\kappa(n)}(u(\mathbf{x}, t) - \mathbf{y}) + \phi_{\kappa(n)}(u(\sigma_\alpha(\mathbf{x}), t) - \mathbf{y})) \tau_1(\mathbf{y}) \, d\mathbf{y} \\ & = \int_{\mathbb{R}^{N+1}} \phi_{\kappa(n)}(\mathbf{y}) (\tau_1(u(\mathbf{x}, t) - \mathbf{y}) + \tau_1(u(\sigma_\alpha(\mathbf{x}), t) - \mathbf{y})) \, d\mathbf{y} \\ & \leq 2 \int_{\mathbb{R}^{N+1}} \phi_{\kappa(n)}(\mathbf{y}) \max(\tau_1(u(\mathbf{x}, t) - \mathbf{y}), \tau_1(u(\sigma_\alpha(\mathbf{x}), t) - \mathbf{y})) \, d\mathbf{y}, \end{aligned}$$

then use (4.8) with $\mathbf{a} = u(\mathbf{x}, t) - \mathbf{y}$ and $\mathbf{b} = u(\sigma_\alpha(\mathbf{x}), t) - \mathbf{y}$. Consequently,

$$\begin{aligned} & \int_{\mathbb{R}^{N+1}} \phi_{\kappa(n)}(u(\mathbf{x}, t) - \mathbf{y}) \tau_1(\mathbf{y}) \, d\mathbf{y} \\ & \leq 2^7 \int_{\mathbb{R}^{N+1}} \phi_{\kappa(n)}(\mathbf{y}) \int_0^1 s \tau_1(su(\mathbf{x}, t) + (1-s)u(\sigma_\alpha(\mathbf{x}), t) - \mathbf{y}) \, ds \, d\mathbf{y}, \end{aligned}$$

which leads us to

$$\begin{aligned} & \sum_{\alpha \in R} k(\alpha) \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty v_\varepsilon(t) \left(\int_{\mathbb{R}^{N+1}} \phi_{\kappa(n)}(u(\mathbf{x}, t) - \mathbf{y}) \tau_1(\mathbf{y}) d\mathbf{y} \right) \\ & \qquad \qquad \qquad \times \left| \frac{P_t f(\mathbf{x}) - P_t f(\sigma_\alpha(\mathbf{x}))}{\langle \mathbf{x}, \alpha \rangle} \right|^2 dt d\omega(\mathbf{x}) \\ & \leq 2^7 \sum_{\alpha \in R} k(\alpha) \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty v_\varepsilon(t) \\ & \qquad \qquad \qquad \times \left(\int_0^1 s(\tau_1 \star \phi_{\kappa(n)})(su(\mathbf{x}, t) + (1-s)u(\sigma_\alpha(\mathbf{x}), t)) ds \right) \\ & \qquad \qquad \qquad \times \left| \frac{P_t f(\mathbf{x}) - P_t f(\sigma_\alpha(\mathbf{x}))}{\langle \mathbf{x}, \alpha \rangle} \right|^2 dt d\omega(\mathbf{x}). \end{aligned}$$

Now, by (4.6) with

$$\begin{cases} N_1 = 1, N_2 = N, \\ \eta = sP_t f(\mathbf{x}) + (1-s)P_t f(\sigma_\alpha(\mathbf{x})), \\ \zeta = sP_t \mathbf{g}(\mathbf{x}) + (1-s)P_t \mathbf{g}(\sigma_\alpha(\mathbf{x})), \\ \omega = \rho_\alpha u(\mathbf{x}, t), \end{cases}$$

where $s \in [0, 1]$ and $\alpha \in R$ (see (4.10), (4.11), and (4.16)) we get

$$\begin{aligned} & 2^7 \sum_{\alpha \in R} k(\alpha) \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty v_\varepsilon(t) \\ & \qquad \qquad \qquad \times \left(\int_0^1 s(\tau_1 \star \phi_{\kappa(n)})(su(\mathbf{x}, t) + (1-s)u(\sigma_\alpha(\mathbf{x}), t)) ds \right) \\ (5.9) \qquad \qquad \qquad & \qquad \qquad \qquad \times \left| \frac{P_t f(\mathbf{x}) - P_t f(\sigma_\alpha(\mathbf{x}))}{\langle \mathbf{x}, \alpha \rangle} \right|^2 dt d\omega(\mathbf{x}) \\ & \leq \frac{2^8}{\gamma} \sum_{\alpha \in R} k(\alpha) \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty v_\varepsilon(t) \\ & \qquad \qquad \qquad \times \left(\int_0^1 s(\text{Hess}(s\tilde{u}(\mathbf{x}, t) + (1-s)\tilde{u}(\mathbf{x}, t))\rho_\alpha u(\mathbf{x}, t), \rho_\alpha u(\mathbf{x}, t)) ds \right) dt d\omega(\mathbf{x}). \end{aligned}$$

Now the claim is a direct consequence of (5.7), (5.8), (5.9), and Proposition 4.10.

Finally, in order to prove (5.4), note that, by the definition of the Poisson semi-group (see Definition 3.1) and Lemma 3.3 (D), for G -invariant f , the function $P_t f$ is also G -invariant for all $t > 0$. Therefore, for all $1 \leq j \leq N$ we have

$$T_j P_t f = \partial_j P_t f$$

and the summand I_2 in (5.5) is equal to zero. Moreover, by (4.6), the third summand in (4.15) is nonnegative, so (5.4) follows. □

5.3 Upper estimate of $I(n, \varepsilon, \kappa)$.

Lemma 5.3. *Let $p \geq 2$ and $q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $f, g_j \in \mathcal{S}(\mathbb{R}^N)$, $1 \leq j \leq N$, and $\varepsilon > 0$. For $n \in \mathbb{N}$ we set*

$$(5.10) \quad \kappa(n) = \left(\frac{1}{n} \max(1, w(B(0, 2n)))^{-1} \right)^{1/q}.$$

Then we have

$$(5.11) \quad \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty v_\varepsilon(t) \Delta_{k,\mathbf{x}}(b_{\kappa(n)})(\mathbf{x}, t) dt d\omega(\mathbf{x}) = 0.$$

Proof. Recall that $\text{supp } \Phi \subseteq B(0, 2)$. Therefore, by Lemma 4.9 (A), Corollary 4.11, and the fact that for fixed $\varepsilon > 0$ we have $\int_0^\infty t^{-2} v_\varepsilon(t) dt < \infty$, we can change the order of integration in (5.11). Note that by the fact that $\int_0^\infty t^{-2} |v_\varepsilon(t)| dt < \infty$, it is enough to show that for fixed $t > 0$ we have

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \Delta_{k,\mathbf{x}}(b_{\kappa(n)})(\mathbf{x}, t) d\omega(\mathbf{x}) = 0.$$

Recall that $b_{\kappa(n)} \in C^\infty(\mathbb{R}^N \times (0, \infty))$. Integrating by parts (see Theorem 2.1 and Remark 2.2), for any $t > 0$ we get

$$\int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \Delta_{k,\mathbf{x}}(b_{\kappa(n)})(\mathbf{x}, t) d\omega(\mathbf{x}) = \int_{\mathbb{R}^N} \Delta_k \Phi(\mathbf{x}/n)(b_{\kappa(n)})(\mathbf{x}, t) d\omega(\mathbf{x}).$$

Recall that $\text{supp } \Phi(\cdot/n) \subseteq B(0, 2n)$. Then, it follows from Lemma 2.3 that there is a constant $C > 0$ independent of Φ and n such that for all $\mathbf{x} \in \mathbb{R}^N$ and $n \in \mathbb{N}$ we have

$$(5.12) \quad \begin{aligned} |(\Delta_k \Phi)(\mathbf{x}/n)| &\leq C \sup_{\mathbf{y} \in \mathbb{R}^N} \sum_{\beta \in \mathbb{N}_0^N, |\beta| \leq 2} |\partial_{\mathbf{x}}^\beta \Phi(\mathbf{y}/n)| \\ &\leq C \sum_{\beta \in \mathbb{N}_0^N, |\beta| \leq 2} \|\partial^\beta \Phi\|_{L^\infty} \leq C_\Phi. \end{aligned}$$

Moreover, by (2.14) and the fact that $\Phi(\mathbf{x}/n) = 1$ for all $\mathbf{x} \in B(0, n)$ we have

$$\Delta_k \Phi(\mathbf{x}/n) = 0 \quad \text{for all } \mathbf{x} \in B(0, n).$$

Consequently, by (4.5) there is a constant $C_p > 0$, which depends just on p , such that for all $n \in \mathbb{N}$ we have

$$(5.13) \quad \left| \int_{\mathbb{R}^N} (\Delta_k \Phi)(\mathbf{x}/n)(b_{\kappa(n)})(\mathbf{x}, t) d\omega(\mathbf{x}) \right| \leq C_p C_\Phi \int_{B(0, 2n) \setminus B(0, n)} (|P_t f(\mathbf{x})|^p + \|P_t \mathbf{g}(\mathbf{x})\|^q + \kappa(n)^q) d\omega(\mathbf{x}).$$

Since $f, g_j \in \mathcal{S}(\mathbb{R}^N)$, by Lemma 3.5 we get $P_t f \in L^p(dw)$ and $P_t g_j \in L^q(dw)$ for all $t > 0$ and $1 \leq j \leq N$. Hence,

$$\lim_{n \rightarrow \infty} \left(\int_{B(0,2n) \setminus B(0,n)} (|P_t f(\mathbf{x})|^p + \|P_t \mathbf{g}(\mathbf{x})\|^q) dw(\mathbf{x}) \right) = 0.$$

Moreover, by the choice of $\kappa(n)$ (see (5.10)) we get

$$\lim_{n \rightarrow \infty} \left(\int_{B(0,2n) \setminus B(0,n)} \kappa(n)^q dw(\mathbf{x}) \right) = \lim_{n \rightarrow \infty} \left(\frac{1}{n} \frac{w(B(0, 2n) \setminus B(0, n))}{w(B(0, 2n))} \right) = 0.$$

Therefore,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \Delta_{k,\mathbf{x}}(b_{\kappa(n)})(\mathbf{x}, t) dw(\mathbf{x}) \\ = \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} \Delta_k \Phi(\mathbf{x}/n)(b_{\kappa(n)})(\mathbf{x}, t) dw(\mathbf{x}) = 0. \end{aligned} \quad \square$$

Lemma 5.4. *Assume that $f, g_j \in \mathcal{S}(\mathbb{R}^N)$, $1 \leq j \leq N$, $\kappa \in (0, 1]$, and $\varepsilon > 0$. Then for all $\mathbf{x} \in \mathbb{R}^N$ we have*

$$(5.14) \quad \lim_{t \rightarrow 0} v_\varepsilon(t) |\partial_t b_\kappa(\mathbf{x}, t)| = 0, \quad \lim_{t \rightarrow \infty} v_\varepsilon(t) |\partial_t b_\kappa(\mathbf{x}, t)| = 0,$$

$$(5.15) \quad \lim_{t \rightarrow 0} |v'_\varepsilon(t)| |b_\kappa(\mathbf{x}, t)| = 0, \quad \lim_{t \rightarrow \infty} |v'_\varepsilon(t)| |b_\kappa(\mathbf{x}, t)| = 0.$$

Recall that v_ε is defined in (5.1).

Proof. Note that (5.14) is a consequence of Lemma 4.9 (B) and the fact that for fixed $\varepsilon > 0$ we have

$$\lim_{t \rightarrow 0} \frac{1}{t} v_\varepsilon(t) = \lim_{t \rightarrow \infty} \frac{1}{t} v_\varepsilon(t) = 0.$$

The proof of (5.15) is similar. Indeed, since $f, g_j \in \mathcal{S}(\mathbb{R}^N)$, by (3.6) there is a constant $C = C_{f,g} > 0$ such that for all $\mathbf{x} \in \mathbb{R}^N$ and $t > 0$ we have

$$|P_t f(\mathbf{x})| \leq C \quad \text{and} \quad |P_t g_j(\mathbf{x})| \leq C.$$

Consequently, by (4.5), there is a constant $C' > 0$, which depends on f and g_j and is independent of $\kappa \in (0, 1]$, such that for all $\mathbf{x} \in \mathbb{R}^N$ and $t > 0$ we have

$$0 \leq b_\kappa(\mathbf{x}, t) \leq C',$$

so the claim is a consequence of an elementary fact that for fixed $\varepsilon > 0$ we have

$$\lim_{t \rightarrow 0} |v'_\varepsilon(t)| = \lim_{t \rightarrow \infty} |v'_\varepsilon(t)| = 0. \quad \square$$

Lemma 5.5. *Recall that v_ε is defined in (5.1). We have*

$$\limsup_{\varepsilon \rightarrow 0^+} \int_0^\infty |v_\varepsilon''(t)| dt \leq 2(1 + e^{-2}).$$

Proof. It follows from an elementary calculation (see, e.g., [20, (3.30)]). \square

Lemma 5.6. *Let $p \geq 2$ and $q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $f, g_j \in \mathcal{S}(\mathbb{R}^N)$, $1 \leq j \leq N$, and $\varepsilon > 0$. For $n \in \mathbb{N}$ we set*

$$(5.16) \quad \kappa(n) = \left(\frac{1}{n} \max(1, w(B(0, 2n))) \right)^{1/q}.$$

Then we have

$$(5.17) \quad \limsup_{\varepsilon \rightarrow 0^+} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty v_\varepsilon(t) \partial_t^2 (b_{\kappa(n)})(\mathbf{x}, t) dt d\mathbf{w}(\mathbf{x}) \leq 3(1 + \gamma)(\|f\|_{L^p(dw)}^p + \|\mathbf{g}\|_{L^q(dw)}^q).$$

Proof. Integrating by parts with respect to t without boundary terms (it is possible thanks to Lemma 5.4) we get

$$(5.18) \quad \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty v_\varepsilon(t) \partial_t^2 (b_{\kappa(n)})(\mathbf{x}, t) dt d\mathbf{w}(\mathbf{x}) = \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty v_\varepsilon''(t) (b_{\kappa(n)})(\mathbf{x}, t) dt d\mathbf{w}(\mathbf{x}).$$

Then, by (4.5), we have

$$\left| \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty v_\varepsilon''(t) (b_{\kappa(n)})(\mathbf{x}, t) dt d\mathbf{w}(\mathbf{x}) \right| \leq (1 + \gamma) \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty |v_\varepsilon''(t)| ((|P_t f(\mathbf{x})| + \kappa(n))^p + (\|P_t \mathbf{g}(\mathbf{x})\| + \kappa(n))^q) dt d\mathbf{w}(\mathbf{x}).$$

For fixed $t > 0$ let $A_t := \{\mathbf{x} \in \mathbb{R}^N : \varepsilon |P_t f(\mathbf{x})| \geq \kappa(n)\}$. Then

$$\begin{aligned} & \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty |v_\varepsilon''(t)| ((|P_t f(\mathbf{x})| + \kappa(n))^p) dt d\mathbf{w}(\mathbf{x}) \\ &= \int_0^\infty \int_{A_t} \cdots + \int_0^\infty \int_{A_t^c} \cdots \\ &\leq (1 + \varepsilon)^p \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty |v_\varepsilon''(t)| |P_t f(\mathbf{x})|^p dt d\mathbf{w}(\mathbf{x}) \\ &\quad + (1 + \varepsilon^{-1})^p \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty |v_\varepsilon''(t)| |\kappa(n)|^p dt d\mathbf{w}(\mathbf{x}). \end{aligned}$$

Recall that $\text{supp } \Phi(\cdot/n) \subseteq B(0, 2n)$ and $0 \leq \Phi(\cdot/n) \leq 1$ for all $n \in \mathbb{N}$. Consequently, by the choice of $\kappa(n)$ (see (5.10)) we get

$$\begin{aligned} & \limsup_{n \rightarrow \infty} (1 + \varepsilon^{-1})^p \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty |v_\varepsilon''(t)| |\kappa(n)|^p dt d\mathbf{w}(\mathbf{x}) \\ & \leq \lim_{n \rightarrow \infty} \left(\int_0^\infty |v_\varepsilon''(t)| dt \right) (1 + \varepsilon^{-1})^p \omega(B(0, 2n)) \frac{1}{n^{p/q} \omega(B(0, 2n))^{p/q}} = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} (5.19) \quad & \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty |v_\varepsilon''(t)| (|P_t f(\mathbf{x})| + \kappa(n))^p dt d\mathbf{w}(\mathbf{x}) \\ & = (1 + \varepsilon)^p \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty |v_\varepsilon''(t)| |P_t f(\mathbf{x})|^p dt d\mathbf{w}(\mathbf{x}). \end{aligned}$$

Similarly,

$$\begin{aligned} (5.20) \quad & \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty |v_\varepsilon''(t)| (\|P_t \mathbf{g}(\mathbf{x})\| + \kappa(n))^q dt d\mathbf{w}(\mathbf{x}) \\ & = \limsup_{n \rightarrow \infty} (1 + \varepsilon)^q \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty |v_\varepsilon''(t)| \|P_t \mathbf{g}(\mathbf{x})\|^q dt d\mathbf{w}(\mathbf{x}). \end{aligned}$$

Note that by Hölder's inequality and Lemma 3.3 (B) and (C) for all $\mathbf{x} \in \mathbb{R}^N$ and $t > 0$ we have

$$|P_t f(\mathbf{x})|^p \leq P_t(|f(\cdot)|^p)(\mathbf{x}), \quad \|P_t \mathbf{g}(\mathbf{x})\|^q \leq P_t(\|\mathbf{g}(\cdot)\|^q)(\mathbf{x}).$$

Furthermore, by Lemma 3.3 we have

$$\int_{\mathbb{R}^N} |P_t f(\mathbf{x})|^p d\mathbf{w}(\mathbf{x}) \leq \|f\|_{L^p(dw)}^p, \quad \int_{\mathbb{R}^N} P_t(\|\mathbf{g}(\cdot)\|^q)(\mathbf{x}) d\mathbf{w}(\mathbf{x}) \leq \|\mathbf{g}\|_{L^q(dw)}^q.$$

Consequently, by the Fubini theorem, the fact that $0 \leq \Phi(\cdot/n) \leq 1$, and Lemma 4.9, we get

$$\begin{aligned} (5.21) \quad & \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty |v_\varepsilon''(t)| (|P_t f(\mathbf{x})|^p + \|P_t \mathbf{g}(\mathbf{x})\|^q) dt d\mathbf{w}(\mathbf{x}) \\ & \leq \limsup_{n \rightarrow \infty} \int_0^\infty |v_\varepsilon''(t)| \int_{\mathbb{R}^N} (|P_t f(\mathbf{x})|^p + \|P_t \mathbf{g}(\mathbf{x})\|^q) d\mathbf{w}(\mathbf{x}) dt \\ & \leq \left(\int_0^\infty |v_\varepsilon''(t)| dt \right) (\|f\|_{L^p(dw)}^p + \|\mathbf{g}\|_{L^q(dw)}^q). \end{aligned}$$

Finally, by (5.18), (5.19), (5.20), (5.21), and Lemma 5.5 we obtain

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0^+} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} \Phi(\mathbf{x}/n) \int_0^\infty v_\varepsilon(t) \partial_t^2 (b_{\kappa(n)})(\mathbf{x}, t) dt d\mathbf{w}(\mathbf{x}) \\ & \leq (1 + \gamma) 2(1 + e^{-2}) (\|f\|_{L^p}^p + \|\mathbf{g}\|_{L^q}^q) \\ & \leq 3(1 + \gamma) (\|f\|_{L^p(dw)}^p + \|\mathbf{g}\|_{L^q(dw)}^q). \end{aligned}$$

□

As a direct consequence of Lemmas 5.3 and 5.6 we obtain the following corollary.

Corollary 5.7. *Let $p \geq 2$ and $q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $f, g_j \in \mathcal{S}(\mathbb{R}^N)$, $1 \leq j \leq N$. For $n \in \mathbb{N}$ we set*

$$(5.22) \quad \kappa(n) = \left(\frac{1}{n} \max(1, w(B(0, 2n))) \right)^{-1/q}.$$

Then we have

$$\liminf_{\varepsilon \rightarrow 0^+} \liminf_{n \rightarrow \infty} I(n, \varepsilon, \kappa) \leq 3(1 + \gamma)(\|f\|_{L^p(dw)}^p + \|\mathbf{g}\|_{L^q(dw)}^q),$$

where $I(n, \varepsilon, \kappa)$ is defined in (5.2).

5.4 Proof of Theorem 1.3.

Proof of Theorem 1.3. We will prove (1.4) first. Assume first that $p \geq 2$. Take $f \in L^p(dw)$. Thanks to Theorem 1.2 and the fact that $\mathcal{S}(\mathbb{R}^N)$ is dense in $L^p(dw)$, without loss of generality we can assume $f \in \mathcal{S}(\mathbb{R}^N)$. Let $\kappa: \mathbb{N} \rightarrow (0, 1]$ be defined by (5.10). By Corollary 3.7 we get

$$(5.23) \quad \begin{aligned} & \|\mathcal{R}f\|_{L^p(dw)} \\ &= 4 \sup_{g_j \in \mathcal{S}(\mathbb{R}^N), \|\mathbf{g}(y)\|_{L^q(dw(y))} \leq 1} \left| \sum_{j=1}^N \int_{\mathbb{R}^N} \int_0^\infty t \partial_t P_t g_j(\mathbf{x}) T_j P_t f(\mathbf{x}) dt dw(\mathbf{x}) \right|. \end{aligned}$$

Next, by Lemma 5.2 and Corollary 5.7,

$$(5.24) \quad \begin{aligned} & 4 \sup_{g_j \in \mathcal{S}(\mathbb{R}^N), \|\mathbf{g}(y)\|_{L^q(dw(y))} \leq 1} \left| \sum_{j=1}^N \int_{\mathbb{R}^N} \int_0^\infty t \partial_t P_t g_j(\mathbf{x}) T_j P_t f(\mathbf{x}) dt dw(\mathbf{x}) \right| \\ & \leq \frac{8}{\gamma} \left(\sum_{\alpha \in R} k(\alpha) + 2^7 \right) \liminf_{\varepsilon \rightarrow 0^+} \liminf_{n \rightarrow \infty} I(n, \varepsilon, \kappa) \\ & \leq \frac{24(1 + \gamma)}{\gamma} \left(\sum_{\alpha \in R} k(\alpha) + 2^7 \right) (\|f\|_{L^p(dw)}^p + \|\mathbf{g}\|_{L^q(dw)}^q). \end{aligned}$$

Finally, we use a polarization argument. Let $s > 0$. We replace $f(\cdot)$ by $sf(\cdot)$ and $\mathbf{g}(\cdot)$ by $s^{-1}\mathbf{g}(\cdot)$ in (5.24). Then, the left hand side of (5.24) is unchanged, and minimizing the right-hand side by $s > 0$ we obtain

$$\begin{aligned} & \|\mathcal{R}f\|_{L^p(dw)} \\ & \leq \frac{24(1 + \gamma)}{\gamma} ((p/q)^{1/p} + (q/p)^{1/q}) \left(\sum_{\alpha \in R} k(\alpha) + 2^7 \right) \|f\|_{L^p(dw)} \|\mathbf{g}(y)\|_{L^q(dw(y))}. \end{aligned}$$

It was shown in [60, proof of the main theorem] that

$$\frac{(1 + \gamma)}{\gamma}((p/q)^{1/p} + (q/p)^{1/q}) \leq 6(p^* - 1),$$

which ends the proof for $p \geq 2$. The proof in case $1 < p < 2$ is analogous: we switch $P_t f$ and $P_t g$ in the definition of b_κ . The proof of (1.5) is similar (we use (5.4) instead of (5.3) in (5.24)). □

6 One-dimensional case

This section is devoted to the proof of Theorem 1.4. We will work in the one-dimensional setting, i.e., we assume $N = 1$. We would like to emphasize that in this case we have

$$R = \{\sqrt{2}, -\sqrt{2}\}, \quad G = \{\text{id}, \sigma_{-\sqrt{2}}\},$$

where $\sigma_{-\sqrt{2}}(x) = -x$ for all $x \in \mathbb{R}$. Consequently, the multiplicity function k takes just one value, which, for simplicity of the notation, will be denoted by k . In this case, the associated measure $d\omega$ is of the form

$$(6.1) \quad d\omega(x) = 2|x|^{2k} dx.$$

The Dunkl operator in the one-dimensional case is

$$(6.2) \quad Tf(x) := \partial_x f(x) + k \frac{f(x) - f(-x)}{x}.$$

In this section, we will use the same notation as in the previous sections unless specified otherwise. We will also assume $k > 1$ (otherwise, the claim follows by Theorem 1.3).

We will slightly modify the proof of Theorem 1.3 to obtain Theorem 1.4. The main point is to prove a modified version of Lemma 5.2. We will also need the following version of Proposition 3.6. We state and prove it for the convenience of the reader.

Proposition 6.1. *For all $f, g \in \mathcal{S}(\mathbb{R})$ we have*

$$(6.3) \quad \left| \int_{\mathbb{R}} \mathcal{H}f(x)g(x) d\omega(x) \right| = 4 \left| \int_0^\infty \int_{\mathbb{R}} t \partial_t P_t f(x) T P_t g(x) d\omega(x) dt \right|.$$

Proof. By Plancherel’s identity (see (2.10)) and the definition of the Dunkl Hilbert transform (see (1.6)) we have

$$\begin{aligned} \int_{\mathbb{R}} \mathcal{H}f(x)g(x) d\omega(x) &= \int_{\mathbb{R}} \left(\frac{-i\xi}{|\xi|} \mathcal{F}f(\xi) \right) \mathcal{F}g(\xi) d\omega(\xi) \\ &= \int_{\mathbb{R}} \mathcal{F}f(\xi) \left(\frac{-i\xi}{|\xi|} \mathcal{F}g(\xi) \right) d\omega(\xi) = - \int_{\mathbb{R}} \mathcal{H}g(x)f(x) d\omega(x). \end{aligned}$$

Consequently, the rest of the proof is the same as in the proof of Proposition 3.6 (with g instead of f and f instead of g). □

Now we are ready to state and prove the modified version of Lemma 5.2. Recall that $I(n, \varepsilon, \kappa)$ is defined in Subsection 5.1 (see (5.1)).

Lemma 6.2. *Assume that $p \geq 2, q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, and $f, g \in \mathcal{S}(\mathbb{R})$. Assume that g is odd. Then for any $\kappa: \mathbb{N} \rightarrow (0, 1]$ we have*

$$(6.4) \quad \int_{\mathbb{R}} \int_0^\infty t |\partial_t P_t f(x)| |TP_t g(x)| dt dw(x) \leq \frac{8}{\gamma} \liminf_{\varepsilon \rightarrow 0^+} \liminf_{n \rightarrow \infty} I(n, \varepsilon, \kappa) + \liminf_{\varepsilon \rightarrow 0^+} \liminf_{n \rightarrow \infty} e_1(n, \varepsilon, \kappa) + \liminf_{\varepsilon \rightarrow 0^+} \liminf_{n \rightarrow \infty} e_2(n, \varepsilon, \kappa),$$

where

$$(6.5) \quad e_1(n, \varepsilon, \kappa) := 6\kappa(n)^{2-q} \int_{\mathbb{R}} \Phi(x/n) \int_0^\infty v_\varepsilon(t) |\partial_t P_t f(x)|^2 dt dw(x),$$

$$(6.6) \quad e_2(n, \varepsilon, \kappa) := -n^{-1} \int_{\mathbb{R}} (\partial_x \Phi)(x/n) \int_0^\infty \frac{4k}{q} (|P_t g(x)|^2 + \kappa(n)^2)^{q/2} x^{-1} dt dw(x).$$

Remark 6.3. Let us note that in Lemma 6.2 we do not have the factor “ k ” in front of “ $\liminf_{\varepsilon \rightarrow 0^+} \liminf_{n \rightarrow \infty} I(n, \varepsilon, \kappa)$ ”, which appears in Lemma 5.2. This is a crucial difference. However, there are two additional error terms $e_1(n, \varepsilon, \kappa)$ and $e_2(n, \varepsilon, \kappa)$, but we will show that they are negligible (see Lemmas 6.6 and 6.7). The terms $e_1(n, \varepsilon, \kappa)$ and $e_2(n, \varepsilon, \kappa)$ appear in (6.4) just for technical reasons.

Proof of Lemma 6.2. Fix $\kappa: \mathbb{N} \rightarrow (0, 1]$. By the monotone convergence theorem we have

$$\int_0^\infty \int_{\mathbb{R}} |t \partial_t P_t f(x)| |TP_t g(x)| dw(x) dt \leq \lim_{\varepsilon \rightarrow 0^+} \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \Phi(x/n) \int_0^\infty v_\varepsilon(t) |\partial_t P_t f(x)| |TP_t g(x)| dt dw(x).$$

Recall that Φ and v_ε are defined in Definition 5.1. Fix $n \in \mathbb{N}$ and $\varepsilon > 0$. It follows by the definition of the Poisson semigroup (see Definition 3.1) and Lemma 3.3 (D) that if g is odd, then $P_t g$ is also odd for all $t > 0$. Consequently, by (6.2),

$$TP_t g(x) = \partial_x P_t g(x) + 2k \frac{P_t g(x)}{x}.$$

For $(y_1, y_2) \in \mathbb{R} \times \mathbb{R}$ we set

$$(6.7) \quad \tau_2(y_1, y_2) = (|y_2|^2 + \kappa(n)^2)^{(2-q)/2}$$

(cf. (5.6)). By the inequality between the geometric and arithmetic mean we get

$$\begin{aligned} & \int_{\mathbb{R}} \Phi(x/n) \int_0^\infty v_\varepsilon(t) |\partial_t P_t f(x)| \left| \partial_x P_t g(x) + 2k \frac{P_t g(x)}{x} \right| dt dw(x) \\ & \leq \int_{\mathbb{R}} \Phi(x/n) \int_0^\infty v_\varepsilon(t) |\partial_t P_t f(x)|^2 \tau_2(u(x, t)) dt dw(x) \\ & \quad + \int_{\mathbb{R}} \Phi(x/n) \int_0^\infty v_\varepsilon(t) \left| \partial_x P_t g(x) + 2k \frac{P_t g(x)}{x} \right|^2 \tau_2^{-1}(u(x, t)) dt dw(x) \\ & =: I_1 + I_2. \end{aligned}$$

We will estimate I_1 and I_2 separately.

Estimate of I_1 . Recall that

$$\int_{\mathbb{R} \times \mathbb{R}} \phi_{\kappa(n)}(y_1, y_2) dy_1 dy_2 = 1 \quad \text{and} \quad \text{supp } \phi_{\kappa(n)} \subseteq B(0, \kappa(n))$$

($\phi_{\kappa(n)}$ is defined in (4.3)). Moreover, $q \in (1, 2]$, so $(2 - q)/2 \geq 0$. Hence,

$$\begin{aligned} \tau_2(u(x, t)) &= \int_{\mathbb{R} \times \mathbb{R}} \phi_{\kappa(n)}(y_1, y_2) \tau_2(u(x, t)) dy_1 dy_2 \\ &= \int_{\mathbb{R} \times \mathbb{R}} \phi_{\kappa(n)}(y_1, y_2) (|P_t g(x)|^2 + \kappa(n)^2)^{(2-q)/2} dy_1 dy_2 \\ &\leq \int_{\mathbb{R} \times \mathbb{R}} \phi_{\kappa(n)}(y_1, y_2) (2|P_t g(x) - y_2|^2 + 2|y_2|^2 + \kappa(n)^2)^{(2-q)/2} dy_1 dy_2 \\ &\leq \int_{\mathbb{R} \times \mathbb{R}} \phi_{\kappa(n)}(y_1, y_2) (2|P_t g(x) - y_2|^2 + 3\kappa(n)^2)^{(2-q)/2} dy_1 dy_2 \\ &\leq \int_{\mathbb{R} \times \mathbb{R}} \phi_{\kappa(n)}(y_1, y_2) (4|P_t g(x) - y_2|^{(2-q)} + 6\kappa(n)^{(2-q)}) dy_1 dy_2 \\ &= 4\phi_{\kappa(n)} \star \tau_1(u(x, t)) + 6\kappa(n)^{2-q}, \end{aligned}$$

where τ_1 and $u(x, t)$ are defined in (5.6) and (4.10) respectively. Therefore,

$$\begin{aligned} (6.8) \quad I_1 &\leq 4 \int_{\mathbb{R}} \Phi(x/n) \int_0^\infty v_\varepsilon(t) (\tau_1 \star \phi_{\kappa(n)})(u(x, t)) |\partial_t P_t f(x)|^2 dt dw(x) \\ &\quad + 6\kappa(n)^{2-q} \int_{\mathbb{R}} \Phi(x/n) \int_0^\infty v_\varepsilon(t) |\partial_t P_t f(x)|^2 dt dw(x) \\ &= 4 \int_{\mathbb{R}} \Phi(x/n) \int_0^\infty v_\varepsilon(t) (\tau_1 \star \phi_{\kappa(n)})(u(x, t)) |\partial_t P_t f(x)|^2 dt dw(x) \\ &\quad + e_1(n, \varepsilon, \kappa). \end{aligned}$$

Estimate of I_2 . We split I_2 into three parts:

$$\begin{aligned}
 & \int_{\mathbb{R}} \Phi(x/n) \int_0^\infty v_\varepsilon(t) \left| \partial_x P_t g(x) + 2k \frac{P_t g(x)}{x} \right|^2 \tau_2^{-1}(u(x, t)) dt dw(x) \\
 &= \int_{\mathbb{R}} \Phi(x/n) \int_0^\infty v_\varepsilon(t) |\partial_x P_t g(x)|^2 \tau_2^{-1}(u(x, t)) dt dw(x) \\
 (6.9) \quad &+ \int_{\mathbb{R}} \Phi(x/n) \int_0^\infty v_\varepsilon(t) 4k^2 \frac{|P_t g(x)|^2}{x^2} \tau_2^{-1}(u(x, t)) dt dw(x) \\
 &+ \int_{\mathbb{R}} \Phi(x/n) \int_0^\infty v_\varepsilon(t) 4k (\partial_x P_t g)(x) \frac{P_t g(x)}{x} \tau_2^{-1}(u(x, t)) dt dw(x) \\
 &=: J_1 + J_2 + J_3.
 \end{aligned}$$

We will estimate J_1 and $J_2 + J_3$ separately.

Estimate of J_1 . Recall that

$$\int_{\mathbb{R} \times \mathbb{R}} \phi_{\kappa(n)}(y_1, y_2) dy_1 dy_2 = 1 \quad \text{and} \quad \text{supp } \phi_{\kappa(n)} \subseteq B(0, \kappa(n)).$$

Therefore, by the definitions of τ_2 and τ_1 (see (6.7) and (5.6) respectively), the fact that $q \in (1, 2]$, and the triangle inequality we get

$$\begin{aligned}
 & \int_{\mathbb{R}} \Phi(x/n) \int_0^\infty v_\varepsilon(t) |\partial_x P_t g(x)|^2 \tau_2^{-1}(u(x, t)) dt dw(x) \\
 &= \int_{\mathbb{R}} \Phi(x/n) \int_0^\infty v_\varepsilon(t) |\partial_x P_t g(x)|^2 \\
 & \quad \times \left(\int_{\mathbb{R} \times \mathbb{R}} \phi_{\kappa(n)}(y_1, y_2) \tau_2^{-1}(u(x, t)) dy_1 dy_2 \right) dt dw(x) \\
 (6.10) \quad &\leq 2 \int_{\mathbb{R}} \Phi(x/n) \int_0^\infty v_\varepsilon(t) |\partial_x P_t g(x)|^2 \\
 & \quad \times \left(\int_{\mathbb{R} \times \mathbb{R}} \phi_{\kappa(n)}(y_1, y_2) \tau_1^{-1}(u(x, t) - (y_1, y_2)) dy_1 dy_2 \right) dt dw(x) \\
 &= 2 \int_{\mathbb{R}} \Phi(x/n) \int_0^\infty v_\varepsilon(t) (\tau_1^{-1} \star \phi_{\kappa(n)})(u(x, t)) |\partial_x P_t g(x)|^2 dw(x) dt.
 \end{aligned}$$

Estimate of $J_2 + J_3$. By the definition of τ_2 (see (6.7)), we get

$$4k (\partial_x P_t g)(x) \frac{P_t g(x)}{x} \tau_2^{-1}(u(x, t)) = x^{-1} \frac{4k}{q} \partial_x ((|P_t g(x)|^2 + \kappa(n)^2)^{q/2}).$$

Therefore, by (6.1) and the integration by parts (with respect to the Lebesgue measure), for any $t > 0$ we get

$$\begin{aligned}
 J_3 &= \int_{\mathbb{R}} \Phi(x/n) 4k(\partial_x P_t g)(x) \frac{P_t g(x)}{x} \tau_2^{-1}(u(x, t)) \, d\omega(x) \\
 &= \int_{\mathbb{R}} \frac{4k}{q} \partial_x ((|P_t g(x)|^2 + \kappa(n)^2)^{q/2}) (x^{-1} 2|x|^{2k} \Phi(x/n)) \, dx \\
 (6.11) \quad &= - \int_{\mathbb{R}} (|P_t g(x)|^2 + \kappa(n)^2)^{q/2} \frac{4k(2k-1)}{q} \frac{2|x|^{2k}}{x^2} \Phi(x/n) \, dx \\
 &\quad - \int_{\mathbb{R}} (|P_t g(x)|^2 + \kappa(n)^2)^{q/2} \frac{4k}{q} \frac{2|x|^{2k}}{x} (n^{-1}(\partial_x \Phi)(x/n)) \, dx \\
 &= J_{3,1} + e_2(n, \varepsilon, \kappa).
 \end{aligned}$$

Recall that $q \in (1, 2]$, so

$$0 \leq 4k^2 - \frac{4k(2k-1)}{q} \leq \frac{4k}{q}.$$

Hence, by the definitions of J_2 and $J_{3,1}$ and the assumption $k > 1$, we get

$$\begin{aligned}
 &J_2 + J_{3,1} \\
 &= \int_{\mathbb{R}} \Phi(x/n) \int_0^\infty v_\varepsilon(t) \\
 (6.12) \quad &\times \left(4k^2 |P_t g(x)|^2 - \frac{4k(2k-1)}{q} (|P_t g(x)|^2 + \kappa(n)^2) \right) \frac{\tau_2^{-1}(u(x, t))}{x^2} \, dt \, d\omega(x) \\
 &\leq \int_{\mathbb{R}} \Phi(x/n) \int_0^\infty v_\varepsilon(t) \frac{4k}{q} |P_t g(x)|^2 \frac{\tau_2^{-1}(u(x, t))}{x^2} \, dt \, d\omega(x).
 \end{aligned}$$

Note that, by the fact that $q \in (1, 2]$ and the triangle inequality, for all $(y_1, y_2) \in B(0, \kappa(n))$ we have

$$(6.13) \quad \tau_2^{-1}(u(x, t)) \leq 2 \max(|P_t g(x) - y_2|, |-P_t g(x) - y_2|)^{q-2}.$$

Recall that $P_t g$ is odd for all $t > 0$, so

$$\frac{2P_t g(x)}{x} = \frac{P_t g(x) - P_t g(-x)}{x}.$$

Therefore, by (6.13) and (4.9) with

$$\mathbf{a} = P_t g(x) - y_2 \quad \text{and} \quad \mathbf{b} = -P_t g(x) - y_2 = P_t g(-x) - y_2,$$

we get

$$\begin{aligned}
 & \int_{\mathbb{R}} \Phi(x/n) \int_0^\infty v_\varepsilon(t) \frac{4k}{q} |P_t g(x)|^2 \frac{\tau_2^{-1}(u(x, t))}{x^2} dt dw(x) \\
 &= \int_{\mathbb{R}} \Phi(x/n) \int_0^\infty v_\varepsilon(t) \frac{4k}{q} \\
 &\quad \times \left(\int_{\mathbb{R} \times \mathbb{R}} \phi_{\kappa(n)}(y_1, y_2) \tau_2^{-1}(u(x, t)) dy_1 dy_2 \right) \frac{|P_t g(x)|^2}{x^2} dt dw(x) \\
 &\leq 2 \int_{\mathbb{R}} \Phi(x/n) \int_0^\infty v_\varepsilon(t) \frac{4k}{q} \\
 &\quad \times \left(\int_{\mathbb{R} \times \mathbb{R}} \phi_{\kappa(n)}(y_1, y_2) \max(|P_t g(x) - y_2|, |P_t g(-x) - y_2|)^{q-2} dy_1 dy_2 \right) \\
 &\quad \times \frac{|P_t g(x)|^2}{x^2} dt dw(x) \\
 (6.14) \quad &\leq 4 \int_{\mathbb{R}} \Phi(x/n) \int_0^\infty v_\varepsilon(t) \frac{4k}{q} \\
 &\quad \times \left(\int_{\mathbb{R} \times \mathbb{R}} \left(\int_0^1 \phi_{\kappa(n)}(y_1, y_2) \tau_1^{-1}(su(x, t) + (1-s)u(-x, t) - y_2) ds \right) dy_1 dy_2 \right) \\
 &\quad \times \frac{|P_t g(x)|^2}{x^2} dt dw(x) \\
 &= 4 \int_{\mathbb{R}} \Phi(x/n) \int_0^\infty v_\varepsilon(t) \frac{4k}{q} \\
 &\quad \times \left(\int_0^1 s(\phi_{\kappa(n)} \star \tau_1^{-1})(su(x, t) + (1-s)u(-x, t)) ds \right) \\
 &\quad \times \frac{|P_t g(x)|^2}{x^2} dt dw(x).
 \end{aligned}$$

Finally, by (6.9), (6.10), (6.11), and (6.14),

$$\begin{aligned}
 I_2 &\leq 2 \int_{\mathbb{R}} \Phi(x/n) \int_0^\infty v_\varepsilon(t) (\phi_{\kappa(n)} \star \tau_1^{-1})(u(x, t)) |\partial_x P_t g(x)|^2 dw(x) dt \\
 &\quad + 4k \int_{\mathbb{R}} \Phi(x/n) \int_0^\infty v_\varepsilon(t) \\
 (6.15) \quad &\quad \times \left(\int_0^1 s(\phi_{\kappa(n)} \star \tau_1^{-1})(su(x, t) + (1-s)u(-x, t)) ds \right) \\
 &\quad \times \frac{|P_t g(x) - P_t g(-x)|^2}{x^2} dt dw(x) \\
 &\quad + e_2(n, \varepsilon, \kappa).
 \end{aligned}$$

Now we are ready to apply the same argument as in the proof of Lemma 5.2.

Indeed, by (4.6) with

$$\begin{cases} N_1 = N_2 = 1, \\ \eta = P_t f(x), \\ \zeta = P_t g(x), \\ \omega = \partial_t u(x, t) \text{ or } \omega = \partial_x u(x, t) \end{cases}$$

(see (4.10)), we get

$$\begin{aligned} & 4 \int_{\mathbb{R}} \Phi(x/n) \int_0^\infty v_\varepsilon(t) (\tau_1 \star \phi_{\kappa(n)})(u(x, t)) |\partial_t P_t f(x)|^2 dt dw(x) \\ & + 2 \int_{\mathbb{R}} \Phi(x/n) \int_0^\infty v_\varepsilon(t) (\tau_1^{-1} \star \phi_{\kappa(n)})(u(x, t)) |\partial_x P_t g(x)|^2 dt dw(x) \\ (6.16) \quad & \leq \frac{8}{\gamma} \int_{\mathbb{R}} \Phi(x/n) \int_0^\infty v_\varepsilon(t) \langle \text{Hess}(B_{\kappa(n)})(\tilde{u}(x, t)) \partial_t u(x, t), \partial_t u(x, t) \rangle dt dw(x) \\ & + \frac{8}{\gamma} \int_{\mathbb{R}} \Phi(x/n) \\ & \quad \times \int_0^\infty v_\varepsilon(t) \langle \text{Hess}(B_{\kappa(n)})(\tilde{u}(x, t)) \partial_x u(x, t), \partial_x u(x, t) \rangle dt dw(x). \end{aligned}$$

Then, by (4.6) with

$$\begin{cases} N_1 = N_2 = 1, \\ \eta = sP_t f(x) + (1-s)P_t f(-x), \\ \zeta = sP_t g(x) + (1-s)P_t g(-x), \\ \omega = \rho_{-\sqrt{2}}u(x, t) = \left(\frac{P_t f(x) - P_t f(-x)}{x}, \frac{P_t g(x) - P_t g(-x)}{x} \right), \end{cases}$$

where $s \in [0, 1]$, we obtain

$$\begin{aligned} & 4k \int_{\mathbb{R}} \Phi(x/n) \int_0^\infty v_\varepsilon(t) \left(\int_0^1 s(\phi_{\kappa(n)} \star \tau_1^{-1})(su(x, t) + (1-s)u(-x, t)) ds \right) \\ & \quad \times \frac{|P_t g(x) - P_t g(-x)|^2}{x^2} dt dw(x) \\ (6.17) \quad & \leq \frac{8k}{\gamma} \int_{\mathbb{R}} \Phi(x/n) \int_0^\infty v_\varepsilon(t) \\ & \quad \times \left(\int_0^1 s \langle \text{Hess}(B_{\kappa(n)})(s\tilde{u}(x, t) \right. \\ & \quad \left. + (1-s)\tilde{u}(-x, t)) \rho_{-\sqrt{2}}u(x, t), \rho_{-\sqrt{2}}u(x, t) \rangle ds \right) dt dw(x). \end{aligned}$$

Finally, the claim is a consequence of (6.8), (6.15), (6.16) (6.17), and Proposition 4.10. □

Remark 6.4. We would like to emphasize that we get rid of the “ k^2 ” factor in (6.11) and (6.12), which is a crucial difference between the proofs of Lemma 5.2 and Lemma 6.2.

For the sake of completeness, we also formulate an analogue of Lemma 6.2 for even functions. Its proof is identical to the proof of Lemma 5.2.

Lemma 6.5. *Assume that $f, g \in \mathcal{S}(\mathbb{R})$ and g is even. Then for any $\kappa: \mathbb{N} \rightarrow (0, 1]$ we have*

$$(6.18) \quad \int_0^\infty \int_{\mathbb{R}} |t\partial_t P_t f(x)| |TP_t g(x)| \, dw(x) \, dt \leq \frac{2}{\gamma} \liminf_{\varepsilon \rightarrow 0^+} \liminf_{n \rightarrow \infty} I(n, \varepsilon, \kappa).$$

Proof. Note that by the definition of the Poisson semigroup (see Definition 3.1) and Lemma 3.3 (D), if g is even, then $P_t g$ is also even for all $t > 0$. Hence, by (6.2), for all $t > 0$ and $x \in \mathbb{R}$ we get

$$TP_t g(x) = \partial_x P_t g(x).$$

Therefore, the remainder of the proof is the same as in the proof of Lemma 5.2 since $I_2 = 0$ in (5.5). □

Lemma 6.6. *Assume that $f \in \mathcal{S}(\mathbb{R})$, $\kappa: \mathbb{N} \rightarrow (0, 1]$, $\varepsilon > 0$, and $e_1(n, \varepsilon, \kappa)$ is defined in (6.5). If $\lim_{n \rightarrow \infty} \kappa(n) = 0$, then*

$$(6.19) \quad \limsup_{n \rightarrow \infty} e_1(n, \varepsilon, \kappa) = 0.$$

Proof. It follows by the definition of the Poisson semigroup (see Definition 3.1) and (3.6) that there is a constant $C > 0$ independent of f such that for all $t > 0$ and $x \in \mathbb{R}$ we have

$$|(\partial_t P_t f)(x)|^2 \leq Ct^{-2} P_t |f|^2(x).$$

Therefore, by the fact that $0 \leq \Phi \leq 1$ and, by definition, $\{P_t\}_{t \geq 0}$ are contractions on $L^2(dw)$, we get

$$(6.20) \quad \begin{aligned} |e_1(n, \varepsilon, \kappa)| &\leq 6C\kappa(n)^{2-q} \int_0^\infty v_\varepsilon(t) \left(\int_{\mathbb{R}} \Phi(x/n)t^{-2} |P_t f(x)|^2 \, dw(x) \right) dt \\ &\leq 6C\kappa(n)^{2-q} \|f\|_{L^2(dw)}^2 \left(\int_0^\infty t^{-2} v_\varepsilon(t) \, dt \right). \end{aligned}$$

Recall that $f \in \mathcal{S}(\mathbb{R})$, so $\|f\|_{L^2(dw)} < \infty$. Moreover, by the definition of v_ε (see (5.1)), for fixed $\varepsilon > 0$ we have $\int_0^\infty t^{-2} v_\varepsilon(t) \, dt < \infty$. Finally, the claim is a consequence of (6.20) and the assumption $\lim_{n \rightarrow \infty} \kappa(n) = 0$. □

Lemma 6.7. *Let $p \geq 2$ and $q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $g \in \mathcal{S}(\mathbb{R})$ and $\varepsilon > 0$. For $n \in \mathbb{N}$ we set*

$$(6.21) \quad \kappa(n) = \left(\frac{1}{n} \max(1, w(B(0, 2n))) \right)^{1/q}.$$

Then we have

$$(6.22) \quad \limsup_{n \rightarrow \infty} |e_2(n, \varepsilon, \kappa)| = 0,$$

where $e_2(n, \varepsilon, \kappa)$ is defined in (6.6).

Proof. We split $e_2(n, \varepsilon, \kappa)$ as follows:

$$\begin{aligned} |e_2(n, \varepsilon, \kappa)| &\leq \frac{16k}{q} n^{-1} \int_{\mathbb{R}} |(\partial_x \Phi)(x/n)| x^{-1} \left(\int_0^\infty v_\varepsilon(t) |P_t f(x)|^q dt \right) dw(x) \\ &\quad + \frac{16k}{q} n^{-1} \kappa(n)^q \int_{\mathbb{R}} |(\partial_x \Phi)(x/n)| x^{-1} \left(\int_0^\infty v_\varepsilon(t) dt \right) dw(x) \\ &=: e_{2,1}(n, \varepsilon, \kappa) + e_{2,2}(n, \varepsilon, \kappa). \end{aligned}$$

We will estimate $e_{2,1}(n, \varepsilon, \kappa)$ and $e_{2,2}(n, \varepsilon, \kappa)$ separately.

Estimate of $e_{2,1}(n, \varepsilon, \kappa)$. In this case, we use the same argument as in Lemma 6.6. Recall that $\Phi \in C_c^\infty(\mathbb{R})$. Therefore, there is $C > 0$ such that

$$(6.23) \quad |(\partial_x \Phi)(x/n)| \leq C \quad \text{for all } x \in \mathbb{R}.$$

Moreover, $\text{supp } \Phi \subseteq B(0, 2)$ and $\Phi \equiv 1$ on $B(0, 1)$, so

$$(6.24) \quad (\partial_x \Phi)(x/n) = 0 \quad \text{for all } x \in B(0, n).$$

Consequently,

$$(6.25) \quad e_{2,1}(n, \varepsilon, \kappa) \leq \frac{16k}{q} C n^{-1} \int_{B(0, 2n) \setminus B(0, n)} x^{-1} \left(\int_0^\infty v_\varepsilon(t) |P_t g(x)|^q dt \right) dw(x).$$

Next, since $|x^{-1}| \leq n^{-1}$ for all $x \notin B(0, n)$ and, by Lemma 3.5, $\{P_t\}_{t \geq 0}$ are uniformly bounded on $L^q(dw)$, we get

$$\begin{aligned} \frac{16k}{q} C n^{-1} \int_{B(0, 2n) \setminus B(0, n)} x^{-1} \left(\int_0^\infty v_\varepsilon(t) |P_t g(x)|^q dt \right) dw(x) \\ \leq C' n^{-2} \|f\|_{L^q(dw)}^q \int_0^\infty v_\varepsilon(t) dt. \end{aligned}$$

Now, $\limsup_{n \rightarrow \infty} e_{2,1}(n, \varepsilon, \kappa) = 0$ follows by the fact that $\|f\|_{L^q(dw)} < \infty$, for fixed $\varepsilon > 0$ we have $\int_0^\infty v_\varepsilon(t) dt < \infty$, and $\lim_{n \rightarrow \infty} C' n^{-2} = 0$.

Estimate of $e_{2,2}(n, \varepsilon, \kappa)$. In this case, we utilize (6.21). By (6.23) and (6.24) we get

$$e_{2,2}(n, \varepsilon, \kappa) \leq \frac{16k}{q} Cn^{-1}\kappa(n)^q \int_{B(0,2n)\setminus B(0,n)} x^{-1} \left(\int_0^\infty v_\varepsilon(t) dt \right) dw(x).$$

Then, by the fact that $|x^{-1}| \leq n^{-1}$ for $x \notin B(0, n)$ and by (6.21),

$$\begin{aligned} & \frac{16k}{q} Cn^{-1}\kappa(n)^q \int_{B(0,2n)\setminus B(0,n)} x^{-1} \left(\int_0^\infty v_\varepsilon(t) dt \right) dw(x) \\ & \leq \frac{16k}{q} Cn^{-3} \frac{1}{w(B(0, 2n))} \left(\int_{B(0,2n)\setminus B(0,n)} dw(x) \right) \left(\int_0^\infty v_\varepsilon(t) dt \right) \\ & \leq \frac{16k}{q} \frac{Cw(B(0, 2n) \setminus B(0, n))}{n^3w(B(0, 2n))} \left(\int_0^\infty v_\varepsilon(t) dt \right). \end{aligned}$$

Finally, the claim is a consequence of the fact that for fixed $\varepsilon > 0$ we have $\int_0^\infty v_\varepsilon(t) dt < \infty$. □

As a direct consequence of Lemmas 6.2, 6.6, and 6.7, we obtain the following corollary.

Corollary 6.8. *Assume that $p \geq 2, f, g \in \mathcal{S}(\mathbb{R})$, and g is odd. Let $\kappa: \mathbb{N} \rightarrow (0, 1]$ be defined in (5.10). Then we have*

$$(6.26) \quad \int_0^\infty \int_{\mathbb{R}} |t\partial_t P_t f(x)| |TP_t g(x)| dw(x) dt \leq \frac{8}{\gamma} \liminf_{\varepsilon \rightarrow 0^+} \liminf_{n \rightarrow \infty} I(n, \varepsilon, \kappa).$$

Proof of Theorem 1.4. Assume first that $p \geq 2$. Take $f \in L^p(dw)$. Thanks to Theorem 1.2 and the fact that $\mathcal{S}(\mathbb{R})$ is dense in $L^p(dw)$, without loss of generality we can assume $f \in \mathcal{S}(\mathbb{R})$. Let $\kappa: \mathbb{N} \rightarrow (0, 1]$ be defined by (5.10) and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. By Proposition 6.1 we have

$$\begin{aligned} \| \mathcal{H}f \|_{L^p(dw)} &= \sup_{g \in \mathcal{S}(\mathbb{R}), \|g\|_{L^q(dw)}=1} \left| \int_{\mathbb{R}} \mathcal{H}f(x)g(x) dw(x) \right| \\ &= 4 \sup_{g \in \mathcal{S}(\mathbb{R}), \|g\|_{L^q(dw)}=1} \left| \int_{\mathbb{R}} \int_0^\infty t\partial_t P_t f(x) TP_t g(x) dt dw(x) \right|. \end{aligned}$$

Then we split g into even and odd parts g_1 and g_2 respectively, so

$$(6.27) \quad \begin{aligned} \| \mathcal{H}f \|_{L^p(dw)} &\leq 4 \sup_{g \in \mathcal{S}(\mathbb{R}), \|g\|_{L^q(dw)}=1} \int_0^\infty \int_{\mathbb{R}} |t\partial_t P_t f(x) TP_t g_1(x)| dw(x) dt \\ &\quad + 4 \sup_{g \in \mathcal{S}(\mathbb{R}), \|g\|_{L^q(dw)}=1} \int_0^\infty \int_{\mathbb{R}} |t\partial_t P_t f(x) TP_t g_2(x)| dw(x) dt. \end{aligned}$$

For $j \in \{1, 2\}$, $n \in \mathbb{N}$, and $\varepsilon > 0$ let us denote

$$I_j(n, \varepsilon, \kappa) := \int_{\mathbb{R}} \Phi(x/n) \int_0^\infty v_\varepsilon(t) (\partial_t^2 + \Delta_k)(b_{\kappa(n)}^{(j)})(\mathbf{x}, t) dt dw(x),$$

where

$$b_{\kappa(n)}^{(j)}(x, t) = B_{\kappa(n)}(P_t f(x), P_t g_j(x)).$$

By Lemma 6.5 and Corollary 5.7 we have

$$\begin{aligned} (6.28) \quad 4 \sup_{g \in \mathcal{S}(\mathbb{R}), \|g\|_{L^q(dw)}=1} & \left| \int_{\mathbb{R}} \int_0^\infty t \partial_t P_t f(x) TP_t g_1(x) dt dw(x) \right| \\ & \leq \frac{8}{\gamma} \liminf_{\varepsilon \rightarrow 0^+} \liminf_{n \rightarrow \infty} I_1(n, \varepsilon, \kappa) \\ & \leq \frac{24(\gamma + 1)}{\gamma} (\|f\|_{L^p(dw)}^p + \|g_1\|_{L^q(dw)}^q). \end{aligned}$$

Then, by Lemma 6.2, Corollary 6.8, and Corollary 5.7,

$$\begin{aligned} (6.29) \quad 4 \sup_{g \in \mathcal{S}(\mathbb{R}), \|g\|_{L^q(dw)}=1} & \left| \int_{\mathbb{R}} \int_0^\infty t \partial_t P_t f(x) TP_t g_2(x) dt dw(x) \right| \\ & \leq \frac{32}{\gamma} \liminf_{\varepsilon \rightarrow 0^+} \liminf_{n \rightarrow \infty} I_2(n, \varepsilon, \kappa) \\ & \leq \frac{96(\gamma + 1)}{\gamma} (\|f\|_{L^p(dw)}^p + \|g_2\|_{L^q(dw)}^q). \end{aligned}$$

By the triangle inequality and (2.5), for $j \in \{1, 2\}$ we have $\|g_j\|_{L^q(dw)} \leq \|g\|_{L^q(dw)}$. Therefore, by (6.27), (6.28), and (6.29),

$$\|\mathcal{H}f\|_{L^p(dw)} \leq \frac{240(\gamma + 1)}{\gamma} (\|f\|_{L^p(dw)}^p + \|g\|_{L^q(dw)}^q).$$

Finally, applying the same polarization arguments as in the proof of Theorem 1.3, we obtain the claim. In case $1 < p < 2$, one can use the duality argument. \square

Appendix A Proof of (4.6)

We will consider first the function $B(\eta, \zeta)$ defined in (4.2).

Proposition A.1. *The function $(\eta, \zeta) \mapsto B(\eta, \zeta)$ is C^2 on the set $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \setminus \Upsilon$, where*

$$(A.1) \quad \Upsilon = \{(\eta, \zeta) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} : \|\eta\|^p = \|\zeta\|^q \text{ or } \zeta = \mathbf{0}\}.$$

Moreover, for all $(\eta, \zeta) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \setminus \Upsilon$ and $\omega = (\omega_1, \omega_2) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ we have

$$(A.2) \quad \langle \text{Hess } B(\eta, \zeta)\omega, \omega \rangle \geq \frac{\gamma}{2}(\tau(\eta, \zeta)\|\omega_1\|^2 + \tau(\eta, \zeta)^{-1}\|\omega_2\|^2),$$

where

$$(A.3) \quad \tau(\eta, \zeta) = \|\zeta\|^{2-q}.$$

Proof. We repeat the proof of [37, Proposition 6.2]. The regularity properties of B follows directly by the definition of B , so we will prove just (A.2). First, we observe that $\langle \text{Hess } B(\eta, \zeta)\omega, \omega \rangle$ is the sum of three summands as follows:

$$\begin{aligned} &\langle \text{Hess } B(\eta, \zeta)\omega, \omega \rangle \\ &= \sum_{i,j=1}^{N_1} \partial_{\eta_i} \partial_{\eta_j} B(\eta, \zeta)(\omega_1)_i (\omega_1)_j + 2 \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \partial_{\eta_i} \partial_{\zeta_j} B(\eta, \zeta)(\omega_1)_i (\omega_2)_j \\ &\quad + \sum_{i,j=1}^{N_2} \partial_{\zeta_i} \partial_{\zeta_j} B(\eta, \zeta)(\omega_2)_i (\omega_2)_j =: B_1 + B_2 + B_3. \end{aligned}$$

We will estimate $\langle \text{Hess } B(\eta, \zeta)\omega, \omega \rangle$ in two regions:

$$R_1 = \{(\eta, \zeta) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} : \|\eta\|^p < \|\zeta\|^q, \zeta \neq \mathbf{0}\}$$

and

$$R_2 = \{(\eta, \zeta) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} : \|\eta\|^p > \|\zeta\|^q, \zeta \neq \mathbf{0}\}.$$

Estimate in R_1 . In this case we have

$$B(\eta, \zeta) = \frac{1}{2}(\|\eta\|^p + \|\zeta\|^q + \gamma\|\eta\|^2\|\zeta\|^{2-q}).$$

Hence, we calculate

$$\begin{aligned} \partial_{\eta_i} \partial_{\eta_j} B(\eta, \zeta) &= \begin{cases} \frac{p(p-1)}{2} \|\eta\|^{p-4} \eta_i \eta_j & \text{if } i \neq j, \\ \frac{p(p-1)}{2} \|\eta\|^{p-4} \eta_i \eta_j + \frac{p}{2} \|\eta\|^{p-2} + \gamma \|\zeta\|^{2-q} & \text{if } i = j, \end{cases} \\ \partial_{\eta_i} \partial_{\zeta_j} B(\eta, \zeta) &= \gamma(2-q)\|\zeta\|^{-q} \eta_i \zeta_j, \\ \partial_{\zeta_i} \partial_{\zeta_j} B(\eta, \zeta) &= \begin{cases} \frac{q(q-2)}{2} \|\zeta\|^{q-4} \zeta_i \zeta_j + \frac{\gamma}{2} q(q-2)\|\eta\|^2 \|\zeta\|^{-q-2} & \text{if } i \neq j, \\ \frac{q(q-2)}{2} \|\zeta\|^{q-4} \zeta_i \zeta_j + \frac{q}{2} \|\zeta\|^{q-2} & \text{if } i = j. \\ + \frac{\gamma}{2} q(q-2)\|\eta\|^2 \|\zeta\|^{-q-2} + \frac{\gamma(2-q)}{2} \|\eta\|^2 \|\zeta\|^{-q} & \end{cases} \end{aligned}$$

Thus, in R_1 ,

$$B_1 = \frac{p(p-2)}{2} \|\eta\|^{p-4} \langle \eta, \omega_1 \rangle^2 + \frac{p}{2} \|\eta\|^{p-2} \|\omega_1\|^2 + \gamma \|\zeta\|^{2-q} \|\omega_1\|^2 \geq \gamma \|\zeta\|^{2-q} \|\omega_1\|^2.$$

Next, note that the condition $\|\eta\|^p < \|\zeta\|^q$ implies $\|\eta\| \|\zeta\|^{1-q} < 1$. Therefore,

$$\begin{aligned} B_2 &= 2\gamma(2-q) \|\zeta\|^{-q} \langle \eta, \omega_1 \rangle \langle \zeta, \omega_2 \rangle \\ &\geq -2\gamma(2-q) \|\zeta\|^{-q} \|\omega_1\| \|\omega_2\| \|\eta\| \|\zeta\| \\ &\geq -\gamma \left(\frac{\|\zeta\|^{2-q}}{2} \|\omega_1\|^2 + 2\|\zeta\|^{q-2} \|\omega_2\|^2 \right). \end{aligned}$$

In order to estimate B_3 , note that $\|\eta\| \|\zeta\|^{1-q} < 1$ implies $\|\eta\|^2 \|\zeta\|^{-q} < \|\zeta\|^{q-2}$. Recall that $\gamma = \frac{q(q-1)}{8}$. Consequently,

$$\begin{aligned} B_3 &\geq \frac{\gamma}{2} \|\zeta\|^{q-2} ((q-2) \|\zeta\|^2 \langle \zeta, \omega_2 \rangle^2 + \|\omega_2\|^2) \\ &\quad + \frac{\gamma}{2} (2-q) \|\eta\|^2 \|\zeta\|^{-q} (-q \|\zeta\|^{-2} \langle \zeta, \omega_2 \rangle^2 + \|\omega_2\|^2) \\ &\geq \frac{\gamma}{2} (8 \|\zeta\|^{q-2} + (2-q)(1-q) \|\eta\|^2 \|\zeta\|^{-q}) \|\omega_2\|^2 \\ &\geq \frac{\gamma}{2} (8 + (2-q)(1-q)) \|\zeta\|^{q-2} \|\omega_2\|^2. \end{aligned}$$

Combining the estimates for B_1, B_2 , and B_3 we get

$$\begin{aligned} \langle \text{Hess } B(\eta, \zeta) \omega, \omega \rangle &\geq \frac{\gamma}{2} (\|\zeta\|^{2-q} \|\omega_1\|^2 + (q^2 - 3q + 6) \|\zeta\|^{q-2} \|\omega_2\|^2) \\ &\geq \frac{\gamma}{2} (\|\zeta\|^{2-q} \|\omega_1\|^2 + \|\zeta\|^{q-2} \|\omega_2\|^2), \end{aligned}$$

so (A.2) follows with $\tau(\eta, \zeta) = \|\zeta\|^{2-q}$.

Estimate in R_2 . In this case we have

$$B(\eta, \zeta) = \frac{1}{2} \left(\|\eta\|^p + \|\zeta\|^q + \gamma \left(\frac{2}{p} \|\eta\|^p + \left(\frac{2}{q} - 1 \right) \|\zeta\|^q \right) \right),$$

so the second derivatives are

$$\partial_{\eta_i} \partial_{\eta_j} B(\eta, \zeta) = \begin{cases} \frac{1}{2} (p + 2\gamma)(p - 2) \|\eta\|^{p-4} \eta_i \eta_j & \text{if } i \neq j, \\ \frac{1}{2} (p + 2\gamma)(p - 2) \|\eta\|^{p-4} \eta_i \eta_j + \frac{1}{2} (p + 2\gamma) \|\eta\|^{p-2} & \text{if } i = j, \end{cases}$$

$$\partial_{\eta_i} \partial_{\zeta_j} B(\eta, \zeta) = 0,$$

$$\partial_{\zeta_i} \partial_{\zeta_j} B(\eta, \zeta) = \begin{cases} \frac{1}{2} (q + \gamma(2 - q))(q - 2) \|\zeta\|^{q-4} \zeta_i \zeta_j & \text{if } i \neq j, \\ \frac{1}{2} (q + \gamma(2 - q))(q - 2) \|\zeta\|^{q-4} \zeta_i \zeta_j \\ \quad + \frac{1}{2} (q + \gamma(2 - q)) \|\zeta\|^{q-2} & \text{if } i = j. \end{cases}$$

Hence,

$$\begin{aligned} &\langle \text{Hess} B(\eta, \zeta)\omega, \omega \rangle \\ &\geq \frac{1}{2}(p + 2\gamma)(p - 2)\|\eta\|^{p-4}\langle \eta, \omega_1 \rangle^2 + \frac{1}{2}(p + 2\gamma)\|\eta\|^{p-2}\|\omega_1\|^2 \\ &\quad + \frac{1}{2}(q + \gamma(2 - q))(q - 2)\|\zeta\|^{q-4}\langle \zeta, \omega_2 \rangle^2 + \frac{1}{2}(q + \gamma(2 - q))\|\zeta\|^{q-2}\|\omega_2\|^2. \end{aligned}$$

Recall that $\gamma = \frac{q(q-1)}{8}$. Hence, we have $p + 2\gamma \geq 1$ and $q + \gamma(2 - q) \geq 1$. Therefore,

$$\langle \text{Hess} B(\eta, \zeta)\omega, \omega \rangle \geq \frac{1}{2}((p - 1)\|\eta\|^{p-2}\|\omega_1\|^2 + (q - 1)\|\zeta\|^{q-2}\|\omega_2\|^2).$$

Note that $\|\eta\|^p > \|\zeta\|^q$ implies

$$\|\eta\|^{p-2} = \|\eta\|^{p(p-2)/p} > \|\zeta\|^{q(p-2)/p} = \|\zeta\|^{2-q},$$

so

$$\langle \text{Hess} B(\eta, \zeta)\omega, \omega \rangle \geq \frac{1}{2}((p - 1)\|\zeta\|^{2-q}\|\omega_1\|^2 + (q - 1)\|\zeta\|^{q-2}\|\omega_2\|^2).$$

Note that $p - 1 \geq 1$ and $q - 1 \geq \gamma$. Therefore,

$$\langle \text{Hess} B(\eta, \zeta)\omega, \omega \rangle \geq \frac{\gamma}{2}(\|\zeta\|^{2-q}\|\omega_1\|^2 + \|\zeta\|^{q-2}\|\omega_2\|^2)$$

and we can take $\tau(\eta, \zeta) = \|\zeta\|^{2-q}$ in (A.2). □

Proof of (4.6). We repeat the argument from [20, Theorem 4] and [37, Proposition 6.3]. It follows by the formulas for the second derivatives of B which are given above that they are C^2 on $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \setminus \Upsilon$ and they are locally integrable. Moreover, B is C^1 on $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$. That means that the distributional derivatives of B exist and they coincide with the usual ones on $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \setminus \Upsilon$. Hence, we get the identity

$$\langle \text{Hess} B_\kappa(\eta, \zeta)\omega, \omega \rangle = \int_{\mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \setminus \Upsilon} \phi_\kappa(\eta - \eta_1, \zeta - \zeta_1)\langle \text{Hess} B(\eta_1, \zeta_1)\omega, \omega \rangle d\eta_1 d\zeta_1$$

for all $(\eta, \zeta) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ and $\omega \in \mathbb{R}^{N_1+N_2}$. By (A.2) we obtain

$$\begin{aligned} &\int_{\mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \setminus \Upsilon} \phi_\kappa(\eta - \eta_1, \zeta - \zeta_1)\langle \text{Hess} B(\eta_1, \zeta_1)\omega, \omega \rangle d\eta_1 d\zeta_1 \\ &\geq \frac{\gamma}{2} \int_{\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}} \phi_\kappa(\eta - \eta_1, \zeta - \zeta_1)\tau(\eta_1, \zeta_1)\|\omega_1\|^2 d\eta_1 d\zeta_1 \\ &\quad + \frac{\gamma}{2} \int_{\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}} \phi_\kappa(\eta - \eta_1, \zeta - \zeta_1)\tau^{-1}(\eta_1, \zeta_1)\|\omega_2\|^2 d\eta_1 d\zeta_1, \end{aligned}$$

so (4.6) is proved. □

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